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ABSTRACT

The concept of diakoptic analysis is derived rigorously using only elementary circuit-theoretic ideas. It is shown that all the published versions of diakoptic analysis —— including the dual concept of codiakoptic analysis and the generalized concept of multi-stage diakoptic analysis —— are no more than special cases of a much simpler, yet more general, method called generalized hybrid analysis to be derived in this paper. This identification of diakoptic analysis with generalized hybrid analysis allows a direct generalization to the analysis of large-scale nonlinear resistive network by tearing the network into several small subnetworks and then analyzing each subnetwork separately. Since the subnetworks are uncoupled, this method of analysis is particularly suited for parallel computation where several small computers are used concurrently instead of one much larger computer.

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## 1. Introduction

The concept of diakoptic analysis was originally conceived by Kron more than two decades ago [1]. The significance of this concept, however, has not been widely appreciated mainly because Kron's original writings [1 - 3] were vague and extremely difficult to understand.[1] Branin [4], Baty [5], Brameller [6] and Happ [7,8][2] were among the few who appreciated Kron's work and attempted to clarify the concept. Others, such as Roth [9], Amari [10] and Wang [11] had responded to Kron's appeal [2] and attempted to derive diakoptic analysis rigorously using concepts from <u>algebraic topology</u>. In spite of all these efforts, however, the exact domain of applicability of diakoptic analysis remains unknown.[3] Indeed, since almost all published works [1 - 14] on diakoptic analysis had been exclusively addressed to linear networks, it is not even clear whether the concept is applicable to nonlinear networks. Our objectives in this paper are twofold. First, we will derive diakoptic analysis via a very simple <u>circuit-theoretic approach</u> and show rigorously that it is no more than a special case of the generalized hybrid analysis[4] to be derived in Section 4. In fact, we will show in the Appendix that all published variants of diakoptic analysis ——

[1] Even Kron himself implied that many of his results are intuitive and appealed to mathematicians for rigorous justifications [2].

[2] These references are listed in chronological order. However, many of Happ's earlier works actually predate the reference cited here.

[3] A simple derivation of a modified form of diakoptic analysis using only concepts from linear algebra was given recently by Wu in a seminar at Berkeley. An earlier version of Wu's derivation will appear in [12]. Our derivation is based on the familiar concept of equivalent circuits in contrast to Wu's derivation which is based strictly on linear algebraic manipulations.

[4] Any method of analyzing resistive networks which involves solving a system of linear or nonlinear algebraic equations involving both voltage and current variables will be referred to in this paper as <u>hybrid analysis</u>.

including the dual "codiakoptic analysis" and the generalized "multi-stage diakoptic analysis" proposed by Onodera [13,14], as well as the more recent "modified nodal method" by Ho, Ruehli and Brennen [15], and the "linear algebraic elimination method" by Wu [12] —— are also special cases of the generalized hybrid analysis.

Having established the generality and simplicity of the generalized hybrid analysis in Section 4, our second objective is to generalize diakoptic analysis to large-scale nonlinear resistive networks in Section 5. Not only is diakoptic analysis directly applicable to nonlinear resistive networks, but it is also possible to formulate the problem in such a way that each stage of the Newton-Raphson iteration is equivalent to that of solving a linearized resistive network by hybrid analysis. In other words, each nonlinear resistor can be linearized at the branch level, thereby obviating the expensive process of numerically evaluating the Jacobian matrix associated with the Newton-Raphson method.

In order to motivate the concept of an "open loop" and its "associated generator" to be introduced in Section 2, it is perhaps instructive to give first a somewhat intuitive explanation of what diakoptic analysis entails. The clues to finding the "domain of applicability" as well as for uncovering the many "subtle" points that needed justification will surface from this brief introduction.

Let N be a network containing two-terminal elements and let $N_0$ be a subset of branches of N —— henceforth called the torn branches —— whose removal will separate N into "m" separable subnetworks[5] $N_1, N_2, \ldots, N_m$ as

[5] A graph is separable if it is disconnected or hinged [16]. Although the following derivation of diakoptic analysis is illustrated with disconnected subnetworks, our results are valid also for hinged subnetworks.

shown in Fig. 1(a). Observe that those "connection branches" $N_{0-k} \subset N_0$ which share a common node with subnetwork $N_k$ (e.g., in Fig. 1(a), $N_{0-1} = \{b_1(N_1), b_2(N_1), \ldots, b_{n_1}(N_1)\}$[6]) must necessarily form a cutset. Suppose for the moment that the current solution waveform of each connection branch is given a priori[7], then we can apply the current source substitution theorem [17] and replace each connection branch by a current source whose terminal current is prescribed to be identical to its solution waveform, as shown in Fig. 1(b), without affecting the original solution of the remaining branches. Since the connection branches in $N_{0-k}$ are all current sources, the network in Fig. 1(b) can be transformed into the "hinged" network shown in Fig. 1(c) without affecting the original solutions. Since each group $N_{0-k}$ of substituted current sources form a cutset, one source is clearly redundant and may be replaced by a short circuit.[8] The "separated" network shown in Fig. 1(d) remains equivalent to the original network N in the sense that the solution waveforms of corresponding branches are identical. Hence the solution of N can be found by solving the component subnetworks separately — provided that each current source waveform $I_j(N_k)$ is given *a priori*. Unfortunately, this hypothetical situation seldom occurs in practice and the waveforms characterizing each substituted current source must be treated as a variable for the time being. The basic concept of diakoptic analysis is to transform each subnetwork $N_k$ into an equivalent

[6] $b_i(N_1)$ denotes the branch $b_i$ in subnetwork $N_1$. Similar notations will be used in the sequel.

[7] We assume throughout this paper that the network under consideration has a unique solution for all time t.

[8] Observe that each separated node $n_{\alpha}$ and its associated internal node $n'_{\alpha}$ coalesced into a single node in Fig. 1(d).

acyclic subnetwork[9] $N_{k_{eq}}$ containing the original set of nodes while preserving the identity of the substituted current sources. This may be achieved (in section 3) by removing all links with respect to some tree inside $N_k$ while replacing the tree branches with "coupled branches" (see Fig. 1(e)).[10] Since, except for the deleted branches, the topology remains unchanged, we will prove that the earlier transformation steps in going from Fig. 1(a) to Fig. 1(d) can be reversed. Eventually, an equivalent interconnected network $N_{eq}$ (Fig. 1(f)) can be obtained upon replacing the substituted current sources by their original branches. Since each equivalent subnetwork $N_{k_{eq}}$ contains no loops, the equivalent interconnected network $N_{eq}$ is a greatly simplified network which can be analyzed efficiently by conventional loop analysis.[11] A rigorous derivation of diakoptic analysis based on the above equivalent network transformation technique will be presented in Section 3.

## 2. Concept of an Open Loop and its Associated Generator

Let $\mathcal{G}$ be a directed graph with "n" nodes and "b" branches, and let $\mathcal{T}$ be a tree of $\mathcal{G}$ and $\mathcal{L}$ its associated cotree (i.e., complement of the tree). Elements of $\mathcal{T}$ and $\mathcal{L}$ will henceforth be called twigs and links, respectively. For reasons that will be obvious soon, we will augment $\mathcal{G}$ with (n-1) external branches which also form a tree $\mathcal{T}_g$, henceforth called a generator tree, of $\mathcal{G}$. For example, consider the graph $\mathcal{G}$ shown in Fig. 2(a) with n = 4 and b = 6. Let $\mathcal{T}$ = {4,5,6} and $\mathcal{L}$ = {1,2,3} be a

9 A graph is acyclic if it contains no loops.

10 This will be referred to in Section 3 as the fundamental-tree case.

11 Terminologies not defined in this paper can be found in [16] or [18].

prescribed tree and cotree. We can augment $\mathcal{G}$ with several distinct generator trees having n-1 = 3 external branches. For example, we can pick the augmented tree $\mathcal{T}_g$ = {4',5',6'} (dotted branches) shown in Fig. 2(b). Observe that this particular augmented tree forms a node-to-datum "star-tree" structure and will henceforth be referred to as the <u>node-to-datum generator tree</u>. Another tree that we will often encounter in the sequel is obtained by adding an oppositely directed branch in parallel with each twig of the given tree $\mathcal{T}$ as shown in Fig. 2(c). This particular choice of $\mathcal{T}_g$ will henceforth be referred to as the <u>fundamental generator tree</u>. For complete generality, however, we will allow the augmented tree $\mathcal{T}_g'$ to be arbitrarily chosen, such as the dotted branches shown in Fig. 2(d).

Using the above notation, we can now define an <u>open loop</u> of $\mathcal{G}$ with respect to any generator tree $\mathcal{T}_g$ to be any set of branches of $\mathcal{G}$ which form a closed loop with a branch, henceforth called the <u>associated open loop generator</u>, of $\mathcal{T}_g'$.[12] For example, the sets of branches {1,3}, {3,4,5}, {5,6}, {2} all form closed loop with augmented branch 4' of the node-to-datum generator tree in Fig. 2(b). They are, therefore, open loops induced by the same generator 4'. Out of the many possible open loops induced by the same generator, we will often choose the unique open loop consisting exclusively of twigs (of the prescribed tree $\mathcal{T}'$) and call it a <u>fundamental open loop</u>. Hence, {5,6}, {4,6} and {6} are the three fundamental open loops induced by the open loop generators 4', 5' and 6', respectively, in Fig. 2(b).

[12] An open loop is referred to elsewhere [1 - 9] as an <u>open path</u> with no mention whatsoever of its associated generator. We introduce the idea of an open loop and its associated generator in order that the concept of <u>codiakoptics</u> may be derived in a dual manner as shown in Appendix C.

Similarly, {4}, {5} and {6} are the fundamental open loops induced by the open loop generators 4', 5' and 6', respectively, in Fig. 2(c). From now on, we will assign a positive orientation to each open loop to be that induced by its generator. The orientation and branches of an open loop can be precisely represented by a 1×b row vector $\underset{\sim}{p}$. For example, several open loops induced by the node-to-datum generator tree branches in Fig. 2(b) are represented by:

$$
\begin{array}{l c cccccc c c}
 & & \multicolumn{6}{c}{\text{branches in original graph } \mathcal{G}} & & \\
 & & 1 & 2 & 3 & 4 & 5 & 6 & & \\
\text{fundamental open loop induced by } 4' & = ( & 0 & 0 & 0 & 0 & 1 & -1 & ) & \triangleq \underset{\sim}{p}_{4'} \\
\text{fundamental open loop induced by } 5' & = ( & 0 & 0 & 0 & 1 & 0 & -1 & ) & \triangleq \underset{\sim}{p}_{5'} \\
\text{fundamental open loop induced by } 6' & = ( & 0 & 0 & 0 & 0 & 0 & -1 & ) & \triangleq \underset{\sim}{p}_{6'} \\
\text{open loop induced by } 4' & = ( & 0 & 0 & -1 & -1 & 1 & 0 & ) & \triangleq \hat{\underset{\sim}{p}}_{4'} \\
\text{open loop induced by } 5' & = ( & 1 & 1 & 0 & 0 & 0 & 0 & ) & \triangleq \hat{\underset{\sim}{p}}_{5'} \\
\text{open loop induced by } 6' & = ( & 1 & 1 & 0 & -1 & 0 & 0 & ) & \triangleq \hat{\underset{\sim}{p}}_{6'}
\end{array}
$$

It is important to observe that the <u>elements of the 1×b open loop row vector belong exclusively to the original graph $\mathcal{G}$</u>. Observe also that the set of all open loops of $\mathcal{G}$ does not form a vector space. In fact, the difference between two open loops does not necessarily give rise to another open loop, e.g.,

$$\underset{\sim}{p}_{4'} - \hat{\underset{\sim}{p}}_{4'} = (\,0 \quad 0 \quad 1 \quad 1 \quad 0 \quad -1\,) \triangleq \underset{\sim}{z}_4$$

where $\underset{\sim}{z}_4$ is actually a closed loop! This observation turns out to be a basic property of open loops as shown by the following lemma.

<u>Lemma 1</u> <u>The difference between any two distinct open loop vectors induced by the same generator is always a closed loop</u>.

Proof: Let $\underset{\sim}{p}_a$ and $\underset{\sim}{p}_b$ be any two distinct open loops of $\mathcal{G}$ induced by the same augmented generator g. Define an augmented graph $\bar{\mathcal{G}} \triangleq \mathcal{G} \cup \{g\}$ and three $1\times(b+1)$ augmented vectors as follows:

| | | | branches in $\mathcal{G}$ | | augmented branch g | |
|---|---|---|---|---|---|---|
| $\bar{\underset{\sim}{p}}_a$ | $\triangleq$ | ( | $\underset{\sim}{p}_a$ | , | 0 | ) |
| $\bar{\underset{\sim}{p}}_b$ | $\triangleq$ | ( | $\underset{\sim}{p}_b$ | , | 0 | ) |
| $\bar{\underset{\sim}{g}}$ | $\triangleq$ | ( | $\underset{\sim}{0}$ | , | 1 | ) |

By the definition of an open loop, $\bar{\underset{\sim}{z}}_a \triangleq \bar{\underset{\sim}{p}}_a + \bar{\underset{\sim}{g}}$ and $\bar{\underset{\sim}{z}}_b \triangleq \bar{\underset{\sim}{p}}_b + \bar{\underset{\sim}{g}}$ are both closed loops. The observation that the difference between two distinct closed loops is another closed loop yields

$$\bar{\underset{\sim}{z}}_a - \bar{\underset{\sim}{z}}_b = (\bar{\underset{\sim}{p}}_a + \bar{\underset{\sim}{g}}) - (\bar{\underset{\sim}{p}}_b + \bar{\underset{\sim}{g}}) = \bar{\underset{\sim}{p}}_a - \bar{\underset{\sim}{p}}_b \triangleq \bar{\underset{\sim}{z}}$$

where $\bar{\underset{\sim}{z}}$ is a closed loop in $\bar{\mathcal{G}}$. Let $\underset{\sim}{z}$ be the $1\times b$ row vector containing the first b components of $\bar{\underset{\sim}{z}}$, then since $\bar{\underset{\sim}{z}}$ does not contain the augmented branch g, $\underset{\sim}{z}$ is a closed loop of the original graph $\mathcal{G}$. Hence, we have

$$\underset{\sim}{p}_a - \underset{\sim}{p}_b = \underset{\sim}{z}$$

where $\underset{\sim}{z}$ is a closed loop of $\mathcal{G}$. ■

It follows from Lemma 1 that an arbitrary collection of open loop vectors need not be linearly independent. The following important lemma shows us how to construct a maximal set of linearly independent open loop vectors.

Lemma 2. Let $\mathcal{G}$ be a connected graph with "n" nodes and "b" branches and let $\mathcal{T}'_g$ be any augmented generator tree. Then any collection of (n-1) open loops induced by the (n-1) distinct open loop generators in $\mathcal{T}'_g$ are linearly independent.

Proof: See Appendix A.

In view of Lemma 2, it makes sense to define a full-ranked $(n-1)\times b$ matrix $\hat{T}$, henceforth called the *open loop matrix*, where each element $\hat{T}_{ij}$ is defined as follows:

$$\hat{T}_{ij} = \begin{cases} 1, & \text{if open loop } i \text{ contains branch } j \text{ and has the same orientation as branch } j; \\ -1, & \text{if open loop } i \text{ contains branch } j \text{ but has opposite orientation as branch } j; \\ 0, & \text{if open loop } i \text{ does not contain branch } j. \end{cases}$$

Observe that the number of columns in $\hat{T}$ is equal to the number of branches of $\mathcal{G}$. In the special case where all open loops are chosen to be *fundamental* open loops, the associated matrix will be called the *fundamental open loop matrix* and denoted by $T$, without the "hat". An important property of the fundamental open loop matrix $T$ is given by:

*Lemma 3*. *If the columns of the fundamental open loop matrix $T$ are ordered such that the links appear before the twigs, i.e., $T = [T_{\mathcal{L}}\ T_{\mathcal{T}}]$, where $T_{\mathcal{L}}$ denotes the first $(b-n+1)$ columns representing links in $\mathcal{L}$ and $T_{\mathcal{T}}$ denotes the remaining $(n-1)$ columns representing twigs in $\mathcal{T}$, then $T_{\mathcal{L}} = 0_{\mathcal{L}}$ and $T_{\mathcal{T}}$ is an $(n-1)\times(n-1)$ nonsingular matrix.*

*Proof*: By definition, the branches in a fundamental open loop consists exclusively of twigs in $\mathcal{T}$, hence, $T_{\mathcal{L}} = 0_{\mathcal{L}}$. Moreover, it follows from Lemma 2 that the $(n-1)$ fundamental open loops are linearly independent, hence, $T_{\mathcal{T}}$ is nonsingular. ■

The significance of the open loop matrix $\hat{T}$ is summarized by the following main result of this section:

*Theorem 1*. *Let $\mathcal{G}$ be a connected graph with "n" nodes and "b" branches* and let $\mathcal{T}_g$ denote an augmented generator tree. Then there exists a unique coboundary matrix[13] $\hat{Q}$, henceforth called the associated coboundary matrix, such that

$$\boxed{\hat{Q}\,\hat{T}^t = 1_{\mathcal{T}}} \tag{1}$$

where $\hat{T}$ is any open loop matrix induced by the (n-1) distinct open loop generators in $\mathcal{T}_g$.

Proof: Let $\mathcal{T}$ be any tree of the given graph $\mathcal{G}$. From Lemma 1, the difference between an arbitrary open loop $\hat{p}_j$ and a fundamental open loop $p_j$ is a loop $z_j$, provided that both $\hat{p}_j$ and $p_j$ are induced by the same generator. Let $\hat{T}$ be an arbitrary open loop matrix induced by $\mathcal{T}_g$ and let $T$ be the fundamental open loop matrix induced by $\mathcal{T}_g$. Since each row vector of $\hat{T} - T$ defines a loop, we can always decompose an arbitrary open loop matrix $\hat{T}$ as the sum of a fundamental open loop matrix $T$ and some loop matrix[14] $\overline{B}$; namely,

$$\hat{T} = T + \overline{B} \tag{2}$$

It follows from Eq. (2) and Tellegen's theorem [16] that, if $\hat{Q}$ is any coboundary matrix, then

$$\hat{Q}\,\hat{T}^t = \hat{Q}\,T^t + \hat{Q}\,\overline{B}^t = \hat{Q}\,T^t \tag{3}$$

[13] A coboundary (resp., cycle) matrix $\hat{Q}$ (resp., $\hat{B}$) is a generalized cutset (resp., loop) matrix whose rows represent linear combinations of fundamental cutsets (resp., loops) [18]. Hence, any coboundary (resp., cycle) matrix $\hat{Q}$ (resp., $\hat{B}$) may be decomposed into $\hat{Q} = P_Q Q$ (resp., $\hat{B} = P_B B$) where $Q$(resp., $B$) is a fundamental cutset (resp., loop) matrix and $P_Q$ (resp., $P_B$) is some nonsingular transformation matrix.

[14] Note that the row vectors of the (n-1)×b matrix $\overline{B}$ is, in general, linearly dependent.

From Lemma 3, $\underset{\sim}{T}$ is given by

$$\underset{\sim}{T} = [\underset{\sim}{0}_{\mathcal{L}} \; \underset{\sim}{T}_{\mathcal{T}}] \tag{4}$$

where $\underset{\sim}{T}_{\mathcal{T}}$ is an $(n-1)\times(n-1)$ nonsingular matrix. If we choose $\hat{\underset{\sim}{Q}} = \underset{\sim}{P}_Q Q$, where $\underset{\sim}{Q} = [\underset{\sim}{Q}_{\mathcal{L}} \; \underset{\sim}{1}_{\mathcal{T}}]$ is the fundamental cutset matrix and

$$\underset{\sim}{P}_Q \triangleq (\underset{\sim}{T}^t_{\mathcal{T}})^{-1} \tag{5}$$

then it follows from Eqs. (3) - (5) that

$$\hat{\underset{\sim}{Q}} \; \hat{\underset{\sim}{T}}^t = \underset{\sim}{P}_Q \; \underset{\sim}{Q} \; \underset{\sim}{T}^t = (\underset{\sim}{T}^t_{\mathcal{T}})^{-1} [\underset{\sim}{Q}_{\mathcal{L}} \; \underset{\sim}{1}_{\mathcal{T}}] \begin{bmatrix} \underset{\sim}{0}^t_{\mathcal{L}} \\ \\ \underset{\sim}{T}^t_{\mathcal{T}} \end{bmatrix} = (\underset{\sim}{T}^t_{\mathcal{T}})^{-1} (\underset{\sim}{T}^t_{\mathcal{T}}) = \underset{\sim}{1}_{\mathcal{T}}$$

Hence, the coboundary matrix

$$\hat{\underset{\sim}{Q}} = \underset{\sim}{P}_Q \; \underset{\sim}{Q} \;\; \triangleq \; (\underset{\sim}{T}^t_{\mathcal{T}})^{-1} \; \underset{\sim}{Q} \tag{6}$$

satisfies the property of Theorem 1. It remains to prove that the coboundary matrix $\hat{\underset{\sim}{Q}}$ is uniquely defined (i.e., it is independent of the choice of the tree $\mathcal{T}$ chosen in Eq. (6)). Suppose we choose another tree $\mathcal{T}'$ and obtain a different coboundary matrix $\hat{\underset{\sim}{Q}}'$ such that

$$\hat{\underset{\sim}{Q}}' \; \hat{\underset{\sim}{T}}^t = \underset{\sim}{1}_{\mathcal{T}} \tag{7}$$

By definition, any row of a coboundary matrix can be expressed as a linear combination of the rows of the fundamental cutset matrix $\underset{\sim}{Q}$ with respect to the tree $\mathcal{T}$. Hence, there exists a <u>nonsingular</u> matrix $\underset{\sim}{P}'_Q$ such that $\hat{\underset{\sim}{Q}}' = \underset{\sim}{P}'_Q \; \underset{\sim}{Q}$. Equations (2), (4) and (7) imply that

$$\hat{\underset{\sim}{Q}}' \hat{\underset{\sim}{T}}^t = \hat{\underset{\sim}{Q}}' \underset{\sim}{T}^t = \underset{\sim}{P}'_Q \; [\underset{\sim}{Q}_{\mathcal{L}} \; \underset{\sim}{1}_{\mathcal{T}}] \begin{bmatrix} \underset{\sim}{0}^t_{\mathcal{L}} \\ \\ \underset{\sim}{T}^t_{\mathcal{T}'} \end{bmatrix} = \underset{\sim}{P}'_Q \; \underset{\sim}{T}^t_{\mathcal{T}'} = \underset{\sim}{1}_{\mathcal{T}'} \tag{8}$$

therefore, we have

$$\underset{\sim}{P}'_Q = (\underset{\sim}{T}^t_{\mathcal{T}})^{-1} = \underset{\sim}{P}_Q \tag{9}$$

Hence $\hat{Q}' = \hat{Q}$ and the associated coboundary matrix is unique and depends only on the augmented generator tree $\mathcal{T}_g$. ■

The following corollary gives us the algorithm[15] for constructing the unique coboundary matrix associated with a prescribed augmented generator tree $\mathcal{T}_g$:

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Corollary. Algorithm for constructing $\hat{Q}$.

Let $\mathcal{G}$ be a connected graph with "n" nodes and "b" branches and let $\mathcal{T}_g$ be an augmented generator tree of $\mathcal{G}$. Then the unique coboundary matrix $\hat{Q}$ associated with $\mathcal{T}_g$ can be constructed as follows:

Step 1. Choose any tree $\mathcal{T}$ of $\mathcal{G}$.

Step 2. Construct the $(n-1)\times b$ fundamental open loop matrix $T = [0_{\mathcal{L}}\ T_{\mathcal{T}}]$ induced by $\mathcal{T}_g$.

Step 3. Construct the $(n-1)\times b$ fundamental cutset matrix $Q$ of $\mathcal{G}$ with respect to $\mathcal{T}$.

Step 4. Obtain $\hat{Q}$ from

$$\boxed{\hat{Q} = (T^t_{\mathcal{T}})^{-1}\, Q} \qquad (6)$$

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3. Derivation of Diakoptic Analysis

We are now ready to return to the separated subnetworks $N_0, N_1, N_2, \ldots, N_m$ in Fig. 1(d) and pick up the pieces. It suffices to consider the first subnetwork $N_1$ since the same derivations apply to the remaining subnetworks $N_2, N_3, \ldots, N_m$.

Recall that the branch connecting node $n_{\alpha}$ and node $n'_{\alpha}$ is a short circuit; hence nodes $n_{\alpha}$ and $n'_{\alpha}$ can be coalesced into a single node as

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[15] In the literature [1,2,3,7,8], the associated coboundary matrix is found by inverting a nonsingular matrix $\Omega \triangleq \begin{bmatrix} \hat{T} \\ \hat{B} \end{bmatrix}$, where $\hat{B}$ is some cycle matrix.

shown in Fig. 3(a). In other words, the external current sources do not generate any new node in subnetwork $N_1$. If we let "n" and "b" denote respectively the number of nodes and branches inside $N_1$ (i.e., not counting the external current sources), then it follows from Fig. 3(a) that the number of external current sources associated with $N_1$ is less than or equal to (n-1).[16] There is no loss of generality, however, to assume that there are exactly (n-1) linearly independent external current sources associated with $N_1$ since we can always introduce additional "zero-valued" current sources, if necessary, without affecting the original solutions of $N_1$ as shown in Figs. 3(b) or 3(c). Observe that since these (n-1) external current sources are linearly independent (i.e., they can be individually specified without violating KCL) and since $N_1$ is a connected network with "n" nodes, these (n-1) external current sources must necessarily form a tree of $N_1$. Our first step in deriving diakoptic analysis is to choose this tree to be the augmented generator tree $\mathcal{T}_g$ introduced in the preceding section. If a zero-valued current source is introduced between each of the remaining nodes and the datum node (Fig. 3(b)), the resulting generator tree assumes a star-tree structure and corresponds, therefore, to the node-to-datum case (Fig. 2(b)). For complete generality, we will allow the set of (n-1) external current sources to assume any configuration (such as that shown in Fig. 3(c)) so long as they form a tree of $N_1$.

Our next step in the derivation of diakoptic analysis is to apply the i-shift theorem [17] and shift each of the (n-1) external current sources around any open loop induced by the current source acting as the associated

[16] We assumed here that each node is associated with at most one external current source. When this is not the case, we can always combine the current sources into the same node as one single current source.

open loop generator. After this is done, the original set of external current sources became open circuits. Each of the <u>shifted current sources</u> is now in parallel with an <u>internal branch</u> of $N_1$, which we assume to be a <u>composite branch</u> as shown in Fig. 4(a). Observe that an external current source $i_{s_j}$ will be shifted in parallel with an internal composite branch k whenever the open loop induced by $i_{s_j}$ contains branch k (Fig. 4(b)). Hence, after all external current sources have been shifted, each composite branch k may have up to (n-1) external current sources connected in parallel (Fig. 4(c)).[17] The next step of our derivation will be to combine all those shifted current sources in parallel with the composite branch k by an equivalent external source $j_k'$ (Fig. 4(d)). We will show that the properties of the open loop matrix presented in the preceding section can be used to implement the above sequence of external source transformations in an algebraic form which automatically "monitors" the direction of each shifted current source. However, before we do this, let us pause to consider a specific example in order that the reader may form a concrete picture of the ensuing development.

Consider the network graph N shown in Fig. 5(a), where each branch is assumed to represent a composite branch (Fig. 4(a)). Let the set {5,6} denote the torn branches $N_0$ whose removal results in the two separated subnetworks $N_1$ = {1,2,3,4} and $N_2$ = {7,8,9,10}. Replacing each branch in $N_0$ by its equivalent current source (Fig. 5(b)), we obtain the equivalent hinged network shown in Fig. 5(c). This network can be separated into the two equivalent subnetworks $N_1$ and $N_2$ shown in Fig. 5(d). Since each pair

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[17] It must be emphasized that although the current sources in Fig. 3(c) are shown all pointed in the same direction, this need not be the case in general.

of current sources form a cutset, the current source $I_6(t) = -I_5(t)$ can be replaced by a short circuit, thereby coalescing nodes $n_{\alpha}$ with $n'_{\alpha}$ in $N_1$ and $n_{\beta}$ with $n'_{\beta}$ in $N_2$ (Figs. 5(e) and (f), resp.). Let us now focus our attention on subnetwork $N_1$ which has n=4 nodes and b=4 branches. Since we have only one external current source $i_{s_1} = I_5$ but n-1 = 3, we will introduce two additional "zero-valued" current sources $i_{s_2} = 0$ and $i_{s_3} = 0$ so that the three sources form an augmented tree $\mathcal{T}_g$ of $N_1$. For this example, let us choose to connect these sources in a node-to-datum star-tree structure as shown in Fig. 5(g) (a similar augmentation on $N_2$ will give the star-tree structure shown in Fig. 5(h)). Our next step is to apply the i-shift theorem and shift each external current source in parallel with the internal branches of $N_1$ in order that it may be combined with the composite branches. We can achieve this goal by shifting each external source $i_{s_j}$ around any open loop induced by $i_{s_j}$. For example, we can shift $i_{s_1}$ around open loop {1,2,3}, $i_{s_2}$ around open loop {1,3,4} and $i_{s_3}$ around open loop {3,4} (Fig. 5(i)). Two or more shifted sources which are in parallel with each other can be combined as one source $j'_1$, $j'_2$, $j'_3$ and $j'_4$ (Fig. 5(j)). Notice that <u>the objective of this transformation is to get rid of the source $i_{s_3}$ which is not across a branch of $N_1$</u> (Fig. 5(g)). The choice of the open loops in Fig. 5(i) was, therefore, unnecessarily complicated and was done only for the sake of generality. We could have accomplished the same objective by choosing a simpler set of open loops. For example, we could shift $i_{s_1}$ around open loop {4}, $i_{s_2}$ around open loop {2} and $i_{s_3}$ around open loop {1,2} (Figs. 5(k) and 5(l)). Clearly, there are in general many different open loops that can be chosen. However, one surprising result that will be proved shortly with the help of Theorem 1 is that <u>one choice</u>

of open loops is as good as another in so far as the subsequent analysis is concerned.

