WEAK MARTINGALES AND STOCHASTIC INTEGRALS IN THE PLANE

by

Eugene Wong and Moshe Zakai

Memorandum No. ERL-M496

6 February 1975

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Eugene Wong and Moshe Zakai

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract

This paper continues the development of a stochastic calculus for two-parameter martingale. It is shown that such a calculus is complete only if one introduces a mixed area integral in addition to the ordinary integral and the two types of stochastic integrals which were introduced earlier. In particular, the mixed integral is necessary for an elucidation and representation of weak martingales which were introduced by Cairoli and Walsh. As a preliminary development of differentiation formulas of the Itô type, representation of products of integrals of the various types is derived.

Stopping times are introduced for two-parameter processes, and a characterization of strong martingales in terms of stopping times is given. Finally, some brief results on path-independent variation and on two-parameter Markov processes are presented.

Research sponsored by U. S. Army Research Office, Durham, N.C., Grant DAHC04-74-G0087.

0. Introduction

This paper continues recent work toward the development of a stochastic calculus in the plane (i.e. for the case where the time parameter is two dimensional) for continuous martingales in general and for the two parameter Wiener process in particular.

The basic references for this work are the fundamental paper by Cairoli and Walsh [3] and a previous paper by the present authors [4]. The reader is referred to [3] and [4] for further references in this field.

In order to describe the contents of this paper we give, first, an incomplete definition for two parameter martingales, weak, 1- and 2-martingales. Precise definitions and references will be given in the next section. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{F}_{s,t}$, $0 \leq s \leq s_0$, $0 \leq t \leq t_0$, sub σ-fields of $\mathcal{F}$ such that $\mathcal{F}_{s_1,t} \subset \mathcal{F}_{s_2,t_2}$ if $s_1 \leq s_2$ and $t_1 \leq t_2$. In what follows assume $0 \leq s_1 \leq s_2 \leq s_0$, $0 \leq t_1 \leq t_2 \leq t_0$, and $X_{s,t}$ to be $\mathcal{F}_{st}$ measurable. Then $X_{s,t}$ is a martingale if

$E(X_{s_2,t_2} | \mathcal{F}_{s_1,t_1}) = X_{s_1,t}$, a 2-martingale if for all fixed s

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$E(X_{s,t_2} | \mathcal{F}_{s,t_1}) = X_{s,t_1}$ (there is some difference between the definition of 1- and 2- martingales used in this paper and [3] as will be pointed out in the next section). $X_{s,t}$ is a weak martingale if

$$E\{X_{s_2,t_2} + X_{s_1,t_1} - X_{s_2,t_1} - X_{s_1,t_2} | \mathcal{F}_{s_1,t_1}\} = 0$$

In section 2 we show that $X_{s,t}$ is a weak martingale if and only if it is the sum of a martingale, a 1-martingale and a 2-martingale (a discrete version of this result appears in [1]). A one (or two) martingale $X_{s,t}$ is said to be proper if for a fixed s (resp.t) it is of bounded variation in t (resp.s). It is shown that weak martingales satisfying certain restrictions can be decomposed uniquely into the sum of a proper martingale, a proper one martingale and a proper two martingale. In section 3 we introduce a mixed area integral $\iint \psi(z,z')dM_z d\mu(z')$ where $\mu(z)$ is a (possibly random) function of bounded variation and M_z is a martingale. It is shown that such integrals are proper 1 or 2 martingales. In some special cases this integral reduces to the mixed integral introduced by Cairoli and Walsh [3]. In section 4 it is shown that every proper 1 or 2 martingale of the Wiener Process satisfying a suitable differentiability condition can be represented as a mixed area integral.

In section 5 we consider a stochastic (Itô type) calculus in terms of area integrals including the mixed area integrals of section 3. A partial multiplication table representing the product of two stochastic integrals as sum of stochastic integrals is constructed, a complete stochastic calculus will be presented in a later report. Stopping times are introduced in section 6 and used to give a characterization of strong martingales of the Wiener process. We also give a characterization of path independent martingales in this section. The possibility of constructing two parameter Markov processes is discussed briefly in section 7.

1. Preliminaries and Notation

Let $z=(s,t), 0 \le s \le s_0,\ 0 \le t \le t_0$ denote points on a rectangle in the positive quadrant of the plane. $z_1 \prec z_2$ will denote $s_1 \le s_2$ and $t_1 \le t_2$. R_{z_0} will denote the rectangle $\{z: 0 \prec z \prec z_0\}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_z, z \in R_{z_0}\}$ be a family of sub σ-fields of $\mathcal{F}$ such that [3]:

F_1) $z \prec z'$ implies $\mathcal{F}_z \subset \mathcal{F}_{z'}$

F_2) $\mathcal{F}_0$ contains all the null sets of $\mathcal{F}$

F_3) for all z, $\mathcal{F}_z = \bigcap \mathcal{F}_{z'}$, $s' > s,\ t' > t$.

F_4) for each z, $\mathcal{F}_z^1$ and $\mathcal{F}_z^2$ are conditionally independent given $\mathcal{F}_z$, where $\mathcal{F}_z^1 = \mathcal{F}_{s,t_0}$, $\mathcal{F}_z^2 = \mathcal{F}_{s_0,t}$.

<u>Definition</u>: A process $\{M_z, z \in R_{z_0}\}$ is a martingale if (1) M_z is z adapted (2) for each z, M_z is integrable, (3) for each $z \prec z'$, $E(M_{z'}|\mathcal{F}_z) = M_z$.

Let $z=(s,t)$, $z'=(s',t')$, the condition $s < s'$, $t < t'$ will be denoted by $z \prec\prec z'$. If $z \prec\prec z'$, $(z,z']$ will denote the rectangle $(s,s'] \times (t,t']$ and if X_z is a random process, $X(z,z']$ will denote $X_{s',t'} + X_{s,t} - X_{s',t} - X_{s,t'}$.

Several other notions of martingales were introduced in [3]. We follow here these definitions with the exception of the definitions of 1- and 2-martingales which differ from those given in [3], as will be pointed out later. In the following definitions $X = \{X_z, z \in R_{z_0}\}$ is assumed, for each $z \in R_{z_0}$, to be integrable and $\mathcal{F}_z$ adapted.

<u>Definitions</u>: (a) X_z is a <u>weak martingale</u> if $E\{X(z,z']|\mathcal{F}_z\} = 0$ for

(b) X_z is an i-martingale, $i = 1,2$, if $E\{X(z,z')|\mathcal{F}_z^i\} = 0$ for every $z \prec\prec z'$, and $x_{s,0}$ is a one parameter martingale for $i = 1$ and $X_{0,t}$ is a one parameter martingale for $i = 2$.

(c) X_z is a strong martingale if it vanishes at the axes and $E\{X(z,z')|\mathcal{F}_z^1 \vee \mathcal{F}_z^2\} = 0$ for every $z \prec\prec z'$.

Remark: The definition of an i-martingale given here differs from the one given in [3] by the requirement that X_z be $\mathcal{F}_z$ adapted and that $X_{s,0}$ or $X_{0,t}$ be a one parameter martingale while in [3] it was only required that X_z be $\mathcal{F}_z^i$ adapted.

Some additional notational conventions

(a) The letters z,ζ,η will be used to denote points in R_{z_0} whenever these letters appear with or without primes. It will always be assumed that $z_0 = (s_0,t_0)$, $0 < s_0 < \infty$, $0 < t_0 < \infty$ is a fixed point in the plane.

(b) $z_1 \curlywedge z_2$ will denote that $s_1 < s_2$ and $t_2 < t_1$, and $z_1 \wedge z_2$ will denote the point (s_1,t_2).

(c) $z_1 \vee z_2$ will denote the point $(\max(s_1,s_2),\max(t_1,t_2))$.

(d) The function $h(z,z')$ is defined as

$$h(z,z') = \begin{cases} 1 & \text{if } z \curlywedge z' \\ 0 & \text{otherwise} \end{cases}$$

(e) Unless otherwise specified, if the time parameter in the integrand is ζ then the integration is over R_z or $R_z \times R_z$, i.e.,

$$\int \psi_\zeta dM_\zeta = \int_{R_z} \psi_\zeta dM_\zeta$$

$$\iint \psi(\zeta,\zeta')dM_\zeta dM_{\zeta'} = \int_{R_z \times R_z}\int \psi(\zeta,\zeta')dM_\zeta dM_{\zeta'}$$

and if the time parameter in the integrand is z then the integration is over R_{z_0} or $R_{z_0} \times R_{z_0}$, namely.

$$\int \psi(z)\,dM_z = \int_{R_{z_0}} \psi(z)\,dM_z$$

2. The Decomposition of Weak Martingales

Recall that throughout this paper a 1-martingale M_z^1 and a 2-martingale M_z^2 are as defined by Cairoli and Walsh [3] with the additional assumptions that M_z^1 and M_z^2 be $\mathcal{F}_z$ measurable and $M_{s,0}^1$ and $M_{0,t}^2$ be one parameter martingales.

Proposition 2-1 X_2 is a weak martingale on R_{z_0} if and only if it is expressible as $X_z = M_z^1 + M_z^2$ where M_z^1 is a 1-martingale, M_z^2 is a 2-martingale.

Proof: Every 1- or 2-martingale is by definition a weak martingale. Let

$$M_{s,t}^1 = E(X_{s_0,t} \mid \mathcal{F}_{s,t}).$$

Note that $E(X_{s_0,t} \mid \mathcal{F}_{s,t}) = E(X_{s_0,t} \mid \mathcal{F}_{s,t_0})$ by assumption (F-4) on the conditional independence property of the σ-fields. Therefore $M_{s,t}^1$ is a 1-martingale.

