A QUALITATIVE ANALYSIS OF THE BEHAVIOR OF DYNAMIC NONLINEAR NETWORKS: STEADY-STATE SOLUTIONS OF NONAUTONOMOUS NETWORKS

by

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Abstract

Several theorems are presented which predict in a qualitative manner the behavior of dynamic nonlinear networks. In particular, conditions are given which assure that the voltage and current waveforms of a dynamic nonlinear network $\mathcal{N}$ are bounded or eventually uniformly bounded, or converge to a unique steady-state solution. The periodic nature of the solutions is examined, and conditions are given which guarantee that the solutions are periodic, almost periodic or asymptotically almost periodic. These conditions are the best possible for the class of networks considered here.

The theorems are significant in that they apply to a large class of networks. Furthermore, their hypotheses are simple and easily verifiable. The hypotheses are of two types: First, very general conditions on the network state equations, and second, conditions on the individual element characteristics and their interconnection. The latter type of theorems use graph-theoretic results of [14] and involve solely the examination of the global nature of each network element and the verification of a topological "loop-cutset" condition.

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I. Introduction

Much of the analysis of dynamic nonlinear networks has been in the area of the formulation of networks equations [1]-[5], and in the area of numerically solving these equations [6]-[7]. There are results concerning the behavior of networks containing specific nonlinear elements such as transistors or iron-core inductors [8]-[10] but there are relatively few results which examine in a qualitative way the behavior of general non-linear dynamic networks [1], [4], [11], [12], [13]. This paper is the third of three papers which develop methods for predicting in a qualitative way the behavior of dynamic nonlinear networks. The other two papers are titled "Graph-Theoretic Properties of Dynamic Nonlinear Networks" [14], and "A Qualitative Analysis of the Behavior of Dynamic Nonlinear Networks: Stability of Autonomous Networks," [15]. In [14] graph-theoretic methods are used to determine properties of network equations. We combine these results with the mathematical analysis of the equations to determine the behavior of autonomous networks in [15], and to determine the behavior of nonautonomous networks in this paper. We answer the following types of questions in these papers: Let $\mathcal{N}$ be a dynamic nonlinear network. Under what condition may we conclude all network voltage and current waveforms are bounded, or eventually uniformly bounded (Def. 5)? If $\mathcal{N}$ contains a T-periodic source, when is there a T-periodic solution of $\mathcal{N}$, or a sub-harmonic solution? If $\mathcal{N}$ contains constant independent voltage and current sources, when does $\mathcal{N}$ have a unique, globally asymptotically stable equi-librium point? When $\mathcal{N}$ has time-varying sources, under what conditions does $\mathcal{N}$ have a unique steady-state solution (in the same sense as with linear networks)? In this case, do the transients decay exponentially? While answer to some of these questions have been published for various classes of nonlinear differential equations [16]-[18], they are strictly mathematical in nature and often contain conditions which are too strong or impractical when applied to networks. The main feature of our results is that many of the theorems are couched in graph- and circuit theoretic terms so that they can easily be verified by examining only the network topology and the elements' constitutive relations. The graph theoretic methods have been presented in [14], and applied to autonomous networks in [15]. In this paper we study nonautonomous networks.

In Sec. II, a very general class of dynamic nonlinear networks is defined along with a characterization of the various types of resistive n-ports to be considered in the sequel. Various properties of functions such as the passivity property, the increasing property, the strictly increasing property, etc., are defined. The properties have been discussed extensively in [14]. The graph-theoretic results of [14] which are needed later are presented and discussed here. Qualitative properties of the nonlinear, nonautonomous differential Eq. (15) are defined. Of special interest is the definition of a unique steady-state solution. This concept is well-known in the study of linear systems and is important in the study of nonlinear networks.

In Sec. III the mathematical results used in this paper are presented. In Theorem A.1 conditions are given such that the solutions of the nonlinear differential Eq. (15) are bounded or eventually uniformly bounded. In Theorem A.2 we define an Incremental Lyapunov Function which is used to show that (15) has a unique steady-state solution. This is a natural extension of Lyapunov's direct method and has been used in [9], [12] and [18]. Variations of Theorem A.2 are presented in Corollaries A.1-A.4. In Theorems B.1-B.3 we define and discuss almost periodic functions. Theorem

C prescribes properties of a C^1-strictly increasing diffeomorphic state function.