If we combine each equivalent shifted current source $j'_k$ (Fig. 5(1)) with the original composite branch, we would obtain the modified subnetwork $N'_1$ as shown in Fig. 5(m), where each branch in $N'_1$ represents an augmented composite branch (Fig. 4(d)). In other words, after going through all these transformations, we end up with a subnetwork $N'_1$ which is identical to the original subnetwork $N_1$ except for an augmented current source $j'_k$ in parallel with each of the original composite branches.

Returning now to the general case of the modified subnetwork $N'_1$, let us state a lemma which relate the "combined shifted" current source vector $\underset{\sim}{j}' \triangleq [j'_1, j'_2, \ldots, j'_b]^t$ with the "unshifted" current source vector $\underset{\sim}{i}_s \triangleq [i_{s_1}, i_{s_2}, \ldots i_{s_{n-1}}]^t$ :

<u>Lemma 4. Let $\hat{\underset{\sim}{T}}$ be any open loop matrix induced by the (n-1) external current sources acting as the open loop generators and let $\underset{\sim}{j}'$ denote the "combined shifted" current source vector, then</u>

$$\underset{\sim}{j}' = \hat{\underset{\sim}{T}}^t \, \underset{\sim}{i}_s \tag{10}$$

<u>Proof</u>: Follows immediately from i-shift theorem and the definition of an open loop. ■

Our next step in the derivation of diakoptic analysis is to carry out a <u>coboundary analysis</u> (i.e., a generalized cutset analysis) [18] of the modified subnetwork $N'_1$. Using the notation associated with the augmented composite branch shown in Fig. 4(d), the KCL equation becomes

$$\hat{\underset{\sim}{Q}} \, \underset{\sim}{I}' = \hat{\underset{\sim}{Q}} \, \underset{\sim}{I} - \hat{\underset{\sim}{Q}} \, \underset{\sim}{j}' = \underset{\sim}{0} \tag{11}$$

where

$$\underset{\sim}{I}' \triangleq \begin{bmatrix} I_1' \\ I_2' \\ \cdot \\ \cdot \\ \cdot \\ I_b' \end{bmatrix}, \quad \underset{\sim}{I} \triangleq \begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ \cdot \\ I_b \end{bmatrix}, \text{ and } \underset{\sim}{j}' \triangleq \begin{bmatrix} j_1' \\ j_2' \\ \cdot \\ \cdot \\ \cdot \\ j_b' \end{bmatrix} \tag{12}$$

and $\hat{\underset{\sim}{Q}}$ is any coboundary matrix. But

$$\underset{\sim}{I} = \underset{\sim}{i} - \underset{\sim}{j} = \underset{\sim}{G}\underset{\sim}{v} - \underset{\sim}{j} = \underset{\sim}{G}\underset{\sim}{V} + \underset{\sim}{G}\underset{\sim}{e} - \underset{\sim}{j} \tag{13}$$

where

$$\underset{\sim}{i} \triangleq \begin{bmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ \cdot \\ i_b \end{bmatrix}, \quad \underset{\sim}{j} \triangleq \begin{bmatrix} j_1 \\ j_2 \\ \cdot \\ \cdot \\ \cdot \\ j_b \end{bmatrix}, \quad \underset{\sim}{V} \triangleq \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_b \end{bmatrix}, \quad \underset{\sim}{e} \triangleq \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_b \end{bmatrix} \tag{14}$$

and $\underset{\sim}{G}$ is the b×b branch conductance matrix of $N_1'$. The composite voltage vector $\underset{\sim}{V}$ can, in turn, be expressed via the following coboundary transformation[18] [18]:

$$\underset{\sim}{V} = \hat{\underset{\sim}{Q}}^t \, \hat{\underset{\sim}{V}} \tag{15}$$

where $\hat{\underset{\sim}{V}}$ is an (n-1)×1 vector, called the <u>generalized voltage coordinate</u> vector.[19] Substitute Eqs. (10), (13) and (15) into Eq. (11) and we get

$$(\hat{\underset{\sim}{Q}}\underset{\sim}{G}\hat{\underset{\sim}{Q}}^t)\hat{\underset{\sim}{V}} + \hat{\underset{\sim}{Q}}(\underset{\sim}{G}\underset{\sim}{e} - \underset{\sim}{j}) - (\hat{\underset{\sim}{Q}}\,\hat{\underset{\sim}{T}}^t)\,\underset{\sim}{i}_s = \underset{\sim}{0} \tag{16}$$

Now suppose we <u>choose</u> $\hat{\underset{\sim}{Q}}$ <u>to be the unique coboundary matrix associated with the augmented generator tree</u> $\mathcal{T}_g$ made up of the (n-1) unshifted external

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[18]This is a generalized form of "node transformation" and is equivalent to KVL equations.

[19]The precise interpretation of the generalized voltage coordinate vector will be given shortly in Lemma 5.

current sources, then it follows from Theorem 1 that $\hat{Q}\hat{T}^t = 1$ and Eq. (16) reduces to[20]

$$(\hat{Q}G\hat{Q}^t)\hat{V} + \hat{Q}(Ge - j) - i_s = 0 \qquad (17)$$

The generalized voltage coordinate vector $\hat{V}$ can now be interpreted as follows:

<u>Lemma 5</u>. <u>The generalized voltage coordinate vector $\hat{V}$ in Eq. (17) is equal to the negative of the (n-1)×1 voltage vector $V_s$ associated with the (n-1) external current sources in the augmented generator tree $\mathcal{T}_g$,[21] namely,</u>

$$\hat{V} = -V_s \qquad (18)$$

where $$V_s \triangleq [V_{s_1}, V_{s_2}, \ldots, V_{s_{n-1}}]^t. \qquad (19)$$

<u>Proof</u>: It follows from Eqs. (1) and (15) that

$$\hat{V} = (\hat{T}\,\hat{Q}^t)\hat{V} = \hat{T}\,V = -V_s \qquad (20)$$

■

Lemma 5 will now provide us with the key to transform the modified subnetwork $N_1'$ into an equivalent "acylic" network $N_{1_{eq}}$. Observe that Eq. (17) can be recast as follows:

$$\hat{V} = R_{eq}(\hat{j} + i_s) = -V_s \qquad (21)$$

where $$R_{eq} \triangleq (\hat{Q}\,G\,\hat{Q}^t)^{-1} \qquad (22)$$

and $$\hat{j} \triangleq \hat{Q}(j - Ge) \qquad (23)$$

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[20]The remarkable thing about Eq. (17) is that we have managed to recover the external current source vector $i_s$ intact, after all these manipulations!

[21]We assume the associated reference convention [16] throughout this paper.

Now suppose we construct an acyclic network $N_{1_{eq}}$ by simply adding a current source $\hat{j}_k$ (i.e., the kth component of $\hat{\underset{\sim}{j}}$) and a coupled linear resistor defined by the kth row of $\underset{\sim}{R}_{eq}$ in parallel with each external current source $i_{s_k}$ of the generator tree $\mathcal{T}_g$ (Figs. 6(a) and (b)), then it follows from Eq. (18) that the network equation describing $N_{1_{eq}}$ is precisely given by Eq. (21)! For example, corresponding to the star-tree generator of Fig. 5(g) (which is redrawn for convenience in Fig. 6(c)), we construct the network $N_{1_{eq}}$ shown in Fig. 6(d). It follows from the above observation that the subnetwork $N_1$ in Fig. 5(g) and the acyclic "coupled" subnetwork $N_{1_{eq}}$ in Fig. 6(d) are equivalent in the sense that they have identical governing equations. Applying the same transformation procedure to subnetwork $N_2$ (Fig. 5(h)), we obtain the equivalent acyclic network $N_{2_{eq}}$ shown in Fig. 6(e). Now recall that $i_{s_2} = i_{s_3} = 0$ (Fig. 5(g)) and $i_{s_5} = i_{s_6} = 0$ (Fig. 5(h)). Hence, the two equivalent networks $N_{1_{eq}}$ and $N_{2_{eq}}$ (Figs. 6(d) and (e), resp.) can be redrawn as in Fig. 6(f). We can now reverse the steps implemented earlier in going from Fig. 5(a) to Figs. 5(e) and (f) and apply the sequence of equivalent "inverse" transformations shown in Figs. 6(g) and (h) to obtain the equivalent interconnected network $N_{eq}$ shown in Fig. 6(i). Observe that this equivalent interconnected network $N_{eq}$ contains only one loop. In the general case, the resulting equivalent interconnected network $N_{eq}$ will contain as many loops as there are in the original torn network $N_0$ (Fig. 1(a)).

Note that the equivalent interconnected network $N_{eq}$ can be characterized by a "coupled" branch impedance matrix (Eq. (21)). Hence, we can analyze this network by the classical loop analysis method (or by any convenient cycle analysis [18]). If we let $\underset{\sim}{V}_0$ denote the composite voltage vector of the torn branches $N_0$ and order these branches first, then the KVL equation for the equivalent interconnected network $N_{eq}$ is given by

$$\underset{\sim}{B}_{eq}\, \underset{\sim}{V}_{eq} \triangleq [\underset{\sim}{B}_o\ \underset{\sim}{B}_a] \begin{bmatrix} \underset{\sim}{V}_o \\ \hat{\underset{\sim}{V}} \end{bmatrix} = \underset{\sim}{0} \tag{24}$$

where $\underset{\sim}{B}_{eq}$ is the fundamental loop matrix (with respect to any tree[22] of $N_{eq}$) of $N_{eq}$, $\underset{\sim}{B}_o$ and $\underset{\sim}{B}_a$ are the submatrices corresponding to the partition of the torn branches and the acyclic coupled branches, respectively, and $\hat{\underset{\sim}{V}}$ denotes the branch voltage vector of the coupled resistors in $N_{eq}$.[23] Now, carrying out the usual loop analysis, we write

$$\begin{aligned} \underset{\sim}{V}_o &= \underset{\sim}{v}_o - \underset{\sim}{e}_o = \underset{\sim}{R}_o\, \underset{\sim}{i}_o - \underset{\sim}{e}_o = \underset{\sim}{R}_o\, \underset{\sim}{I}_o + \underset{\sim}{R}_o\, \underset{\sim}{j}_o - \underset{\sim}{e}_o \\ &= \underset{\sim}{R}_o\, \underset{\sim}{B}_o^t\, \hat{\underset{\sim}{I}}_o + \underset{\sim}{R}_o\, \underset{\sim}{j}_o - \underset{\sim}{e}_o \end{aligned} \tag{25}$$

where $\underset{\sim}{v}_o$, $\underset{\sim}{e}_o$, $\underset{\sim}{i}_o$ and $\underset{\sim}{I}_o$ are defined as in Fig. 4(d), $\underset{\sim}{R}_o$ and $\hat{\underset{\sim}{I}}_o$ denote the branch resistance matrix and the link currect vector of $N_o$, respectively. Substituting Eq. (25) into Eq. (24), we obtain

$$(\underset{\sim}{B}_o\, \underset{\sim}{R}_o\, \underset{\sim}{B}_o^t)\hat{\underset{\sim}{I}}_o + \underset{\sim}{B}_o(\underset{\sim}{R}_o\, \underset{\sim}{j}_o - \underset{\sim}{e}_o) + \underset{\sim}{B}_a \hat{\underset{\sim}{V}} = \underset{\sim}{0} \tag{26}$$

Observe that Eq. (26) is the classical loop analysis equation [16] of the equivalent interconnected network $N_{eq}$, provided $\hat{\underset{\sim}{V}}$ is given and considered as sources. The branch voltage vector $\hat{\underset{\sim}{V}}$, in turn, must satisfy the associated coboundary analysis equation (Eq. (17)) derived earlier for

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[22] We assume that any tree of $N_{eq}$ contain all the acyclic coupled branches as twigs.

[23] We abuse our notation slightly here since $\hat{\underset{\sim}{V}}$, as defined later in Eq. (29), is actually made up of the collection of vectors defined in Eq. (21) for the m acyclic subnetworks $N_{1_{eq}}, N_{2_{eq}}, \ldots, N_{m_{eq}}$.

a single subnetwork $N_1$. We will once again abuse our notation slightly by using the same Eq. (17) to denote the associated coboundary analysis of the "m" separate subnetworks $N_1, N_2, \ldots, N_m$, provided the coboundary matrix $\hat{\underset{\sim}{Q}}$ and the branch conductance matrix $\underset{\sim}{G}$ are defined by:

$$\hat{\underset{\sim}{Q}} \triangleq \begin{bmatrix} \hat{\underset{\sim}{Q}}(N_1) & & \underset{\sim}{0} \\ & \hat{\underset{\sim}{Q}}(N_2) & \\ & & \ddots \\ \underset{\sim}{0} & & \hat{\underset{\sim}{Q}}(N_m) \end{bmatrix}, \quad \underset{\sim}{G} \triangleq \begin{bmatrix} \underset{\sim}{G}(N_1) & & \underset{\sim}{0} \\ & \underset{\sim}{G}(N_2) & \\ & & \ddots \\ \underset{\sim}{0} & & \underset{\sim}{G}(N_m) \end{bmatrix} \tag{28}$$

where $\hat{\underset{\sim}{Q}}(N_k)$ and $\underset{\sim}{G}(N_k)$ denote respectively the unique <u>associated coboundary matrix</u> and the <u>branch conductance matrix</u> of subnetwork $N_k$. Similarly, the vectors $\underset{\sim}{e}$, $\underset{\sim}{j}$, $\underset{\sim}{i}_s$ and $\underset{\sim}{V}$ are defined as follows:

$$\underset{\sim}{e} \triangleq \begin{bmatrix} \underset{\sim}{e}(N_1) \\ \underset{\sim}{e}(N_2) \\ \vdots \\ \underset{\sim}{e}(N_m) \end{bmatrix}, \quad \underset{\sim}{j} \triangleq \begin{bmatrix} \underset{\sim}{j}(N_1) \\ \underset{\sim}{j}(N_2) \\ \vdots \\ \underset{\sim}{j}(N_m) \end{bmatrix}, \quad \underset{\sim}{i}_s \triangleq \begin{bmatrix} \underset{\sim}{i}_s(N_1) \\ \underset{\sim}{i}_s(N_2) \\ \vdots \\ \underset{\sim}{i}_s(N_m) \end{bmatrix} \quad \text{and} \quad \hat{\underset{\sim}{V}} \triangleq \begin{bmatrix} \hat{\underset{\sim}{V}}(N_1) \\ \hat{\underset{\sim}{V}}(N_2) \\ \vdots \\ \hat{\underset{\sim}{V}}(N_m) \end{bmatrix} \tag{29}$$

Now, recall that $\underset{\sim}{i}_s$ is the current vector of the torn branches $N_{o-k}$ connecting subnetwork $N_k$ to $N_o$. This current vector is uniquely determined by the link current vector $\hat{\underset{\sim}{I}}_o$ of $N_o$. Hence, there exists some matrix $\underset{\sim}{C}$ such that[24]

$$\underset{\sim}{i}_s = \underset{\sim}{C}\, \hat{\underset{\sim}{I}}_o \tag{30}$$

Substitute Eq. (30) into Eq. (17) and we have

$$(\hat{\underset{\sim}{Q}}\, \underset{\sim}{G}\, \hat{\underset{\sim}{Q}}^t)\hat{\underset{\sim}{V}} + \hat{\underset{\sim}{Q}}(\underset{\sim}{G}\underset{\sim}{e} - \underset{\sim}{j}) - \underset{\sim}{C}\, \hat{\underset{\sim}{I}}_o = \underset{\sim}{0} \tag{31}$$

---

[24]Actually, the matrices $\underset{\sim}{C}$ and $\underset{\sim}{B}_a$ are related by $\underset{\sim}{C} = \underset{\sim}{B}_a^t$, as will be shown in the next section.

Eqs. (31) and (26) can now be combined into a single matrix equation

$$\boxed{\begin{bmatrix} \hat{Q}G\hat{Q}^t & -C \\ B_a & B_oR_oB_o^t \end{bmatrix} \begin{bmatrix} \hat{V} \\ \hat{I}_o \end{bmatrix} = \begin{bmatrix} \hat{Q}\,(j - Ge) \\ B_o(e_o - R_oj_o) \end{bmatrix}} \tag{32}$$

The process of formulating and solving Eq. (32) is called diakoptic analysis by Kron [1 - 3]. The main advantage of this analysis is that through a judicious choice of the torn network $N_o$, the matrix $(B_oR_oB_o^t)$ will have a relatively small dimension. The matrix $(\hat{Q}G\hat{Q}^t)$ has a large dimension but it is in block diagonal form in view of Eq. (28). Hence, the structure of the "sparse portion" of Eq. (32) assumes the computationally desirable block diagonal form shown in Fig. 7. So far, our derivation assumes no coupling among the resistors just for simplicity. It should now be clear that the entire derivation remains valid so long as couplings are restricted to among branches within the same subnetwork (as required by the nonzero structure of $G$ defined in Eq. (28)).

Observe that Eq. (32) is independent of the choice of the open loops (recall the two choices made in Figs. 5(i) and (k)) for shifting the (n-1) external current sources. This is because the key identity —— $\hat{Q}\hat{T}^t = 1$ —— given by Theorem 1 remains valid for any choice of the open loop matrix $\hat{T}$, so long as $\hat{Q}$ is chosen to be the unique coboundary matrix defined in Eq. (6). This coboundary matrix is determined as soon as the augmented "current source" generator tree $\mathcal{T}_g$ is prescribed (see corollary to Theorem 1). In fact, the next two lemmas show that $\hat{Q}$ can be obtained by inspection for two particular choices of $\mathcal{T}_g$.

Lemma 6. The unique coboundary matrix $\hat{Q}$ associated with each subnetwork $N_k$ for the node-to-datum case (i.e., when $\mathcal{T}_g$ is chosen to be a node-to-datum star-tree) <u>is simply the reduced incidence matrix</u> (relative to the same datum node) <u>of subnetwork $N_k$</u>. <u>Moreover, the generalized voltage coordinate vector $\hat{V}$ is simply equal to the node-to-datum voltage vector $\underset{\sim}{V}_n$</u>.

<u>Proof</u>. Since the current reference direction of each external current source in the star-tree $\mathcal{T}_g$ is assigned leaving the datum node (see Figs. 2(b) and 5(g)), it follows that the (n-1)×1 voltage vector $\underset{\sim}{V}_s$ associated with the current sources is

$$\underset{\sim}{V}_s = -\underset{\sim}{V}_n \tag{33}$$

where $\underset{\sim}{V}_n$ denotes the standard node-to-datum voltage vector of subnetwork $N_k$. Substituting Eq. (33) into Eq. (18), we obtain

$$\hat{\underset{\sim}{V}} = \underset{\sim}{V}_n$$

Eq. (15) then becomes

$$\underset{\sim}{V} = \hat{\underset{\sim}{Q}}^t \underset{\sim}{V}_n \tag{34}$$

Since $\underset{\sim}{V}$ is the b×1 composite branch voltage vector of $N_k$, it is also given by $\underset{\sim}{V} = \underset{\sim}{A}^t \underset{\sim}{V}_n$ [16]. Hence, $\hat{\underset{\sim}{Q}} = \underset{\sim}{A}$. ■

It follows from Lemma 6 that the associated coboundary analysis (defined by Eq. (31)) for the node-to-datum case is simply the classical nodal analysis. Since the <u>node admittance matrix</u> $\underset{\sim}{A}\underset{\sim}{G}\underset{\sim}{A}^t$ is known to be extremely sparse [18,19], from the computational efficiency point of view, the augmented current source generator tree for each subnetwork $N_k$ should be chosen to be a node-to-datum star-tree.

<u>Lemma 7</u>. <u>Let $\mathcal{T}$ be any tree of the graph $\mathcal{G}_k$ of subnetwork $N_k$ and let the augmented generator tree $\mathcal{T}_g$ be the fundamental generator tree with respect to $\mathcal{T}$</u> (i.e., each generator branch is in parallel but oppositely oriented with a twig in $\mathcal{T}$ as shown in Fig. 2(c)), <u>then the unique associated</u>

coboundary matrix $\hat{Q}$ is simply the fundamental cutset matrix $Q$ with respect to $\mathcal{T}$. Moreover, the generalized voltage coordinate vector $\hat{V}$ is simply equal to the twig voltage vector $V_{\mathcal{T}}$.

Proof. For the fundamental generator tree $\mathcal{T}_g$, the fundamental open loop matrix induced by $\mathcal{T}_g$ is simply $T = [0_{\mathcal{L}} \; 1_{\mathcal{T}}]$. Hence the associated nonsingular submatrix $T_{\mathcal{T}}$ is simply the identity matrix. It follows from Eq. (6) that $\hat{Q} = Q$ and $\hat{V} = V_{\mathcal{T}}$. ∎

## 4. Relationship Between Diakoptic and Hybrid Analysis

Various forms of hybrid analysis have been reported in the literature [19 - 22]. Our first task in this section is to derive a generalized form of hybrid analysis which includes all existing forms as special cases. Following this, we will prove rigorously that diakoptic analysis is also a special case.

Let $\mathcal{G}$ be the graph of a linear resistive network N, let $\mathcal{T}$ denote some tree of $\mathcal{G}$ and let $\mathcal{L}$ denote its associated cotree. Partition the tree $\mathcal{T}$ into two arbitrary sets $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$.[25] Let $\mathcal{L}_1$ denote any subset of the cotree $\mathcal{L}$ which forms loops exclusively with branches in $\mathcal{T}_1$ and let $\mathcal{L}_2$ denote the remaining branches such that $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. If we relabel the branches in the order $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{T}_1$ and $\mathcal{T}_2$, then the fundamental loop matrix $B$ and the fundamental cutset matrix $Q$ with respect to $\mathcal{T}$ are given respectively by[26]

[25] The subsets $\mathcal{T}_1$ and $\mathcal{T}_2$ need not be connected and each component therefore represents a forest of $\mathcal{G}$.

[26] The relation $BQ^t = 0$ [16] was used in describing $Q$.

$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad\mathcal{L}_1 & \quad\quad\mathcal{L}_2 & \quad\quad\mathcal{T}_1 & \quad\quad\mathcal{T}_2 \end{array} \\
\mathbf{B} = & \begin{bmatrix} \mathbf{1}_{\mathcal{L}_1\mathcal{L}_1} & \mathbf{0}_{\mathcal{L}_1\mathcal{L}_2} & \mathbf{B}_{\mathcal{L}_1\mathcal{T}_1} & \mathbf{0}_{\mathcal{L}_1\mathcal{T}_2} \\ \mathbf{0}_{\mathcal{L}_2\mathcal{L}_1} & \mathbf{1}_{\mathcal{L}_2\mathcal{L}_2} & \mathbf{B}_{\mathcal{L}_2\mathcal{T}_1} & \mathbf{B}_{\mathcal{L}_2\mathcal{T}_2} \end{bmatrix}
\end{array}
\tag{35}
$$

$$
\mathbf{Q} = \begin{bmatrix} -\mathbf{B}^t_{\mathcal{L}_1\mathcal{T}_1} & -\mathbf{B}^t_{\mathcal{L}_2\mathcal{T}_1} & \mathbf{1}_{\mathcal{T}_1\mathcal{T}_1} & \mathbf{0}_{\mathcal{T}_1\mathcal{T}_2} \\ \mathbf{0}^t_{\mathcal{L}_1\mathcal{T}_2} & -\mathbf{B}^t_{\mathcal{L}_2\mathcal{T}_2} & \mathbf{0}_{\mathcal{T}_2\mathcal{T}_1} & \mathbf{1}_{\mathcal{T}_2\mathcal{T}_2} \end{bmatrix}
\tag{36}
$$

where $\mathbf{1}_{XX}$ and $\mathbf{0}_{XY}$ denote respectively the unit and zero matrix of dimension $|X|\times|X|$ and $|X|\times|Y|$.[27] Similarly, $\mathbf{B}_{XY}$ and $\mathbf{Q}_{XY}$ denote respectively some submatrix of $\mathbf{B}$ and $\mathbf{Q}$ of dimension $|X|\times|Y|$. It is important to observe that the upper right-hand corner submatrix of $\mathbf{B}$ is equal to $\mathbf{0}_{\mathcal{L}_1\mathcal{T}_2}$ because, by construction, branches in $\mathcal{L}_1$ form loops only with branches $\mathcal{T}_1$.

Using the composite branch notation shown in Fig. 4(a), KVL and KCL equations assume the following form:

$$
\mathbf{V}_{\mathcal{L}_1} + \mathbf{B}_{\mathcal{L}_1\mathcal{T}_1}\mathbf{V}_{\mathcal{T}_1} = \mathbf{0}_{\mathcal{L}_1} \tag{37}
$$

$$
\mathbf{V}_{\mathcal{L}_2} + \mathbf{B}_{\mathcal{L}_2\mathcal{T}_1}\mathbf{V}_{\mathcal{T}_1} + \mathbf{B}_{\mathcal{L}_2\mathcal{T}_2}\mathbf{V}_{\mathcal{T}_2} = \mathbf{0}_{\mathcal{L}_2} \tag{38}
$$

$$
-\mathbf{B}^t_{\mathcal{L}_1\mathcal{T}_1}\mathbf{I}_{\mathcal{L}_1} - \mathbf{B}^t_{\mathcal{L}_2\mathcal{T}_1}\mathbf{I}_{\mathcal{L}_2} + \mathbf{I}_{\mathcal{T}_1} = \mathbf{0}_{\mathcal{T}_1} \tag{39}
$$

$$
-\mathbf{B}^t_{\mathcal{L}_2\mathcal{T}_2}\mathbf{I}_{\mathcal{L}_2} + \mathbf{I}_{\mathcal{T}_2} = \mathbf{0}_{\mathcal{T}_2} \tag{40}
$$

[27] We use the notation $|X|$ to denote the number of elements in the set X.

where $\left\{\underset{\sim}{V}_{\mathcal{L}_1}, \underset{\sim}{V}_{\mathcal{L}_2}, \underset{\sim}{V}_{\mathcal{T}_1}, \underset{\sim}{V}_{\mathcal{T}_2}\right\}$ and $\left\{\underset{\sim}{I}_{\mathcal{L}_1}, \underset{\sim}{I}_{\mathcal{L}_2}, \underset{\sim}{I}_{\mathcal{T}_1}, \underset{\sim}{I}_{\mathcal{T}_2}\right\}$ denote the composite voltage and current vectors of the branches in $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let

$$\begin{bmatrix} \underset{\sim}{i}_{\mathcal{L}_1} \\ \\ \underset{\sim}{i}_{\mathcal{T}_1} \end{bmatrix} = \underset{\sim}{G}_1 \begin{bmatrix} \underset{\sim}{v}_{\mathcal{L}_1} \\ \\ \underset{\sim}{v}_{\mathcal{T}_1} \end{bmatrix} \triangleq \begin{bmatrix} \underset{\sim}{G}_{\mathcal{L}_1\mathcal{L}_1} & \underset{\sim}{G}_{\mathcal{L}_1\mathcal{T}_1} \\ \\ \underset{\sim}{G}_{\mathcal{T}_1\mathcal{L}_1} & \underset{\sim}{G}_{\mathcal{T}_1\mathcal{T}_1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_{\mathcal{L}_1} \\ \\ \underset{\sim}{v}_{\mathcal{T}_1} \end{bmatrix} \tag{41}$$

denote the branch conductance matrix of branches in $\mathcal{L}_1 \cup \mathcal{T}_1$ and let

$$\begin{bmatrix} \underset{\sim}{v}_{\mathcal{L}_2} \\ \\ \underset{\sim}{v}_{\mathcal{T}_2} \end{bmatrix} = \underset{\sim}{R}_2 \begin{bmatrix} \underset{\sim}{i}_{\mathcal{L}_2} \\ \\ \underset{\sim}{i}_{\mathcal{T}_2} \end{bmatrix} \triangleq \begin{bmatrix} \underset{\sim}{R}_{\mathcal{L}_2\mathcal{L}_2} & \underset{\sim}{R}_{\mathcal{L}_2\mathcal{T}_2} \\ \\ \underset{\sim}{R}_{\mathcal{T}_2\mathcal{L}_2} & \underset{\sim}{R}_{\mathcal{T}_2\mathcal{T}_2} \end{bmatrix} \begin{bmatrix} \underset{\sim}{i}_{\mathcal{L}_2} \\ \\ \underset{\sim}{i}_{\mathcal{T}_2} \end{bmatrix} \tag{42}$$

denote the <u>branch resistance matrix</u> of branches in $\mathcal{L}_2 \cup \mathcal{T}_2$. Using the relations $\underset{\sim}{v}_X = \underset{\sim}{V}_X + \underset{\sim}{e}_X$ and $\underset{\sim}{i}_X = \underset{\sim}{I}_X + \underset{\sim}{j}_X$ (see Fig. 4(a)), we obtain from Eqs. (39), (41) and (37) the following equation:

$$\left[-\underset{\sim}{B}^t_{\mathcal{L}_1\mathcal{T}_1} \;\; \underset{\sim}{1}_{\mathcal{T}_1\mathcal{T}_1}\right]\left\{\underset{\sim}{G}_1\left(\begin{bmatrix} -\underset{\sim}{B}_{\mathcal{L}_1\mathcal{T}_1} \\ \\ \underset{\sim}{1}_{\mathcal{T}_1\mathcal{T}_1} \end{bmatrix} \underset{\sim}{V}_{\mathcal{T}_1} + \begin{bmatrix} \underset{\sim}{e}_{\mathcal{L}_1} \\ \\ \underset{\sim}{e}_{\mathcal{T}_1} \end{bmatrix}\right) - \begin{bmatrix} \underset{\sim}{j}_{\mathcal{L}_1} \\ \\ \underset{\sim}{j}_{\mathcal{T}_1} \end{bmatrix}\right\} -$$

$$\underset{\sim}{B}^t_{\mathcal{L}_2\mathcal{T}_1} \underset{\sim}{I}_{\mathcal{L}_2} = \underset{\sim}{0}_{\mathcal{T}_1} \tag{43}$$

Eq. (43) can be rewritten in the more compact form

$$(\underset{\sim}{Q}_1 \underset{\sim}{G}_1 \underset{\sim}{Q}_1^t) \underset{\sim}{V}_{\mathcal{T}_1} + \underset{\sim}{Q}_1(\underset{\sim}{G}_1 \underset{\sim}{e}_1 - \underset{\sim}{j}_1) - \underset{\sim}{B}^t_{\mathcal{L}_2\mathcal{T}_1} \underset{\sim}{I}_{\mathcal{L}_2} = \underset{\sim}{0}_{\mathcal{T}_1} \tag{44}$$

where

$$\underset{\sim}{Q}_1 \triangleq [-\underset{\sim}{B}^t_{\mathcal{L}_1\mathcal{T}_1} \;\; \underset{\sim}{1}_{\mathcal{T}_1\mathcal{T}_1}], \quad \underset{\sim}{e}_1 \triangleq \begin{bmatrix} \underset{\sim}{e}_{\mathcal{L}_1} \\ \\ \underset{\sim}{e}_{\mathcal{T}_1} \end{bmatrix} \text{ and } \underset{\sim}{j}_1 \triangleq \begin{bmatrix} \underset{\sim}{j}_{\mathcal{L}_1} \\ \\ \underset{\sim}{j}_{\mathcal{T}_1} \end{bmatrix}. \tag{45}$$

Since $\underset{\sim}{Q}_1$ can be obtained from $\underset{\sim}{Q}$ upon deleting all columns corresponding to branches in $\mathcal{L}_2 \cup \mathcal{T}_2$ and by deleting the last $|\mathcal{T}_2|$ rows, it can be interpreted as the <u>fundamental cutset matrix of the subnetwork $\mathcal{N}_1$ made up of branches in $\mathcal{L}_1 \cup \mathcal{T}_1$, with all branches in $\mathcal{L}_2 \cup \mathcal{T}_2$ replaced by open circuits</u>.