Let $Y_z = X_z - M_z^1$, then for $h > 0$, $(s,t+h) \prec z_0$,

$$E(Y_{s,t+h}-Y_{s,t} \mid \mathcal{F}_{s_0,t}) = E(Y_{s,t+h}-Y_{s,t} \mid \mathcal{F}_{s,t})$$

$$= E\{X_{s,t+h} + X_{s,t} - E(X_{s_0,t+h} \mid \mathcal{F}_{s,t+h}) - E(X_{s_0,t} \mid \mathcal{F}_{s,t}) \mid \mathcal{F}_{s,t}\}$$

$$= E\{X_{s,t+h} + X_{s,t} - X_{s_0,t+h} - X_{s_0,t} \mid \mathcal{F}_{s,t}\}$$

$$= 0$$

since $X_{s,t}$ is a weak martingale. Therefore $Y_z = M_z^2$ is a 2-martingale.

Remarks: (a) Note that if X_z is right continuous, so are M_z^1 and M_z^2. (b) If the σ-fields $\mathcal{F}_{0,\infty}$ or $\mathcal{F}_{\infty,0}$ are trivial and $X_{0,0} = 0$, then $M_{0,t}^1 = M_{s,0}^1 = M_{0,t}^2 = 0$. (c) The decomposition of proposition 1 is not unique. However, if $X_z = M_z^1 + M_z^2$ and also $X_z = N_z^1 + N_z^2$ then $M_z^1 - N_z^1$ and $M_z^2 - N_z^2$ are both 1 and 2-martingales. Therefore, by the converse to proposition 1.1 of [1] (see the proof of proposition 1.1 of [3]) $M_z^1 - N_z^1 = N_z^2 - M_z^2$ is a martingale.

__Definition__ A weak martingale X_z will be said to be __regular__ on R_{z_0} if for every fixed t, $X_{s,t}$ as a function of s is a one-parameter semimartingale (namely the sum of a martingale and a function of bounded variation) and for every fixed s, $X_{s,t}$ as a function of t is a one-parameter semimartingale, for almost all ω.

__Definition__ A 1-martingale M_z^1 (2-martingale M_z^2) is said to be __proper__ if, for almost all ω M_{st}^1 ($M_{s,t}^2$) is of bounded variation in the t direction for all fixed $s \in [0,s_0]$ (in the s direction for all fixed t).

__Proposition 2-2__ Let M_z^1 be a 1-martingale on R_{z_0}, if $M_{s_0,t}^1$ is of bounded variation as a function of t then $M_{s,t}^1$ is proper on R_{z_0}.

__Proof__: Let $\lambda(t) = M_{s_0,t}$, then

$$M_{s,t}^1 = E(\lambda(t) | \mathcal{F}_{s,t}) = E(\lambda(t) | \mathcal{F}_{s,t_0})$$

$$\sum \left| M_{s,t_{i+1}}^1 - M_{s,t_i}^1 \right| \leq E\{\sum |\lambda(t_{i+1}) - \lambda(t_i)| \,\big|\, \mathcal{F}_{s,t_0}\}$$

and, taking the supremum over both sides, it follows that $M_{s,t}^1$ is of bounded variation in the t direction for all s.

Proposition 2-3 Let M_z^1 be a regular and continuous 1 martingale then $M_z^1 = M_z^{1,P} + M_z$ where $M_z^{1,P}$ is a proper 1-martingale and M_z is a martingale. If $M_{s,0}^1 = 0$ then the decomposition is unique.

Proof: Let $M_{s_0,t}^1 = \lambda(t) + m(t)$ where $\lambda(t)$ is of bounded variation and continuous and $m(t)$ is a one parameter martingale. Let

$$X_z = E(\lambda(t) \mid \mathcal{F}_{s,t})$$

$$Y_z = E(m(t) \mid \mathcal{F}_{s,t})$$

Then X_z is a proper 1-martingale, Y_z is a martingale and $M_z^1 = X_z + Y_z$.

Theorem 2-4 Every regular and continuous weak martingale X_z can be decomposed as

$$X_z = M_z^{1,P} + M_z^{2,P} + M_z$$

where $M_z^{1,P}$ is a proper 1-martingale, $M^{2,P}$ is a proper 2-martingale and M_z is a martingale. Moreover, if $X_{s,0} = X_{0,t} = 0$, then the decomposition is unique.

Proof: The result follows directly from propositions 1 and 3.

Let $M_z^{1,P}$ be a proper and continuous martingale on R_{z_0} and $M_{s_0,t}^{1,P} = \lambda(t)$. Since $\lambda(t)$ is of bounded variation, we can write

$\lambda(t) = \lambda(0) + \lambda^+(t) - \lambda^-(t)$ where $\lambda^+(t)$ and $\lambda^-(t)$ are nondecreasing nonnegative and $\lambda^+(0) = \lambda^-(0) = 0$. From now on we will assume that $\lambda(0) = 0$. Let

$$M_z^{1+} = E(\lambda^+(t)|\mathcal{F}_{s,t})$$

$$M_z^{1-} = E(\lambda^-(t)|\mathcal{F}_{s,t})$$

Then M_z^{1+} and M_z^{1-} are both proper 1-martingales and $M_z^{1,P} = M_z^{1+} + M_z^{1-}$. Since $\lambda^+(t)$ is nonnegative and nondecreasing, M_z^{1+} is nonnegative and

$$\sup_{0<z<z_0} |M_z^{1+}| = \sup_{0\leq s\leq s_0} |M_{s,t_0}^{1+}|$$

Since M_{s,t_0}^{1+} is a one parameter martingale, we have by the one parameter maximal inequality

$$E(\sup_z |M_z^{1+}|)^q \leq (\frac{q}{q-1})^q \sup E|M_{s,t_0}^{1+}|^q,$$

$$\leq (\frac{q}{q-1})^q E(\lambda^+(t_0))^q \quad , \quad q > 1$$

Therefore, by the Minkowsky inequality

$$E^{1/q}(\sup_z |M_z^{1,P}|)^q \leq \frac{q}{q-1} (E^{1/q}(\lambda^+(t_0))^q + E^{1/q}(\lambda^-(t_0))^q)$$

$$\leq 2\frac{q}{q-1} E^{1/q}(\lambda^+(t_0)+\lambda^-(t_0))^q$$

To summarize, we have

<u>Theorem 2-4</u> If $M_z^{1,P}$ is a proper and continuous 1-martingale, then

$$E(\sup_{z\in R_{z_0}} |M_z^{1,P}|)^q \leq (\frac{2q}{q-1})^q E(Var(M_{z_0}^{1,P}))^q$$

where $Var(M^{1,P}_{z_0}) = \lambda^+_{t_0} + \lambda^-_{t_0}$ is the variation of $\lambda(t)$, and similarly for a proper and continuous 2-martingale

$$E(\sup_{z\in R_{z_0}} |M^{2,P}_z|)^q \le (\frac{q}{q-1}) E(Var(M^{2,P}_{z_0}))^q$$

where

$$M^{2,P}_{s,t_0} = \rho(s) = \rho^+(s) - \rho^-(s)$$

and

$$Var\ (M^{2,P}_{s,t_0}) = \rho^+(s) + \rho^-(s).$$

Let X_z be a regular and continuous weak martingale such that $X_{0,t} = X_{s,0} = 0$. Let $q > 1$

$$\|X\|_q = [EM^q_{z_0} + E(Var\ M^{1,P}_{z_0})^q + E(Var\ M^{2,P}_{z_0})^q]^{1/q}$$

Let $\mathcal{X}^q(z_0)$ be the class of regular and continuous weak martingales such that $X_{0,t} = X_{s,0} = 0$ and $\|X\|_q < \infty$.

<u>Theorem 2-5</u> $\mathcal{X}^q(z_0)$, $q > 1$ with this norm is a Banach space.

Proof: We have to show that $X^q(z_0)$ is complete.

Let X^k_z be a Cauchy sequence. Then by taking a subsequence we can get

$$\|X^{n+1}_z - X^n_z\|_q \le 2^{-n}$$

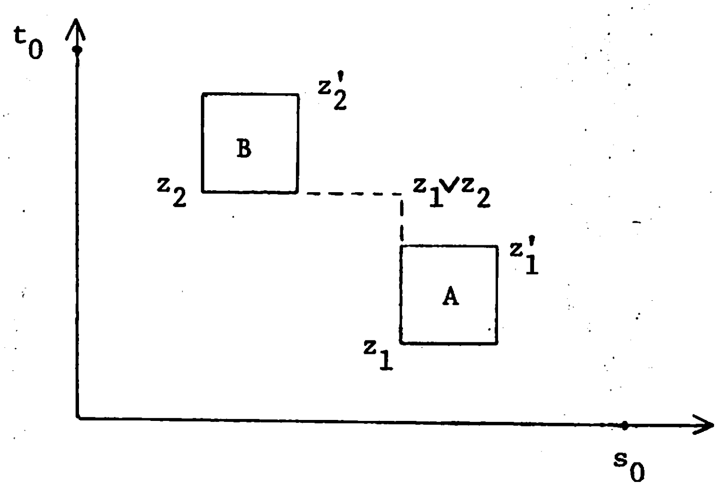
then by the maximal inequality and the Borel Cantelli lemma, X^n converges uniformly in $z \prec z_0$ to a process X_z and if $X^n_z = M^n_z + M^{1,P,n}_z + M^{2,P,n}_z$ then the components converge to a martingale, a proper 1-martingale and a proper 2-martingale respectively.