In Secs. IV and V, theorems are given for analyzing the qualitative behavior of nonlinear dynamic networks. The hypotheses of these theorems are of two types; namely, conditions upon the network state equations, and conditions on the constitutive relations of the network elements and their interconnection. The difference between these two types of hypotheses is discussed in a general way in Sec. III. These conditions are used in Theorems 1-11 to show (i) that the waveforms of $\mathcal{N}$ are bounded or eventually uniformly bounded, or (ii) that the waveforms converge (possibly exponentially) to a unique steady-state solution. The important aspect of our results is that the hypotheses apply to a large class of networks and that they are easily verifiable. In their final form, the hypotheses involve simply investigating the passive or increasing nature of each network element, and satisfying an easily verifiable topological "loop-cutset" condition on the interconnection of the elements. As illustrated in the examples in Secs. IV and V, the results may be applied to a variety of nonlinear dynamic networks.

II. Characterization of State Equations

Consider the dynamic nonlinear network $\mathcal{N}$ shown in Fig. 1. It contains n_C (possibly coupled) one-port capacitors, and n_L (possibly coupled) one-port inductors.[1] Let $\underset{\sim}{v}_C$, $\underset{\sim}{i}_C, \underset{\sim}{q}_C \in \mathbb{R}^{n_C}$ and $\underset{\sim}{v}_L$, $\underset{\sim}{i}_L$, $\underset{\sim}{\phi}_L \in \mathbb{R}^{n_L}$ denote

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[1] There is no loss of generality in our choice of this network model, since any multi-port of multi-terminal capacitor (resp., inductor) can always be modeled as a system of "coupled" one-port capacitors (resp., inductors). Observe also that an (n+1)-terminal element can always be modeled as a "grounded" n-port.

respectively the capacitor voltages, currents, charges, and the inductor voltages, currents and fluxes. The constitutive relations of a charge-controlled capacitor and a flux-controlled inductor are given respectively by:

$$\begin{aligned} \underset{\sim}{v}_C &= \underset{\sim}{h}_C(\underset{\sim}{q}_C) \\ \underset{\sim}{i}_L &= \underset{\sim}{h}_L(\underset{\sim}{\phi}_L) \end{aligned} \tag{1}$$

where $\underset{\sim}{h}_C: \mathbb{R}^{n_C} \to \mathbb{R}^{n_C}$ and $\underset{\sim}{h}_L: \mathbb{R}^{n_L} \to \mathbb{R}^{n_L}$. Define n_p-vectors ($n_p = n_C + n_L$) (the subscript "p" denotes a "port variable")

$$\underset{\sim}{v}_p = \begin{pmatrix} \underset{\sim}{v}_C \\ \underset{\sim}{v}_L \end{pmatrix}; \quad \underset{\sim}{i}_p = \begin{pmatrix} \underset{\sim}{i}_C \\ \underset{\sim}{i}_L \end{pmatrix}; \quad \underset{\sim}{x}_p = \begin{pmatrix} \underset{\sim}{v}_C \\ \underset{\sim}{i}_L \end{pmatrix}$$

$$\underset{\sim}{y}_p = \begin{pmatrix} \underset{\sim}{i}_C \\ \underset{\sim}{v}_L \end{pmatrix}; \quad \underset{\sim}{z}_p = \begin{pmatrix} \underset{\sim}{q}_C \\ \underset{\sim}{\phi}_L \end{pmatrix} \tag{2}$$

then (1) becomes

$$\underset{\sim}{x}_p = \underset{\sim}{h}_p(\underset{\sim}{z}_p) \tag{3}$$

$\underset{\sim}{h}_p(\cdot) = \left[\underset{\sim}{h}_C^T(\cdot), \underset{\sim}{h}_L^T(\cdot)\right]^T$ (where the superscript "T" denotes transpose).