Applying the same manipulation to Eqs. (40), (42) and (38), we obtain the dual equation

$$(\underset{\sim}{B}_2\underset{\sim}{R}_2\underset{\sim}{B}_2^t)\ \underset{\sim}{I}_{\mathcal{L}_2} + \underset{\sim}{B}_2(\underset{\sim}{R}_2\underset{\sim}{j}_2 - \underset{\sim}{e}_2) + \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}\ \underset{\sim}{V}_{\mathcal{T}_1} = \underset{\sim}{0}_{\mathcal{L}_2} \tag{46}$$

where

$$\underset{\sim}{B}_2 \triangleq [\underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2}\ \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2}],\quad \underset{\sim}{e}_2 \triangleq \begin{bmatrix} \underset{\sim}{e}_{\mathcal{L}_2} \\ \underset{\sim}{e}_{\mathcal{T}_2} \end{bmatrix} \text{ and } \underset{\sim}{j}_2 \triangleq \begin{bmatrix} \underset{\sim}{j}_{\mathcal{L}_2} \\ \underset{\sim}{j}_{\mathcal{T}_2} \end{bmatrix} \tag{47}$$

Since $\underset{\sim}{B}_2$ can be obtained from $\underset{\sim}{B}$ upon deleting all columns corresponding to branches in $\mathcal{L}_1 \cup \mathcal{T}_1$ and by deleting the first $|\mathcal{L}_1|$ rows, it can be <u>interpreted as the fundamental loop matrix of the subnetwork $\mathcal{N}_2$ made up of branches in $\mathcal{L}_2 \cup \mathcal{T}_2$, with all branches in $\mathcal{L}_1 \cup \mathcal{T}_1$ replaced by short circuits</u>.

Eqs. (44) and (46) constitute a system of $|\mathcal{T}_1| + |\mathcal{L}_2|$ hybrid equations with $|\mathcal{T}_1|$ voltage variables in $\underset{\sim}{V}_{\mathcal{T}_1}$ and $|\mathcal{L}_2|$ current variables in $\underset{\sim}{I}_{\mathcal{L}_2}$. They may be combined into a single matrix equation

$$\boxed{\begin{bmatrix} \underset{\sim}{Q}_1\underset{\sim}{G}_1\underset{\sim}{Q}_1^t & -\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}^t \\ \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1} & \underset{\sim}{B}_2\underset{\sim}{R}_2\underset{\sim}{B}_2^t \end{bmatrix} \begin{bmatrix} \underset{\sim}{V}_{\mathcal{T}_1} \\ \underset{\sim}{I}_{\mathcal{L}_2} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Q}_1(\underset{\sim}{j}_1 - \underset{\sim}{G}_1\underset{\sim}{e}_1) \\ \underset{\sim}{B}_2(\underset{\sim}{e}_2 - \underset{\sim}{R}_2\underset{\sim}{j}_2) \end{bmatrix}} \tag{48}$$

Equation (48) represents one form of hybrid analysis [19,20]. We will now derive a generalized form of hybrid analysis by defining the generalized voltage and current vectors

$$\hat{\underset{\sim}{V}}_{\mathcal{T}_1} \triangleq \underset{\sim}{T}_Q \underset{\sim}{V}_{\mathcal{T}_1} \tag{49}$$

$$\hat{\underset{\sim}{I}}_{\mathcal{L}_2} \triangleq \underset{\sim}{T}_B \underset{\sim}{I}_{\mathcal{L}_2} \tag{50}$$

where $\underset{\sim}{T}_Q$ is any $|\mathcal{T}_1|\times|\mathcal{T}_1|$ nonsingular matrix and $\underset{\sim}{T}_B$ is any $|\mathcal{L}_2|\times|\mathcal{L}_2|$ nonsingular matrix. Observe that each element of the generalized voltage vector $\hat{\underset{\sim}{V}}_{\mathcal{T}_1}$ is simply an algebraic sum of composite twig voltages, and each element of the generalized current vector $\hat{\underset{\sim}{I}}_{\mathcal{L}_2}$ is simply an algebraic sum of composite link currents.

Premultiply Eq. (44) by $(\underset{\sim}{T}_Q^t)^{-1}$ and Eq. (46) by $(\underset{\sim}{T}_B^t)^{-1}$, and combine the resulting equations with Eqs. (49) and (50), we obtain

$$\boxed{\begin{aligned}
&\begin{bmatrix} \hat{\underset{\sim}{Q}}_1 \underset{\sim}{G}_1 \hat{\underset{\sim}{Q}}_1^t & -\hat{\underset{\sim}{C}}^t \\ \hat{\underset{\sim}{C}} & \hat{\underset{\sim}{B}}_2 \underset{\sim}{R}_2 \hat{\underset{\sim}{B}}_2^t \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}_{\mathcal{L}_2} \end{bmatrix} = \begin{bmatrix} \hat{\underset{\sim}{Q}}_1 (\underset{\sim}{j}_1 - \underset{\sim}{G}_1 \underset{\sim}{e}_1) \\ \hat{\underset{\sim}{B}}_2 (\underset{\sim}{e}_2 - \underset{\sim}{R}_2 \underset{\sim}{j}_2) \end{bmatrix} && (51) \\
&\text{where} \quad \hat{\underset{\sim}{Q}}_1 = (\underset{\sim}{T}_Q^t)^{-1} \underset{\sim}{Q}_1 \triangleq \underset{\sim}{P}_Q \underset{\sim}{Q}_1 && (52) \\
&\hat{\underset{\sim}{B}}_2 = (\underset{\sim}{T}_B^t)^{-1} \underset{\sim}{B}_2 \triangleq \underset{\sim}{P}_B \underset{\sim}{B}_2 && (53) \\
&\hat{\underset{\sim}{C}} = (\underset{\sim}{T}_B^t)^{-1} \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_1} (\underset{\sim}{T}_Q)^{-1} = \underset{\sim}{P}_B \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_1} \underset{\sim}{P}_Q^t && (54)
\end{aligned}}$$

Equation (51) will henceforth be referred to as the generalized hybrid equation.

Our next objective is to show that the diakoptic analysis derived earlier in Eq. (32) is no more than a special case of Eq. (51) with a particular choice of the two transformation matrices $\underset{\sim}{T}_Q$ and $\underset{\sim}{T}_B$! This can be shown by first identifying $N_0 = \mathcal{L}_2 \cup \mathcal{T}_2$ and $N_1 \cup N_2 \cup \ldots \cup N_m = \mathcal{L}_1 \cup \mathcal{T}_1$, where $N_0$ is the "torn branches" and $N_k$, $k = 1,2,\ldots,m$, is the kth separated subnetwork defined in Fig. 1. For simplicity, we will assume throughout the following proof that m = 1 (i.e., $N_1 = \mathcal{L}_1 \cup \mathcal{T}_1$), since the generalization to the m > 1 case follows trivially - mutatis mutandis.

Consider first the case of diakoptic analysis where the augmented current source generator $\mathcal{T}_g$ is chosen to be the fundamental tree with respect to $\mathcal{T}$. Then it follows from Lemma 7 that the unique associated coboundary matrix $\hat{\underset{\sim}{Q}}$ in Eq. (32) is precisely the fundamental cutset matrix $\underset{\sim}{Q}_1$ of subnetwork $N_1 = \mathcal{L}_1 \cup \mathcal{T}_1$ with respect to the tree $\mathcal{T}_1$. Hence $\hat{\underset{\sim}{Q}}$ can be obtained simply from the fundamental cutset matrix $\underset{\sim}{Q}$ in Eq. (36) of the complete network upon deleting those columns corresponding to the branches in $\mathcal{L}_2 \cup \mathcal{T}_2$ and by deleting the last $|\mathcal{T}_2|$ rows. It follows from Eq. (45) that

$$\hat{\underset{\sim}{Q}} = \underset{\sim}{Q}_1 = [-\underset{\sim}{B}^t_{\mathcal{L}_1\mathcal{T}_1} \quad \underset{\sim}{1}_{\mathcal{T}_1\mathcal{T}_1}] \tag{55}$$

Now let us return to our earlier derivation of diakoptic analysis and rewrite Eq. (11) with the help of Eqs. (10) and (1) as

$$\hat{\underset{\sim}{Q}}\underset{\sim}{I} - \hat{\underset{\sim}{Q}}\underset{\sim}{j}' = \hat{\underset{\sim}{Q}}\underset{\sim}{I} - \underset{\sim}{i}_s = \underset{\sim}{0} \tag{56}$$

Substituting Eq. (55) for $\hat{\underset{\sim}{Q}}$ in Eq. (56) and observing that $\underset{\sim}{I} = \begin{bmatrix} \underset{\sim}{I}_{\mathcal{L}_1} \\ \underset{\sim}{I}_{\mathcal{T}_1} \end{bmatrix}$,

we obtain

$$-\underset{\sim}{B}^{t}_{\mathcal{L}_1\mathcal{T}_1}\ \underset{\sim}{I}_{\mathcal{L}_1} + \underset{\sim}{I}_{\mathcal{T}_1} - \underset{\sim}{i}_s = \underset{\sim}{0}_{\mathcal{T}_1} \tag{57}$$

Comparing Eq. (57) with Eq. (39) and recalling that $\underset{\sim}{i}_s$ corresponds to the original set of substituted current sources connecting subnetwork $N_1$ to $N_0$, it follows that the effect accounted for by $\underset{\sim}{i}_s$ is precisely given by

$$\underset{\sim}{i}_s = \underset{\sim}{B}^{t}_{\mathcal{L}_2\mathcal{T}_1}\ \underset{\sim}{I}_{\mathcal{L}_2}\ . \tag{58}$$

Since $N_1 = \mathcal{L}_1 \cup \mathcal{T}_1$, we can examine the terms in Eq. (17) with the help of Eq. (45) and identify $\underset{\sim}{e} = \underset{\sim}{e}_1$, $\underset{\sim}{j} = \underset{\sim}{j}_1$ and $\underset{\sim}{G} = \underset{\sim}{G}_1$. Moreover, it follows from Lemma 7 that $\hat{\underset{\sim}{V}} = \underset{\sim}{V}_{\mathcal{T}_1}$. If we substitute Eqs. (55) and (58) along with the preceding identities, we would obtain Eq. (44).

In order to identify Eq. (26) of the equivalent interconnected network $N_{eq}$ of diakoptic analysis with Eq. (46) of hybrid analysis, we need to make the following observation. Since $\mathcal{T}_g$ is chosen to be the fundamental tree in this case, the linear graph associated with $N_{eq}$ is precisely those branches in $\mathcal{T}_g \cup N_o = \mathcal{T}_1 \cup N_o = \mathcal{T}_1 \cup \mathcal{L}_2 \cup \mathcal{T}_2$. Hence the fundamental loop matrix $\underset{\sim}{B}_{eq}$ of $N_{eq}$ can be obtained from Eq. (35) upon deleting the first $|\mathcal{L}_1|$ columns and the first $|\mathcal{L}_1|$ rows, namely,

$$\begin{array}{cccc} & \mathcal{L}_2 & \mathcal{T}_1 & \mathcal{T}_2 \\ \underset{\sim}{B}_{eq} = & \Big[\ \underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2}\ \Big] \end{array} \tag{59}$$

If we reorder the columns of $\underset{\sim}{B}_{eq}$ so that the branches in $N_0 = \mathcal{L}_2 \cup \mathcal{T}_2$ are grouped together, we obtain

$$\begin{array}{cccccc} & \mathcal{L}_2 & \mathcal{T}_2 & \mathcal{T}_1 & \mathcal{L}_2 \cup \mathcal{T}_2 = N_0 & \mathcal{T}_1 = \mathcal{T}_g \\ \underset{\sim}{B}_{eq} = & \Big[\ \underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}\ \Big] \triangleq \Big[ & \underset{\sim}{B}_2 & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}\ \Big] \end{array} \tag{60}$$

where $\underset{\sim}{B}$ is defined earlier in Eq. (47). Since $\underset{\sim}{B}_{eq}$ is also defined earlier in Eq. (24), we can identify

$$\underset{\sim}{B}_0 = \underset{\sim}{B}_2 \tag{61}$$

$$\underset{\sim}{B}_a = \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_1} \tag{62}$$

If we substitute Eqs. (61) and (62) into Eq. (26) and identify $\underset{\sim}{R}_0 = \underset{\sim}{R}_2$, $\underset{\sim}{j}_0 = \underset{\sim}{j}_2$, $\underset{\sim}{e}_0 = \underset{\sim}{e}_2$ and $\underset{\sim}{\hat{I}}_0 = \underset{\sim}{I}_{\mathcal{L}_2}$, we would obtain Eq. (46). Hence, we have proved that the above form of diakoptic analysis is indeed a special case of the generalized hybrid analysis (Eq. (51)) with <u>unit transformation matrices</u> $\underset{\sim}{T}_Q = \underset{\sim}{1}_{\mathcal{T}_1 \mathcal{T}_1}$ and $\underset{\sim}{T}_B = \underset{\sim}{1}_{\mathcal{L}_2 \mathcal{L}_2}$.

Let us now prove the general case where the "augmented-current-source" generator tree $\mathcal{T}_g$ may be arbitrarily chosen. To facilitate the identification of Eq. (17) with the first equation of Eq. (51), let us first substitute Eq. (6) for the unique associated coboundary matrix $\underset{\sim}{\hat{Q}}$ into Eq. (56) and obtain

$$\left(\underset{\sim}{T}^t_{\mathcal{T}}\right)^{-1} \begin{bmatrix} -\underset{\sim}{B}^t_{\mathcal{L}_1 \mathcal{T}_1} & \underset{\sim}{1}_{\mathcal{T}_1 \mathcal{T}_1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{I}_{\mathcal{L}_1} \\ \underset{\sim}{I}_{\mathcal{T}_1} \end{bmatrix} - \underset{\sim}{i}_s = \underset{\sim}{0}_{\mathcal{T}_1} \tag{63}$$

where $\underset{\sim}{T}_{\mathcal{T}}$ is the nonsingular submatrix of the fundamental open loop matrix $\underset{\sim}{T}$. If we premultiply Eq. (39) by $(\underset{\sim}{T}^t_{\mathcal{T}})^{-1}$, we obtain

$$\left(\underset{\sim}{T}^t_{\mathcal{T}}\right)^{-1} \begin{bmatrix} -\underset{\sim}{B}^t_{\mathcal{L}_1 \mathcal{T}_1} & \underset{\sim}{1}_{\mathcal{T}_1 \mathcal{T}_1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{I}_{\mathcal{L}_1} \\ \underset{\sim}{I}_{\mathcal{T}_1} \end{bmatrix} - \left(\underset{\sim}{T}^t_{\mathcal{T}}\right)^{-1} \underset{\sim}{B}^t_{\mathcal{L}_2 \mathcal{T}_1} \underset{\sim}{I}_{\mathcal{L}_2} = \underset{\sim}{0}_{\mathcal{T}_1} \tag{64}$$

It follows from Eqs. (63) and (64) that the external current source vector $\underset{\sim}{i}_s$ which accounts for the effect of the currents from the branches connecting $N_1$ to $N_0$ is simply given by:

$$\underset{\sim}{i}_s = (\underset{\sim}{T}^t_{\mathcal{T}})^{-1}\ \underset{\sim}{B}^t_{\mathcal{L}_2\mathcal{T}_1}\ \underset{\sim}{I}_{\mathcal{L}_2} \tag{65}$$

Since $N_1 = \mathcal{L}_1 \cup \mathcal{T}_1$, we can identify $\underset{\sim}{G} = \underset{\sim}{G}_1$, $\underset{\sim}{e} = \underset{\sim}{e}_1$ and $\underset{\sim}{j} = \underset{\sim}{j}_1$ in Eq. (17). Substituting Eqs. (6), (65) and the above identities into Eq. (17), we obtain

$$((\underset{\sim}{T}^t_{\mathcal{T}})^{-1}\ \underset{\sim}{Q}_1)\ \underset{\sim}{G}_1((\underset{\sim}{T}^t_{\mathcal{T}})^{-1}\ \underset{\sim}{Q}_1)^t\ \hat{\underset{\sim}{V}} + ((\underset{\sim}{T}^t_{\mathcal{T}})^{-1}\ \underset{\sim}{Q}_1)\ (\underset{\sim}{G}_1\underset{\sim}{e}_1 - \underset{\sim}{j}_1) - (\underset{\sim}{T}^t_{\mathcal{T}})^{-1}\ \underset{\sim}{B}^t_{\mathcal{L}_2\mathcal{T}_1}\ \underset{\sim}{I}_{\mathcal{L}_2} = \underset{\sim}{0}_{\mathcal{T}_1} \tag{66}$$

Observe that Eq. (66) is precisely the first equation of Eq. (51) if we identify

$$\underset{\sim}{T}_Q = \underset{\sim}{T}_{\mathcal{T}} \tag{67}$$

$$\underset{\sim}{T}_B = \underset{\sim}{1}_{\mathcal{L}_2} \tag{68}$$

$$\hat{\underset{\sim}{V}}_{\mathcal{T}_1} \triangleq \underset{\sim}{T}_Q\underset{\sim}{V}_{\mathcal{T}_1} = \underset{\sim}{T}_{\mathcal{T}}\underset{\sim}{V}_{\mathcal{T}_1} = \hat{\underset{\sim}{V}} \tag{69}$$

Hence, we have proved that the "diakoptic" Eq. (17) is a special case of the first equation of the generalized hybrid equation (51). It remains for us to identify Eq. (31) with the second equation in Eq. (51).

Since the "augmented-current-source" generator tree $\mathcal{T}_g$ in this case may be arbitrarily chosen, the equivalent interconnected network $N_{eq}$ is given by $N_{eq} = \mathcal{T}_g \cup N_0 = \mathcal{T}_g \cup \mathcal{L}_2 \cup \mathcal{T}_2$. Observe that although $\mathcal{T}_g$ connects all nodes of $N_1 = \mathcal{L}_1 \cup \mathcal{T}_1$, it need not be a subgraph of $N_1$ since some branches of $\mathcal{T}_g$ may not be connected in parallel with branches of $N_1$ (See Fig. 5(g), for example). The fundamental loop matrix of $N_{eq}$ with respect to the new tree $\mathcal{T}'_g \cup \mathcal{T}'_2$ is given by

$$\begin{array}{cccc} & \mathcal{L}_2 & \mathcal{T}_g & \mathcal{T}_2 \\ \underset{\sim}{B}_{eq} = & \Big[ \underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2} \Big] \end{array} \tag{70}$$

If we reorder the columns of $\underset{\sim}{B}_{eq}$ so that the branches in $N_0 = \mathcal{L}_2 \cup \mathcal{T}_2$ are grouped together, we obtain

$$\begin{array}{cccccc} & \mathcal{L}_2 & \mathcal{T}_2 & \mathcal{T}_g & \mathcal{L}_2 \cup \mathcal{T}_2 = N_0 & \mathcal{T}_g \\ \underset{\sim}{B}_{eq} = & \Big[ \underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g} \Big] \triangleq \Big[ & \underset{\sim}{B}_2 & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g} \Big] \end{array} \tag{71}$$

where $\underset{\sim}{B}_2$ is defined earlier in Eq. (47). Note that $\underset{\sim}{B}_{eq}$ is earlier defined in Eq. (24), hence we can identify

$$\underset{\sim}{B}_0 = \underset{\sim}{B}_2 \tag{72}$$

$$\underset{\sim}{B}_a = \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g} \tag{73}$$

Our next step will be to state the following lemma which expresses $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g}$ in terms of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}$:

<u>Lemma 8</u>. <u>The $|\mathcal{L}_2|\times|\mathcal{T}_g|$ submatrix $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g}$ of the fundamental loop matrix of $N_{eq} = \mathcal{T}_g \cup \mathcal{L}_2 \cup \mathcal{T}_2$ with respect to the tree $\mathcal{T}_g \cup \mathcal{T}_2$ is related to the $|\mathcal{L}_2|\times|\mathcal{T}_1|$ submatrix of the fundamental loop matrix of $N_{eq} = \mathcal{T}_1 \cup \mathcal{L}_2 \cup \mathcal{T}_2$ with respect to the tree $\mathcal{T}_1 \cup \mathcal{T}_2$ by</u>

$$\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g} = \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1} \underset{\sim}{T}_{\mathcal{T}}^{-1} \tag{74}$$

<u>Proof</u>. See Appendix B.

If we substitute Eqs. (67)-(69) and (72)-(74) into Eq. (26) and identify $\underset{\sim}{R}_0 = \underset{\sim}{R}_2$, $\underset{\sim}{j}_0 = \underset{\sim}{j}_2$, $\underset{\sim}{e}_0 = \underset{\sim}{e}_2$ and $\hat{\underset{\sim}{i}}_0 = \underset{\sim}{I}_{\mathcal{L}_2}$, we would obtain the second equation of Eq. (51). Hence, we have proved that any form of diakoptic equation (Eq. (32)) is indeed a special case of the generalized diakoptic equation (Eq. (51)) with $\underset{\sim}{T}_Q = \underset{\sim}{T}_{\mathcal{T}}$ and $\underset{\sim}{T}_B = \underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2}$. This important result can be summarized as follows:

Theorem 2. There exists a one-to-one correspondence that equates any form of diakoptic analysis (as given by Eq. (32)) involving variables $\hat{\underset{\sim}{V}}$ and $\hat{\underset{\sim}{I}}_0$ to a special case of generalized hybrid analysis (as given by Eq. (51)) involving variables $\hat{\underset{\sim}{V}}_{\mathcal{T}_1} = \underset{\sim}{T}_{\mathcal{T}}\underset{\sim}{V}_{\mathcal{T}_1}$ and $\hat{\underset{\sim}{I}}_{\mathcal{L}_2} = \underset{\sim}{I}_{\mathcal{L}_2}$.

Now that we have proved diakoptic analysis is a special case of generalized hybrid analysis, there is no point pursuing any further the algorithm derived in the preceding section since hybrid analysis is much simpler to implement in practice. From the computational efficiency point of view, we will choose $\mathcal{T}_g$ to be the node-to-datum case so that

$$\hat{\underset{\sim}{V}}_{\mathcal{T}_1} = \underset{\sim}{T}_Q\,\underset{\sim}{V}_{\mathcal{T}_1} = \underset{\sim}{T}_{\mathcal{T}}\,\underset{\sim}{V}_{\mathcal{T}_1} = \underset{\sim}{V}_n \tag{77}$$

where $\underset{\sim}{T}_{\mathcal{T}}$ represents the "node-to-datum" open loops with branches restricted to any tree of $\mathcal{N}_1 = \mathcal{L}_1 \cup \mathcal{T}_1$, and where $\underset{\sim}{V}_n$ denotes the "node-to-datum" voltage vector.

Observe that since $\mathcal{L}_1 \cup \mathcal{T}_1 = N_1 \cup N_2 \cup \ldots \cup N_m$, a datum node must be assigned to each subnetwork $N_k$ and the vector $\underset{\sim}{V}_n$ will consist of the union of the node-to-datum voltage vectors of the m subnetworks. With this choice of $\mathcal{T}_1$, the coboundary matrix $\hat{\underset{\sim}{Q}}_1$ in Eq. (51) is just the direct sum of the reduced incidence matrix $\underset{\sim}{A}(N_1)$, $\underset{\sim}{A}(N_2)\ldots,\underset{\sim}{A}(N_m)$ of the individual subnetworks $N_1, N_2,\ldots,N_m$; namely,

$$\hat{\mathbf{Q}}_1 = \begin{bmatrix} \mathbf{A}(N_1) & & & \mathbf{0} \\ & \mathbf{A}(N_2) & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{A}(N_m) \end{bmatrix} \triangleq \mathbf{A}_1 \tag{78}$$

The corresponding hybrid equation is then given by

$$\boxed{\begin{bmatrix} \mathbf{A}_1\mathbf{G}_1\mathbf{A}_1^t & -\hat{\mathbf{C}}^t \\ \hat{\mathbf{C}} & \mathbf{B}_2\mathbf{R}_2\mathbf{B}_2^t \end{bmatrix} \begin{bmatrix} \mathbf{V}_n \\ \mathbf{I}_{\mathcal{L}_2} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1(\mathbf{j}_1 - \mathbf{G}_1\mathbf{e}_1) \\ \mathbf{B}_2(\mathbf{e}_2 - \mathbf{R}_2\mathbf{j}_2) \end{bmatrix}} \tag{79}$$

where $\mathbf{G}_1$ is defined earlier in Eq. (28). An efficient algorithm for constructing the matrices $\mathbf{A}_1$, $\mathbf{B}_2$ and $\hat{\mathbf{C}}$ in Eq. (79) as well as some special cases [8,12,15], including the tableau implementation, of Eqs. (51) and (79) are given in Appendix E.

We close this section by pointing out that if, instead of $\mathcal{L}_2 \cup \mathcal{T}_2$, we choose $N_0 = \mathcal{L}_1 \cup \mathcal{T}_1$ as the torn branches, and interpret Eq. (46) before Eq. (44), then the resulting hybrid analysis will be shown in Appendix C to be the <u>codiakoptic analysis</u> as proposed by Onodera [13]. Moreover, a further generalization by Onodera [14] - the so-called <u>multi-stage diakoptic analysis</u> - will also be shown in Appendix D to be no more than another special case of our generalized hybrid analysis.

## 5. <u>Nonlinear Diakoptic Analysis and Solution Algorithm</u>

So far we have proved rigorously that diakoptic analysis of linear resistive networks is but a special case of <u>generalized hybrid analysis</u>. To extend diakoptic analysis to large scale <u>nonlinear resistive</u> networks, it suffices to formulate an efficient algorithm for implementing a generalized hybrid analysis of nonlinear resistive networks.