3. Mixed Area Integrals

In [4] we introduced a stochastic integral over $\mathbb{R}_+ \times \mathbb{R}_+$ $\iint \psi(z,z')dW(z)dW(z')$ (see also [3]). It seems that for the full development of a stochastic calculus in the plane still another integral is necessary. This integral will be of the form $\iint \psi(z,z')dW(z')dz$ where $z \curlywedge z'(z' \curlywedge z)$ and will be a proper 1 martingale (2 martingale). A related integral has been introduced by Cairoli and Walsh in [3] and termed a mixed integral. The relation between the mixed integral of Cairoli and Walsh and the mixed area integral so defined in this section will be pointed out later.

Let μ_z, $z \in R_{z_0}$ be a continuous random function of bounded variation adapted to $\mathcal{F}_z$, and let $\mu(A)$ be the signed measure induced on the Borel sets A of R_{z_0} by μ_z. Let $|\mu|(A)$ denote the variation of the μ measure, namely, $\mu(A) = \mu^+(A) - \mu^-(A)$ is the Jordan decomposition of μ and $|\mu|(A) = \mu^+(A) + \mu^-(A)$. We assume that the total variation of μ is bounded by a constant $\mu_0 < \infty$, i.e., $|\mu|(R_{z_0}) \leq \mu_0$ a.s.

Let M_z be a continuous martingale and let $A = (z_1,z_1']$, $B = (z_2,z_2']$ be rectangles such that if $z \in B$ and $z' \in A$, then $z \curlywedge z'$



Define, now, the process

$$X_z = \alpha M(A \cap R_z)\mu(B \cap R_z) \tag{3.1}$$

where α is $\mathcal{F}_{z_1 \vee z_2}$ measurable. Then

a) X_z is a continuous proper 1-martingale

b) The variation of X_z is $\alpha \cdot |M(A)| \cdot \int_0^{t_0} |d_t \mu(B \cap R_{s_0,t})| \leq |M(A)| \cdot |\mu|(B)$

Let

$$\begin{aligned} \psi(z,z') &= \alpha \quad \text{if } z \in B,\ z' \in A \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and define

$$\iint \psi(\zeta,\zeta')dM_{\zeta'}d\mu_\zeta = X_z \tag{3.2}$$

where X_z is as defined by (3.1).

To simplify notation assume $z_0 = (1,1)$. Fix an integer n and introduce a grid on R_{z_0}

$$z_{ij} = (2^{-n}i, 2^{-n}j)$$

where i,j are integers $0 \leq i,j \leq 2^n$. Define the rectangle $\Delta_{ij} = (z_{ij}, z_{i+1,j+1}]$. Let $I_{\Delta_{ij}}(z)$ denote the indicator function of Δ_{ij}. Define

$$\begin{aligned} \psi_{ij,k\ell}(z,z') &= \alpha I_{\Delta_{ij}}(z) I_{\Delta_{k\ell}}(z') \quad \text{if } z_{ij} \curlywedge z_{k\ell} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and α is bounded and $\mathcal{F}_{z_{ij}\vee z_{k\ell}}$ measurable. A function $\psi(z,z')$ is said to be a simple function if it is a finite sum of functions of the form $\psi_{ij,k\ell}(z,z')$ for some n. The extension of (3-2) to simple functions is obvious, and the resulting X_z is a proper 1-martingale.

Let ψ be a simple function and for $\Delta_{ij} = (z_{ij}, z_{i+1,j+1}]$, let

$M(\Delta_{ij}) = z_{i+1,j+1} + z_{ij} - z_{i+1j} - z_{ij+1}$

Then

$$X_{z_0} = \sum_{ij,k\ell} \psi_{ijk\ell}\,\mu(\Delta_{ij})M(\Delta_{k\ell}). \tag{3.3}$$

If M_z is a strong martingale then we have

$$EX_{z_0}^2 = E\left\{\sum_{ij,k\ell,i'j'} \psi_{ij,k\ell}\psi_{i'j',k\ell}\,\mu(\Delta_{ij})\mu(\Delta_{i'j'})M^2(\Delta_{k\ell})\right\}$$

$$= E\int_{R_{z_0}}\int_{\times R_{z_0}}\int_{\times R_{z_0}} \psi(z,z')\psi(\eta,z')d\mu_\eta d\mu_z d[M]^1_{z'}$$

$$= E\int_{R_{z_0}} \left(\int_{R_{z_0}}\psi(z,z')d\mu_z\right)^2 d[M]^1_{z'} \tag{3.4}$$

where $[M]^1_z$ is the unique $\mathcal{F}^1_{st}$ predictable process such that $\left\{M_z^2 - [M]^1_z, \mathcal{F}^1_{st}\right\}$ is a martingale in s for t fixed, and the passage from (3.3) to (3.4) follows from Proposition 1.7 of [3].

The variation of $X_{s_0,\theta}$, $0 \le \theta \le t_0$ is upper bounded by

$$\mathrm{Var}(X_{s_0,\theta}, 0\le\theta\le t_0) \le \sum_{ij} |\mu|(\Delta_{ij}) \cdot \left|\sum_{k,\ell} \psi_{ij,k\ell}M(\Delta_{k,\ell})\right| \tag{3.5}$$

Setting $|\mu|(\Delta_{ij}) = \sqrt{|\mu|}\cdot\sqrt{|\mu|}$ we have by the Schwarz inequality

$$E(\mathrm{Var}(X_{s_0,\theta},\ 0\le\theta\le t_0))^2$$

$$\le E\left\{\sum_{ij}|\mu|(\Delta_{ij})\cdot\sum_{ij}|\mu|(\Delta_{ij})\Big(\sum_{k,\ell}\psi_{ijk\ell}M(\Delta_{k\ell})\Big)^2\right\} \qquad (3.6)$$

And since M_z is a square integrable strong martingale, we have by 1.7 of [3]

$$E(\mathrm{Var}(X_{s_0,\theta},0\le\theta\le t_0))^2$$

$$\le \mu_0 E\sum_{ij}|\mu|(\Delta_{ij})\Big(\sum_{k\ell}\psi^2_{ij,k\ell}M^2(\Delta_{k\ell})\Big) \qquad (3.7)$$

$$= \mu_0 E\int_{R_{z_0}}\int_{\times R_{z_0}}\psi^2(z,z')d|\mu|(z)d[M]^1_{z'}, \qquad (3.8)$$

Consider now the special case where $\mu(z)$ is a product measure $\mu(s,t) = \mu^{(1)}(s)\mu^{(2)}(t)$. For simplicity we will assume that μ is a positive measure, $\mu^{(1)}(d_i)$ will denote $\mu^{(1)}(2^{-n}(i+1)) - \mu^{(1)}(2^{-n}i)$ and similarly for $\mu^{(2)}(d_j)$. In this case we can write instead of (3.5)

$$\mathrm{Var}(X_{s_0},\theta\ 0\le\theta\le t_0) \le\sum_j\mu^{(2)}(d_j)\Big|\sum_{i,k\ell}\psi_{ijk\ell}\mu^{(1)}(d_i)M(\Delta_{k\ell})\Big|$$

Setting $\mu^{(2)} = \sqrt{\mu^{(2)}}\ \sqrt{\mu^{(2)}}$ yields

$$E(\mathrm{Var}\ X)^2 \le E\left\{\sum_j\mu^{(2)}(d_j)\quad\sum_j\mu^{(2)}(d_j)\left(\sum_{ik\ell}\right)^2\right\}$$

$$\le \mu_0^{(2)}E\int_0^{t_0}\left(\int_0^{s'}\psi(\sigma,\tau,z')d\mu^{(1)}(\sigma)\right)^2 d[M]^1_{z'}d\mu^{(2)}(\tau) \qquad (3.9)$$

If μ is not positive, then (3.9) holds with $\mu^{(2)}(t)$ replaced by $|\mu^{(2)}|(t)$.

The requirement that M_z be a strong martingale was needed to pass from (3.7) to (3.8); in the following particular case this is not necessary.

Let $\psi(z,z') = h(z,z')\pi(z\vee z')$ where $h(z,z') = 1$ whenever $z \curlywedge z'$ and zero otherwise. Then

$$\psi_{ij,k\ell} = \pi_{k,j} \cdot I(i<k) \cdot I(\ell<j) \tag{3.10}$$

where I( ) denotes the indicator function. Substituting (3.10) in (3.3) and summing over ℓ we have

$$X_{z_0} = \sum_{ij} \mu(\Delta_{ij}) \sum_{k>i} \pi_{kj}(M(k+1,j)-M(k,j)) \tag{3.11}$$

Setting $\mu = \sqrt{\mu}\ \sqrt{\mu}$ we have

$$EX^2_{z_0} \leq \mu_0 E\left\{\sum_{ij}|\mu|(\Delta_{ij}) \sum_{k>i} \pi^2_{kj}(M(k+1,j)-M(k,j))^2\right\}$$

$$= \mu_0 E\left\{\int_{R_{z_0}}^{s_0} d|\mu|(s,t) \int_{s}^{s_0} \pi^2_{\theta,t} d_\theta [M]^1_{\theta,t}\right\}$$

where $[M]^1_z$ is as in (3.4) chosen to be measurable in (s,t). Integration by parts with respect to s yields

$$EX^2_{z_0} \leq \mu_0 E \int_0^{t_0}\int_0^{s_0} \pi^2_{s,t} d_s [M]^1_{s,t} d_t|\mu|(s,t). \tag{3.12}$$

Furthermore

$$\mathrm{Var}(X_{s_0,\theta}, 0\leq\theta\leq t_0) \leq \sum_{ij} |\mu|(\Delta_{ij})| \sum_{k>i} \pi_{kj}(M(k+1,j)-M(k,j))$$