Remark: In [14], the capacitors and inductors are respectively voltage-controlled and current-controlled; i.e., instead of (3), we have $\underset{\sim}{z}_p = \underset{\sim}{f}_p(\underset{\sim}{x}_p)$. We use $\underset{\sim}{f}_p$ in [14], and we use $\underset{\sim}{h}_p$ here and in [15] purely for ease of notation in each paper. In some of the theorems in this paper and in [15], $\underset{\sim}{h}_p$ is bijective; hence $\underset{\sim}{f}_p = \underset{\sim}{h}_p^{-1}$ exists, and either $\underset{\sim}{h}_p$ or $\underset{\sim}{f}_p$ may be considered as the capacitor-inductor function. See Example 1.

We view the n_C capacitors and the n_L inductors as attached to an

(n_p+n_S)-port N. Time-varying,[2] independent voltage and current sources are attached to the remaining n_S ports of N. Let $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$ denote the voltages of the independent voltage sources, and currents of the independent current sources. Let $\underset{\sim}{w}_S \in \mathbb{R}^{n_S}$ denote the currents of the independent voltage sources, and voltages of the independent current sources. The vectors $\underset{\sim}{x}_p$, $\underset{\sim}{y}_p$, $\underset{\sim}{u}_S$ and $\underset{\sim}{w}_S$ are port variables of N as well as capacitor, inductor and source variables. N contains (nonlinear) one-port resistors, (nonlinear) multi-port resistors,[3] and constant independent voltage and current sources.

Assume resistor R_α of N is an n_α-port resistor. Its voltage and current are, respectively, $\underset{\sim}{v}_{R_\alpha}$, $\underset{\sim}{i}_{R_\alpha} \in \mathbb{R}^{n_\alpha}$. In defining its constitutive relations (when it exists) we assume that for each port of the n_α-port resistor either the port voltage or the port current is an independent resistor variable, and the remaining port variable is a dependent resistor variable. Let $\underset{\sim}{x}_{R_\alpha}$, $\underset{\sim}{y}_{R_\alpha} \in \mathbb{R}^{n_\alpha}$ denote respectively the independent and dependent resistor vectors. The constitutive relation is therefore

$$\underset{\sim}{y}_{R_\alpha} = \underset{\sim}{g}_{R_\alpha}(\underset{\sim}{x}_{R_\alpha}) \tag{4}$$

Let m_R be the number of resistors of N, and let n_R be the number of all internal resistor ports of N ($m_R=n_R$ if, and only if, all resistors are two-terminal elements). The composite resistor vectors are $\underset{\sim}{v}_R$, $\underset{\sim}{i}_R \in \mathbb{R}^{n_R}$

[2] Here, a source is considered time-varying if it indeed varies with time, or if it is a constant source which is to be represented by the source vector $\underset{\sim}{u}_S$.

[3] N also contains controlled voltage and current sources in the following sense: We assume every controlled source of N is represented by "coupling" within multi-port resistors. For example, although transistors, FET, and operational amplifiers are multi-terminal elements which are often modeled using controlled sources, they can also be represented as multi-port resistors. Hence, a transistor can be characterized as a 2-port with the constitutive relation (57) of Example 1.

representing respectively all internal voltages and currents. Let the m_R resistors be described by their constitutive relation $\underset{\sim}{g}_{R_1}(\cdot), \underset{\sim}{g}_{R_2}(\cdot), \ldots, \underset{\sim}{g}_{R_{m_R}}(\cdot)$, and let $\underset{\sim}{x}_R, \underset{\sim}{y}_R \in \mathbb{R}^{n_R}$ denote, respectively, the independent and dependent resistors vectors. Then

$$\underset{\sim}{y}_R = \underset{\sim}{g}_R(\underset{\sim}{x}_R) \tag{5}$$

is the composite resistor constitutive relation representing all internal resistors, where $\underset{\sim}{g}_R(\cdot) = \left[\underset{\sim}{g}^T_{R_1}(\cdot), \underset{\sim}{g}^T_{R_2}(\cdot), \ldots, \underset{\sim}{g}^T_{R_\alpha}(\cdot), \ldots, \underset{\sim}{g}^T_{R_{m_R}}(\cdot)\right]^T$. The constitutive relation of the "overall resistor" (n_p+n_S)-port N (when it exists) is given by