Let N denote a nonlinear resistive network and let $\mathcal{G}$ be its associated graph, where each branch of $\mathcal{G}$ represents a "composite branch" as shown in Fig. 8(a). Partition the branches of $\mathcal{G}$ into two subgraphs $\mathcal{G}_{vc}$ and $\mathcal{G}_{cc}$, where $\mathcal{G}_{vc}$ denotes all branches that are voltage-controlled (Fig. 8(b)) and $\mathcal{G}_{cc}$ denotes all branches that are current-controlled (Fig. 8 (c)).[28] Let $\mathcal{T}$ be a tree — henceforth called a <u>hybrid tree</u> — of $\mathcal{G}$ chosen in such a way that it contains as many $\mathcal{G}_{vc}$ (i.e., voltage-controlled) branches as possible. Denote the $\mathcal{G}_{vc}$ branches in $\mathcal{T}$ by $\mathcal{T}_1$, and the remaining $\mathcal{G}_{vc}$ branches, which must necessarily form loops with branches in $\mathcal{T}_1$ , by $\mathcal{L}_1$. Branches of $\mathcal{G}$ which are not already in $\mathcal{L}_1 \cup \mathcal{T}_1$ must necessarily be current-controlled branches and therefore belong to $\mathcal{G}_{cc}$. Denote the $\mathcal{G}_{cc}$ branches in $\mathcal{T}$ by $\mathcal{T}_2$ and the remaining $\mathcal{G}_{cc}$ branches by $\mathcal{L}_2$. For this particular choice[29] of the "hybrid tree" $\mathcal{T}$, the fundamental loop matrix $\underset{\sim}{B}$ and the fundamental cutset matrix $\underset{\sim}{Q}$ with respect to $\mathcal{T}$ would assume the same block structure as in Eqs. (35) and (36), respectively. Besides, the <u>constitutive relations</u> of the branches can be represented by the following two nonlinear vector equations:

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[28] A composite branch is said to be voltage-controlled (resp., current-controlled) if the nonlinear resistor in the composite branch is voltage-controlled, i.e., $i_k = g_k(v_k)$ (resp., current-controlled, i.e., $v_k = r_k(i_k)$), where the nonlinear function $g_k(\cdot)$ (resp., $r_k(\cdot)$) denotes the $v_k$-$i_k$ curve of the kth resistor. In the case where the nonlinear function is <u>one-to-one</u> and <u>onto</u> (as in Fig. 8(d)), the associated composite branch can be classified as either voltage-controlled or current-controlled, depending on whether it is more advantageous to include it as an element of $\mathcal{G}_{vc}$ or $\mathcal{G}_{cc}$.

[29] In Section 4, the choice of the tree $\mathcal{T}$ for the linear case is much simpler than our choice here. This is because each linear resistor is both voltage-controlled and current controlled and hence Eqs. (80) and (81) are well defined under all possible choices of the hybrid tree.

$$\begin{bmatrix} i_{\mathcal{L}_1} \\ \\ i_{\mathcal{T}_1} \end{bmatrix} = g_1\left(\begin{bmatrix} v_{\mathcal{L}_1} \\ \\ v_{\mathcal{T}_1} \end{bmatrix}\right) \tag{80}$$

$$\begin{bmatrix} v_{\mathcal{L}_2} \\ \\ v_{\mathcal{T}_2} \end{bmatrix} = r_2\left(\begin{bmatrix} i_{\mathcal{L}_2} \\ \\ i_{\mathcal{T}_2} \end{bmatrix}\right) \tag{81}$$

To simplify our notation in the following derivation, we shall assume that $\mathcal{L}_1 \cup \mathcal{T}_1$ contains only one connected component. The same result, of course, applies to the more general case where $\mathcal{L}_1 \cup \mathcal{T}_1$ contains several separable components. If we define $Q_1$, $\hat{Q}_1$, $\hat{V}_{\mathcal{T}_1}$, $B_2$, $\hat{I}_{\mathcal{L}_2}$, $\hat{C}$, $e_1$, etc. in the same way as we defined them in the preceding section[30], and follow the same procedure leading to Eq. (51), we would obtain the following <u>generalized nonlinear hybrid equations</u>:

$$\boxed{\begin{aligned} \hat{Q}_1 g_1 \circ (\hat{Q}_1^t \hat{V}_{\mathcal{T}_1} + e_1) - \hat{C}^t \hat{I}_{\mathcal{L}_2} - \hat{Q}_1 j_1 &= 0 \\ \hat{C} \hat{V}_{\mathcal{T}_1} + \hat{B}_2 r_2 \circ (\hat{B}_2^t \hat{I}_{\mathcal{L}_2} + j_2) - \hat{B}_2 e_2 &= 0 \end{aligned}} \tag{82}$$

where "$\circ$" denotes the "composition" operation. Observe that if N is a linear resistive network, then the nonlinear vector function $g_1(\cdot)$ reduces to $g_1(v_1) = G_1 v_1$, where $G_1$ is the associated branch conductance matrix, and the nonlinear vector function $r_2(\cdot)$ reduces to $r_2(i_2) = R_2\, i_2$, where $R_2$ is the associated branch resistance matrix. In this case, Eq. (82) clearly reduces to Eq. (51). Notice that Eq. (82) remains valid if two or more nonlinear resistors belonging to $\mathcal{L}_1 \cup \mathcal{T}_1$ are "coupled" with each other ---

[30] See Eqs. (45), (47), (49), (50), (52), (53) and (54).

such as the Ebers-Moll equation of a transistor. The same observation applies to all resistors belonging to $\mathcal{L}_2 \cup \mathcal{T}_2$.

Although Eq. (82) can be solved by any efficient algorithm for finding solution to nonlinear algebraic equations, we will propose a Newton-Raphson iteration scheme such that each stage of the iteration is equivalent to that of solving an associated linearized resistive network N', having the same topology as N, by the generalized hybrid analysis. Once this goal is achieved, highly efficient sparse matrix techniques for solving linear algebraic equations may be brought to bear on the linearized network N'.

Let us first recast Eq. (82) into a single vector equation:

$$\underset{\sim}{\mathcal{H}}\left(\begin{bmatrix} \hat{\underset{\sim}{V}}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}_{\mathcal{L}_2} \end{bmatrix}\right) \triangleq \begin{bmatrix} \hat{\underset{\sim}{Q}}_1 \underset{\sim}{g}_1 \circ \{\hat{\underset{\sim}{Q}}_1^t \hat{\underset{\sim}{V}}_{\mathcal{T}_1} + \underset{\sim}{e}_1\} - \hat{\underset{\sim}{C}}^t \hat{\underset{\sim}{I}}_{\mathcal{L}_2} - \hat{\underset{\sim}{Q}}_1 \underset{\sim}{j}_1 \\ \hat{\underset{\sim}{C}}\, \hat{\underset{\sim}{V}}_{\mathcal{T}_1} + \hat{\underset{\sim}{B}}_2 \underset{\sim}{r}_2 \circ \{\hat{\underset{\sim}{B}}_2^t \hat{\underset{\sim}{I}}_{\mathcal{L}_2} + \underset{\sim}{j}_2\} - \hat{\underset{\sim}{B}}_2 \underset{\sim}{e}_2 \end{bmatrix} = \underset{\sim}{0} \qquad (83)$$

Applying the Newton-Raphson algorithm, we obtain

$$\begin{bmatrix} \hat{\underset{\sim}{V}}^{n+1}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}^{n+1}_{\mathcal{L}_2} \end{bmatrix} = \begin{bmatrix} \hat{\underset{\sim}{V}}^{n}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}^{n}_{\mathcal{L}_2} \end{bmatrix} - \left[ \left. \frac{\partial \underset{\sim}{\mathcal{H}}\left(\begin{bmatrix} \hat{\underset{\sim}{V}}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}_{\mathcal{L}_2} \end{bmatrix}\right)}{\partial \begin{bmatrix} \hat{\underset{\sim}{V}}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}_{\mathcal{L}_2} \end{bmatrix}} \right|_{\begin{bmatrix} \hat{\underset{\sim}{V}}^{n}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}^{n}_{\mathcal{L}_2} \end{bmatrix}} \right]^{-1} \underset{\sim}{\mathcal{H}}\left(\begin{bmatrix} \hat{\underset{\sim}{V}}^{n}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}^{n}_{\mathcal{L}_2} \end{bmatrix}\right) \qquad (84)$$

If we define

$$\underset{\sim}{G}_1^n \triangleq \left. \frac{\partial \underset{\sim}{g}_1(\underset{\sim}{v}_1)}{\partial \underset{\sim}{v}_1} \right|_{\underset{\sim}{v}_1^n \triangleq \hat{\underset{\sim}{Q}}_1^t \hat{\underset{\sim}{V}}^n_{\mathcal{T}_1} + \underset{\sim}{e}_1} \qquad (85)$$

and

$$\underset{\sim}{R}_2^n \triangleq \left.\frac{\partial\, \underset{\sim}{r}_2(\underset{\sim}{i}_2)}{\partial \underset{\sim}{i}_2}\right|_{\underset{\sim}{i}_2^n \triangleq \hat{\underset{\sim}{B}}_2^t \hat{\underset{\sim}{I}}^n_{\mathcal{L}_2} + \underset{\sim}{j}_2} \tag{86}$$

then the Jacobian matrix in Eq. (84) can be rewritten as follows:

$$\underset{\sim}{J}\left(\begin{bmatrix} \hat{\underset{\sim}{V}}^n_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}^n_{\mathcal{L}_2} \end{bmatrix}\right) \triangleq \left[\left.\frac{\partial \underset{\sim}{\mathcal{H}}\left(\begin{bmatrix} \hat{\underset{\sim}{V}}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}_{\mathcal{L}_2} \end{bmatrix}\right)}{\partial \begin{bmatrix} \hat{\underset{\sim}{V}}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}_{\mathcal{L}_2} \end{bmatrix}}\right|_{\begin{bmatrix} \hat{\underset{\sim}{V}}^n_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}^n_{\mathcal{L}_2} \end{bmatrix}}\right] = \begin{bmatrix} \hat{\underset{\sim}{Q}}_1 \underset{\sim}{G}_1^n \hat{\underset{\sim}{Q}}_1^t & -\hat{\underset{\sim}{C}}^t \\ \hat{\underset{\sim}{C}} & \hat{\underset{\sim}{B}}_2 \underset{\sim}{R}_2^n \hat{\underset{\sim}{B}}_2^t \end{bmatrix} \tag{87}$$

Premultiplying both sides of Eq. (85) by this Jacobian matrix and making use of Eq. (85), we obtain

$$\begin{bmatrix} \hat{\underset{\sim}{Q}}_1 \underset{\sim}{G}_1^n \hat{\underset{\sim}{Q}}_1^t & -\hat{\underset{\sim}{C}}^t \\ \hat{\underset{\sim}{C}} & \hat{\underset{\sim}{B}}_2 \underset{\sim}{R}_2^n \hat{\underset{\sim}{B}}_2^t \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}^{n+1}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}^{n+1}_{\mathcal{L}_2} \end{bmatrix} = \begin{bmatrix} \hat{\underset{\sim}{Q}}_1 \underset{\sim}{G}_1^n \hat{\underset{\sim}{Q}}_1^t & -\hat{\underset{\sim}{C}}^t \\ \hat{\underset{\sim}{C}} & \hat{\underset{\sim}{B}}_2 \underset{\sim}{R}_2^n \hat{\underset{\sim}{B}}_2^t \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}^n_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}^n_{\mathcal{L}_2} \end{bmatrix} - \mathcal{H}\left(\begin{bmatrix} \hat{\underset{\sim}{V}}^n_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}^n_{\mathcal{L}_2} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \hat{\underset{\sim}{Q}}_1 \underset{\sim}{G}_1^n \hat{\underset{\sim}{Q}}_1^t \hat{\underset{\sim}{V}}^n_{\mathcal{T}_1} - \hat{\underset{\sim}{C}}^t \hat{\underset{\sim}{I}}^n_{\mathcal{L}_2} - \hat{\underset{\sim}{Q}}_1 \underset{\sim}{g}_1 \circ (\hat{\underset{\sim}{Q}}_1^t \hat{\underset{\sim}{V}}^n_{\mathcal{T}_1} + \underset{\sim}{e}_1) + \hat{\underset{\sim}{C}}^t \hat{\underset{\sim}{I}}^n_{\mathcal{L}_2} + \hat{\underset{\sim}{Q}}_1 \underset{\sim}{j}_1 \\ \hat{\underset{\sim}{C}} \hat{\underset{\sim}{V}}^n_{\mathcal{T}_1} + \hat{\underset{\sim}{B}}_2 \underset{\sim}{R}_2 \hat{\underset{\sim}{B}}_2^t \hat{\underset{\sim}{I}}^n_{\mathcal{L}_2} - \hat{\underset{\sim}{C}} \hat{\underset{\sim}{V}}^n_{\mathcal{T}_1} - \hat{\underset{\sim}{B}}_2 \underset{\sim}{r}_2 \circ (\hat{\underset{\sim}{B}}_2^t \hat{\underset{\sim}{I}}^n_{\mathcal{L}_2} + \underset{\sim}{j}_2) + \hat{\underset{\sim}{B}}_2 \underset{\sim}{e}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\underset{\sim}{Q}}_1 \underset{\sim}{G}_1^n (\underset{\sim}{v}_1^n - \underset{\sim}{e}_1) - \hat{\underset{\sim}{Q}}_1 \underset{\sim}{i}_1^n + \hat{\underset{\sim}{Q}}_1 \underset{\sim}{j}_1 \\ \hat{\underset{\sim}{B}}_2 \underset{\sim}{R}_2^n (\underset{\sim}{i}_2^n - \underset{\sim}{j}_2) - \hat{\underset{\sim}{B}}_2 \underset{\sim}{v}_2^n + \hat{\underset{\sim}{B}}_2 \underset{\sim}{e}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\underset{\sim}{Q}}_1 (\underset{\sim}{j}_1^n - \underset{\sim}{G}_1^n \underset{\sim}{e}_1^n) \\ \hat{\underset{\sim}{B}}_2 (\underset{\sim}{e}_2^n - \underset{\sim}{R}_2^n \underset{\sim}{j}_2^n) \end{bmatrix}$$

where

$$\underset{\sim}{v}_1^n \triangleq \hat{\underset{\sim}{Q}}_1^t \hat{\underset{\sim}{v}}^n_{\mathcal{T}_1} + \underset{\sim}{e}_1,\ \underset{\sim}{i}_1^n \triangleq \underset{\sim}{g}_1(\underset{\sim}{v}_1^n),\ \underset{\sim}{i}_2^n \triangleq \hat{\underset{\sim}{B}}_2^t \hat{\underset{\sim}{I}}^n_{\mathcal{L}_2} + \underset{\sim}{j}_2,\ \underset{\sim}{v}_2^n \triangleq \underset{\sim}{r}_2(\underset{\sim}{i}_2^n) \tag{88}$$

$$\underset{\sim}{j}_1^n \triangleq \underset{\sim}{j}_1 - \underset{\sim}{i}_1^n,\ \underset{\sim}{e}_1^n \triangleq \underset{\sim}{e}_1 - \underset{\sim}{v}_1^n,\ \underset{\sim}{e}_2^n \triangleq \underset{\sim}{e}_2 - \underset{\sim}{v}_2^n,\ \underset{\sim}{j}_2^n \triangleq \underset{\sim}{j}_2 - \underset{\sim}{i}_2^n \tag{89}$$

Hence, each stage of the above Newton-Raphson iteration scheme can be rewritten as

$$\boxed{\begin{bmatrix} \hat{\underset{\sim}{Q}}_1 \underset{\sim}{G}_1^n \hat{\underset{\sim}{Q}}_1^t & -\hat{\underset{\sim}{C}}^t \\ \hat{\underset{\sim}{C}} & \hat{\underset{\sim}{B}}_2 \underset{\sim}{R}_2^n \hat{\underset{\sim}{B}}_2^t \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}^{n+1}_{\mathcal{T}_1} \\ \hat{\underset{\sim}{I}}^{n+1}_{\mathcal{L}_2} \end{bmatrix} = \begin{bmatrix} \hat{\underset{\sim}{Q}}_1(\underset{\sim}{j}_1^n - \underset{\sim}{G}_1^n \underset{\sim}{e}_1^n) \\ \hat{\underset{\sim}{B}}_2(\underset{\sim}{e}_2^n - \underset{\sim}{R}_2^n \underset{\sim}{j}_2^n) \end{bmatrix}} \tag{90}$$

where the vectors $\underset{\sim}{j}_1^n$, $\underset{\sim}{e}_1^n$, etc. are defined via Eqs. (88) and (89).

Comparing Eq. (90) with Eq. (51), we conclude that each stage of the Newton-Raphson iteration of Eq. (83) is <u>equivalent</u> to that of solving a linearized resistive network $N^n$ —— having <u>the same topology</u> as N but with <u>updated</u> branch conductance matrix $\underset{\sim}{G}_1^n$, branch resistance matrix $\underset{\sim}{R}_2^n$, current source vectors $\underset{\sim}{j}_1^n$ and $\underset{\sim}{j}_2^n$, voltage source vectors $\underset{\sim}{e}_1^n$ and $\underset{\sim}{e}_2^n$ as defined by Eqs. (85), (86), (88) and (89) —— by generalized hybrid analysis.[31] It is important to observe that instead of evaluating the large Jacobian matrix $\underset{\sim}{J}(\cdot)$ of the nonlinear vector function $\underset{\sim}{\mathcal{H}}(\cdot)$ directly by numerical differentiation, we first <u>linearize</u> each nonlinear resistor and characterize the resistors in $\mathcal{L}_1 \cup \mathcal{T}_1$ by an <u>incremental conductance</u>

[31] This linearized resistive network $N^n$ is also known as a "companion model"[23] and as a "discretized circuit model" [19].

matrix $G_1^n$ (Eq. (85)) and the resistors in $\mathcal{L}_2 \cup \mathcal{T}_2$ by an incremental resistance matrix $R_2^n$ (Eq. (86)).[32] The Jacobian matrix is then obtained from Eq. (87). In other words, our algorithm calls for linearizing the resistors at the branch level. Of course, after each iteration, the matrices and vectors defined in Eqs. (85), (86), (88) and (89) must be updated to account for the change in the operating point $\begin{bmatrix} \hat{V}_{\mathcal{T}_1} \\ \hat{I}_{\mathcal{L}_2} \end{bmatrix}$.

In the general case where $\mathcal{L}_1 \cup \mathcal{T}_1$ consists of several separable components, the submatrix $\hat{Q}_1 G_1^n \hat{Q}_1^t$ in Eq. (90) assumes the same block diagonal structure as in Fig. 7. Hence, it remains for us to devise an efficient algorithm for solving Eq. (90) which takes full advantage of this highly desirable sparse structure.

We shall proceed by first defining the standard LU-decomposition method [19] and, then, by utilizing the block diagonal structure of Eq. (90), we shall define an efficient solution algorithm for Eq. (82).

For convenience, let us define a matrix $H^n$ as

$$H^n \triangleq \begin{bmatrix} H_{11}^n & H_{12}^n \\ H_{21}^n & H_{22}^n \end{bmatrix} \triangleq \begin{bmatrix} \hat{Q}_1 G_1^n \hat{Q}_1^t & -\hat{C}^t \\ \hat{C} & \hat{B}_2 R_2^n \hat{B}_2^t \end{bmatrix} \tag{91}$$

Substituting Eq. (91) into Eq. (90), we obtain

$$\begin{bmatrix} H_{11}^n & H_{12}^n \\ H_{21}^n & H_{22}^n \end{bmatrix} \begin{bmatrix} \hat{V}_{\mathcal{T}_1}^{n+1} \\ \hat{I}_{\mathcal{L}_2}^{n+1} \end{bmatrix} = \begin{bmatrix} S_1^n \\ S_2^n \end{bmatrix} \tag{92}$$

[32] The matrices $G_1^n$ and $R_2^n$ are diagonal matrices if the resistors are uncoupled. However, if some resistors are coupled to each other, or if N contains linear or nonlinear controlled sources, then $G_1^n$ and $R_1^n$ will contain off diagonal elements.

where

$$\underset{\sim}{S}_1^n \triangleq \hat{\underset{\sim}{Q}}_1(\underset{\sim}{j}_1^n - \underset{\sim}{G}_1^n\underset{\sim}{e}_1^n) \text{ and } \underset{\sim}{S}_2^n \triangleq \hat{\underset{\sim}{B}}_2(\underset{\sim}{e}_2^n - \underset{\sim}{R}_2^n\underset{\sim}{j}_2^n) \tag{93}$$

Let us LU-factorize the matrix $\underset{\sim}{H}^n$, then

$$\begin{bmatrix} \underset{\sim}{H}_{11}^n & \underset{\sim}{H}_{12}^n \\ \underset{\sim}{H}_{21}^n & \underset{\sim}{H}_{22}^n \end{bmatrix} = \begin{bmatrix} \underset{\sim}{L}_{11}^n & \underset{\sim}{0}_{12} \\ \underset{\sim}{L}_{21}^n & \underset{\sim}{L}_{22}^n \end{bmatrix} \begin{bmatrix} \underset{\sim}{U}_{11}^n & \underset{\sim}{U}_{12}^n \\ \underset{\sim}{0}_{21} & \underset{\sim}{U}_{22}^n \end{bmatrix} = \begin{bmatrix} \underset{\sim}{L}_{11}^n\underset{\sim}{U}_{11}^n & \underset{\sim}{L}_{11}^n\underset{\sim}{U}_{12}^n \\ \underset{\sim}{L}_{21}^n\underset{\sim}{U}_{11}^n & \underset{\sim}{L}_{21}^n\underset{\sim}{U}_{12}^n + \underset{\sim}{L}_{22}^n\underset{\sim}{U}_{22}^n \end{bmatrix} \tag{94}$$

where $\underset{\sim}{L}_{11}^n$ and $\underset{\sim}{L}_{22}^n$ are <u>lower triangular matrices</u>, $\underset{\sim}{U}_{11}^n$ and $\underset{\sim}{U}_{22}^n$ are <u>unit upper triangular matrices</u>. From Eq. (94), we can identify the following relations:

$$\underset{\sim}{H}_{11}^n = \underset{\sim}{L}_{11}^n\underset{\sim}{U}_{11}^n \tag{95}$$

$$\underset{\sim}{H}_{12}^n = \underset{\sim}{L}_{11}^n\underset{\sim}{U}_{12}^n \tag{96}$$

$$\underset{\sim}{H}_{21}^n = \underset{\sim}{L}_{21}^n\underset{\sim}{U}_{11}^n \tag{97}$$

$$\underset{\sim}{H}_{22}^n = \underset{\sim}{L}_{21}^n\underset{\sim}{U}_{12}^n + \underset{\sim}{L}_{22}^n\underset{\sim}{U}_{22}^n \tag{98}$$

Given $\underset{\sim}{H}^n$, these relations will be used later on to find the matrices $\underset{\sim}{L}_{11}^n$, $\underset{\sim}{U}_{11}^n$, etc.

Substituting Eq. (94) into Eq. (92), we obtain

$$\begin{bmatrix} \underset{\sim}{L}_{11}^n & \underset{\sim}{0}_{12} \\ \underset{\sim}{L}_{21}^n & \underset{\sim}{L}_{22}^n \end{bmatrix} \begin{bmatrix} \underset{\sim}{U}_{11}^n & \underset{\sim}{U}_{12}^n \\ \underset{\sim}{0}_{21} & \underset{\sim}{U}_{22}^n \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_{\mathcal{T}_1}^{n+1} \\ \hat{\underset{\sim}{I}}_{\mathcal{L}_2}^{n+1} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{S}_1^n \\ \underset{\sim}{S}_2^n \end{bmatrix} \tag{99}$$

Defining the intermediate variables

$$\begin{bmatrix} \underset{\sim}{x}^{n+1} \\ \underset{\sim}{y}^{n+1} \end{bmatrix} \triangleq \begin{bmatrix} \underset{\sim}{U}_{11}^n & \underset{\sim}{U}_{12}^n \\ \underset{\sim}{0}_{21} & \underset{\sim}{U}_{22}^n \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_{\mathcal{T}_1}^{n+1} \\ \hat{\underset{\sim}{I}}_{\mathcal{L}_2}^{n+1} \end{bmatrix} \tag{100}$$

and substituting Eq. (100) into Eq. (99), we obtain

$$\begin{bmatrix} L_{11}^n & 0_{12} \\ L_{21}^n & L_{22}^n \end{bmatrix} \begin{bmatrix} x^{n+1} \\ y^{n+1} \end{bmatrix} = \begin{bmatrix} S_1^n \\ S_2^n \end{bmatrix} \tag{101}$$

Hence, Eq. (99) can be solved via two simple substitutions - <u>forward substitution</u> via Eq. (101) and <u>backward substitution</u> via Eq. (100). In particular, Eq. (101) can be rewritten in the form

$$L_{11}^n \, x^{n+1} = S_1^n \tag{102}$$

$$L_{22}^n \, y^{n+1} = S_2^n - L_{21}^n \, x^{n+1} \tag{103}$$

to emphasize the "forward substitution procedure." Similarly, Eq. (100) can be rewritten in the form

$$U_{22}^n \, \hat{I}_{\mathcal{L}_2}^{n+1} = y^{n+1} \tag{104}$$

$$U_{11}^n \, \hat{V}_{\mathcal{J}_1}^{n+1} = x^{n+1} - U_{12}^n \, \hat{I}_{\mathcal{L}_2}^{n+1} \tag{105}$$

to emphasize the "backward substitution procedure."

Observe that since the matrix $H_{11}^n$ (as in Fig. 7) has a block diagonal structure, the associated factorized matrices $L_{11}^n$ and $U_{11}^n$ will also have the same block diagonal structure. In other words, the LU-factorization of all blocks of $H_{11}^n$ can be done simultaneously. Besides, Eqs. (102) and (105) each will contain a set of "uncoupled" blocks of equations.

The preceding observations can now be used to devise the following <u>efficient nonlinear diakoptic algorithms</u>:

Assume $\mathcal{L}_1 \cup \mathcal{T}_1$ consist of "m" separable subnetworks $N_1, N_2, \ldots, N_m$.

Step 0. Initialization: Given $\hat{\underset{\sim}{I}}^0_{\mathcal{L}_2}$ and $\hat{\underset{\sim}{V}}^0_{\mathcal{T}_1}(N_k)$. Find topological matrices $\hat{\underset{\sim}{Q}}_1(N_k)$, $\hat{\underset{\sim}{C}}(N_k)$ and $\hat{\underset{\sim}{B}}_2$, $k = 1,2,\ldots,m$. Set n=0.

Step 1. Construct $\underset{\sim}{H}^n_{11}(N_k) \triangleq \hat{\underset{\sim}{Q}}_1(N_k)\ \underset{\sim}{G}^n_1\ \hat{\underset{\sim}{Q}}^t_1(N_k)$ and LU-factorize $\underset{\sim}{H}^n_{11}(N_k)$ into $\underset{\sim}{L}^n_{11}(N_k)\ \underset{\sim}{U}^n_{11}(N_k)$, $k = 1,2,\ldots,m$.

Step 2. Obtain $\underset{\sim}{U}^n_{12}(N_k)$ and $\underset{\sim}{L}^n_{21}(N_k)$ by solving Eqs. (96) and (97), respectively, where $\underset{\sim}{H}^n_{12}(N_k) \triangleq -\hat{\underset{\sim}{C}}^t(N_k)$ and $\underset{\sim}{H}^n_{21}(N_k) \triangleq \hat{\underset{\sim}{C}}(N_k)$, $k = 1, 2,\ldots,m$.

Step 3. Obtain $\underset{\sim}{x}^{n+1}(N_k)$ by forward substitution via Eq. (102), $k = 1,2, \ldots,m$.

Step 4. Compute $\underset{\sim}{D}^n(N_k) \triangleq \underset{\sim}{L}^n_{21}(N_k)\ \underset{\sim}{U}^n_{12}(N_k)$ and $\underset{\sim}{S}^n(N_k) \triangleq \underset{\sim}{L}^n_{21}(N_k)\ \underset{\sim}{x}^{n+1}(N_k)$, $k = 1,2,\ldots,m$.

Step 5. Construct $\underset{\sim}{H}^n_{22} \triangleq \hat{\underset{\sim}{B}}_2 \underset{\sim}{R}^n_2 \hat{\underset{\sim}{B}}^t_2$ and compute $\hat{\underset{\sim}{D}}^n \triangleq \underset{\sim}{H}^n_{22} - \sum_{k=1}^{m} \underset{\sim}{D}^n(N_k)$, $\hat{\underset{\sim}{S}}^n_2 \triangleq \underset{\sim}{S}^n_2 - \sum_{k=1}^{m} \underset{\sim}{S}^n(N_k)$.

Step 6. LU-factorize $\hat{\underset{\sim}{D}}^n$ into $\underset{\sim}{L}^n_{22}\underset{\sim}{U}^n_{22}$.

Step 7. Obtain $\underset{\sim}{y}^{n+1}$ by forward substitution via $\underset{\sim}{L}^n_{22}\ \underset{\sim}{y}^{n+1} = \hat{\underset{\sim}{S}}^n_2$.

Step 8. Obtain $\hat{\underset{\sim}{I}}^{n+1}_{\mathcal{L}_2}$ by backward substitution via Eq. (104).

Step 9. Compute $\hat{\underset{\sim}{S}}^n_1(N_k) \triangleq \underset{\sim}{x}^{n+1}(N_k) - \underset{\sim}{U}^n_{12}(N_k)\ \hat{\underset{\sim}{I}}^{n+1}_{\mathcal{L}_2}$, $k = 1,2,\ldots,m$.

Step 10. Obtain $\hat{\underset{\sim}{V}}^{n+1}_{\mathcal{T}_1}(N_k)$ by backward substitution via $\underset{\sim}{U}^n_{11}(N_k)\ \hat{\underset{\sim}{V}}^{n+1}_{\mathcal{T}_1}(N_k) = \hat{\underset{\sim}{S}}_1(N_k)$, $k = 1,2,\ldots,m$.