Therefore by the same arguments as those leading from (3.11) to (3.12) we have

$$E(\mathrm{Var}\ X_{s_0,\theta},\ 0\leq\theta\leq t_0)^2 \leq \mu_0 E\left\{\int_0^{t_0}\int_0^{s_0}\pi_{s,t}^2 d_s[M]_{st}^1 d_t|\mu|(s,t)\right\} \qquad (3.13)$$

In addition to (3.10) assume, now, that μ is a product measure namely $\mu(s,t) = \mu^{(1)}(s)\mu^{(2)}(t)$. For simplicity assume that $\mu^{(1)}$ and $\mu^{(2)}$ are positive measures, then

$$X_{z_0} = \sum_j \mu_j^{(2)}\left(\sum_i \mu_i^{(1)}\left(\sum_{k>i} \pi_{kj}(M_{k+1,j}-M_{k,j})\right)\right)$$

Let

$$a_j = \sum_i \mu_i^{(1)}\left(\sum_{k>i} \pi_{kj}(M_{k+1,j}-M_{k,j})\right)$$

then $\mathrm{Var}(X_{s_0,\theta},\ 0\leq\theta\leq t_0) \leq \sum_j \mu_j^{(2)}|a_j|$

Setting $\mu_j^{(2)} = \sqrt{\mu_j^{(2)}}\sqrt{\mu_j^{(2)}}$

$$E(\mathrm{Var}\ X)^2 \leq \mu^{(2)}(t_0)E\left(\sum_j \mu_j^{(2)}a_j^2\right)$$

Now, a_j can also be written as

$$a_j = \sum_k \left(\pi_{kj}(M_{k+1,j}-M_{k,j})\sum_{i>k}\mu_i^{(1)}\right)$$

Therefore

$$E(\mathrm{Var}\ X)^2 \leq \mu^{(2)}(t_0)\int_0^{t_0}\int_0^{t_0}(\mu^{(1)}(s))\pi_{s,t}^2\, d_s[M]_{s,t}^1 d_t\mu^{(2)}(t) \qquad (3.14)$$

Let B_a be the class of all processes $\{\psi(\zeta,\zeta'),\ \zeta,\zeta' \prec z_0\}$ satisfying

1) ψ is predictable as defined in section 2 of [3]

2) $\psi(\zeta,\zeta') = 0$ unless $\zeta \curlywedge \zeta'$

3) $E \int_{R_{z_0} \times R_{z_0}} \int \psi^2(\zeta,\zeta') d|\mu| d[M]^1_z < \infty$ or, if μ_z is a product measure, the right-hand side of (3.9) is finite, and let M_z be a square integrable strong martingale.

Since simple functions are dense in B_a, the mixed area integral $\iint \psi d\mu dM$ can be extended by continuity to all ψ in B_a. In view of Theorem 3 of section 2 the integral will be a continuous proper 1 martingale satisfying (3.4) and (3.8). Similarly, let B_b be the class of all $\psi(\zeta,\zeta') = h(\zeta,\zeta')\pi(\zeta\ \zeta')$ satisfying

1) $\pi(\zeta)$ is F_ζ predictable

2) $E\left\{\int_0^{t_0} \int_0^{s_0} \pi^2_{s,t} d_s [M]^1_{st} d_t |\mu|(s,t)\right\} < \infty$, or if μ is a product measure, the right-hand side of (3.9) is finite,

and let M_z be a square integrable martingale then the mixed surface integral can be extended to B_b. To summarize,

<u>Theorem 3-1</u> 1) Let M_z be a continuous strong square integrable martingale and $\psi \in B_1$ then

(a) $\iint \psi(\zeta,\zeta') d\mu(\zeta) dM_{\zeta'}$, is a proper square integrable continuous 1-martingale

(b) the integral is linear in ψ

(c) EX^2_z is as given by (3.4) and $E(\text{Var } X_{s,}\ \theta,\ 0 \leq \theta \leq t)^2$ satisfies the upper bound (3.8), and if μ is a product measure, (3.9) holds.

2) Let M_z be a continuous square integrable martingale and $\pi \in B_b$ then (a) and (b) hold with $\psi(\zeta,\zeta') = h(\zeta,\zeta')\pi(\zeta \vee \zeta')$. EX_z^2 and $E(\text{Var } X_{s,\theta}, 0 \leq \theta \leq t)^2$ satisfy the bounds (3.12) and (3.13) respectively. If μ is a product measure then (3.14) is satisfied.

Remarks (a) The definition of the mixed area integral can be extended in an obvious way to the case where instead of

$$E \iint \psi^2(z,z')d|\mu|(z)d[M]^1_{z'} < \infty$$

we require that a.s.

$$\iint \psi^2(z,z')d|\mu|(z)d[M]^1_{z'} < \infty \tag{3.15}$$

And the resulting integral may be termed a "local-martingale." A similar remark for the case where the boundedness of the expectations in (3.9), (3.13) and (3.14) is replaced by a.s. boundedness and also to the stochastic integrals $\int \psi dM$ and $\iint \psi dM_\zeta dM_{\zeta'}$. These extensions will be used without further reference in the following sections. The related stopping times are introduced in section 6.

(b) The stochastic integral of the second type [4] was generalized in [3] to $\iint \psi(z,z')dM_z dM_{z'}$ where M_z is a strong martingale. By an argument similar to the one given here $\iint \psi(\zeta,\zeta')dM_\zeta dM_{\zeta'}$ can be defined for martingales which are not strong provided that $\psi(z,z')$ depends on the corner $z \vee z'$ only, i.e., $\psi(z,z') = \pi(z \vee z')h(z,z')$.

(c) In [3] Cairoli and Walsh introduced the mixed integral

$$\int_0^{t_0} \int_0^{s_0} \pi(s,t)\partial_s M_{s,t}dt$$

We now show that the mixed area integral of this section includes the mixed integral of [3] when π_z is $\mathcal{F}_z$ predictable. Let $\mu(t) = st$. Approximate ψ by simple functions. It follows that the area integral $\iint \pi dz dM_{z'}$, can be expressed as

$$\int_{R_{z_0}}\int_{\times R_{z_0}} \pi(z \vee z') dz dM_{z'} = \int_0^{t_0}\int_0^{s_0} s\pi(s,t)\partial_s M_{st} dt$$

and conversely if $E \int_0^{t_0}\int_0^{s_0} \pi^2(s,t) dt d_s [M]^1_{s,t} < \infty$

then

$$\int_0^{t_0}\int_0^{s_0} \pi(s,t)\partial_s M_{s,t} dt = \iint \frac{1}{s'} \pi(z \vee z') dz dM_{z'}$$

and the integrand $\pi(z \vee z')/s'$ is admissible by (3.14). Note that $\pi(z \vee z')/s'$ is also a corner function since we integrate over $z \vee z'$ and $z \vee z' = (s',t)$.

(d) Let $X_z = \iint \psi(\zeta,\zeta') d\mu_\zeta dM_{\zeta'}$, then, in view of (3.4), $X_z = 0$ for all $z \in R_{z_0}$ does not imply that $\psi(\zeta,\zeta') = 0$ in $R_{z_0} \times R_{z_0}$. In particular, for $\zeta = (\sigma,\tau)$, $d\mu_\zeta = d\sigma d\tau$, if

$$\psi(\zeta,\zeta') = \sin \frac{2\pi(\sigma-\sigma')}{\sigma'} \psi(\zeta')h(\zeta,\zeta')$$

then $X_z = 0$ for all z in R_{z_0}. For any $\psi(\zeta,\zeta')$ define

$$\psi(\bar{\zeta},\zeta') = \frac{1}{\sigma'} \int_0^{\sigma'} \psi(\sigma,\tau;\zeta') d\sigma$$

and $\psi(\tilde{\zeta},\zeta') = \psi(\zeta,\zeta') - \psi(\bar{\zeta},\zeta')$

and similarly

$$\psi(\zeta,\bar{\zeta}') = \frac{1}{\tau} \int_0^{\tau} \psi(\zeta,\sigma',\tau') d\tau'$$

Then

$$\iint \psi(\zeta,\tilde{\zeta}')\ dW_{\zeta} d\zeta' = 0$$

$$\iint \psi(\tilde{\zeta},\zeta')\ d\zeta dW_{\zeta'} = 0$$

We can also define $\psi(\bar{\zeta},\bar{\zeta}')$, $\psi(\bar{\zeta},\zeta')$, etc,since the bar and ~ operations on the ζ and ζ' variables commute. Note that $\psi(\bar{\zeta},\bar{\zeta}') = \pi(\sigma',\tau)$ (a corner function) and $\iint \psi_1(\tilde{\zeta},\zeta')\psi_2(\bar{\zeta},\zeta')\ d\zeta d\zeta' = 0$.

4. The Representation of Some Weak Martingales of the Wiener Process

Let $X_z \in \mathcal{X}^2_{z_0}$ be a proper 1-martingale of the Wiener Process and assume that almost all the sample functions of $\lambda(t) = X_{s_0,t}$ are absolutely continuous with respect to some fixed (nonrandom) positive finite measure, i.e.,

$$\lambda(t) = \int_0^t \rho(\theta)dv(\theta) \tag{4.1}$$

Furthermore, we will assume that

$$E\int_0^{t_0} \rho^2(\theta)dv(\theta) < \infty \tag{4.2}$$

It will be shown in this section that 1-martingales satisfying the above conditions can be represented as mixed area integrals. The Wiener process assumption is not used in the following proposition but will be needed later.