$$\underset{\sim}{y}_p = -\underset{\sim}{g}_p(\underset{\sim}{x}_p, \underset{\sim}{u}_S) \tag{6}$$

$$\underset{\sim}{w}_S = -\underset{\sim}{g}_S(\underset{\sim}{x}_p, \underset{\sim}{u}_S) \tag{7}$$

where $\underset{\sim}{g}_p(\cdot,\cdot): \mathbb{R}^{n_p+n_S} \to \mathbb{R}^{n_p}$ and $\underset{\sim}{g}_S(\cdot,\cdot): \mathbb{R}^{n_p+n_S} \to \mathbb{R}^{n_S}$

Remark: Equations (6) and (7) have a negative sign because the port currents (in Fig. 1) are directed away from the ports on "voltage-driven" (i.e., capacitor and voltage source) ports, and the port voltages are reversed on the "current-driven" (i.e., inductor and current source) ports. These reference directions and polarities are chosen to be consistent with those assigned to the capacitors, inductors, and sources.

Using (3) with (6) and (7), we can write the dynamical system representation [19] of $\mathcal{N}$. Note first that $\frac{d}{dt}\underset{\sim}{z}_p(t) = \dot{\underset{\sim}{z}}_p(t) = \underset{\sim}{y}_p(t)$; we have

$$\dot{\underset{\sim}{z}}_p = -\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p), \underset{\sim}{u}_S\right) \tag{8}$$

$$\underset{\sim}{w}_S = -\underset{\sim}{g}_S\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p), \underset{\sim}{u}_S\right) \tag{9}$$

These equations describe the input-output system where $\underset{\sim}{u}_S(\cdot)$ is the input, $\underset{\sim}{w}_S(\cdot)$ is the output, and $\underset{\sim}{z}_p(t)$ denotes the state at time t. An alternative way to view $\mathcal{N}$ is to assume that the source waveform $\underset{\sim}{u}_S(t)$ represents fixed time-varying sources, in which case we are interested only in the capacitor and inductor waveforms described by the state equation (8). In all cases, it is (8) which is of primary importance in determining the behavior of $\mathcal{N}$, and to this differential equation we devote our attention in the sequel.

In Secs. IV and V each result concerning the behavior of $\mathcal{N}$ takes two forms: First, the behavior of the solutions of the network state Eqs. (8) is analyzed using the mathematical methods of Sec. III, and the following definitions. The hypotheses of these theorems are in the form of conditions on the function $\underset{\sim}{h}_p$ describing the capacitors and inductors, and on the function $\underset{\sim}{g}_p$ which describes the overall resistive (n_p+n_S)-port N. In each of the theorems, we make the following assumption: The qualitative behavior of the voltage and current waveforms of each element of $\mathcal{N}$ may be uniquely determined from the behavior of solutions $\underset{\sim}{z}_p(\cdot)$ of (8). In its second form, the conclusions are identical but the hypotheses are in terms of the <u>individual network elements and the interconnection of these elements</u>. The conditions placed upon the elements are those placed on the resistor function $\underset{\sim}{g}_{R_\alpha}$, $\alpha = 1,2,\ldots,m_R$, and upon the capacitor-inductor function $\underset{\sim}{h}_p$. We then use the graph theoretic results of [14]. At this point, it is instructive to state the interconnection assumption of the theorems of [14].

<u>Fundamental Topological Assumption: There is no loop (resp., no cutset) formed exclusively by capacitors, inductors, and/or independent voltage sources (resp., current sources)</u>.

There is an equivalent way to restate the <u>Fundamental Topological Assumption</u>: Upon replacing all voltage sources (resp., current sources) with short circuits, (resp., open circuits), there is no loop and no cut-set formed exclusively by capacitors and/or inductors.