Step 11. Termination: If $\hat{\underset{\sim}{I}}^{n+1}_{\mathcal{L}_2}$ and $\hat{\underset{\sim}{V}}^{n+1}_{\mathcal{T}_1}$ converge, stop! Otherwise, set $n = n+1$, update $\underset{\sim}{G}^n_1$, $\underset{\sim}{R}^n_2$, $\underset{\sim}{j}^n_1$, $\underset{\sim}{j}^n_2$, $\underset{\sim}{e}^n_1$ and $\underset{\sim}{e}^n_2$, and go to Step 1.

Remarks. (i) Steps 1,2,3,4,9 and 10 each involves "m" decoupled computations and this can be carried out by parallel computation on "m" small computers simultaneously. (ii) Steps 2,3,7,8 and 10 each involves only simple substitutions. (iii) Steps 1 and 6, i.e., the LU-factorizations, are the only time consuming steps. However, if we choose the node-to-datum case as in Eq. (79), each block $\underset{\sim}{H}^n_{11}(N_k) \triangleq \underset{\sim}{A}_1(N_k)\ \underset{\sim}{G}^n_1(N_k)\ \underset{\sim}{A}^t_1(N_k)$ is then guaranteed to be very sparse and easy to construct, where $\underset{\sim}{A}_1(N_k)$ is the reduced incidence matrix of $N_k$. Hence, the available sparse matrix techniques [19] can make Step 1 very efficient. (iv) The matrix $\hat{\underset{\sim}{D}}^n$ is obtained through matrix multiplications and additions and is, in general, a "full" matrix (i.e., it contains mostly nonzero elements). Hence, to improve the efficiency of our algorithm, it is desirable to make the dimension of $\hat{\underset{\sim}{D}}^n$ (i.e., the dimension of $\underset{\sim}{H}^n_{22}$) as small as possible. In other words, we want to minimize the number of torn branches.

6. Concluding Remark

We have shown that all published versions[33] of diakoptic analysis can be derived via a sequence of equivalent circuit transformations involving only such basic circuit-theoretic concepts as the substitution theorem, the source-shifting theorem and the properties of open loops.

This analysis always consists of three stages. The first stage involves the choice of the torn branches $N_0$. No efficient algorithm is known for choosing an optimal set of torn branches [24-26] and the current state of art is such that this step must be implemented on an ad hoc basis. The second stage involves an "appropriate" form of coboundary analysis on each separable subnetwork $N_1,N_2,\ldots,N_m$ (The most efficient overall choice remains to be the node-to-datum case). The third stage involves a standard

[33] Our derivation of diakoptic analysis is completely general and should include any future versions as special cases.

loop analysis on an equivalent interconnected network $N_{eq}$ which combines "m" coupled but acyclic equivalent subnetworks $N_{1_{eq}}$, $N_{2_{eq}}$, ..., $N_{m_{eq}}$ with the torn branches $N_0$. This cumbersome three-stage formulation process has been shown to be a special case of the much simpler generalized hybrid analysis. It follows immediately from this derivation that the concept of diakoptic analysis is applicable to both linear and nonlinear large-scale resistive networks.

An efficient algorithm for solving the generalized nonlinear hybrid equation is included in this paper. Two special features of this algorithm is that it deals with the associated linearized circuit model and the nonlinearities appear only at the branch level.

Finally, we must conclude, if somewhat disappointingly, that inspite of its important applications in the analysis of large-scale networks, diakoptic analysis does not represent a new concept in circuit theory.[34] Neither does it represents a new mathematical idea. It is hoped that this conclusion will forever put to rest the misleading impression that diakoptic analysis is a profound concept that can be derived only by using advanced algebraic-topological techniques.

[34] However, full credit should be given to Kron for being the first to choose both voltages and currents as independent variables. As such, his approach can be considered as the forerunner of our generalized hybrid analysis [10], [21].

## Appendix A. Proof of Lemma 2

We shall first derive two important relations and then proceed with the proof. Given a connected graph $\mathcal{G}$ with "n" nodes and "b" branches, and a generator tree $\mathcal{T}_g$ of $\mathcal{G}$, let us consider the augmented graph $\mathcal{G}_a \triangleq \mathcal{G} \cup \mathcal{T}_g$. Observe that it is a connected graph with "n" nodes and "b+n-1" branches. Define $\underset{\sim}{A}_a$ to be the $(n-1)\times(b+n-1)$ <u>reduced incidence matrix</u> of the augmented graph $\mathcal{G}_a$ with respect to an arbitrary datum node. If the columns of the reduced incidence matrix $\underset{\sim}{A}_a$ are ordered such that the original branches (i.e., those branches in $\mathcal{G}$) appear before the augmented branches (i.e., those branches in $\mathcal{T}_g$), then $\underset{\sim}{A}_a$ can be partitioned as $\underset{\sim}{A}_a = [\underset{\sim}{A} \;\; \underset{\sim}{A}_g]$, where $\underset{\sim}{A}$ denotes the first b columns representing branches in $\mathcal{G}$ and $\underset{\sim}{A}_g$ denotes the remaining (n-1) columns representing branches in $\mathcal{T}_g$.[35] Since the branches of $\mathcal{T}_g$ corresponds to a tree of the augmented graph $\mathcal{G}_a$, the submatrix $\underset{\sim}{A}_g$ is nonsingular.

Let each open loop and its associated generator be denoted by a $b\times 1$ vector[36] $\hat{\underset{\sim}{p}}{}_i^t$ and an $(n-1)\times 1$ unit vector[37] $\underset{\sim}{g}_i$, respectively. It follows from the definition of an open loop that the $(b+n-1)\times 1$ vector $\begin{bmatrix} \hat{\underset{\sim}{p}}{}_i^t \\ \underset{\sim}{g}_i \end{bmatrix}$ represents a closed loop in the augmented graph $\mathcal{G}_a$. From Tellegen's theorem [16], we know that any pair of vectors representing respectively a cutset and a loop are orthogonal to each other. In other words, we have the relation

---

[35] Observe that the submatrix $\underset{\sim}{A}$ corresponds to the reduced incidence matrix of the original graph $\mathcal{G}$.

[36] Recall that $\hat{\underset{\sim}{p}}{}^i$ was defined in Section 2 as a row vector. Therefore, we use the transpose here to conform with our earlier notation.

[37] A unit vector has all zero entries except a "1" in the i-th entry.

$$[\underset{\sim}{A} \quad \underset{\sim g}{A}] \begin{bmatrix} \hat{\underset{\sim}{p}}_i^t \\ \underset{\sim}{g}_i \end{bmatrix} = \underset{\sim}{A}\, \hat{\underset{\sim}{p}}_i^t + \underset{\sim g}{A}\, \underset{\sim}{g}_i = \underset{\sim}{0}\ , \quad i = 1,2,\ldots,n-1 \tag{106}$$

Eq. (106) can be rearranged as

$$\underset{\sim}{A}\, \hat{\underset{\sim}{p}}_i^t = -\underset{\sim g}{A}\, \underset{\sim}{g}_i\ , \quad i = 1,2,\ldots,n-1 \tag{107}$$

which will be used in the following proof.

Now that we have shown $\underset{\sim g}{A}$ to be nonsingular and have established Eq. (107), let us suppose that the (n-1) open loops $\hat{\underset{\sim}{p}}_1^t, \hat{\underset{\sim}{p}}_2^t, \ldots, \hat{\underset{\sim}{p}}_{n-1}^t$ are linearly dependent[38], then there exists a set of real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \mathbb{R}$, <u>not all zero</u>, such that $\sum_{i=1}^{n-1} \alpha_i\, \hat{\underset{\sim}{p}}_i^t = \underset{\sim}{0}$. Then

$$\underset{\sim}{A}\Big(\sum_{i=1}^{n-1} \alpha_i\, \hat{\underset{\sim}{p}}_i^t\Big) = \sum_{i=1}^{n-1} \alpha_i (\underset{\sim}{A}\, \hat{\underset{\sim}{p}}_i^t) = -\sum_{i=1}^{n-1} \alpha_i (\underset{\sim g}{A}\, \underset{\sim}{g}_i) = -\underset{\sim g}{A}\Big(\sum_{i=1}^{n-1} \alpha_i\, \underset{\sim}{g}_i\Big) = \underset{\sim}{0} \tag{108}$$

Since the only nonzero entry in $\underset{\sim}{g}_i$ is a "1", it follows that

$$\underset{\sim}{g} \triangleq \sum_{i=1}^{n-1} \alpha_i\, \underset{\sim}{g}_i = [\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]^t \tag{109}$$

Hence, if we denote the i-th column of $\underset{\sim g}{A}$ by $(\underset{\sim g}{A})_i$ and substitute Eq. (109) into Eq. (108), we would then obtain

$$\underset{\sim g}{A}\Big(\sum_{i=1}^{n-1} \alpha_i\, \underset{\sim}{g}_i\Big) = \underset{\sim g}{A}\, \underset{\sim}{g} = \sum_{i=1}^{n-1} \alpha_i (\underset{\sim g}{A})_i = \underset{\sim}{0} \tag{110}$$

since the (n-1)×(n-1) matrix $\underset{\sim g}{A}$ is nonsingular, the columns of $\underset{\sim g}{A}$ are linearly independent, i.e., $\sum_{i=1}^{n-1} \alpha_i (\underset{\sim g}{A})_i = \underset{\sim}{0}$ if, and only if, $\alpha_1 = \alpha_2 = \ldots =$

---

[38] We assume, for convenience, that the concept of linear independence is defined with respect to the "field" of real numbers $\mathbb{R}$.

$\alpha_{n-1} = 0$. It follows from Eqs. (108) and (110) that $\sum_{i=1}^{n-1} \alpha_i \, \hat{\underset{\sim}{p}}_i^t = \underset{\sim}{0}$ if, and only if, $\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 0$, which contradicts our earlier assumption. Hence, the open loops $\hat{\underset{\sim}{p}}_1^t, \hat{\underset{\sim}{p}}_2^t, \ldots, \hat{\underset{\sim}{p}}_{n-1}^t$ are indeed linearly independent. ∎

Appendix B. Proof of Lemma 8

It suffices to prove that $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g} \underset{\sim}{T}_{\mathcal{T}} = \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_1}$. Before we proceed to give the formal proof, let us consider an example first for the sake of clarity. Figure 9 shows a typical graph $\mathcal{G}$ with branches partitioned into four sets $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{T}_1$ and $\mathcal{T}_2$ where $\mathcal{L}_1 = \{1,2\}$, $\mathcal{L}_2 = \{3,4,5,6\}$, $\mathcal{T}_1 = \{7,8,9,10,11,12\}$, $\mathcal{T}_2 = \{13,14,15,16,17\}$. Also shown is an augmented generator tree (in dotted line) $\mathcal{T}_g = \{a,b,c,d,e,f\}$ connecting all nodes of $N_1 = \mathcal{L}_1 \cup \mathcal{T}_1$.

Consider a typical branch of $\mathcal{L}_2$, e.g., link 6 and observe that the fundamental loop generated by link 6 is given by twigs {17,14,7,8,9,10,11,15} with respect to the tree $\mathcal{T}_1 \cup \mathcal{T}_2$ and by twigs {17,14,b,c,f,15} with respect to the tree $\mathcal{T}_g \cup \mathcal{T}_2$. If we let row $r_6$ denote the row of $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_1}$ and $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g}$ corresponding to link 6, then row $r_6$ of $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_1}$ and $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g}$ is given respectively by

$$\text{Row } r_6 \text{ of } \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_1} = \begin{array}{c} \begin{matrix} 7 & 8 & 9 & 10 & 11 & 12 \end{matrix} \\ \begin{bmatrix} 1 & -1 & 1 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

$$\text{Row } r_6 \text{ of } \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g} = \begin{array}{c} \begin{matrix} -a & -b & -c & -d & -e & -f \end{matrix} \\ \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

Now suppose we express each branch "-k"[39] in $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g}$ in terms of the associated unique fundamental open loop,[40] namely,

$$\text{Branch } -b \quad \begin{array}{c} \begin{matrix} 7 & 8 & 9 & 10 & 11 & 12 \end{matrix} \\ \begin{bmatrix} -1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix} \end{array}, \text{ fundamental open loop of b}$$

---

[39]Recall that the reference direction of branches of the equivalent acyclic subnetwork is opposite to those of the generator tree $\mathcal{T}_g$. Since a,b,...,f represent branches of $\mathcal{T}_g$, we must choose -a,-b,...,-f to represent the branches of the equivalent acyclic subnetwork.

[40]By this we mean that both branch "-k" and the corresponding unique fundamental open loop have the same initial and final nodes.

Branch −c  [ 0 0 1 1 −1 −1 ], fundamental open loop of c

Branch −f  [ 0 0 −1 −1 1 0 ], fundamental open loop of f

Observe that if we add the negative of these three row vectors together, we would obtain row $r_6$ of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}$. But the above operation upon applying to all rows of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g}$ is equivalent to

$$
\begin{array}{l}
\text{Row } r_3 \text{ of } \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g} \\
\text{Row } r_4 \text{ of } \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g} \\
\text{Row } r_5 \text{ of } \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g} \\
\text{Row } r_6 \text{ of } \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} -a & -b & -c & -d & -e & -f \end{array} \\
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & -1 & -1 & 0 & 0 & -1
\end{bmatrix}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\begin{bmatrix}
0 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & 1 \\
0 & 0 & 1 & 1 & -1 & -1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 1 & 0
\end{bmatrix}
\end{array}
\begin{array}{l}
-a \\ -b \\ -c \\ -d \\ -e \\ -f
\end{array}
$$

$$
= \begin{array}{c}
\begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & 1 & -1 & 0
\end{bmatrix}
\end{array}
\begin{array}{l}
\text{Row } r_3 \text{ of } \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1} \\
\text{Row } r_4 \text{ of } \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1} \\
\text{Row } r_5 \text{ of } \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1} \\
\text{Row } r_6 \text{ of } \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}
\end{array}
$$

In matrix form, we have $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g} \underset{\sim}{T}_{\mathcal{T}} = \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}$, where $\underset{\sim}{T}_{\mathcal{T}}$ is the nonsingular submatrix of the fundamental open loop $\underset{\sim}{T}$ with respect to the tree $\mathcal{T}_1$. Hence, we have shown via an example that Lemma 8 holds.

We can now formalize the above algorithm to an arbitrary graph $\mathcal{G}$ and observe that each row of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}$ defines a path of branches in $\mathcal{T}_1$. Let the path corresponding to row j of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}$ have end nodes from $n_{\alpha}$ to $n_{\beta}$. Since row j of $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}$ and row j of $\underset{\sim}{B}_2 \triangleq [\underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2} \;\; \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_2}]$ form a closed loop, row j of $\underset{\sim}{B}_2$ must correspond to a path of branches in $\mathcal{L}_2 \cup \mathcal{T}_2$ from node $n_{ⓑ}$ to node $n_{ⓐ}$. Observe that row j of $\underset{\sim}{B}_2$ and row j of $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g}$ also from a closed loop, and hence row j of $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g}$ must correspond to a path of branches in $\mathcal{T}_g$ from node $n_{ⓐ}$ to node $n_{ⓑ}$. Now observe that each row j of $\underset{\sim}{T}_{\mathcal{T}}$ expresses a path of branches in $\mathcal{T}_1$ that has same end nodes as branch j of $\mathcal{T}_g$. Hence, row j of $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g} \underset{\sim}{T}_{\mathcal{T}}$ is a linear combination of those fundamental open loops induced by branches in $\mathcal{T}_g$ of row j of $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g}$. The resulting branches must necessarily form a path from node $n_{ⓐ}$ to node $n_{ⓑ}$. Since $\mathcal{T}_1$ is a tree, there exists a unique path with fixed end nodes $n_{ⓐ}$ and $n_{ⓑ}$. Hence, row j of $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_1}$ and row j of $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g} \underset{\sim}{T}_{\mathcal{T}}$ must coincide and we have $\underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_1} = \underset{\sim}{B}_{\mathcal{L}_2 \mathcal{T}_g} \underset{\sim}{T}_{\mathcal{T}}$. ∎

## Appendix C. Codiakoptic Analysis

We shall proceed to derive codiakoptic analysis via the same equivalent circuit approach. Although our derivation is again completely general, a specific example will be used to illustrate some of the "subtle" points in each step of the equivalent transformations.

Given a network N, let $N_0$ be a subset of branches —— henceforth called the (codiakoptic) torn branches[41] —— such that when they are contracted[42], the original network is reduced to "m" separable subnetworks $N_1, N_2, \ldots, N_m$. In Fig. 10(a), each branch is assumed to represent a composite branch (Fig. 4(a)). Let the set of branches {7,8,9,10} denote the torn branches $N_0$ whose contraction results in the two separated subnetworks $N_1$ = {1,2, 3,4,5,6} and $N_2$ = {11,12,13,14,15,16}.

Assuming the voltage solution waveform of each "connection branch" of $N_{0-k}$ (from Fig. 10(a), we have $N_{0-1}$ = {7} and $N_{0-2}$ = {10}) is given a priori, we can apply the voltage source substitution theorem and replace each connection branch of $N_{0-k}$ by its equivalent voltage source (Fig. 10(b)). Since the substituted voltage sources summarize the "outside" influence upon each subnetwork, the original network N can be separated into (k+1) parts without affecting the solutions of the original network (Fig. 10(c)).

Let us now focus our attention on subnetwork $N_1$ which has "n" nodes and "b" branches (counting the substituted voltage sources as short circuits),

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[41] In order to differentiate the terminologies used in this section from those used in deriving diakoptic analysis, the adjective "codiakoptic" should really precede every terminology in this section. However, to avoid excessive repetitions, we will omit this adjective and simply caution the reader to interpret each terminology and concept in this appendix as "dual" to those introduced in Section 3.

[42] A branch is said to be contracted if it is deleted with its two end nodes coalesced into one node.

and its associated linear graph $\mathcal{G}(N_1)$ is shown in Fig. 10(d). Since the graph $\mathcal{G}(N_1)$ has (b-n+1) linearly independent loops, we know that the substituted voltage sources can be summarized by (b-n+1) "modified" substituted voltage sources, each associated with a loop. In the cases where there are less than (b-n+1) substituted voltage sources, our "dual" procedure calls for inserting additional zero-valued voltage sources (i.e., short circuits) by a "plier-type entry" with the links of $\mathcal{G}(N_1)$. For example, in Fig. 10(e), $\mathcal{G}(N_1)$ has two linearly independent loops but only one substituted voltage source. Hence, a zero-valued voltage source is inserted with link "1" in Fig. 10(f), and an augmented graph $\mathcal{G}_a(N_1)$ (counting each voltage source as a branch) is obtained as shown in Fig. 10(g). Observe that the augmented graph $\mathcal{G}_a(N_1)$ contains "b+1" nodes, "2b-n+1" branches and "b-n+1" linearly independent loops. Since the number of links[43] and the number of substituted voltage sources in the augmented graph $\mathcal{G}_a(N_1)$ are identical, we can always select the set of (b-n+1) substituted voltage sources to form a cotree of $\mathcal{G}_a(N_1)$, henceforth denoted by the generator cotree $\mathcal{L}_g$. For example, in Fig. 10(g), $\mathcal{L}_g = \{1', 2'\}$.

Now, we can define a contracted cutset of $\mathcal{G}$ with respect to any generator cotree $\mathcal{L}_g$ to be any set of branches of $\mathcal{G}$ which form a cutset with a branch, henceforth called the associated contracted cutset generator, of $\mathcal{L}_g$. For example, the sets of branches {5}, {1,4} and {2,4} all form cutsets with the augmented branch 1' in Fig. 10(g). They are, therefore, contracted cutsets induced by the same generator 1'.

---

[43] The number of links is equal to the number of linearly independent loops.

Following a dual procedure used in deriving diakoptic analysis, we have the following lemma:

Lemma 9. Let $\mathcal{G}$(resp., $\mathcal{G}_a$) be a connected graph with "n" nodes and "b" branches (resp., "b+1" nodes and "2b-n+1" branches) and let $\mathcal{L}_g$ be any augmented generator cotree of $\mathcal{G}$. Then any collection of (b-n+1) contracted cutsets induced by the (b-n+1) distinct contracted cutset generators in $\mathcal{L}_g$ are linearly independent.

In view of Lemma 9, we can define a (b-n+1) × b contracted cutset matrix $\hat{\underset{\sim}{K}}$ as follows:

$\hat{K}_{ij}$ = 1, if contracted cutset i contains branch j and has the same orientation as branch j;

= -1, if contracted cutset i contains branch j but has opposite orientation as branch j;

= 0, if contracted cutset i does not contain branch j.

Theorem 3. Let $\mathcal{G}$(resp., $\mathcal{G}_a$) be a connected graph with "n" nodes and "b" branches (resp., "b+1" nodes and "2b-n+1" branches) and let $\mathcal{L}_g$ denote any augmented generator cotree of $\mathcal{G}$. Then there exists a unique cycle matrix $\hat{B}$, henceforth called the associated cycle matrix, such the

$$\hat{\underset{\sim}{B}}\,\hat{\underset{\sim}{K}}^t = \underset{\sim}{1}_{\mathcal{L}}. \qquad (111)$$

Having established the basic property of contracted cutsets as in Theorem 3, let us focus our attention on the augmented subnetwork (see Fig. 10(f)) which now contains (b-n+1) substituted voltage sources (including zero-valued voltage sources). Our next step is to apply the v-shift theorem and shift each substituted voltage source in series with the internal branches of $N_1$ in order that it may be combined with the original composite branches. We can achieve this goal by shifting each substituted voltage source $e_{s_j}$ across any contracted cutset induced by $e_{s_j}$. If we combine all the shifted voltage sources in series with each original composite branch and denote it by $e'_k$, we would obtain a modified subnetwork $N'_1$ having the same topology as $\mathcal{G}(N_1)$ (see Fig. 10(d)), where each branch in $N'_1$ represents an augmented composite branch as shown in Fig. 10(h). In order to find an algebraic expression for $e'_k$ in terms of the $e_{s_j}$'s let us define the "combined shifted" voltage source vector $\underset{\sim}{e}' \triangleq [e'_1, e'_2, \ldots, e'_b]^t$ and the "unshifted" voltage source vector $\underset{\sim}{e}_s \triangleq [e_{s_1}, e_{s_2}, \ldots, e_{s_{b-n+1}}]^t$. If we let $\hat{\underset{\sim}{K}}$ be any contracted cutset matrix induced by the (b-n+1) substituted voltage sources acting as the contracted cutset generators, then

$$\underset{\sim}{e}' = \hat{\underset{\sim}{K}}^t \underset{\sim}{e}_s \,. \tag{112}$$

Let us now carry out an "associated" cycle analysis[44] of the modified subnetwork $N'_1$. Using the notations associated with the augmented composite branch shown in Fig. 10(h) and Eqs. (111) and (112), we have the following sequence of equations:

$$\begin{aligned} \underset{\sim}{0} = \hat{\underset{\sim}{B}}\underset{\sim}{V}' = \hat{\underset{\sim}{B}}\underset{\sim}{V} - \hat{\underset{\sim}{B}}\underset{\sim}{e}' &= \hat{\underset{\sim}{B}}(\underset{\sim}{R}\hat{\underset{\sim}{B}}^t\hat{\underset{\sim}{I}} + \underset{\sim}{R}\underset{\sim}{j} - \underset{\sim}{e}) - \hat{\underset{\sim}{B}}\hat{\underset{\sim}{K}}^t \underset{\sim}{e}_s \\ &= \hat{\underset{\sim}{B}}\underset{\sim}{R}\hat{\underset{\sim}{B}}^t\hat{\underset{\sim}{I}} + \hat{\underset{\sim}{B}}(\underset{\sim}{R}\underset{\sim}{j} - \underset{\sim}{e}) - \underset{\sim}{e}_s \end{aligned} \tag{113}$$

where $\hat{\underset{\sim}{I}}$ is an (b-n+1) × 1 vector, called the <u>generalized current coordinate vector</u>. Observe that <u>the generalized current coordinate vector</u> $\hat{\underset{\sim}{I}}$ <u>is equal to the negative of the (b-n+1) × 1 current vector</u> $\underset{\sim}{I}_s$ <u>through the (b-n+1) substituted voltage sources in the augmented generator cotree</u> $\mathcal{L}_g$, i.e.,

---

[44] A cycle analysis is a <u>generalized loop analysis</u> [18]. By an "associated" cycle analysis, we mean the cycle analysis with respect to the unique associated cycle matrix as defined in Theorem 3.

$\hat{I} = -I_s$. This relation will now provide us with the key to transform the modified subnetwork $N_1'$ into an equivalent "self-loop" network $N_{1_{eq}}$.

Observe that Eq. (113) can be rewritten as

$$\hat{I} = G_{eq}(\hat{e} + e_s) = -I_s \tag{114}$$

where $$G_{eq} = (\hat{B}R\hat{B}^t)^{-1} \tag{115}$$

and

$$\hat{e} \triangleq \hat{B}(e - R\, j) \tag{116}$$

Now suppose we construct a self-loop network $N_{1_{eq}}$ by connecting a voltage source $\hat{e}_k$ (i.e., the k-th component of $\hat{e}$) and a <u>coupled</u> linear resistor defined by the k-th row of $G_{eq}$ in series with each substituted voltage $e_{s_k}$ of the generator cotree $\mathcal{L}_g$ (Fig. 10(i)). Observe that the network equation describing $N_{1_{eq}}$ is precisely given by Eq. (114)! For our example (Fig. 10(f)), the "coupled" self-loop network $N_{1_{eq}}$ is shown in Fig. 10(j). It follows from the above observation that the subnetwork $N_1$ in Fig. 10(f) and the "coupled" self-loop subnetwork $N_{1_{eq}}$ in Fig. 10(j) are equivalent in the sense that they have identical governing equations. Applying the same transformation procedure to subnetwork $N_2$ (Fig. 10(c)), we obtain the equivalent "coupled" self-loop subnetwork $N_{2_{eq}}$ shown in Fig. 10(k).

We can now reverse the steps implemented earlier in going from Fig. 10(a) to Fig. 10(c) and apply the sequence of equivalent "inverse" transformations shown in Figs. 10(l) and (m) to obtain the <u>equivalent interconnected network</u> $N_{eq}$ shown in Fig. 10(n). Observe that this equivalent interconnected network $N_{eq}$ contains very few nodes. Besides, branches of $N_{eq}$ are characterized by a "coupled" <u>branch conductance matrix</u> (Eq. (115)). Hence, we can analyze this network by the standard nodal analysis (or by any convenient coboundary analysis).

If we let $\underset{\sim}{I}_0$ denote the composite current vector of the torn branches $N_0$ and order these branches before the self-loop branches, then the reduced incidence matrix of $N_{eq}$ can be partitioned as

$$\underset{\sim}{A}_{eq} = [\underset{\sim}{A}_o \quad \underset{\sim}{A}_b] \tag{117}$$

And the familiar nodal equation can be derived as[45]

$$\underset{\sim}{0} = \underset{\sim}{A}_{eq}\, \underset{\sim}{I}_{eq} \triangleq [\underset{\sim}{A}_o \quad \underset{\sim}{A}_b] \begin{bmatrix} \underset{\sim}{I}_o \\ \hat{\underset{\sim}{I}} \end{bmatrix} = \underset{\sim}{A}_o \underset{\sim}{I}_o + \underset{\sim}{A}_b \hat{\underset{\sim}{I}}$$

$$= (\underset{\sim}{A}_o \underset{\sim}{G}_o \underset{\sim}{A}_o^t)\hat{\underset{\sim}{V}}_o + \underset{\sim}{A}_o(\underset{\sim}{G}_o \underset{\sim}{e}_o - \underset{\sim}{j}_o) + \underset{\sim}{A}_b \hat{\underset{\sim}{I}} = \underset{\sim}{0} \tag{118}$$

where $\underset{\sim}{G}_o$, $\hat{\underset{\sim}{V}}_o$, $\underset{\sim}{e}_o$ and $\underset{\sim}{j}_o$ denote the branch conductance matrix, the node-to-datum voltage vector, the branch voltage source vector and the branch current source vector of $N_o$, respectively.

Recall that $\underset{\sim}{e}_s$ is the voltage vector of the torn branches $N_o$. This voltage vector can be uniquely determined by the node-to-datum vector $\hat{\underset{\sim}{V}}_o$ of $N_o$. Hence, there exists some matrix $\underset{\sim}{C}'$ such that[46]

$$\underset{\sim}{e}_s = \underset{\sim}{C}' \hat{\underset{\sim}{V}}_o \tag{119}$$

Substituting Eq. (119) into Eq. (113) and together with Eq. (118), we obtain the following block equation:

$$\boxed{\begin{bmatrix} \underset{\sim}{A}_o \underset{\sim}{G}_o \underset{\sim}{A}_o^t & \underset{\sim}{A}_b \\ -\underset{\sim}{C}' & \hat{\underset{\sim}{B}}\underset{\sim}{R}\hat{\underset{\sim}{B}}^t \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_o \\ \hat{\underset{\sim}{I}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}_o(\underset{\sim}{j}_o - \underset{\sim}{G}_o \underset{\sim}{e}_o) \\ \hat{\underset{\sim}{B}}(\underset{\sim}{e} - \underset{\sim}{R}\, \underset{\sim}{j}) \end{bmatrix}} \tag{120}$$

---

[45] We abuse our notation slightly here since $\hat{\underset{\sim}{I}}$ actually represents the composite of the generalize current coordinate vector of subnetworks $N_1, N_2, \ldots, N_m$.