Proposition 4-1 Let $\{f_i\}$ be a complete orthogonal set with respect to the v measure on $[0,t_0]$ (i.e. $\int_0^t f_i(t')f_j(t')dv(t') = \delta_{ij}$). Under the above conditions on X_z there exists a sequence of martingales $M_i(z)$ such that for $z \prec z_0$

$$E(X_z - \sum_i^N \int_0^t f_i(\theta)M_i(s,\theta)dv_\theta)^2 \xrightarrow[N\to\infty]{} 0 \tag{4.3}$$

Proof

$$X_{s,t} = E(\lambda_t | \mathcal{F}_{s,t}) = \int_0^t E(\rho_\theta | \mathcal{F}_{s,t})\, dv_\theta$$

$$= \int_0^t E(\rho_\theta | \mathcal{F}_{s,t})\, dv_\theta$$

Let

$$\alpha_i = \int_0^{t_0} \rho(t) f_i(t) dv_t$$

Therefore α_i are $\mathcal{F}_{z_0}$ measurable and

$$E(\lambda_t - \sum_i^N \alpha_i \int_0^t f_i(\theta) dv_\theta)^2$$

$$= E\left(\int_0^t (\rho(\theta) - \sum_1^N \alpha_i f_i(\theta)) d v_\theta\right)^2$$

$$\leq K \int_0^{t_0} E(\rho(\theta) - \sum_1^N \alpha_i f_i(\theta))^2 d v_\theta$$

which converges to zero by dominated convergence.

Let $M_i(z) = E(\alpha_i | \mathcal{F}_z)$. Then

$$EM_i^2(z) \leq E\alpha_i^2 \tag{4.4}$$

Since $E(\lambda_t | \mathcal{F}_{s_0,t}) = \lambda_t$ and $E(\lambda_t | \mathcal{F}_{s,t}) = X_{s,t}$

$$E\left\{E^2(\lambda_t - \sum_1^N \alpha_i \int_0^t f_i(\theta) dv_\theta) | \mathcal{F}_{s_{0,t}}\right\}$$

$$\leq E(\lambda_t - \sum_1^N \alpha_i \int_0^t f_i(\theta) dv_\theta)^2$$

And if in the above inequality we condition with respect to $\mathcal{F}_{s,t}$ (instead of $\mathcal{F}_{s_0,t}$) we obtain (4.3).

Theorem 4-2 Under the above conditions on X_z, X_z can be written as

$$X_z = \iint_{R_z \times R_z} \psi(\zeta,\zeta')d\mu(\zeta)dW_{\zeta'}, \tag{4.5}$$

where $d\mu(z) = ds\, dv(t)$.

Proof Let $M_i(z)$ be the martingales of Proposition 4-1. Then, by the corollary to Theorem (6-1) of [4]

$$M_i(z) = \int \phi_i(\zeta)dW_\zeta + \iint \psi_i(\zeta,\zeta')dW_\zeta dW_{\zeta'},$$

and by (4.4)

$$E\sum_1^\infty M_i^2(z_0) = E\sum_1^\infty \int_{R_{z_0}} \phi_i^2(\zeta)d\zeta + E\sum_1^\infty \int_{R_{z_0}}\int_{R_{z_0}} \psi_i^2(\zeta,\zeta')d\zeta d\zeta'$$

Let $M_{a,i}(z) = \int \phi(\zeta)dW_\zeta$, and approximate ϕ and f by simple functions. It follows that

$$\int_0^t f_i(\theta)M_{a,i}(s,\theta)dv(\theta) = \iint \psi_{a,i}(\zeta,\zeta')d\mu_\zeta dW_{\zeta'},$$

where $\zeta = (\sigma,\theta)$, $d\mu_\zeta = d\sigma dv(\theta)$,

and

$$\psi_{a,i}(\zeta,\zeta') = h(\zeta,\zeta') \frac{f_i(\theta)}{\sigma'} \phi_i(\zeta').$$

Now, by the orthogonality of $f_i(\theta)$

$$E\iint_{R_{z_0}\times R_{z_0}} \Big(\sum_N^{N+K} f_i(\theta)\phi_i(\zeta')\Big)^2 d\mu_\zeta d\zeta' \le K_1 E \sum_N^{N+K} \int_{R_{z_0}} \phi_i^2(\zeta')d\zeta'$$

where K_1 is independent of N and K. Therefore, by (4.6) $\sum_1^N f_i(\theta)\phi_i(\zeta')$ converges to a function $\phi^{\infty}(\theta, \zeta')$. Set

$$\Phi_1(\zeta,\zeta') = \frac{1}{\sigma'}\phi^{\infty}(\theta,\zeta')$$

then

$$\sum_1^N \int_0^t f_i(\theta)M_{a,i}(s,\theta)dv(\theta) \xrightarrow{q.m.} \iint \Phi_1(\zeta,\zeta')d\mu_\zeta dW_{\zeta'} \tag{4.7}$$

Similarly, let

$$M_{b,i}(z) = \iint \psi_i(\zeta,\zeta')dW_\zeta dW_{\zeta'}$$

and approximate f and ψ by simple functions. It follows that

$$\int f_i(\theta)M_{b,i}(s,\theta)dv(\theta) = \iint \psi_{b,i}(\zeta,\zeta')d\mu_\zeta dW_{\zeta'}$$

where μ_ζ is as before and

$$\psi_{b,i}(\zeta,\zeta') = \frac{f_i(\theta)}{\sigma'}\left(\int_{R_{\zeta\vee\zeta'}} \psi_i(\zeta,\eta)dW_\eta\right)$$

(cf. Theorem 2-6 of [1].) The convergence of $\frac{1}{\sigma'}\sum_1^N \sigma'\psi_{b,i}$ to a function Ψ follows as in the previous case. Hence, by Proposition (4-1)

$$X_z = \iint (\Phi(\zeta,\zeta') + \Psi(\zeta,\zeta'))d\mu_\zeta dW_{\zeta'}$$

which is the desired result.

5. Itô-type Formulas

In the one parameter case the differentiation formula of Ito and its extension to local martingales express functions of semimartingales as semimartingales. The stochastic integrals of [4], the extension of [3] and the mixed area integral of section 3 make available analogous results for the two parameter case. In the simplest case we have in mind the following: Let W_z be a Wiener process and suppose that f is four times continuously differentiable on R then

Proposition 5-1

$$f(W_z) = f(0) + \int f'(W_\zeta)dW_\zeta + \iint f''(W_{\zeta\vee\zeta'})dW_\zeta dW_{\zeta'}$$
$$+ \frac{1}{2}\iint f''' (W_{\zeta\vee\zeta'})dW_\zeta d\zeta' + \frac{1}{2}\iint f''' (W_{\zeta\vee\zeta'})d\zeta dW_{\zeta'}$$
$$+ \frac{1}{4}\iint f''''(W_{\zeta\vee\zeta'})d\zeta d\zeta' + \frac{1}{2}\int f''(W_\zeta)d\zeta \qquad (5.1)$$

Remarks: (1) The integral before the last can be written as

$$\frac{1}{4}\int_0^{t_0}\int_0^{s_0} stf''''(W_{st})dsdt$$

Equation (5.1) is similar to Eq. (6.22) of [3], however the last term in Eq. (6.22) is not an area integral.

Proof: The proof follows by a simple modification of the proof of Eq. (6.22) in [3] as follows. Instead of using (6.21) of [3] to eliminate f''' in (6.20), use it to substitute for f'' and (5.1) follows directly. In (5.1) $f(W_z)$ is expressed as the sum of a martingale, proper 1 and 2 martingales and a function of bounded variation.

The proof of (5.1) via the one parameter Itô formula, the Green formula of [3] and then again the one parameter Itô formula can be applied to more general cases but does not seem to be sufficient for a general differentiation formula. The detailed development of a general formula will be given in a separate paper. Only some special cases of a multiplication table are given here.

<u>Proposition 5-2</u> A partial multiplication table: Let

$$I_i^{(0)}(z) = \int \phi_i(\zeta)d\zeta \quad , \quad I_i^{(1)}(z) = \int \phi_i(\zeta)dW_\zeta$$

$$I_i^{(2)}(z) = \iint \psi_i(\zeta,\zeta')dW_\zeta dW_{\zeta'}$$

where $\int \phi_i^2(z)dz < \infty$ and $\iint \psi_j^2(z,z')dzdz' < \infty$ a.s.