If this assumption is satisfied, we know for example that if each g_{R_α} is strictly increasing (Def. 2), then $g_p(\cdot, u_S)$ in (6) is strictly increasing for each $u_S \in \mathbb{R}^{n_S}$ [14; Theorem 9]. This conclusion and others are used throughout the sequel.

The following definitions characterize the various types of n-ports considered here.

Def. 1: The function f: $\mathbb{R}^n \to \mathbb{R}^n$ is

(i) passive with respect to $x_0 \in \mathbb{R}^n$ if, and only if, for all $x \in \mathbb{R}^n$

$$(x-x_0)^T f(x) \geq 0 \tag{10}$$

(ii) strictly passive with respect to $x_0 \in \mathbb{R}^n$ if, and only if, (10) is true and the left side is positive for all $x \neq x_0$.

(iii) eventually passive with respect to $x_0 \in \mathbb{R}^n$ if, and only if, there exists $k_0 > 0$ so that[4]

$$(x-x_0)^T f(x) \geq 0 , \qquad \forall \, \|x\| > k_0 \tag{11}$$

(iv) eventually strictly passive with respect to $x_0 \in \mathbb{R}^n$ if, and only if, (11) is satisfied where the left side is strictly greater than zero.

Remarks: 1. If $x_0 = 0 \in \mathbb{R}^{n_p}$, we say simply that f is passive, strictly passive, eventually passive, or eventually strictly passive.

2. In (i) and (ii), the domain of f may be an arbitrary connected set $D \subseteq \mathbb{R}^{n_p}$, $x_0 \in D$.

[4]The norm $\|\cdot\|$ we have used in this paper is the Euclidean norm $\|x\| = [(x_1)^2 + \ldots + (x_n)^2]^{1/2}$. Of course, the following results remain valid for any choice of norm in $\mathbb{R}^n$.

Def. 2: [20] Let $D \subseteq \mathbb{R}^n$ be convex. The function $\underset{\sim}{f}: \mathbb{R}^n \to \mathbb{R}^n$ is

(i) increasing on D if, and only if for all $\underset{\sim}{x}', \underset{\sim}{x}'' \in D$

$$(\underset{\sim}{x}'-\underset{\sim}{x}'')^T\left(\underset{\sim}{f}(\underset{\sim}{x}')-\underset{\sim}{f}(\underset{\sim}{x}'')\right) \geq 0 \tag{12}$$

(ii) strictly increasing on D if, and only if, the left side of (12) is positive for all $\underset{\sim}{x}' \neq \underset{\sim}{x}''$.

(iii) uniformly increasing on D if, and only if, there exists $\gamma > 0$ such that for all $\underset{\sim}{x}', \underset{\sim}{x}'' \in D$

$$(\underset{\sim}{x}'-\underset{\sim}{x}'')^T\left(\underset{\sim}{f}(\underset{\sim}{x}')-\underset{\sim}{f}(\underset{\sim}{x}'')\right) \geq \gamma\|\underset{\sim}{x}'-\underset{\sim}{x}''\|^2 \tag{13}$$

(iv) strongly uniformly increasing on D if, and only if, there exists $\bar{\gamma} \geq \underline{\gamma} > 0$ such that for all $\underset{\sim}{x}', \underset{\sim}{x}'' \in D$,

$$\underline{\gamma}\|\underset{\sim}{x}'-\underset{\sim}{x}''\|^2 \leq (\underset{\sim}{x}'-\underset{\sim}{x}'')^T\left(\underset{\sim}{f}(\underset{\sim}{x}')-\underset{\sim}{f}(\underset{\sim}{x}'')\right) \leq \bar{\gamma}\|\underset{\sim}{x}'-\underset{\sim}{x}''\|^2 \tag{14}$$

Def. 3: [20] For any integer $\mu \geq 0$, $\underset{\sim}{f}: \mathbb{R}^n \to \mathbb{R}^n$ is a C^μ-diffeomorphism on $\mathbb{R}^n$ (or is a C^μ-diffeomorphic function on $\mathbb{R}^n$) if, and only if, $\underset{\sim}{f}$ is injective on $\mathbb{R}^n$, and the functions $\underset{\sim}{f}$, $\underset{\sim}{f}^{-1}$ are C^μ. Furthermore, $\underset{\sim}{f}$ is a C^μ-diffeomorphism mapping $\mathbb{R}^n$ onto $\mathbb{R}^n$ if, and only if, $\underset{\sim}{f}$ is a C^μ-diffeomorphism, and $\underset{\sim}{f}$ is surjective. A C^0-diffeomorphism is called a homeomorphism.