[46] Actually, $\underset{\sim}{C}' = \underset{\sim}{A}_b^t$.

where

$$\hat{B} = \begin{bmatrix} \hat{B}(N_1) & & & 0 \\ & \hat{B}(N_2) & & \\ & & \ddots & \\ 0 & & & \hat{B}(N_k) \end{bmatrix}, \quad R = \begin{bmatrix} R(N_1) & & & 0 \\ & R(N_2) & & \\ & & \ddots & \\ 0 & & & R(N_k) \end{bmatrix} \tag{121}$$

and where $\hat{B}(N_j)$ and $R(N_j)$ denote respectively the unique associated cycle matrix and the branch resistance matrix of subnetwork $N_k$.

The process of formulating and solving Eq. (120) is called codiakoptic analysis by Onodera[47] [13]. It offers the same desirable block structure (as shown in Fig. 10(o)) as in diakoptic analysis (see Fig. 7). However, since there exists no cycle analysis dual to that of the computationally efficient nodal analysis, co-diakoptic analysis offers no computational advantage over diakoptic analysis.

We will close this section with the following important theorem:

Theorem 4. There exists a one-to-one correspondence that equates any form of codiakoptic analysis (as given by Eq. (120)) involving variables $\hat{V}_o$ and $\hat{I}$ to a special case of the generalized hybrid analysis (as given by Eq. (51)) involving variables $\hat{V}_{\mathcal{J}_1}$ and $\hat{I}_{\mathcal{L}_2}$.

[47] Onodera did not derive, but did "sketch", Eq. (120).

## Appendix D. Mult-Stage Diakoptic Analysis

The concept of multi-stage diakoptic analysis[48] consists essentially of a repetitive application of both diakoptic and codiakoptic analyses. Our objective in this section is to show that this approach is also a special case of generalized hybrid analysis. Before we start the derivation, however, recall that diakoptic (resp., codiakoptic) analysis always consists of three stages: (i) choose a set of torn branches so as to "tear" the network into "m" subnetworks; (ii) apply an "appropriate" form of coboundary (resp., cycle) analysis on each subnetwork; (iii) apply a conventional loop (resp., nodal) analysis on the diakoptic (resp., codiakoptic) equivalent interconnected network.[49] The basic idea of multi-stage diakoptic analysis is to modify the diakoptic analysis at stage (iii) such that, instead of applying a conventional loop analysis, we apply a codiakoptic analysis on the diakoptic equivalent interconnected network. This process can be further extended if, at stage (iii) of the codiakoptic analysis as applied to the diakoptic equivalent interconnected network, we apply still another diakoptic analysis on the codiakoptic equivalent interconnected network. This algorithm can obviously be extended to any number of "stages" by alternating between a diakoptic and a codiakoptic analysis, each stage involving a smaller interconnected network.

For simplicity, we will derive only the two-stage (or double) diakoptic

[48] In this section, we will derive the version of multi-stage diakoptic analysis as proposed by Onodera [14]. Other slightly different versions [4,7] can also be easily shown as special cases of generalized hybrid analysis.

[49] As we have pointed out earlier, we can actually apply any "convenient" form of cycle (resp., coboundary) analysis on the equivalent interconnected network.

analysis.[50] Furthermore, we will choose the "fundamental generator tree" version of diakoptic analysis since the other methods follow similarly.

Let us partition the branches of the original network N into six subsets $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ with respect to a given tree $\mathcal{T}$ satisfying the following two conditions: (i) we can apply diakoptic analysis on N with $\mathcal{L}_2 \cup \mathcal{T}_2 \cup \mathcal{L}_3 \cup \mathcal{T}_3$ as the diakoptic torn branches; (ii) we can apply codiakoptic analysis on the diakoptic equivalent interconnected network $N_{eq}$ (notice that for the fundamental generator tree case, $N_{eq} = \mathcal{T}_1 \cup \mathcal{L}_2 \cup \mathcal{T}_2 \cup \mathcal{L}_3 \cup \mathcal{T}_3$) with $\mathcal{L}_3 \cup \mathcal{T}_3$ as the codiakoptic torn branches.[51]

From Section 4, we know that the fundamental loops generated by links ($\mathcal{L}_1$) of the diakoptic subnetworks ($\mathcal{L}_1 \cup \mathcal{T}_1$) do not contain twigs ($\mathcal{T}_2 \cup \mathcal{T}_3$) of the diakoptic torn network ($\mathcal{L}_2 \cup \mathcal{T}_2 \cup \mathcal{L}_3 \cup \mathcal{T}_3$). We also know from Appendix C that the fundamental cutsets generated by twigs ($\mathcal{T}_1 \cup \mathcal{T}_2$) of the codiakoptic subnetworks ($\mathcal{T}_1 \cup \mathcal{L}_2 \cup \mathcal{T}_2$) do not contain links ($\mathcal{L}_3$) of the codiakoptic torn network ($\mathcal{L}_3 \cup \mathcal{T}_3$). Now, if we partition the fundamental loop matrix $\underset{\sim}{B}$ and the fundamental cutset matrix $\underset{\sim}{Q}$ with respect to the tree $\mathcal{T}$ as follows:

---

[50] To extend our derivation to more than two stages, it is convenient to choose the <u>fundamental generator tree</u> for each diakoptic analysis, and the <u>fundamental generator cotree</u> for each codiakoptic analysis. The fundamental generator cotree version of codiakoptic analysis is implemented by taking each branch of the augmented contracted cutset generator cotree $\mathcal{L}_g$ to be in series, but oppositely directed, with a link of the original network. The associated cycle matrix for this fundamental generator cotree is simply the fundamental loop matrix.

[51] A simple variation that takes a subset of $\mathcal{T}_1$ together with $\mathcal{L}_3 \cup \mathcal{T}_3$ as the codiakoptic torn branches can also be easily shown to be a special case of generalized hybrid analysis.

$$
\mathbf{B} = \begin{array}{c}
\begin{array}{cccccc} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 \end{array} \\
\begin{bmatrix}
\mathbf{1}_{\mathcal{L}_1\mathcal{L}_1} & \mathbf{0}_{\mathcal{L}_1\mathcal{L}_2} & \mathbf{0}_{\mathcal{L}_1\mathcal{L}_3} & \mathbf{B}_{\mathcal{L}_1\mathcal{T}_1} & \mathbf{B}_{\mathcal{L}_1\mathcal{T}_2} & \mathbf{B}_{\mathcal{L}_1\mathcal{T}_3} \\
\mathbf{0}_{\mathcal{L}_2\mathcal{L}_1} & \mathbf{1}_{\mathcal{L}_2\mathcal{L}_2} & \mathbf{0}_{\mathcal{L}_2\mathcal{L}_3} & \mathbf{B}_{\mathcal{L}_2\mathcal{T}_1} & \mathbf{B}_{\mathcal{L}_2\mathcal{T}_2} & \mathbf{B}_{\mathcal{L}_2\mathcal{T}_3} \\
\mathbf{0}_{\mathcal{L}_3\mathcal{L}_1} & \mathbf{0}_{\mathcal{L}_3\mathcal{L}_2} & \mathbf{1}_{\mathcal{L}_3\mathcal{L}_3} & \mathbf{B}_{\mathcal{L}_3\mathcal{T}_1} & \mathbf{B}_{\mathcal{L}_3\mathcal{T}_2} & \mathbf{B}_{\mathcal{L}_3\mathcal{T}_3}
\end{bmatrix}
\end{array} \tag{122}
$$

$$
\mathbf{Q} = \begin{bmatrix}
-\mathbf{B}^t_{\mathcal{L}_1\mathcal{T}_1} & -\mathbf{B}^t_{\mathcal{L}_2\mathcal{T}_1} & -\mathbf{B}^t_{\mathcal{L}_3\mathcal{T}_1} & \mathbf{1}_{\mathcal{T}_1\mathcal{T}_1} & \mathbf{0}_{\mathcal{T}_1\mathcal{T}_2} & \mathbf{0}_{\mathcal{T}_1\mathcal{T}_3} \\
-\mathbf{B}^t_{\mathcal{L}_1\mathcal{T}_2} & -\mathbf{B}^t_{\mathcal{L}_2\mathcal{T}_2} & -\mathbf{B}^t_{\mathcal{L}_3\mathcal{T}_2} & \mathbf{0}_{\mathcal{T}_2\mathcal{T}_1} & \mathbf{1}_{\mathcal{T}_2\mathcal{T}_2} & \mathbf{0}_{\mathcal{T}_2\mathcal{T}_3} \\
-\mathbf{B}^t_{\mathcal{L}_1\mathcal{T}_3} & -\mathbf{B}^t_{\mathcal{L}_2\mathcal{T}_3} & -\mathbf{B}^t_{\mathcal{L}_3\mathcal{T}_3} & \mathbf{0}_{\mathcal{T}_3\mathcal{T}_1} & \mathbf{0}_{\mathcal{T}_3\mathcal{T}_2} & \mathbf{1}_{\mathcal{T}_3\mathcal{T}_3}
\end{bmatrix} \tag{123}
$$

then it follows from the observation that the fundamental loops generated by $\mathcal{L}_1$ do not contain branches in $\mathcal{T}_2$ and $\mathcal{T}_3$ that the submatrices $\mathbf{B}_{\mathcal{L}_1\mathcal{T}_2}$ and $\mathbf{B}_{\mathcal{L}_1\mathcal{T}_3}$ are zero matrices. Similarly, it follows from the observation that the fundamental cutsets generated by $\mathcal{T}_1 \cup \mathcal{T}_2$ do not contain branches in $\mathcal{L}_3$ that the submatrices $-\mathbf{B}^t_{\mathcal{L}_3\mathcal{T}_1}$ and $-\mathbf{B}^t_{\mathcal{L}_3\mathcal{T}_2}$ are also zero matrices. Hence, the fundamental loop matrix $\mathbf{B}$ and the fundamental cutset matrix $\mathbf{Q}$ associated with the two-stage diakoptic analysis must possess the following structure:

$$
\mathbf{B} = \begin{array}{c}
\begin{array}{cccccc} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 \end{array} \\
\begin{bmatrix}
\mathbf{1}_{\mathcal{L}_1\mathcal{L}_1} & \mathbf{0}_{\mathcal{L}_1\mathcal{L}_2} & \mathbf{0}_{\mathcal{L}_1\mathcal{L}_3} & \mathbf{B}_{\mathcal{L}_1\mathcal{T}_1} & \mathbf{0}_{\mathcal{L}_1\mathcal{T}_2} & \mathbf{0}_{\mathcal{L}_1\mathcal{T}_3} \\
\mathbf{0}_{\mathcal{L}_2\mathcal{L}_1} & \mathbf{1}_{\mathcal{L}_2\mathcal{L}_2} & \mathbf{0}_{\mathcal{L}_2\mathcal{L}_3} & \mathbf{B}_{\mathcal{L}_2\mathcal{T}_1} & \mathbf{B}_{\mathcal{L}_2\mathcal{T}_2} & \mathbf{B}_{\mathcal{L}_2\mathcal{T}_3} \\
\mathbf{0}_{\mathcal{L}_3\mathcal{L}_1} & \mathbf{0}_{\mathcal{L}_3\mathcal{L}_2} & \mathbf{1}_{\mathcal{L}_3\mathcal{L}_3} & \mathbf{0}_{\mathcal{L}_3\mathcal{T}_1} & \mathbf{0}_{\mathcal{L}_3\mathcal{T}_2} & \mathbf{B}_{\mathcal{L}_3\mathcal{T}_3}
\end{bmatrix}
\end{array} \tag{124}
$$

$$
\mathbf{Q} = \begin{bmatrix}
-\mathbf{B}^t_{\mathscr{L}_1\mathscr{T}_1} & -\mathbf{B}^t_{\mathscr{L}_2\mathscr{T}_1} & \mathbf{0}^t_{\mathscr{L}_3\mathscr{T}_1} & \mathbf{1}_{\mathscr{T}_1\mathscr{T}_1} & \mathbf{0}_{\mathscr{T}_1\mathscr{T}_2} & \mathbf{0}_{\mathscr{T}_1\mathscr{T}_3} \\
\mathbf{0}^t_{\mathscr{L}_1\mathscr{T}_2} & -\mathbf{B}^t_{\mathscr{L}_2\mathscr{T}_2} & \mathbf{0}^t_{\mathscr{L}_3\mathscr{T}_2} & \mathbf{0}_{\mathscr{T}_2\mathscr{T}_1} & \mathbf{1}_{\mathscr{T}_2\mathscr{T}_2} & \mathbf{0}_{\mathscr{T}_2\mathscr{T}_3} \\
\mathbf{0}^t_{\mathscr{L}_1\mathscr{T}_3} & -\mathbf{B}^t_{\mathscr{L}_2\mathscr{T}_3} & -\mathbf{B}^t_{\mathscr{L}_3\mathscr{T}_3} & \mathbf{0}_{\mathscr{T}_3\mathscr{T}_1} & \mathbf{0}_{\mathscr{T}_3\mathscr{T}_2} & \mathbf{1}_{\mathscr{T}_3\mathscr{T}_3}
\end{bmatrix} \tag{125}
$$

If we define

$$
\mathbf{Q}_1 \triangleq \begin{bmatrix} \overset{\mathscr{L}_1}{-\mathbf{B}^t_{\mathscr{L}_1\mathscr{T}_1}} & \overset{\mathscr{T}_1}{\mathbf{1}_{\mathscr{T}_1\mathscr{T}_1}} \end{bmatrix}, \quad
\mathbf{B}_2 \triangleq \begin{bmatrix} \overset{\mathscr{L}_2}{\mathbf{1}_{\mathscr{L}_2\mathscr{L}_2}} & \overset{\mathscr{T}_2}{\mathbf{B}_{\mathscr{L}_2\mathscr{T}_2}} \end{bmatrix}, \quad
\mathbf{Q}_3 \triangleq \begin{bmatrix} \overset{\mathscr{L}_3}{-\mathbf{B}^t_{\mathscr{L}_3\mathscr{T}_3}} & \overset{\mathscr{T}_3}{\mathbf{1}_{\mathscr{T}_3\mathscr{T}_3}} \end{bmatrix} \tag{126}
$$

then, by combining KCL and KVL equations together with the following branch relations:

$$
\mathbf{i}_1 = \mathbf{G}_1\mathbf{v}_1, \quad \mathbf{v}_2 = \mathbf{R}_2\mathbf{i}_2, \quad \mathbf{i}_3 = \mathbf{G}_3\mathbf{v}_3 \tag{127}
$$

we would obtain the following <u>two-stage diakoptic equation</u>:

$$
\boxed{
\begin{bmatrix}
\mathbf{Q}_1\mathbf{G}_1\mathbf{Q}_1^t & -\mathbf{B}^t_{\mathscr{L}_2\mathscr{T}_1} & \mathbf{0}^t_{\mathscr{L}_3\mathscr{T}_1} \\
\mathbf{B}_{\mathscr{L}_2\mathscr{T}_1} & \mathbf{B}_2\mathbf{R}_2\mathbf{B}_2^t & \mathbf{B}_{\mathscr{L}_2\mathscr{T}_3} \\
\mathbf{0}_{\mathscr{L}_3\mathscr{T}_1} & -\mathbf{B}^t_{\mathscr{L}_2\mathscr{T}_3} & \mathbf{Q}_3\mathbf{G}_3\mathbf{Q}_3^t
\end{bmatrix}
\begin{bmatrix} \mathbf{V}_{\mathscr{T}_1} \\ \mathbf{I}_{\mathscr{L}_2} \\ \mathbf{V}_{\mathscr{T}_3} \end{bmatrix}
=
\begin{bmatrix} \mathbf{Q}_1(\mathbf{j}_1 - \mathbf{G}_1\mathbf{e}_1) \\ \mathbf{B}_2(\mathbf{e}_2 - \mathbf{R}_2\mathbf{j}_2) \\ \mathbf{Q}_3(\mathbf{j}_3 - \mathbf{G}_3\mathbf{e}_3) \end{bmatrix}
} \tag{128}
$$

Equation (128) can be rearranged as

$$
\begin{bmatrix}
\mathbf{Q}_1\mathbf{G}_1\mathbf{Q}_1^t & \mathbf{0}^t_{\mathscr{L}_3\mathscr{T}_1} & -\mathbf{B}^t_{\mathscr{L}_2\mathscr{T}_1} \\
\mathbf{0}_{\mathscr{L}_3\mathscr{T}_1} & \mathbf{Q}_3\mathbf{G}_3\mathbf{Q}_3^t & -\mathbf{B}^t_{\mathscr{L}_2\mathscr{T}_3} \\
\mathbf{B}_{\mathscr{L}_2\mathscr{T}_1} & \mathbf{B}_{\mathscr{L}_2\mathscr{T}_3} & \mathbf{B}_2\mathbf{R}_2\mathbf{B}_2^t
\end{bmatrix}
\begin{bmatrix} \mathbf{V}_{\mathscr{T}_1} \\ \mathbf{V}_{\mathscr{T}_3} \\ \mathbf{I}_{\mathscr{L}_2} \end{bmatrix}
=
\begin{bmatrix} \mathbf{Q}_1(\mathbf{j}_1 - \mathbf{G}_1\mathbf{e}_1) \\ \mathbf{Q}_3(\mathbf{j}_3 - \mathbf{G}_3\mathbf{e}_3) \\ \mathbf{B}_2(\mathbf{e}_2 - \mathbf{R}_2\mathbf{j}_2) \end{bmatrix} \tag{129}
$$

Observe that Eq. (129) has exactly the same form as Eq. (48). Hence, by rearranging the equations associated with the two-stage diakoptic analysis, we would obtain once again a system of hybrid equations representing a special case of our generalized hybrid analysis.

## Appendix E. Topological Matrices in Eq. (79)

Hybrid analysis in the form of Eq. (48) has not been widely used in computer-aided analysis because there is no efficient method for obtaining the topological matrices involved [27]; namely, the submatrices $\underset{\sim}{Q}_1$, $\underset{\sim}{B}_2$ and $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1}$. This objection, however, can be overcome by choosing the node-to-datum version of our generalized hybrid analysis; namely, Eq. (79). Our objective in this section is to present efficient algorithms for constructing the three matrices $\underset{\sim}{A}_1$, $\underset{\sim}{B}_2$ and $\hat{\underset{\sim}{C}}$ needed to form Eq. (79).

We shall start with the matrix $\underset{\sim}{A}_1$. Recall that $\underset{\sim}{A}_1$ as defined in Eq. (78) is a "composite" matrix containing the reduced incidence matrices of the separable subnetworks of $\mathcal{N}_1$, made up of branches in $\mathcal{L}_1 \cup \mathcal{T}_1$ with all branches in $\mathcal{L}_2 \cup \mathcal{T}_2$ replaced by open circuits. Observe that $\underset{\sim}{A}_1$ can be generated as soon as the separable components of $\mathcal{N}_1$ are identified. In this regard, observe that the highly efficient Tarjan's "depth-first back-tracking search" algorithm [28] can be used to identify all separable components of $\mathcal{N}_1$ in <u>linear</u> time and storage.[52] Hence, the matrix $\underset{\sim}{A}_1$ can be obtained efficiently.

The matrix $\underset{\sim}{B}_2$ as defined in Eq. (47) is the fundamental loop matrix of $\mathcal{N}_2$, made up of branches in $\mathcal{L}_2 \cup \mathcal{T}_2$ with all branches in $\mathcal{L}_1 \cup \mathcal{T}_1$ replaced by short circuits. The following lemma will provide us with an efficient method for obtaining $\underset{\sim}{B}_2$:

<u>Lemma 10</u>. Given a graph $\mathcal{G}$ with reduced incidence matrix $\underset{\sim}{A}$, a tree $\mathcal{T}$ and the node-to-datum fundamental open loop matrix $\underset{\sim}{T}^*$. If the columns of the reduced incidence matrix $\underset{\sim}{A}$, the node-to-datum fundamental open loop matrix $\underset{\sim}{T}^*$ and the fundamental loop matrix $\underset{\sim}{B}$ with respect to $\mathcal{T}$ are ordered such that

[52] Separable components are called biconnected components by Tarjan [28].

the links appear before the twigs, i.e., $\underset{\sim}{A} = [\underset{\sim}{A}_{\mathcal{L}} \;\; \underset{\sim}{A}_{\mathcal{T}}]$, $\underset{\sim}{T}^* = [\underset{\sim}{0}_{\mathcal{L}} \;\; \underset{\sim}{T}^*_{\mathcal{T}}]$ and $\underset{\sim}{B} = [\underset{\sim}{1}_{\mathcal{L}} \;\; \underset{\sim}{B}_{\mathcal{T}}]$, then $\underset{\sim}{B}_{\mathcal{T}} = -\underset{\sim}{A}^t_{\mathcal{L}} \; \underset{\sim}{T}^*_{\mathcal{T}}$.

<u>Proof</u>: Let the fundamental cutset matrix $\underset{\sim}{Q}$ with respect to $\mathcal{T}$ be ordered in the same manner, i.e., $\underset{\sim}{Q} = [\underset{\sim}{Q}_{\mathcal{L}} \;\; \underset{\sim}{1}_{\mathcal{T}}]$. We can express $\underset{\sim}{A}$ in terms of $\underset{\sim}{Q}$ via a nonsingular transformation matrix $\underset{\sim}{P}_Q$ as

$$[\underset{\sim}{A}_{\mathcal{L}} \;\; \underset{\sim}{A}_{\mathcal{T}}] = \underset{\sim}{P}_Q[\underset{\sim}{Q}_{\mathcal{L}} \;\; \underset{\sim}{1}_{\mathcal{T}}] = [\underset{\sim}{P}_Q\underset{\sim}{Q}_{\mathcal{L}} \;\; \underset{\sim}{P}_Q] \tag{130}$$

Therefore, we can identify from Eq. (130) that

$$\underset{\sim}{P}_Q = \underset{\sim}{A}_{\mathcal{T}} \tag{131}$$

Substituting Eq. (131) into Eq. (130) and premultiplying both sides of Eq. (130) by $\underset{\sim}{A}^{-1}_{\mathcal{T}}$, we obtain

$$[\underset{\sim}{A}^{-1}_{\mathcal{T}}\underset{\sim}{A}_{\mathcal{L}} \;\; \underset{\sim}{1}_{\mathcal{T}}] = [\underset{\sim}{Q}_{\mathcal{L}} \;\; \underset{\sim}{1}_{\mathcal{T}}] \tag{132}$$

and, in particular,

$$\underset{\sim}{Q}_{\mathcal{L}} = \underset{\sim}{A}^{-1}_{\mathcal{T}}\underset{\sim}{A}_{\mathcal{L}} \tag{133}$$

From Theorem 1 and Lemma 6, we have

$$\underset{\sim}{A}(\underset{\sim}{T}^*)^t = [\underset{\sim}{A}_{\mathcal{L}} \;\; \underset{\sim}{A}_{\mathcal{T}}]\begin{bmatrix} \underset{\sim}{0}^t_{\mathcal{L}} \\ (\underset{\sim}{T}^*_{\mathcal{T}})^t \end{bmatrix} = \underset{\sim}{A}_{\mathcal{T}}(\underset{\sim}{T}^*_{\mathcal{T}})^t = \underset{\sim}{1}_{\mathcal{T}} \tag{134}$$

Eq. (134) can be rewritten as

$$\underset{\sim}{A}^{-1}_{\mathcal{T}} = (\underset{\sim}{T}^*_{\mathcal{T}})^t \tag{135}$$

From Tellegen's theorem $\underset{\sim}{B}$ and $\underset{\sim}{Q}$ can be related by

$$\underset{\sim}{B}_{\mathcal{T}} = -\underset{\sim}{Q}^t_{\mathcal{L}} \tag{136}$$

Combining Eqs. (136), (133) and (135), we obtain

$$\underset{\sim}{B}_{\mathcal{T}} = -(\underset{\sim}{A}_{\mathcal{T}}^{-1}\underset{\sim}{A}_{\mathcal{L}})^t = -\underset{\sim}{A}_{\mathcal{L}}^t(\underset{\sim}{A}_{\mathcal{T}}^{-1})^t = -\underset{\sim}{A}_{\mathcal{L}}^t\, \underset{\sim}{T}_{\mathcal{T}}^{*}$$ ■

Let us make the following observations regarding the subnetwork $\mathcal{N}_2$: (i) the reduced incidence matrix $\underset{\sim}{A}(\mathcal{N}_2)$ of $\mathcal{N}_2$ can be obtained from the reduced incidence matrix $\underset{\sim}{A}$ of the original network N simply by contracting all branches in $\mathcal{L}_1 \cup \mathcal{T}_1$; (ii) a tree $\mathcal{T}$ of $\mathcal{N}_2$ can be found via Tarjan's "depth-first back-tracking search" algorithm; (iii) the node-to-datum fundamental open loop matrix $\underset{\sim}{T}^{*}(\mathcal{N}_2)$ of $\mathcal{N}_2$ can be obtained easily from the reduced incidence matrix $\underset{\sim}{A}(\mathcal{N}_2)$; (iv) the elements of $\underset{\sim}{T}_{\mathcal{T}}^{*}(\mathcal{N}_2)$ are either -1, 0 or 1. Hence, in view of Lemma 10, $\underset{\sim}{B}_2 \triangleq \underset{\sim}{B}(\mathcal{N}_2) = [\underset{\sim}{1}_{\mathcal{L}}(\mathcal{N}_2) \;\; \underset{\sim}{B}_{\mathcal{T}}(\mathcal{N}_2)]$ can be found by simple additions and subtractions of elements of $\underset{\sim}{A}_{\mathcal{L}}(\mathcal{N}_2)$. In other words, $\underset{\sim}{B}_2$ can be obtained efficiently.

The matrix $\hat{C}$ as defined in Eq. (54) for the "node-to-datum" case (where $\underset{\sim}{P}_B = \underset{\sim}{1}$) is given by

$$\hat{\underset{\sim}{C}} = \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_1} \;\; \underset{\sim}{P}_Q^t \tag{137}$$

Instead of finding $\hat{\underset{\sim}{C}}$ through Eq. (137), however, we shall make use of a special property of the node-to-datum case. From Eqs. (79) and (32), we have the identity

$$\hat{\underset{\sim}{C}} = \underset{\sim}{B}_a \tag{138}$$

Substituting Eq. (73) into Eq. (138), we obtain

$$\hat{\underset{\sim}{C}} = \underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g} \tag{139}$$

Recall that $\underset{\sim}{B}_{\mathcal{L}_2\mathcal{T}_g}$ is a submatrix (corresponding to the columns of branches in $\mathcal{T}_g$) of the fundamental loop matrix $\underset{\sim}{B}_{eq}$ of the equivalent interconnected network $N_{eq}$. It follows from the special structure of the node-to-datum equivalent interconnected network $N_{eq}$ (e.g., Fig. 6(i)) that the (i,j)-th element of the matrix $B_{\mathcal{L}_2\mathcal{T}_g}$ can be easily determined from the following interpretation:

$(B_{\mathcal{L}_2\mathcal{T}_g}(N_k))_{ij}$ = 1 (resp., -1), if fundamental loop i of $\mathcal{N}_2$ enters (resp., leaves) the nondatum node j of subnetworks $N_k$;

= 0, otherwise.

Hence, using this interpretation and Eq. (139), we can obtain $\hat{C}$ efficiently.

Let us examine next some special cases where the submatrices $A_1$, $B_2$ and $\hat{C}$ are even more readily available. Suppose that the branches of $\mathcal{L}_1 \cup \mathcal{T}_1$ span all nodes of the original network N and suppose that a <u>common</u> datum node for each separable subnetwork of $\mathcal{L}_1 \cup \mathcal{T}_1$ is available. Then $\mathcal{T}_2$ becomes an <u>empty</u> set and all torn branches belong to $\mathcal{L}_2$, each of which generates a fundamental loop, i.e., $B_2 = 1_{\mathcal{L}_2}$. If we partition the reduced incidence matrix $A$ of the original network into the form

$$\begin{array}{cc} \mathcal{L}_1 \cup \mathcal{T}_1 & \mathcal{L}_2 \end{array}$$
$$A = [A_{\mathcal{L}_1 \cup \mathcal{T}_1} \quad A_{\mathcal{L}_2}] \tag{140}$$

then we can identify $A_1 = A_{\mathcal{L}_1 \cup \mathcal{T}_1}$, $\hat{C} = -A^t_{\mathcal{L}_2}$ and Eq. (79) reduces to

$$\begin{bmatrix} A_{\mathcal{L}_1 \cup \mathcal{T}_1} G_{\mathcal{L}_1 \cup \mathcal{T}_1} A^t_{\mathcal{L}_1 \cup \mathcal{T}_1} & A_{\mathcal{L}_2} \\ -A^t_{\mathcal{L}_2} & R_{\mathcal{L}_2} \end{bmatrix} \begin{bmatrix} \hat{V}_{\mathcal{T}_1} \\ I_{\mathcal{L}_2} \end{bmatrix} =$$

$$\begin{bmatrix} A_{\mathcal{L}_1 \cup \mathcal{T}_1}(j_{\mathcal{L}_1 \cup \mathcal{T}_1} - G_{\mathcal{L}_1 \cup \mathcal{T}_1} e_{\mathcal{L}_1 \cup \mathcal{T}_1}) \\ e_{\mathcal{L}_2} - R_{\mathcal{L}_2} j_{\mathcal{L}_2} \end{bmatrix} \tag{141}$$

This equations corresponds to Happ's radially attached case [8], as well as to Wu's earlier formulation[53] [12]. If we further restrict the torn branches to be branches containing voltage sources only, i.e., $R_{\mathcal{L}_2} = 0_{\mathcal{L}_2}$, and no other independent sources exists, then Eq. (141) reduces further to

$$\begin{bmatrix} A_{\mathcal{L}_1 \cup \mathcal{T}_1} G_{\mathcal{L}_1 \cup \mathcal{T}_1} A^t_{\mathcal{L}_1 \cup \mathcal{T}_1} & A_{\mathcal{L}_2} \\ -A^t_{\mathcal{L}_2} & 0_{\mathcal{L}_2} \end{bmatrix} \begin{bmatrix} \hat{V}_{\mathcal{T}_1} \\ I_{\mathcal{L}_2} \end{bmatrix} = \begin{bmatrix} 0_{\mathcal{T}_1} \\ e_{\mathcal{L}_2} \end{bmatrix} \tag{142}$$

This equation corresponds to the <u>modified nodal analysis approach</u> proposed in [15].