Then

(a) $$I_1^{(0)}(z)I_2^{(1)}(z) = \int I_2^{(1)}(\zeta)\,\phi_1(\zeta)d\zeta + \int I_1^{(0)}(\zeta)\,\phi_2(\zeta)dW_\zeta$$

$$+ \iint \phi_1(\zeta)\phi_2(\zeta')h(\zeta,\zeta')d\zeta dW_{\zeta'}$$

$$+ \iint \phi_2(\zeta)\phi_1(\zeta')h(\zeta,\zeta')dW_\zeta d\zeta'$$

(b) $$I_1^{(1)}(z)I_2^{(1)}(z) = \int \phi_1(\zeta)\phi_2(\zeta)d\zeta + \int I_1^{(1)}(\zeta)\phi_2(\zeta)dW_\zeta$$

$$+ \int I_2^{(1)}(\zeta)\phi_1(\zeta)dW_\zeta$$

$$+ \iint \phi_1(\zeta)\phi_2(\zeta')h(\zeta,\zeta')dW_\zeta dW_{\zeta'}$$

$$+ \iint \phi_2(\zeta)\phi_1(\zeta')h(\zeta,\zeta')dW_\zeta dW_{\zeta'}$$

$$(c)\quad I_1^{(0)}(z)I_1^{(2)}(z) = \int I_1^{(2)}(\zeta)\phi_1(\zeta)d\zeta + \iint I_1^{(0)}(\zeta\vee\zeta')\psi_1(\zeta,\zeta')dW_\zeta dW_{\zeta'}$$

$$+ \iint\Big(\int_{R_{\zeta\vee\zeta'}} \psi_1(\eta,\zeta')dW_\eta\Big)\phi_1(\zeta)d\zeta dW_{\zeta'}$$

$$+ \iint\Big(\int_{R_{\zeta\vee\zeta'}} \psi_1(\zeta,\eta)dW_\eta\Big)\phi_1(\zeta')dW_\zeta d\zeta'$$

$$(d)\quad I_1^{(1)}(z)I_1^{(2)}(z) = \int I_1^{(2)}(\zeta)\phi_1(\zeta)dW_\zeta + \iint I_1^{(1)}(\zeta\vee\zeta')\psi_1(\zeta,\zeta')dW_\zeta dW_{\zeta'}$$

$$+ \iint\Big(\int_{R_{\zeta\vee\zeta'}} \psi_1(\eta,\zeta')dW_\eta\Big)\phi_1(\zeta)dW_\zeta dW_{\zeta'}$$

$$+ \iint\Big(\int_{R_{\zeta\vee\zeta'}} \psi_1(\zeta,\eta)dW_\eta\Big)\phi_1(\zeta')dW_\zeta dW_{\zeta'}$$

$$+ \iint \psi_1(\zeta,\zeta')\phi_1(\zeta')dW_\zeta d\zeta'$$

$$+ \iint \psi_1(\zeta,\zeta')\phi_1(\zeta)d\zeta dW_{\zeta'}$$

$$+ \iint \psi_1(\zeta,\zeta')\psi_1(\zeta,\zeta')d\zeta d\zeta'$$

$$(e)\quad I_1^{(2)}(z)I_2^{(2)}(z) = \iint I_1^{(2)}(\zeta\vee\zeta')\psi_2(\zeta,\zeta')dW_\zeta dW_{\zeta'}$$

$$+ \iint I_2^{(2)}(\zeta\vee\zeta')\psi_1(\zeta,\zeta')dW_\zeta dW_{\zeta'}$$

$$+ \iint\Big(\int_{R_{\zeta\vee\zeta'}} \psi_1(\zeta,\eta)dW_\eta\Big)\Big(\int_{R_{\zeta\vee\zeta'}} \psi_2(\eta,\zeta')dW_\eta\Big)dW_\zeta dW_{\zeta'}$$

$$+ \iint\Big(\int_{R_{\zeta\vee\zeta'}} \psi_2(\zeta,\eta)dW_\eta\Big)\Big(\int_{R_{\zeta\vee\zeta'}} \psi_1(\eta,\zeta')dW_\eta\Big)dW_\zeta dW_{\zeta'}$$

$$+ \iint \psi_1(\zeta,\zeta')\Big(\int_{R_{\zeta\vee\zeta'}} \psi_2(\zeta,\eta)dW_\eta\Big)d\zeta dW_{\zeta'}$$

$$+ \iint \psi_2(\zeta,\zeta')\left(\int_{R_{\zeta\vee\zeta'}} \psi_1(\zeta,\eta)\,dW_\eta\right) d\zeta\, dW_{\zeta'}$$

$$+ \iint \psi_1(\zeta,\zeta')\left(\int_{R_{\zeta\vee\zeta'}} \psi_2(\eta,\zeta')\,dW_\eta\right) dW_\zeta\, d\zeta'$$

$$+ \iint \psi_2(\zeta,\zeta')\left(\int_{R_{\zeta\vee\zeta'}} \psi_1(\zeta,\zeta')\,dW_\eta\right) dW_\zeta\, d\zeta'$$

Proof: The proofs of (a) and (b) follow along the same lines as the proof of (5.1). We therefore omit the details. We will give the proof of (e), the proofs of (c) and (d) follow along the same lines. Turning to the proof of (e), we assume for simplicity that $\psi_1 = \psi_2 = \psi(z_1,z_2)$ where $\psi(z_1,z_2) = 0$ if $z_1 \curlywedge z_2$ is not satisfied, the proof for $\psi_1 \neq \psi_2$ is exactly the same.

Let $z_0 = (1,1)$, fix n (an integer) and introduce the grid $z_{ij} = (2^{-n}i, 2^{-n}j]$. Denote

$$\Delta_{ij}(W) = W(z_{ij}, z_{i+1,j+1}]$$

Assume first that $\psi(z,z')$ is bounded on R_{z_0} and a simple function with respect to the grid. Let

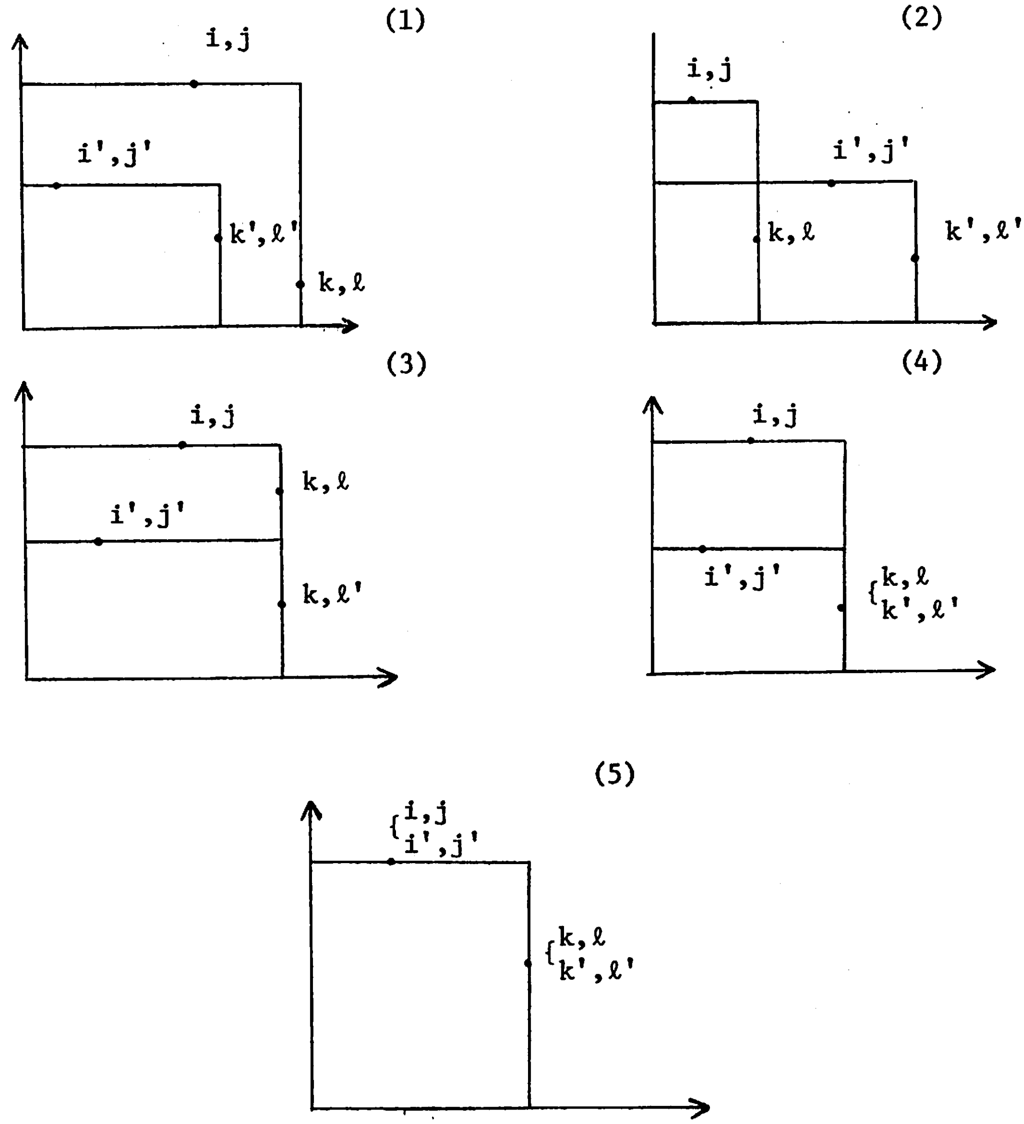
$$\psi_{ij,k\ell} = \psi(z_{ij}, z_{k\ell})$$

Define $[z]$ as follows: if $z \in (z_{ij}, z_{i+1,j+1}]$ then $[z] = z_{ij}$. Under these assumptions

$$(I^{(2)}_{(z)})^2 = \sum \psi_{ij,k\ell}\psi_{i'j',k'\ell'}\Delta_{ij}(W)\Delta_{k\ell}(W)\Delta_{i'j'}(W)\Delta_{k'\ell'}(W) \qquad (5.2)$$

where the summation is over all $i,j,k,\ell,i',j',k',\ell'$, recall however that if $h(z,z') = 0$ then $\psi(z,z') = 0$.

Divide now the summation of (5.2) into the following partial summations:



<u>case (1)</u> $(i,j) \vee (k,\ell) \gg (i',j') \vee (k',\ell')$

and case (1') where the unprimed letters become primed and the primed become unprimed.

<u>case (2)</u> $((i,j)) \vee (k,\ell)) \curlywedge ((i',j') \vee (k',\ell'))$

and case (2') where the primed and unprimed letters are interchanged as in the previous case.