Def. 4: [21] The C^1-function $\underset{\sim}{f}: \mathbb{R}^n \to \mathbb{R}^n$ is a state function if, and only if, its Jacobian $\frac{\partial \underset{\sim}{f}(\underset{\sim}{x})}{\partial \underset{\sim}{x}}$ is symmetric for all $\underset{\sim}{x} \in \mathbb{R}^n$.

In order to develop theorems governing the behavior of the solutions of the network state equation (8), we first examine the solutions of the general nonlinear nonautonomous differential equation

$$\dot{\underset{\sim}{x}} = -\underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{\xi}) \tag{15}$$

where $\underset{\sim}{x} \in \mathbb{R}^n$, $\underset{\sim}{\xi} \in \mathbb{R}^m$, and $\underset{\sim}{f}$: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a C^1-function. The variable $\underset{\sim}{x}$ is not to be confused with the port variable $\underset{\sim}{x}_p$; Eq. (15) is an arbitrary nonlinear differential equation not necessarily used to describe the behavior of a network. The variable $\underset{\sim}{\xi}(\cdot)$ is a continuous function of time. This differential equation is more general than the usual time-varying differential equation $\dot{\underset{\sim}{x}} = -\underset{\sim}{f}(\underset{\sim}{x},t)$ in that (15) reduces to this equation when $m = 1$ and $\xi(t) \equiv t$. We will find (15) useful because the <u>periodic</u> nature of the nonautonomous differential equation is expressed completely by the periodic nature of $\underset{\sim}{\xi}(\cdot)$.

<u>Def. 5</u>: [18] The solutions of (15) are <u>eventually uniformly bounded</u> if, and only if, there exists a compact set $\mathcal{K}_0 \subseteq \mathbb{R}^n$ such that for any solution $\underset{\sim}{x}(\cdot)$ of (15) there is a time $t_0 \in \mathbb{R}^1$ such that

$$\underset{\sim}{x}(t) \in \mathcal{K}_0 \quad , \quad \forall\, t \geq t_0 \tag{16}$$

<u>Remark</u>: When the solutions of the network state equation (8) are eventually uniformly bounded, we know that equivalently all voltage and current vaveforms of $\mathcal{N}$ are eventually uniformly bounded.

The concept of a "unique steady-state solution" is well-known in the study of linear systems. In the following this concept is extended to the study of solutions of (15).

<u>Def. 6</u>: The differential equation (15) has a <u>unique steady-state solution</u> if, and only if, for any pair of solutions $\underset{\sim}{x}'(\cdot)$ and $\underset{\sim}{x}''(\cdot)$ of (15), both solutions are bounded, and

$$\lim_{t \to +\infty} \|\underset{\sim}{x}'(t) - \underset{\sim}{x}''(t)\| = 0 \tag{17}$$

<u>Remarks</u>: 1. If (15) has a unique steady-state solution, then its

solutions are eventually uniformly bounded.

2. It is possible that (17) could be true, and no solution is bounded. For example, for the first order differential equation $\dot{x} = -x + e^{2t}$, (17) is true, but all solutions tend to $+\infty$ as $t \to \infty$, and to say that the solutions converge to a unique steady-state is meaningless.

3. Equation (17) is an <u>incremental</u> criterion. Often, there is a particular <u>choice</u> of a steady-state solution. For example, for the differential equation $\dot{x} = -x$, the solution $x(t) \equiv 0$ is usually called the unique steady-state solution, and by this it is meant that all other solutions asymptotically converge to it. <u>Def. 6</u> does not overtly specify any particular steady-state solution. There are a number of reasons for this:

(i) As we shall see in <u>Theorem A.2</u>, it is possible to conclude that (15) has a unique steady-state solution without knowing any particular solution $\underset{\sim}{x}(\cdot)$.