To implement the generalized hybrid equation in the tableau form [19,29], we simply rearrange Eqs. (37)-(42), (49), (50) (52) and (53) in the following order:[54]

$$\begin{bmatrix}
1 & B_{\mathcal{L}_1\mathcal{T}_1} & & & & & & & & \\
 & 1 & & & & & & & -P^t_Q & \\
 & & 1 & & & & & & & -P^t_B \\
 & & -B^t_{\mathcal{L}_2\mathcal{T}_1} & 1 & & & & & & \\
-G_{\mathcal{L}_1\mathcal{L}_1} & -G_{\mathcal{L}_1\mathcal{T}_1} & & & 1 & & & & & \\
-G_{\mathcal{T}_1\mathcal{L}_1} & -G_{\mathcal{T}_1\mathcal{T}_1} & & & & 1 & & & & \\
 & & -R_{\mathcal{L}_2\mathcal{L}_2} & -R_{\mathcal{L}_2\mathcal{T}_2} & & & 1 & & & \\
 & & -R_{\mathcal{T}_2\mathcal{L}_2} & -R_{\mathcal{T}_2\mathcal{T}_2} & & & & 1 & & \\
 & & -P_Q B^t_{\mathcal{L}_2\mathcal{T}_1} & & -P_Q B^t_{\mathcal{L}_1\mathcal{T}_1} & & P_Q & & & \\
 & P_B B_{\mathcal{L}_2\mathcal{T}_1} & & & & P_B & & P_B B_{\mathcal{L}_2\mathcal{T}_2} & &
\end{bmatrix}
\begin{bmatrix} V_{\mathcal{L}_1} \\ V_{\mathcal{T}_1} \\ I_{\mathcal{L}_2} \\ I_{\mathcal{T}_2} \\ I_{\mathcal{L}_1} \\ I_{\mathcal{T}_1} \\ V_{\mathcal{L}_2} \\ V_{\mathcal{T}_2} \\ V_{\mathcal{T}_1} \\ I_{\mathcal{L}_2} \end{bmatrix}
=
\begin{bmatrix} 0_{\mathcal{L}_1} \\ 0_{\mathcal{T}_1} \\ 0_{\mathcal{L}_2} \\ 0_{\mathcal{T}_2} \\ -j_{\mathcal{L}_1} + G_{\mathcal{L}_1\mathcal{L}_1} e_{\mathcal{L}_1} + G_{\mathcal{L}_1\mathcal{T}_1} e_{\mathcal{T}_1} \\ -j_{\mathcal{T}_1} + G_{\mathcal{T}_1\mathcal{L}_1} e_{\mathcal{L}_1} + G_{\mathcal{T}_1\mathcal{T}_1} e_{\mathcal{T}_1} \\ -e_{\mathcal{L}_2} + R_{\mathcal{L}_2\mathcal{L}_2} j_{\mathcal{L}_2} + R_{\mathcal{L}_2\mathcal{T}_2} j_{\mathcal{T}_2} \\ -e_{\mathcal{T}_2} + R_{\mathcal{T}_2\mathcal{L}_2} j_{\mathcal{L}_2} + R_{\mathcal{T}_2\mathcal{T}_2} j_{\mathcal{T}_2} \\ 0_{\mathcal{T}_1} \\ 0_{\mathcal{L}_2} \end{bmatrix} \tag{143}$$

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[53] To derive the generalized cases of Wu's formulation, we need only add some "open-circuited" and "short-circuited" branches at "appropriate" locations.

[54] For simplicity, all zero submatrices of the tableau matrix in Eqs. (143) and (144) are not shown.

Observe that if we perform Gaussian Elimination along the diagonal of Eq. (143), the last two equations will indeed become Eq. (51). For the node-to-datum case of the generalized hybrid equation, the preceding tableau matrix assumes the following simplified form:

$$\begin{bmatrix} 1 & & & & -A_1^t & \\ & 1 & & & & -B_2^t \\ -G_1 & & 1 & & & \\ & -R_2 & & 1 & & \\ & & A_1 & & & -\hat{C}^t \\ & & & B_2 & \hat{C} & \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \\ I_1 \\ V_2 \\ \hat{V}_n \\ I_{\mathcal{L}_2} \end{bmatrix} = \begin{bmatrix} 0_1 \\ 0_2 \\ -j_1 + G_1 e_1 \\ -e_2 + R_2 j_2 \\ 0_{\mathcal{T}_1} \\ 0_{\mathcal{L}_2} \end{bmatrix} \tag{144}$$

Observe that if we perform Gaussian Elimination along the diagonal of Eq. (144), the last two equations will become Eq. (79).

References

1. Kron, G., "A Set of Principles to Interconnect the Solution of Physical Systems," Journal of Applied Physics, Vol. 24, No. 8, August 1953, pp. 965-980.

2. Kron, G., "Tearing and Interconnecting as a Form of Transformation," Quarterly of Applied Math., Vol. 13, No. 2, 1955, pp. 147-159.

3. Kron, G., Diakoptics - Piecewise Solution of Large-Scale Systems, MacDonald, London, 1963.

4. Branin, F. H., "The Relation Between Kron's Method and the Classical Methods of Network Analysis," The Matrix and Tensor Quarterly, Vol. 12, No. 3, March 1962, pp. 69-105.

5. Baty, J. and K. L. Stewart, "Simple Applications of Dissection Theory," The Matrix and Tensor Quarterly, Vol. 19, No. 3, March 1969, pp. 81-102.

6. Brameller, A., M. M. John and M. R. Scott, Practical Diakoptics for Electrical Networks, Chapman & Hall, London, 1969.

7. Happ, H. H., Diakoptics and Networks, Academic Press, New York, 1971.

8. Happ, H. H., "Diakoptics - The Solution of System Problems by Tearing," Proc. of IEEE, Vol. 62, No. 7, July 1974, pp. 930-940.

9. Roth, J., "An Application of Algebraic Topology : Kron's Method of Tearing," Quarterly of Applied Math., Vol. 17, No. 1, April 1959, pp. 1-24.

10. Amari, S., "Topological Foundations of Kron's Tearing of Electric Networks," RAAG Memoirs, Vol. 3, 1962, pp. 322-350.

11. Wang, K. U. and T. Chao, "An Algebraic Theory of Network Topology," Proc. of the IEEE International Symp. on Circuits and Systems, April 1974, pp. 324-328.

12. Wu, F., "Diakoptic Network Analysis," to appear in the Proc. of the 1975 PICA Conf., June 1975.

13. Onodera, R., "Diakoptics and Codiakoptics of Electric Network," RAAG Memoirs, Vol. 2, 1958, pp. 369-388.

14. Onodera R. and N. Ishiyama, "A Numerical Example of Double Diakoptics," RAAG Memoirs, Vol. 3, 1962, pp. 370-384.

15. Ho, C. W., A. E. Ruehli and P. A. Brennen, "The Modified Nodal Approach to Network Analysis," IEEE Trans. on Circuits and Systems, Vol. CAS-22, No. 6, June 1975, pp. 504-509.

16. Desoer, C. A. and E. S. Kuh, Basic Circuit Theory, McGraw Hill, New York, 1969.

17. Chua, L. O., Introduction to Nonlinear Network Theory, McGraw Hill, New York, 1969.

18. Chua, L. O and L. K. Chen, "On Optimally Sparse Cycle and Coboundary Basis for a Linear Graph," IEEE Trans. on Circuit Theory, Vol. CT-20, No. 5, September 1973, pp. 495-503.

19. Chua, L. O. and P. M. Lin, Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques, Prentice-Hall, Englewood Cliffs, N. J., 1975.

20. Chua, L. O., "Stationary Principles and Potential Functions," Journal of the Franklin Institute, August 1973, pp. 91-114.

21. Ohtsuki, T. and H. Watanabe, "State Variable Analysis of RLC Networks Containing Nonlinear Coupling Elements," IEEE Trans. on Circuit Theory, Vol. CT-16, No. 1, February 1969, pp. 26-38.

22. Branin, F. H., "A Unifying Approach to the Classical Methods of Formulating Network Equations," Proc. of the IEEE International Symp. on Circuits and Systems, April 1974, pp. 750-754.

23. Calahan, D. A., Computer-Aided Network Design, McGraw Hill, New York, 1972.

24. Amari, S., "Information-Theoretical Foundations of Diakoptics and Codiakoptics," RAAG Memoirs, Vol. 3, 1962, pp. 351-371.

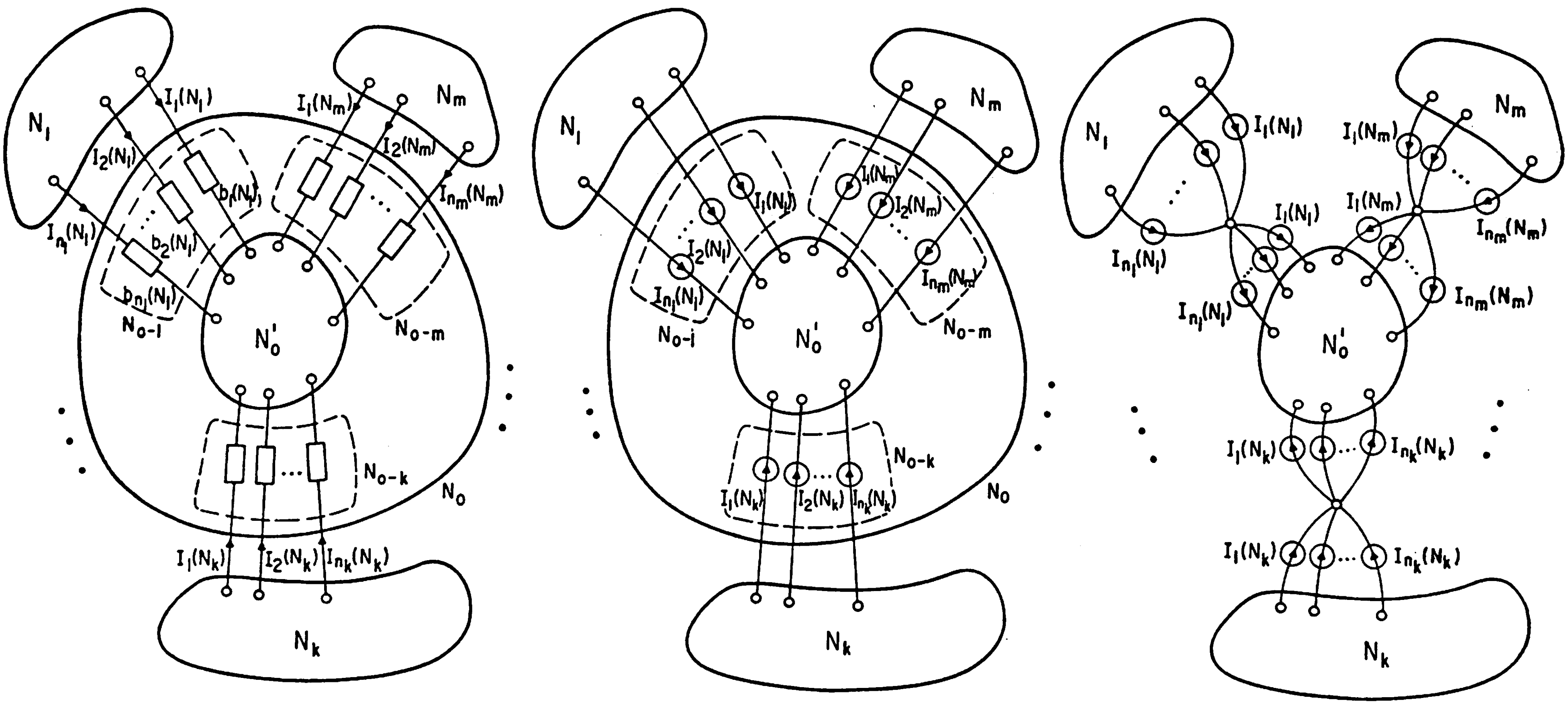
25. Ohtsuki, T., Y. Ishizaki and H. Watanabe, "Topological Degrees of Freedom and Mixed Analysis of Electrical Networks," IEEE Trans. on Circuit Theory, Vol. CT-17, No. 4, November 1970, pp. 491-499.

26. Ogbuobiri, E. C., W. F. Tinney and J. W. Walker, "Sparsity-Directed Decomposition for Gaussian Elimination on Matrices," IEEE Trans. on Power, Apparatus and Systems, Vol. PAS-89, No. 1, January 1970, pp. 141-150.

27. Nagel, L., "SPICE 2: A Computer Program to Simulate Semiconductor Circuits," Ph.D. dissertation, University of California at Berkeley, Berkeley, California, May 1975.

28. Tarjan, R., "Depth-First Search and Linear Graph Algorithms," SIAM Journal on Computing, Vol. 1, No. 2, June 1972, pp. 146-160.

29. Hachtel, G.D., R. K. Brayton and F. G. Gustavson, "The Sparse Tableau Approach to Network Analysis and Design," IEEE Trans. on Circuit Theory, Vol. CT-18, No. 1, January 1971, pp. 101-113.

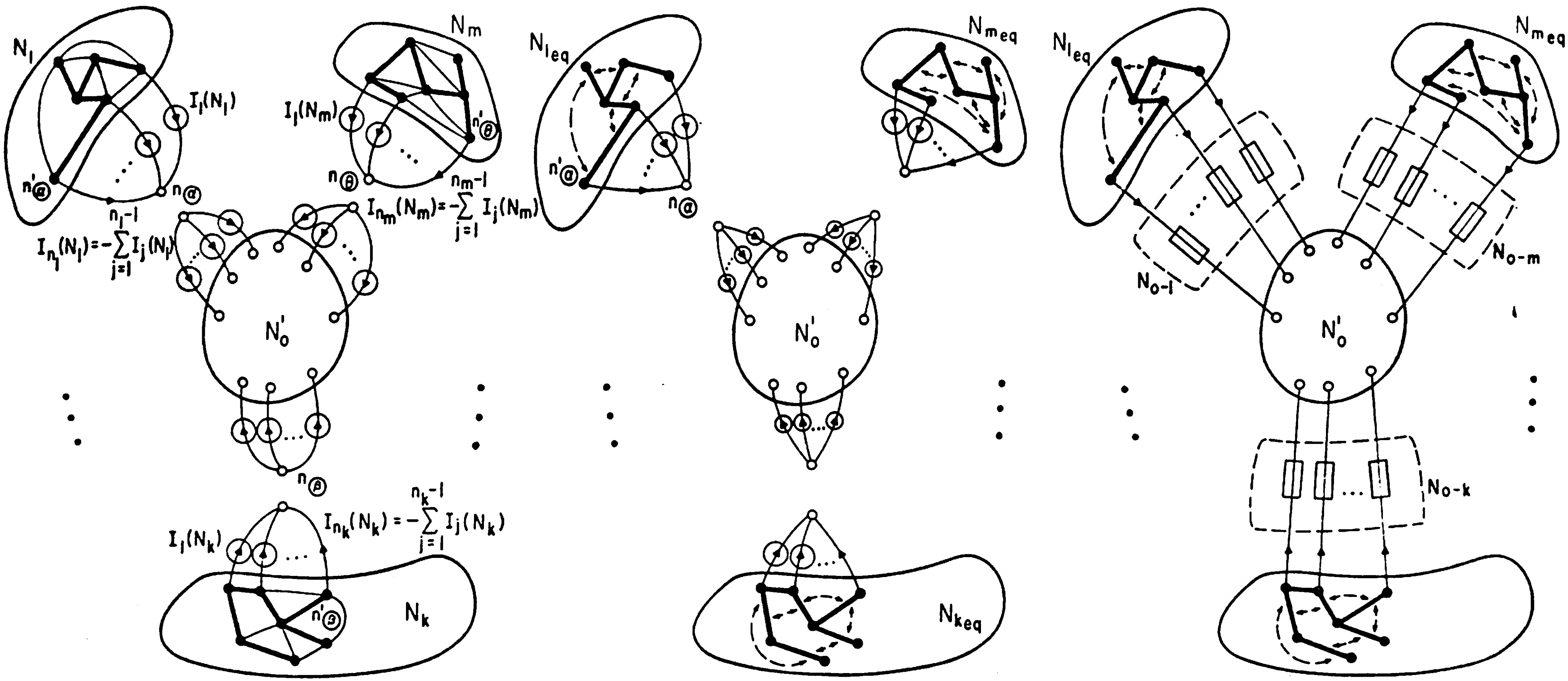



(a) the torn branches $N_0$ and the connection branches $N_{0-k}$

(b) equivalent current source substitution

(c) separation of the subnetworks

Fig. 1(a),(b),(c). A sequence of equivalent transformations.




(d) one current source in each cut set is replaced with a short circuit

(e) the equivalent acyclic subnetworks

(f) the equivalent interconnected network $N_{eq}$

Fig. 1(d),(e),(f). A sequence of equivalent transformations.

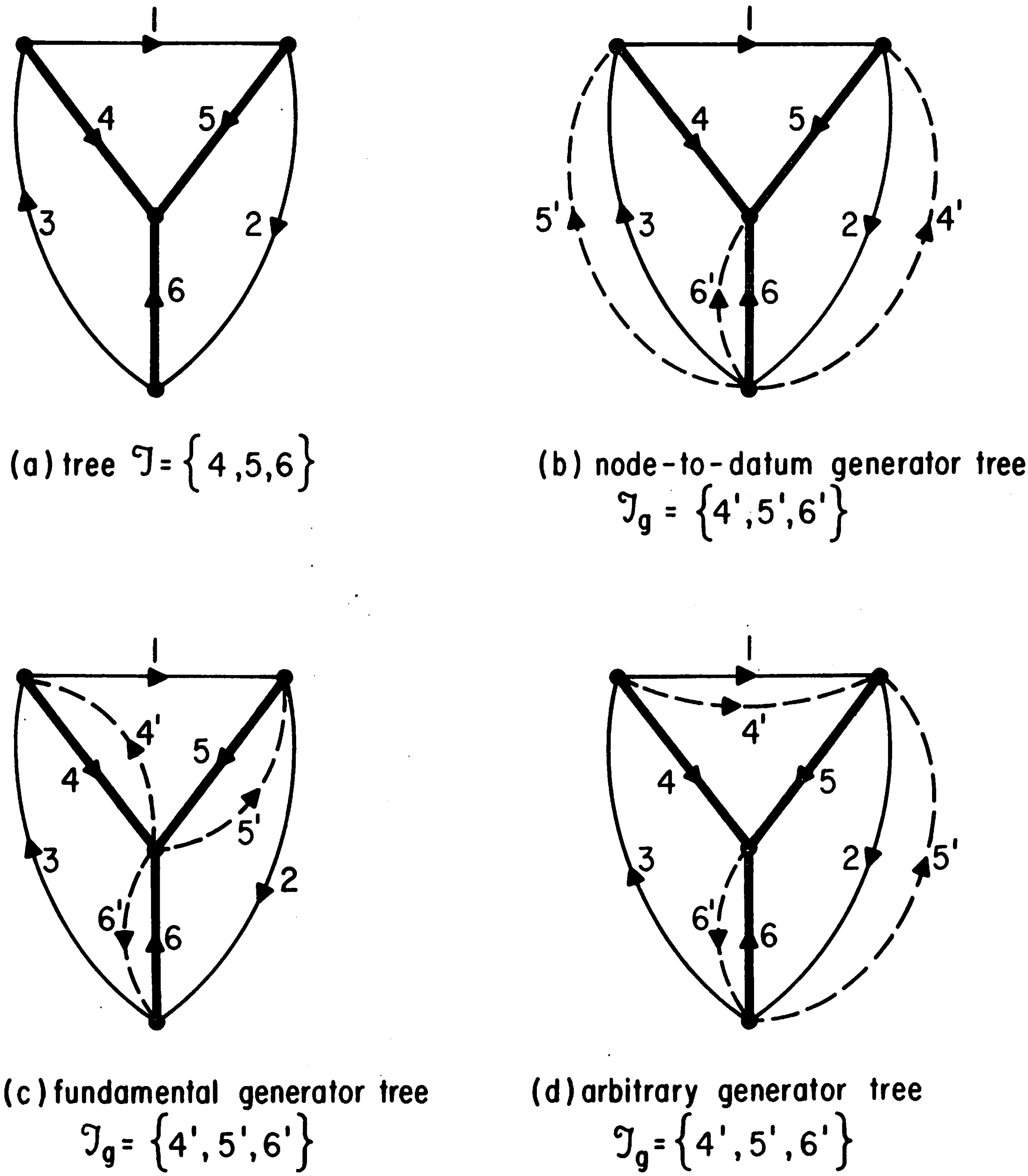
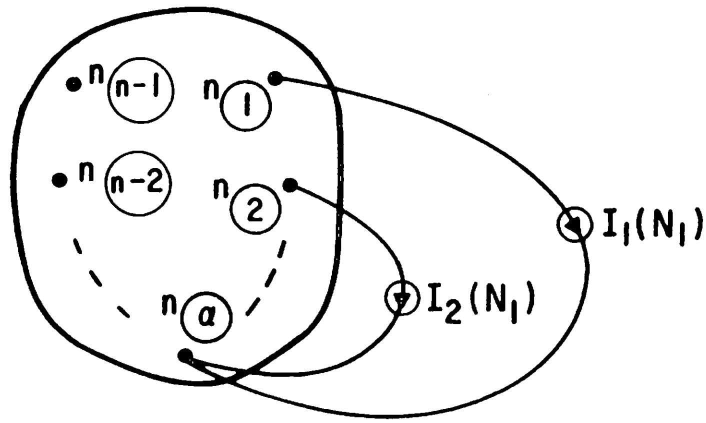
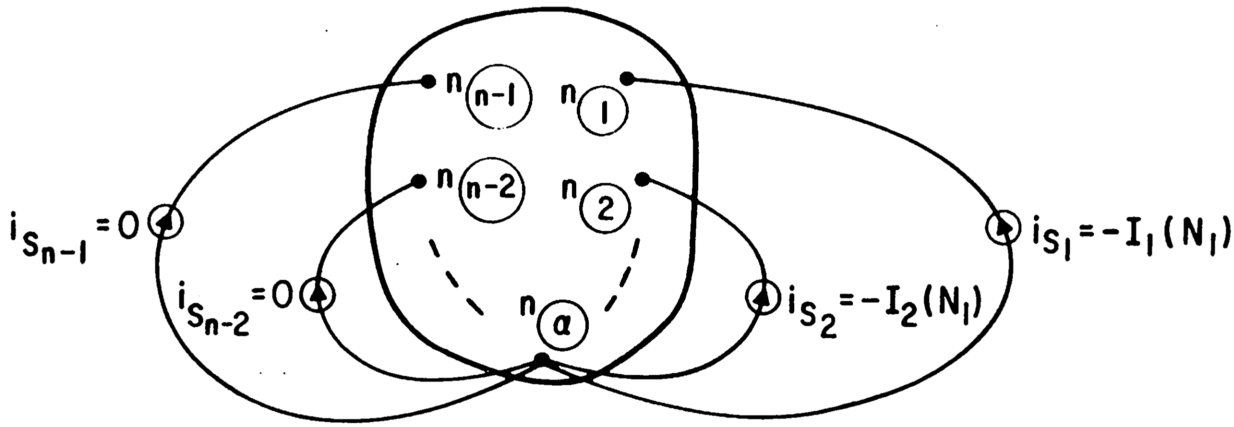


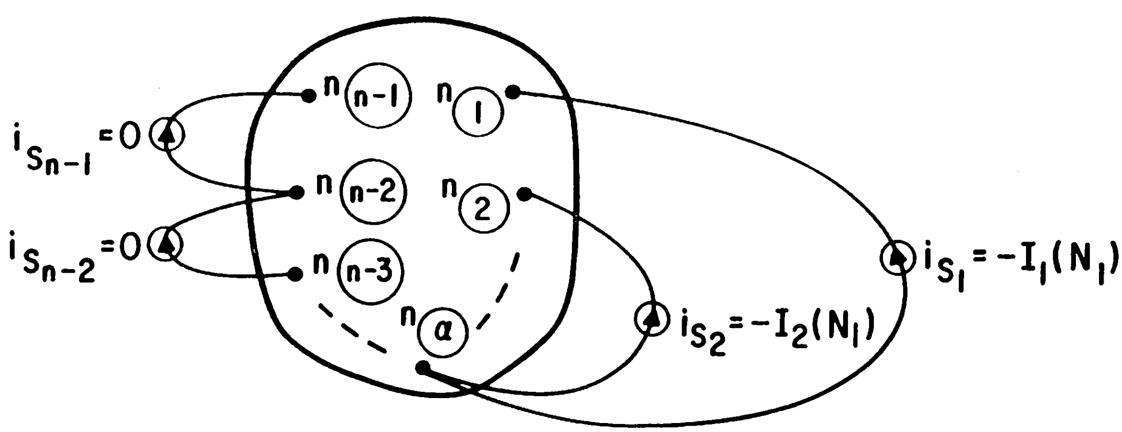

Fig. 2. A graph and several augmented generator trees.




(a) subnetwork with an "incomplete" set of current sources

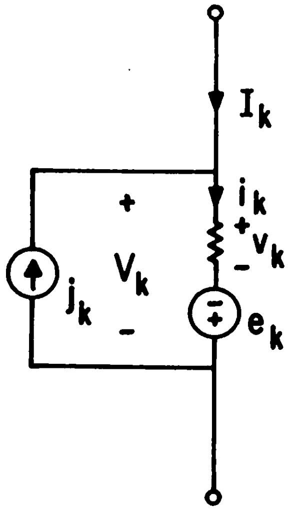



(b) node-to-datum case

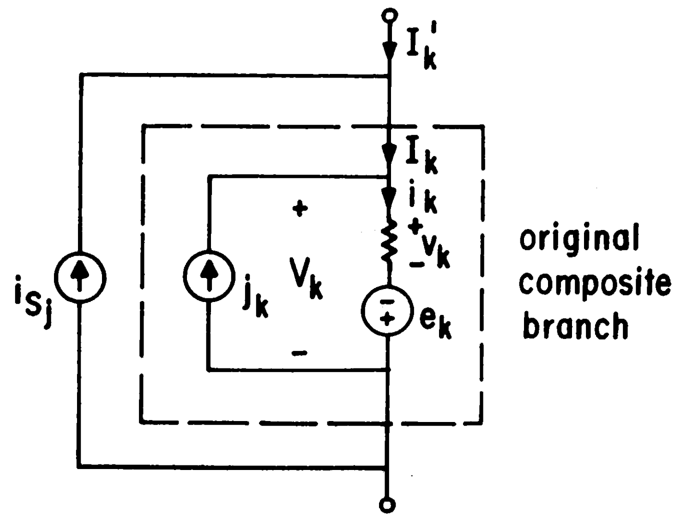



(c) arbitrary case

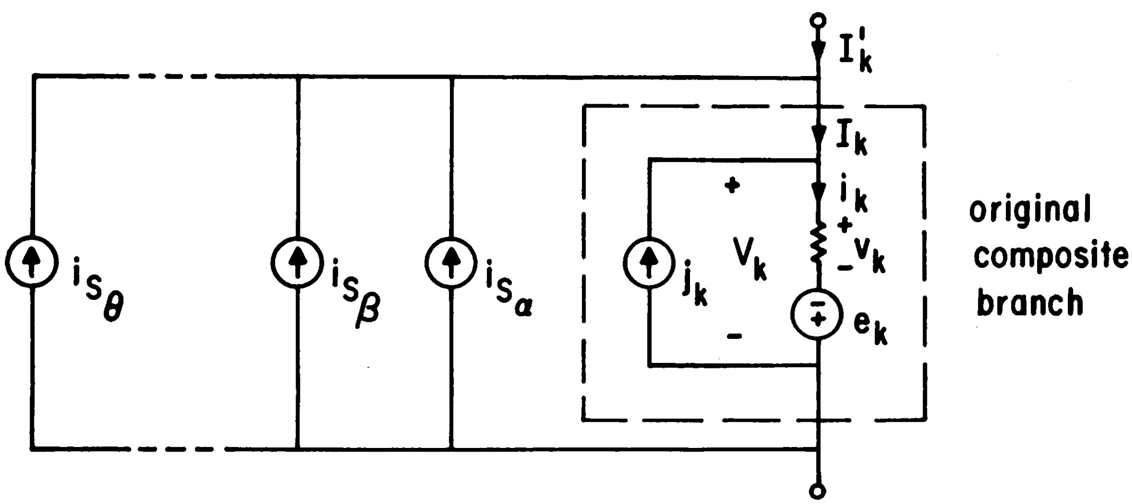
Fig. 3. An illustration of the addition of "zero-valued" current sources to subnetwork $N_1$ to obtain a set of (n-1) external current sources.