<u>case (3)</u> $k = k'$, $\ell \neq \ell'$, $(i,j) \neq (i',j')$

and case (3') $j = j'$, $i \neq i'$, $(k,\ell) \neq (k',\ell')$

<u>case (4)</u> $(i,j) = (i',j')$, $(k,\ell) \neq (k',\ell')$

and case (4') $(k,\ell) = (k',\ell')$, $(i,j) \neq (i',j')$

<u>case (5)</u> $(i,j) = (i',j')$ and $(k,\ell) = (k',\ell')$

<u>case (1)</u> Fixing i,j,k,ℓ and summing over all i',j',k',ℓ' such that $(i',j') \vee (k',\ell')$ is smaller than $(i,j) \vee (k',\ell')$, and then summing over the unprimed indices, the partial sum for case (1) can be written as

$$\iint I([z_2 \vee z_2'])\psi(z_2, z_2')dW_{z_2}dW_{z'_2} \tag{5.3}$$

where the integration is over $R_{z_0} \times R_{z_0}$, $[z]$ is as defined earlier and

$$I(z) = \iint \psi(\zeta,\zeta')dW_\zeta dW_{\zeta'}$$

Keeping ψ unchanged (namely, simple with respect to the original partition) and introducing a refinement of the partition, (5.3) converges to

$$\iint I(z \vee z')\psi(z,z')dW_z dW_{z'}$$

The same result holds for case (1') and thus we get the second and third terms of part (e) of the proposition.

<u>case (2)</u> In this case we have the sum

$$\sum_{i,j,k',\ell'} \left(\sum_{\substack{k<k'\\ \ell<j}} \psi_{ijk\ell}\Delta_{k\ell}W\right)\left(\sum \psi_{i'j'k'\ell'}\Delta_{i'j'}W\right)\Delta_{ij}W\Delta_{k'\ell'}W$$

$$= \iint\left(\int_{[z\vee z']} \psi(z,\eta)dW_\eta\right)\left(\int_{[z\vee z']} \psi(\eta,z')dW_\eta\right) dW_z dW_{z'}$$

which for fixed ψ and refining the partition converges to the fourth term of (e), similarly case (2') leads to the fifth term in (e).

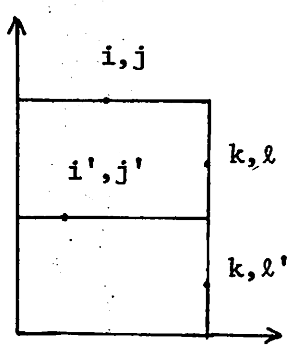
<u>case (3)</u> The expectation of this term is zero, we want to show that its variance tends to zero as the partition is refined $n \to \infty$. In case (3) we consider

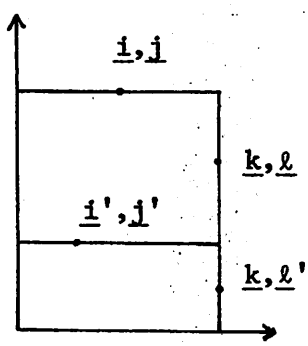
$$\sum \psi_{ij,k\ell}\psi_{i'j'k\ell'}\Delta_{ij}(W)\Delta_{k\ell}(W)\Delta_{i'j'}(W)\Delta_{k\ell'}(W) \tag{5.4}$$

where $k = k'$, $\ell \neq \ell'$, $(i,j) \neq (i',j')$ and assume $\ell' < \ell$ (the result for $\ell < \ell'$ will follow by exactly the same arguments). Multiply (5.4) with

$$\sum \psi_{\underline{ij},\underline{k}\,\underline{\ell}}\psi_{\underline{i}'\underline{j}'\underline{k}\,\underline{\ell}}\Delta_{\underline{ij}}(W)\Delta_{\underline{k}\,\underline{\ell}}(W)\Delta_{\underline{i}'\underline{j}'}(W)\Delta_{\underline{k}\,\underline{\ell}'}(W)$$

and take expectations



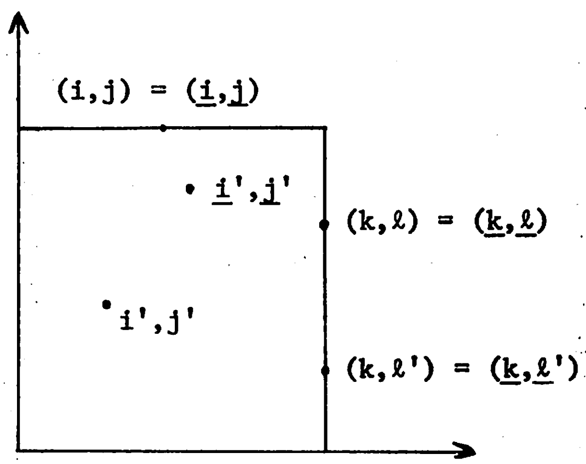


Since $E(\Delta_{ij}(W)|\mathcal{F}^1_{z_{ij}} \vee \mathcal{F}^2_{z_{ij}}) = 0$

and $E(\Delta^2_{ij}(W)|\mathcal{F}^1_{z_{ij}} \vee \mathcal{F}^2_{z_{ij}}) = \Delta_{ij} = 2^{-2n}$

We can eliminate from the product terms with expectation zero. What remain are terms which satisfy all the following conditions

$$\left.\begin{array}{l} k = k' = \underline{k} = \underline{k}' \\ i = \underline{i},\ \ell = \underline{\ell},\ \ell' = \underline{\ell}',\ j = \underline{j} \\ (i',j') \ll (i,\ell) \\ (\underline{i}',\underline{j}') \ll (i,\ell) \end{array}\right\} \tag{5.5}$$



Therefore the expectation of the square of (5.4) is

$$\sum \psi^2_{ijk\ell}\psi_{i'j'k\ell'}\ \psi_{\underline{i}'\underline{j}'k\ell'}\Delta_{ij}\Delta_{k\ell}\Delta_{k\ell'}\Delta_{i',j'}(W)\Delta_{\underline{i}',\underline{j}'}(W) \tag{5.6}$$

where the summation is under the restrictions of the last two lines of (5.5) and $\Delta_{ij} = \Delta_{k\ell} = \Delta_{k\ell'} = 2^{-2n}$. Let $\Delta_{k\ell'} = d_\ell \cdot 2^{-n}$ then (5.6) can be rewritten as

$$2^{-n}\left\{\sum_{ijk\ell}\psi^2_{ijk\ell}\left(\sum_{\ell'<\ell}\left(\sum_{i'j'}\psi_{i',j'k\ell'}\Delta_{i'j'}(W)\right)d_{\ell'}\right)\cdot\left(\sum_{\underline{i}',\underline{j}'}\psi_{\underline{i}',\underline{j}',k,\ell}\Delta_{\underline{i}',\underline{j}'}(W)\right)\Delta_{ij}\Delta_{k\ell}\right\} \tag{5.7}$$

The term in curly brackets has finite variance, therefore (5.7) and consequently (5.4) tend to zero as $n \to \infty$.

Case (4) follows as in cases 1, 2, 3 and 5 and yields the last two entries in (e) and (4') yields the two entries before the two last entries. We omit the details.

Case (5)

$$\sum_{ijk\ell}\psi^2_{ijk\ell}(\Delta_{ij}(W))^2(\Delta_{k\ell}(W))^2 = \sum_{ijk\ell}\psi^2_{ijk\ell}\Delta_{ij}\Delta_{k\ell}$$

$$+\sum_{ijk\ell}\psi^2_{ijk\ell}(\Delta^2_{ij}(W)\Delta^2_{k\ell}(W) - \Delta_{ij}\Delta_{k\ell}) \tag{5.8}$$

where $\Delta_{ij} = \Delta_{k\ell} = 2^{-2n}$. The first term above is

$$\iint \psi^2(z,z')dzdz'$$

which is the first term in (e). It remains to show that the second term in (5.8) converges to zero. Let B denote the second term of (5.8) then. Since $\alpha\beta - \gamma\delta = (\alpha-\gamma)(\beta-\delta) + \gamma(\beta-\delta) + \delta(\alpha-\gamma)$, we have

$$B = \sum\psi^2(\Delta^2_{ij}(W) - \Delta_{ij})(\Delta^2_{k\ell}(W) - \Delta_{k\ell})$$

$$+\sum\psi^2(\Delta^2_{k\ell}(W) - \Delta_{ij})\Delta_{ij} \tag{5.9}$$

$$+\sum\psi^2(\Delta^2_{ij}(W) - \Delta_{ij})\Delta_{k\ell}$$

Note that the expectation of each of the terms is zero. We will show that the variance of each of the terms approaches zero as $n \to \infty$. Consider the first term above. Squaring and taking expectations, we see that all terms vanish except

$$E\left\{\sum_{ijk\ell} \psi^4(\Delta^2_{k\ell}(W) - \Delta_{k\ell})^2(\Delta^2_{ij}(W) - \Delta_{ij})^2\right\}$$

For x Gaussian with $Ex = 0$ and $Ex^2 = \sigma^2$, we have $Ex^4 = 3(\sigma^2)^2$. Since ψ is bounded, the sum above is upper bounded by

$$K \sum_{ijk\ell} 2\Delta^2_{ij}\Delta^2_{k\ell} \leq K_1 \cdot 2^{-4n} \cdot \left(\sum_{ij} \Delta_{ij}\right)^2$$

Therefore the first term in (5.9) tends to zero. The other two terms tend to zero by the same argument.

We have therefore established (e) of Proposition (5-2) for ψ_1 and ψ_2 simple functions. Keeping ψ_1 simple, we can extend the result by continuity to $\iint \psi_2^2(z,z')dzdz' < \infty$ a.s. and then to a general ψ_1.

6. A Characterization of Strong Martingales of the Wiener Process

It was shown by Cairoli and Walsh [3] that a martingale M_z of the Wiener Process W_z is a strong martingale if and only if it is a type-one integral, i.e., $M_z = \int \phi_\zeta dW_\zeta$. A characterization in terms of stopping times will be given here.