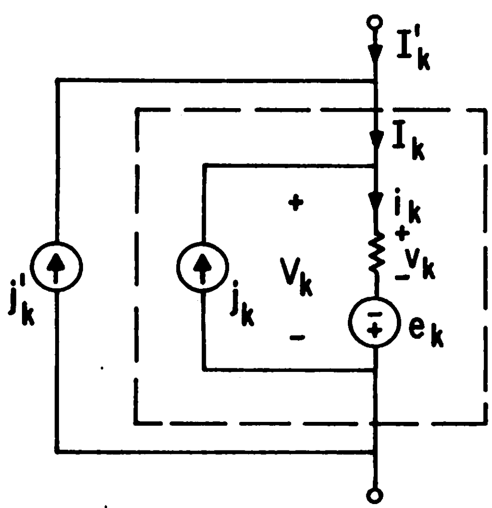

(a) the original composite branch




(b) a shifted current source $i_{s_j}$ in parallel with the original composite branch




(c) up to (n−1) shifted current sources may appear across the original composite branch k: $\alpha, \beta, \ldots, \theta \in \{1, 2, \ldots, n-1\}$




$$I'_k = I_k - j'_k$$
$$I_k = i_k - j_k$$
$$V_k = v_k - e_k$$

(d) the augmented composite branch

Fig. 4. A composite branch and its associated augmented composite branch.

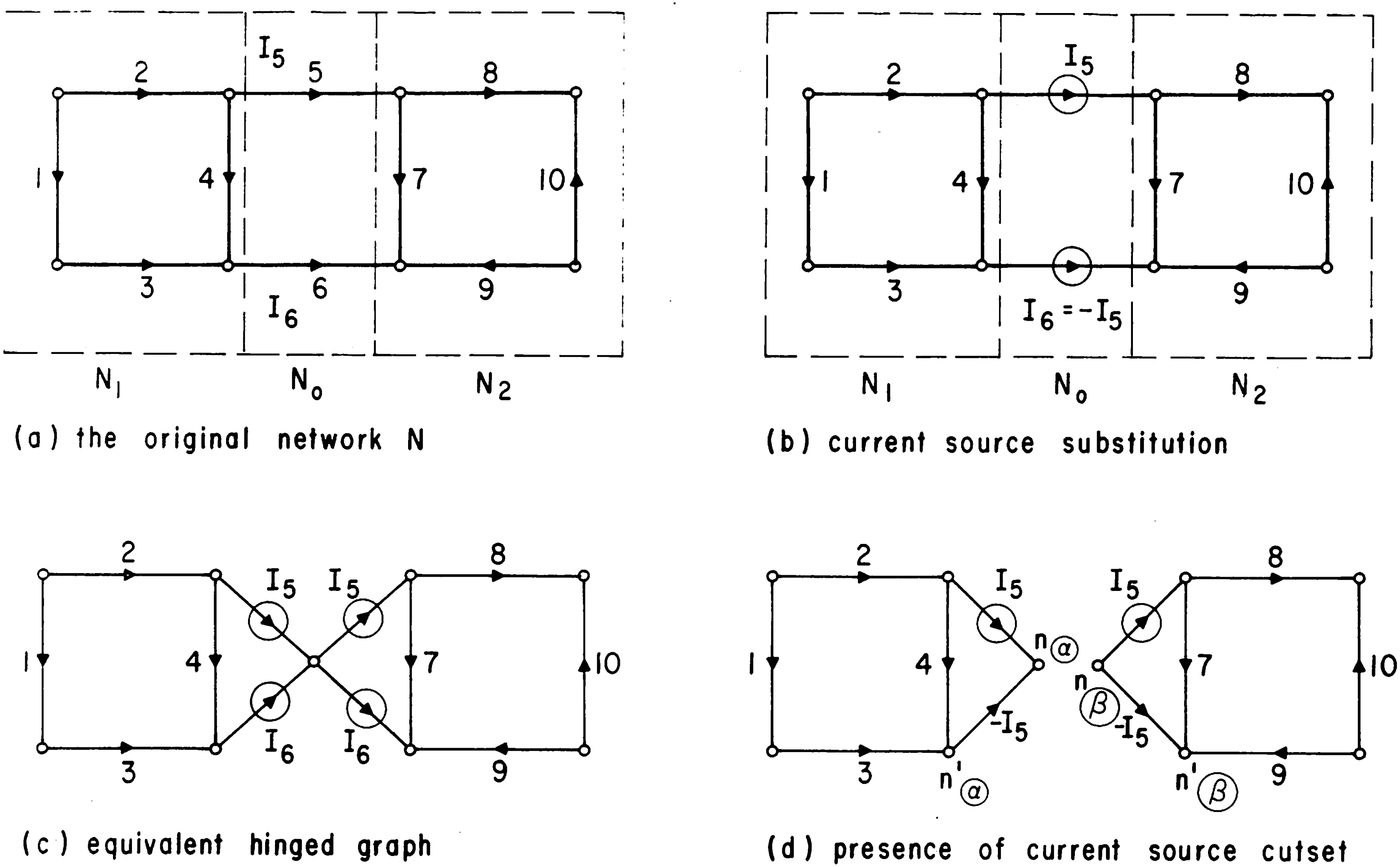
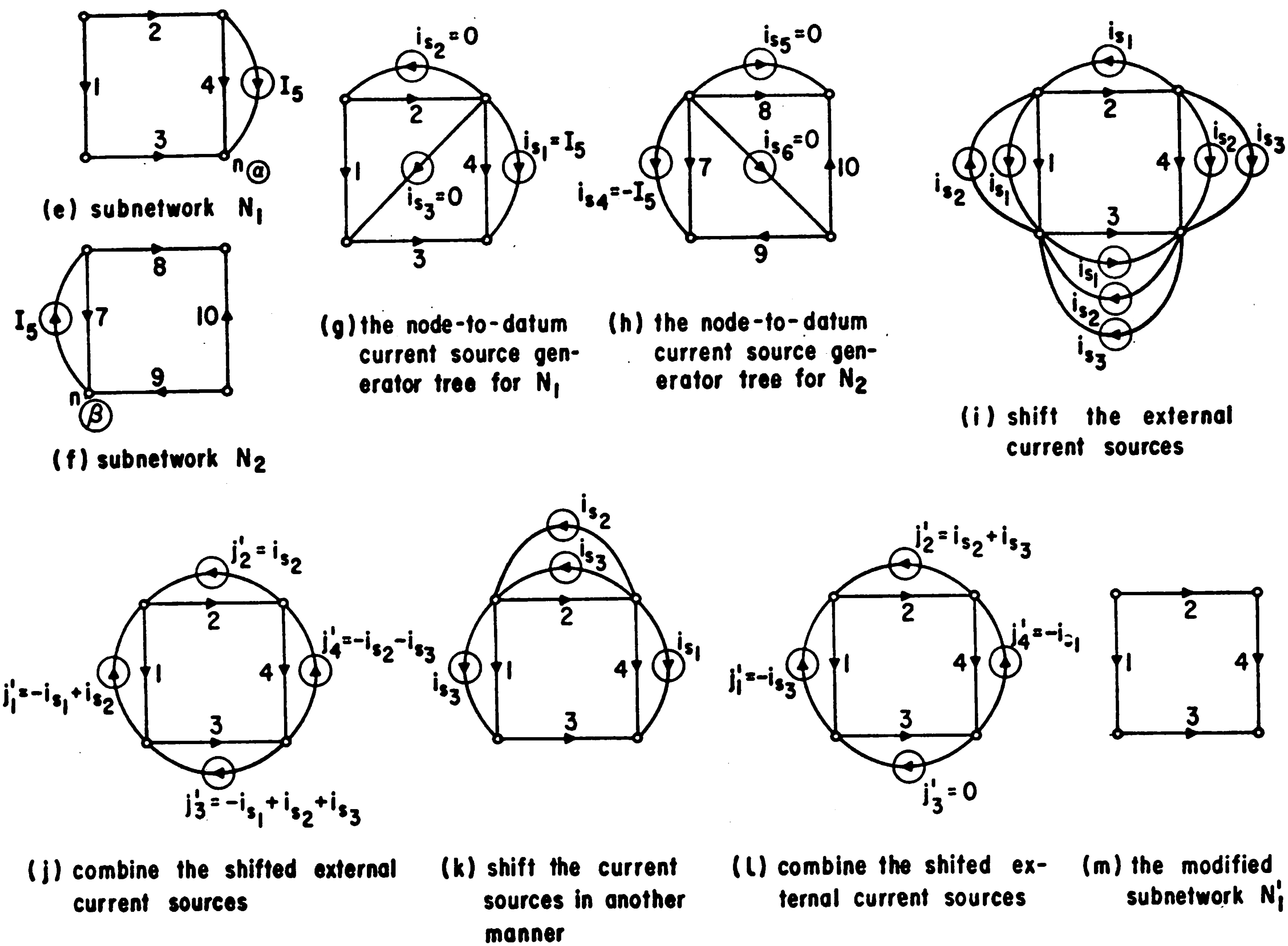



Fig. 5(a), (b), (c), (d). An example illustrating the various stages of the equivalent transformations.




(e) subnetwork $N_1$

(f) subnetwork $N_2$

(g) the node-to-datum current source generator tree for $N_1$

(h) the node-to-datum current source generator tree for $N_2$

(i) shift the external current sources

(j) combine the shifted external current sources

(k) shift the current sources in another manner

(L) combine the shited external current sources

(m) the modified subnetwork $N_1'$

Fig. 5(e), (f), (g), (h), (i), (j), (k), (1), (m). An example illustrating the various stages of the equivalent transformations.

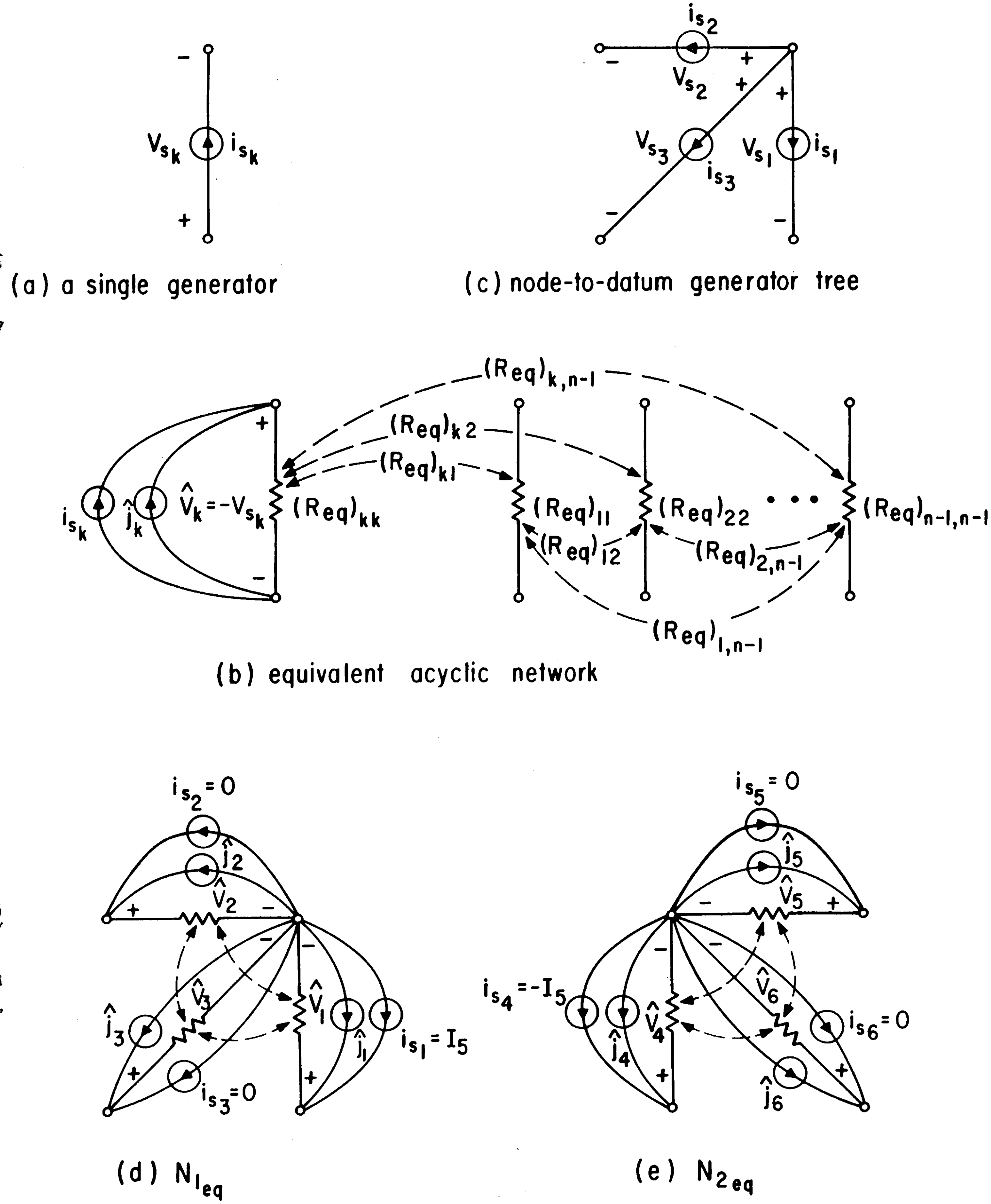



Fig. 6(a), (b), (c), (d), (e). A sequence of "inverse" transformation stages for deriving the equivalent interconnected network $N_{eq}$.

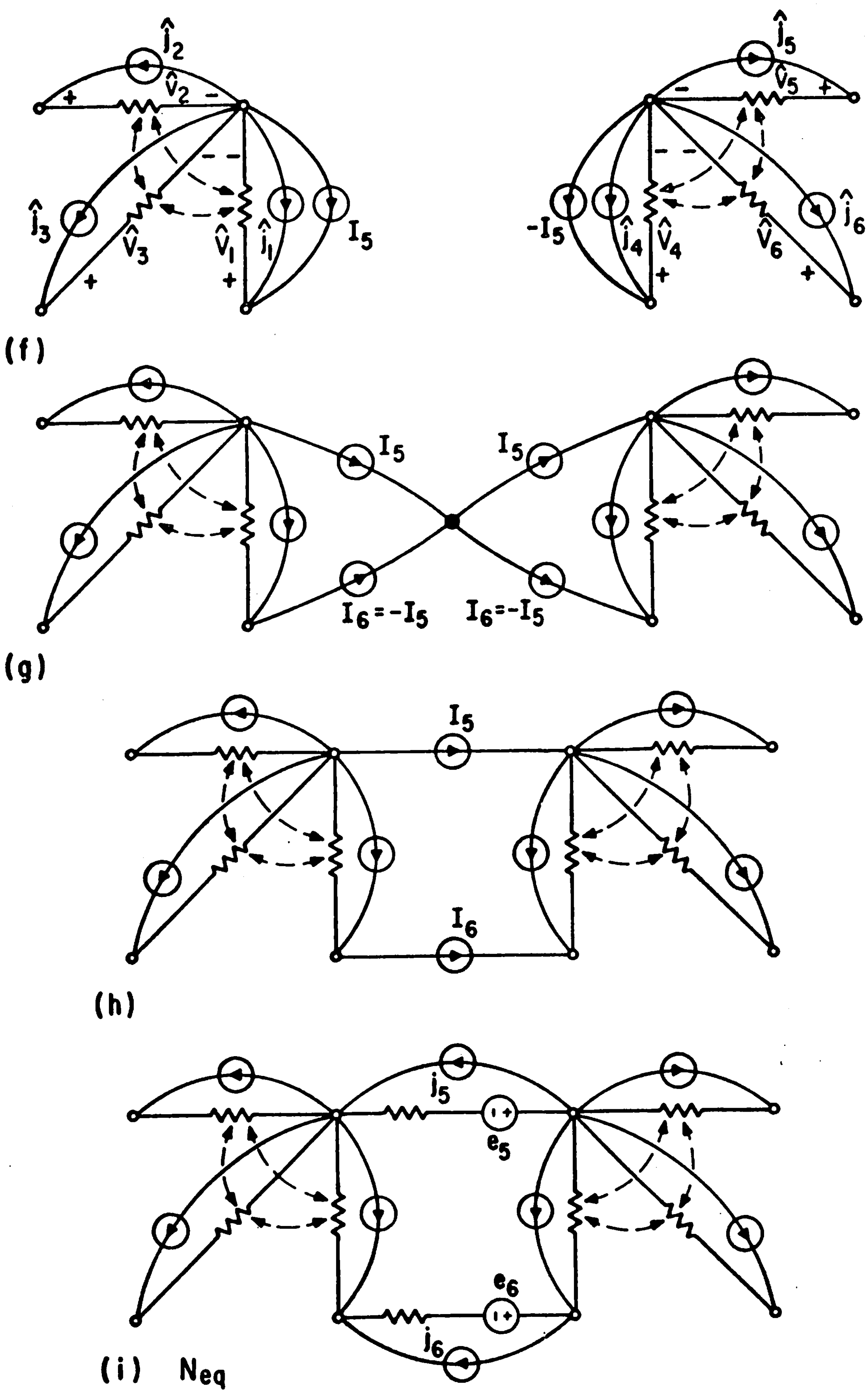



Fig. 6(f), (g), (h), (i). A sequence of "inverse" transformation stages for deriving the equivalent interconnected network $N_{eq}$.

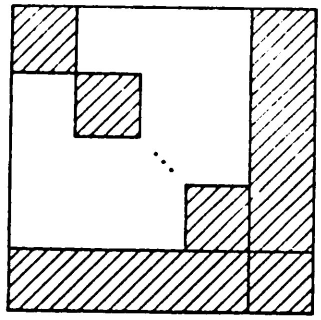


Fig. 7. The bordered block triangular matrix structure characterizing diakoptic analysis.

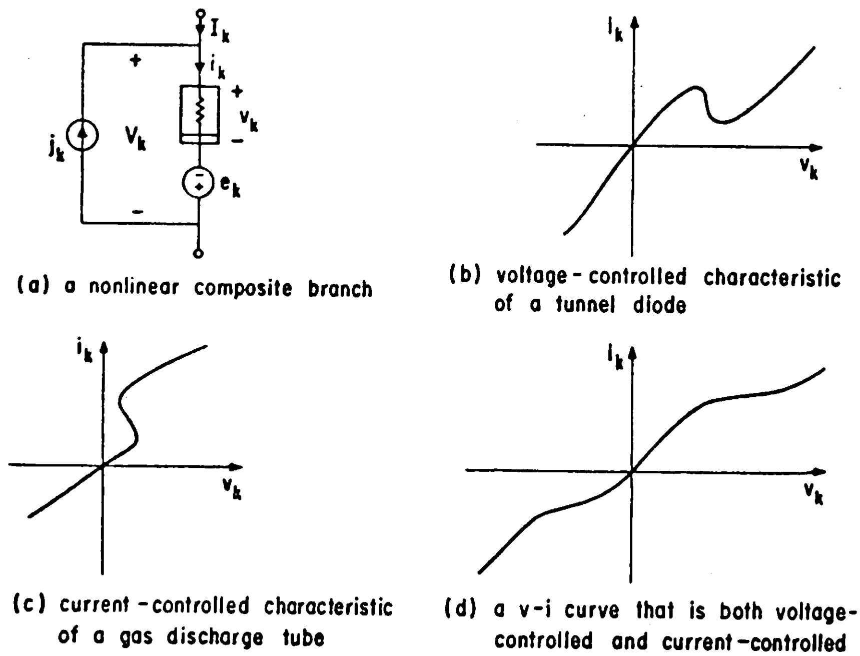



Fig. 8. The constitutive relationship of a nonlinear resistor.

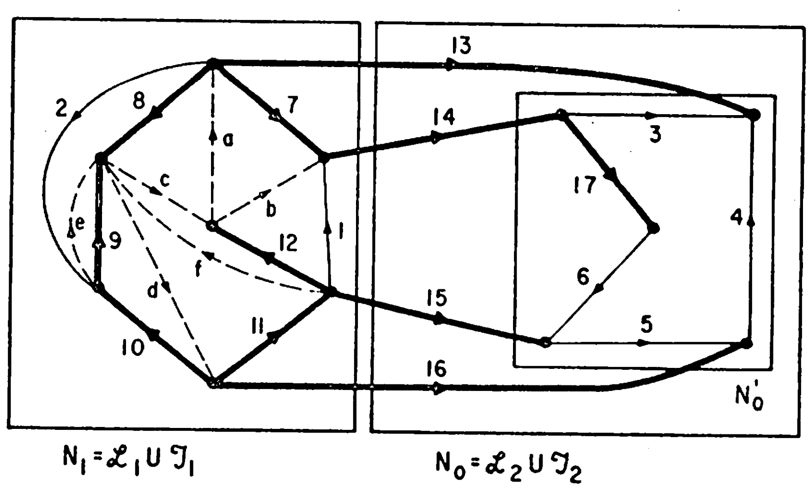



Fig. 9. An illustration of Lemma 8, where $\mathcal{L}_1$ = {1,2}, $\mathcal{L}_2$ = {3,4,5,6}, $\mathcal{T}_1$ = {7,8,9,10,11,12}, $\mathcal{T}_2$ = {13,14,15.16.17} and $\mathcal{T}_g$ = {a,b,c,d,e,f,}

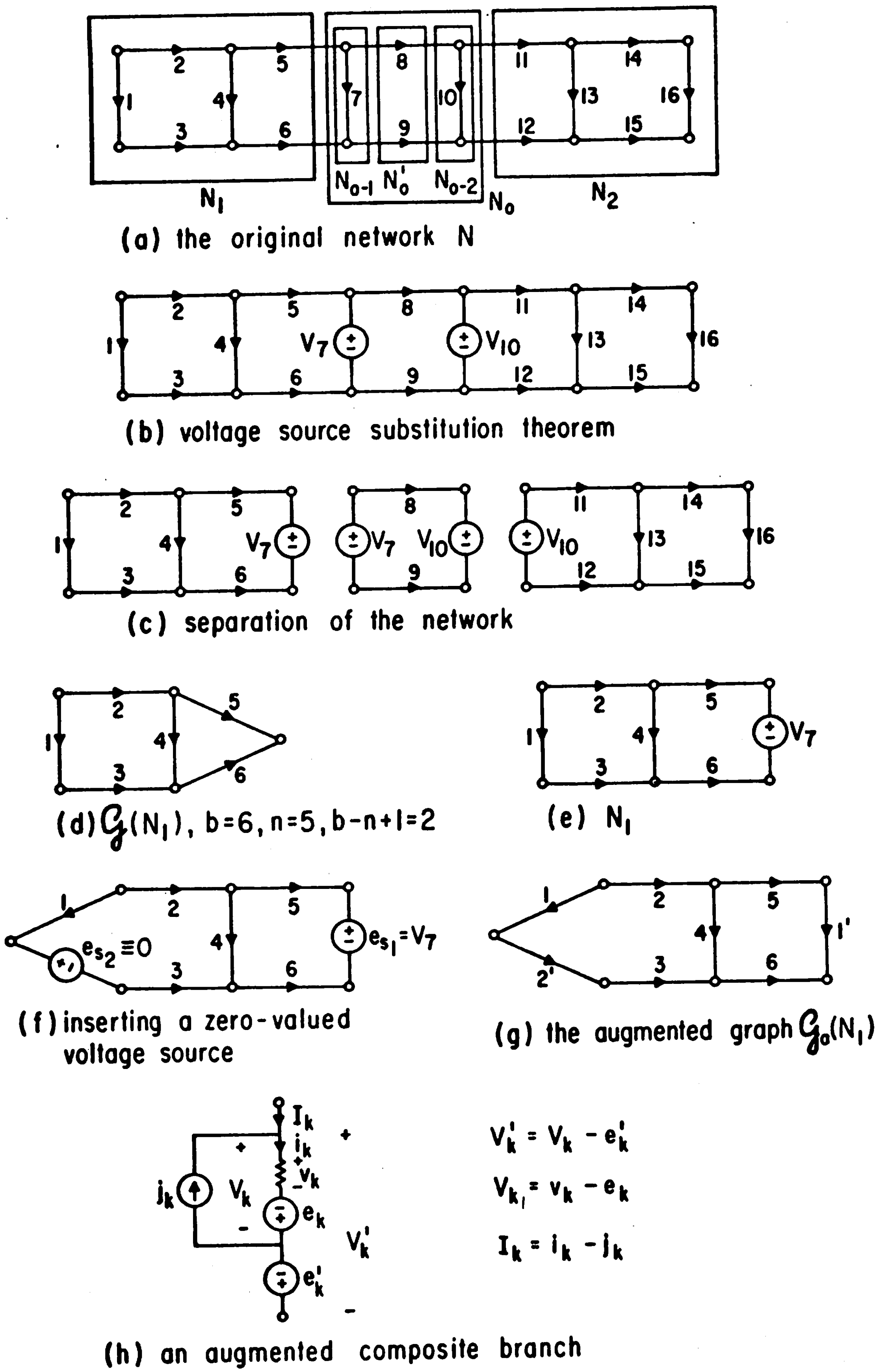



Fig. 10(a), (b), (c), (d), (e), (f), (g), (h). The equivalent transformations in deriving codiakoptic analysis.

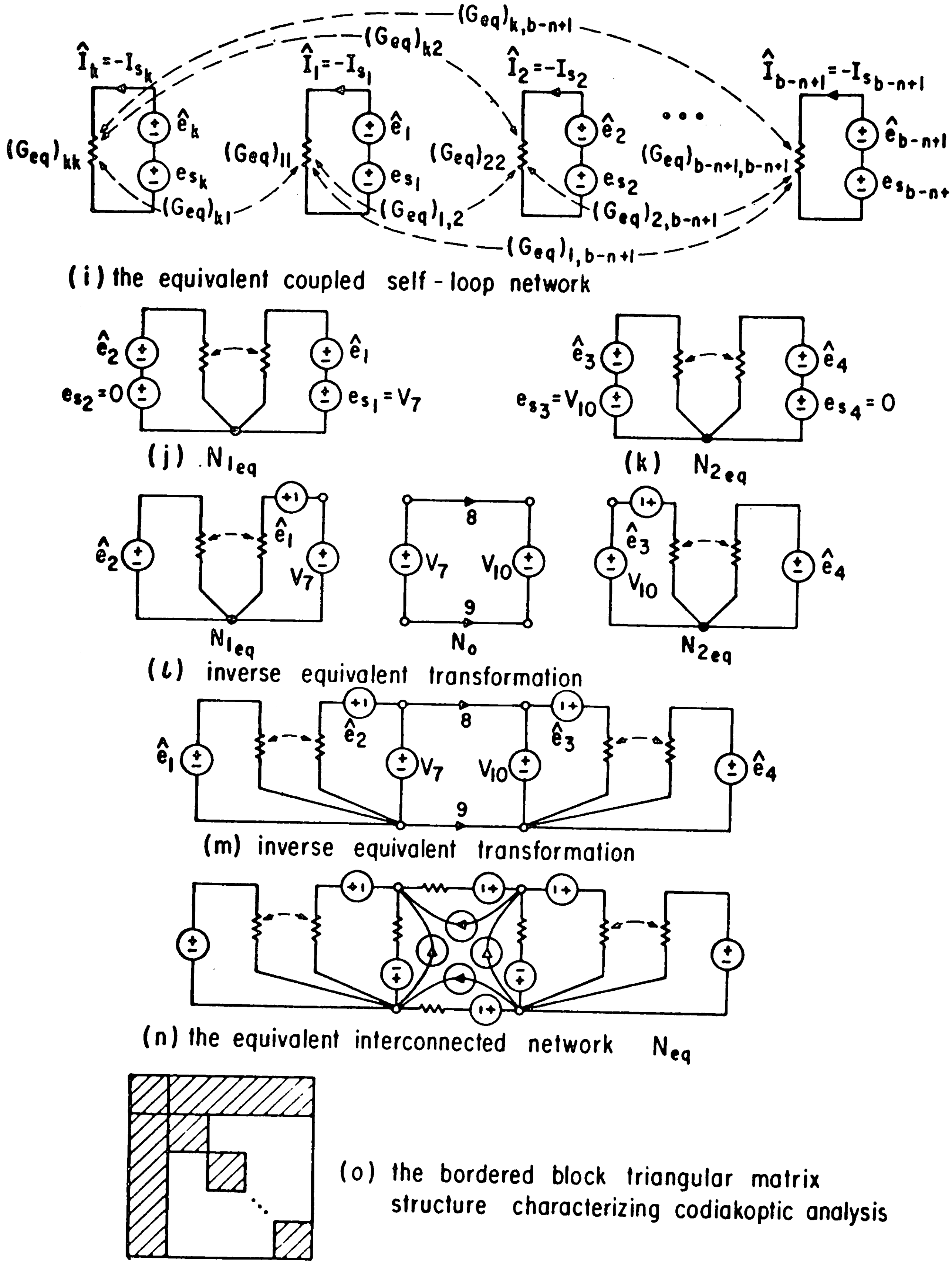



Fig. 10(i), (j), (k), (l), (m), (n), (o). The equivalent transformations in deriving codiakoptic analysis.