Definitions:

1. $T(z,\omega)$ is a stopping time if
   (a) $T(z,\omega)$ is a measurable and adapted random process.
   (b) for almost all ω, $T(z,\omega)$ as a function of z is nonincreasing $(z \succ z' \Rightarrow T_z \leq T_{z'})$ and takes only the values zero or one.
2. $T(z,\omega)$ is a predictable stopping time if it is a stopping time and a predictable process.
3. Let Y_z be a martingale (or a function of bounded variation) and let T be a predictable stopping time. Then $Z_{z\wedge T}$ (Y stopped at T) is defined as

$$Y_{z\wedge T} = \int_{R_z} T(\zeta,\omega) dY(\zeta,\omega)$$

More generally, let Y_z be any adapted process such that

$$\int_{R_z} T_\zeta dY_\zeta$$

is defined and adapted, then $Y_{z\wedge T}$ is defined in the same way.

In order to point out the difference between stopping in the one parameter and the two parameter cases, let T be defined as

$$T(z) = 0 \text{ if } s \geq \frac{1}{2} \text{ and } t \geq \frac{1}{2}$$

$= 1$ otherwise

then if (s,t) is in the region where $T = 0$, $M_{(s,t)\wedge T} = M_{\frac{1}{2},\frac{1}{2}} + (M_{s,\frac{1}{2}} - M_{\frac{1}{2},\frac{1}{2}}) + (M_{\frac{1}{2},t} - M_{\frac{1}{2},\frac{1}{2}})$ therefore in the stopped region $M_{z\wedge T}$ is $M_{\frac{1}{2},\frac{1}{2}}$ plus the sum of two one parameter martingales.

<u>Proposition 6-1</u> Let M_z be a square integrable martingale, T a predictable stopping time and let

$$X_z = \int \phi_\zeta dM_\zeta$$

where a.s.

$$\int \phi_z^2 d\ [M]_z < \infty$$

Also if M_z is a strong martingale, let

$$Y_z = \iint \psi(\zeta,\zeta')dM_\zeta dM_{\zeta'}$$

where a.s.

$$\iint \psi^2(z,z')d\ [M]_z d\ [M]_{z'} < \infty$$

Then

$$X_{z\wedge T} = \int_{R_z} T_\zeta \phi_\zeta dM_\zeta$$

$$Y_{z\wedge T} = \iint_{R_z \times R_z} T(\zeta\ \zeta')\psi(\zeta,\zeta')dM_\zeta dM_{\zeta'}$$

The proof follows directly from the simple function approximation of ϕ and ψ and is therefore omitted. A similar result holds for mixed surface integrals.

In the next theorem we consider the Wiener process case; in this case every stopping time is predictable. Let $\mathcal{F}_z$ be the σ-fields generated by the Wiener process W_ζ, $\zeta \prec z$ and let T be a stopping time and let $\mathcal{F}_{z\wedge T}$ be the σ-fields generated by $W_{\zeta\wedge T}$, $\zeta \prec z$. Let $T_\lambda(z,\omega)$ $0 \le \lambda < \infty$, be a one parameter collection of stopping times such that for almost all ω, $T_{\lambda_2}(z,\omega) \geq T_{\lambda_1}(z,\omega)$ whenever $\lambda_1 \le \lambda_2$. We will call such a collection an increasing collection of stopping times. Let M_z be a martingale of the Wiener process and let z_0 be fixed. We will denote

$$\mathcal{F}_\lambda = \mathcal{F}_{z_0\wedge T_\lambda}$$

$$X_\lambda = M_{z_0\wedge T_\lambda}$$

Theorem 6-2 Let M_z be a square integrable martingale of the Wiener process, then M_z, $z \prec z_0$ is a strong martingale if and only if $\{X_\lambda, \mathcal{F}_\lambda\}$ is a martingale for all increasing families of stopping times.

Proof: If M_z is a strong martingale then $M_z = \int_{R_z} \phi_\zeta dW_\zeta$ [3] and

$$X_\lambda = \int_{R_{z_0}} T_\lambda(\zeta)\phi_\zeta dW_\zeta$$

$$X_{\lambda_2} = X_{\lambda_1} + \int_{R_{z_0}} (T_{\lambda_2} - T_{\lambda_1})\phi_\zeta dW_\zeta$$

approximating T_λ and ϕ_ζ by simple functions and proceeding as in the one parameter case show that X_λ is a martingale in the λ parameter. Conversely, let $\alpha < \beta$ and define

$$A = \{z: \ s + t \leq \alpha\}$$

$$B = \{z: \alpha < s + t \leq \beta\}$$

Let $T_1(z,\omega) = 1$ if $z \in A$

$= 0$ otherwise

$T_2(z,\omega) = 1$ if $z \in A \cup B$

$= 0$ otherwise

Let $M_z = \iint \psi(\zeta,\zeta')dW_\zeta dW_{\zeta'}$,

then $X_{\lambda_2} - X_{\lambda_1} = \int_{R_{z_0}} \int_{\times R_{z_0}} (T_2(\zeta \vee \zeta') - T_1(\zeta \vee \zeta'))\psi(\zeta,\zeta')dW_\zeta dW_{\zeta'}$,

Divide the above integral into five integrals. I_1 is the above integral over $\zeta \vee \zeta' \in A$ hence this integral is zero. I_2 is the above integral over $\zeta \in A$, $\zeta' \in B$, (and $\zeta \vee \zeta' \in B$), I_3 is the above integral over $\zeta' \in A$, $\zeta \in B$, I_4 is over $\zeta \vee \zeta' \in B$, $\zeta \in A$, $\zeta' \in A$, I_5 is over $\zeta' \in B$, $\zeta \in B$. It is easy to see, by simple function approximation that $E(I_i | \mathcal{F}_{T_1}) = 0$ for all i with the exception of i = 4. Consider now $E(I_4 | \mathcal{F}_{T_1})$. If X is to be a martingale, we must, therefore, have a.s.

$$E\left\{\iint_{z\in A,\, z'\in A,\, z\vee z'\in B} \psi(z,z')\,dW_{z\wedge T_1}\,dW_{z'\wedge T_1} \,\middle|\, \mathcal{F}_{T_1}\right\} = 0$$

And, through simple function approximation we must have a.s.

$$\iint E\{\psi(z,z')|\mathcal{F}_{T_1}\}dW_{z\wedge T_1}dW_{z'\wedge T_1} = 0$$

where the region of integration is the same as the previous integral.

Thus $\iint(E\{\psi|\mathcal{F}_{T_1}\})^2 d(z\wedge T_1)d(z'\wedge T_1) = 0$,

and

$$E(\psi(\zeta,\zeta')|\mathcal{F}_{T_1}) = 0 \text{ a.s.}$$

For $\zeta \vee \zeta'$ fixed let $\alpha \nearrow (\zeta \vee \zeta')$, by the continuity of the $\mathcal{F}_\lambda$ σ-fields

$$\psi(\zeta,\zeta') = \lim_{\alpha\to\zeta\wedge\zeta'} E\{\psi(\zeta,\zeta')|\mathcal{F}_\alpha\} = 0$$

which completes the proof.

<u>Remark</u>: The "if" part of the previous theorem holds for the general case where M_z is a strong martingale. That is: If M_z is a strong square integrable martingale and T_λ is a predictable and monotone class, then $X_\lambda = M_{z_0\wedge\lambda}$ is a one parameter martingale.

7. Martingales with Path Independent Variation

A square integrable martingale is said to be of path independent variation (or path independent) in R_{z_0} if for any $z < z_0$ and any two nondecreasing paths with initial point 0 and final point z the increasing functions on the two paths attain the same value at z [4].

Theorem 7-1 M_z is of path independent variation if and only if $(M_z)^2$ is the sum of a martingale and a nondecreasing function.

Proof: In Section 3 of [4] it was shown that M_z is a two parameter martingale if and only if it is a one parameter martingale on every nondecreasing path. Now let M_z be of path independent variation and let A_z be the variation of M_z on any nondecreasing path from (0,0) to z. Then $(M_z)^2 - A_z$ is a one parameter martingale on every nondecreasing path and hence $(M_z)^2 - A_z$ is a two parameter martingale. The converse follows by the same argument.

8. A Remark on Diffusion Processes in the Plane

In [2], Cairoli considered stochastic differential equations of the form

$$X_z = X_o + \int_{R_z} p(X_\zeta)d\zeta + \int_{R_z} q(X_\zeta)dW_\zeta \tag{7.1}$$

and, after defining Markov processes in the plane, showed that the solution to (7.1) is a two parameter Markov process. In an analogous way we can consider stochastic equations of the form

$$X_z = X_o + \int p(X_\zeta)d\zeta + \int q(X_\zeta)dW_\zeta + \iint r_1(X_{\zeta\vee\zeta'})dW_\zeta dW_{\zeta'}$$

$$+ \iint r_2(X_{\zeta\vee\zeta'})dW_\zeta d\zeta' + \iint r_3(X_{\zeta\vee\zeta'})d\zeta dW_{\zeta'} \tag{7.2}$$

The problem arises whether there are solutions (7.2) which are Markov. Proposition 5-1 of Section 5 gives immediately an affirmative answer since if $f(\alpha)$, $\infty < \alpha < \infty$ is a well behaved real valued invertible function and if X_z is a solution to (7.1), then $Y_t = f(X_z)$ is also Markov. However Y_t satisfies an equation of type (7.2) but not of type (7.1). It seems reasonable to conjecture that under suitable conditions the pair (W_z, X_z) is a (vector valued) two parameter Markov process.

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