(ii) <u>Any</u> solution $\underset{\sim}{x}(\cdot)$ may be viewed as <u>the</u> unique steady-state solution if such a solution exists. For example, the solution $x(t) = e^{-t}$ of $\dot{x} = -x$ is the unique steady-state solution because all other solutions (including $x(t) \equiv 0$) converge to it.

(iii) At times, the "natural" choice of the unique steady-state waveform is <u>not</u> a solution at all! For example, the differential equation $\dot{x} = -x + e^{-2t}$ has $x(t) \equiv 0$ as the unique steady-state <u>waveform</u>, but it is not a solution.

<u>Def. 7</u>: [12] Let $\underset{\sim}{\xi}(\cdot)$: $\mathbb{R}^1 \to \mathbb{R}^m$ be continuous. Define

$$\mathcal{R}_T\left(\underset{\sim}{\xi}(\cdot)\right) \triangleq \left\{\hat{\underset{\sim}{\xi}} \in \mathbb{R}^m: \text{ there exists } t \geq T \text{ such that } \underset{\sim}{\xi}(t) = \hat{\underset{\sim}{\xi}}\right\} \quad (18)$$

and[5]

$$\mathcal{R}_\infty\left(\underset{\sim}{\xi}(\cdot)\right) \triangleq \overline{\bigcap_{T\in\mathbb{R}^1} \mathcal{R}_T\left(\underset{\sim}{\xi}(\cdot)\right)} \tag{19}$$

Remark: When $\left(\underset{\sim}{\xi}(\cdot)\right)$ is bounded, $\mathcal{R}_\infty\left(\underset{\sim}{\xi}(\cdot)\right)$ is compact and connected in $\mathbb{R}^m$. We interpret $\mathcal{R}_\infty\left(\underset{\sim}{\xi}(\cdot)\right)$ as the (closure of the) eventual range of $\underset{\sim}{\xi}(t)$ as $t \to \infty$.

III. Mathematical Methods

In [15] a useful theorem [Theorem B.2] is presented which gives conditions under which the solutions of (15) exist for all t as $t \to +\infty$ (i.e., there are no finite escape-time solutions), though solutions may grow arbitrarily large as t tends to $+\infty$. In the following theorem, conditions are given which guarantee that solutions are bounded, or eventually uniformly bounded.

Theorem A.1: [16], [22] Assume there exist constants $k_0 \geq 0$ and $k_1 > 0$ and a C^1-function $\mathcal{V}: \mathbb{R}^n \to \mathbb{R}^1$ such that

$$\lim_{\|\underset{\sim}{x}\|\to\infty} \mathcal{V}(\underset{\sim}{x}) = +\infty \tag{20}$$

and

$$\frac{\partial\mathcal{V}(\underset{\sim}{x})}{\partial\underset{\sim}{x}} \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{\xi}) \geq 0 \quad , \quad \forall \|\underset{\sim}{x}\| > k_0 \quad , \quad \forall \|\underset{\sim}{\xi}\| \leq k_1 \tag{21}$$

where $\dfrac{\partial\mathcal{V}(\underset{\sim}{x})}{\partial\underset{\sim}{x}} \triangleq \left[\dfrac{\partial\mathcal{V}(\underset{\sim}{x})}{\partial x_1}, \dfrac{\partial\mathcal{V}(\underset{\sim}{x})}{\partial x_2}, \ldots, \dfrac{\partial\mathcal{V}(\underset{\sim}{x})}{\partial x_n}\right]$,

then for any continuous $\underset{\sim}{\xi}(\cdot)$ such that

$$\|\underset{\sim}{\xi}(t)\| < k_1 \quad , \quad \forall\ t \in \mathbb{R}^1 \tag{22}$$

the solutions of (15) are bounded.

[5] For any set $D \subseteq \mathbb{R}^m$, $\bar{D}$ denotes its closure.

Furthermore, if

$$\frac{\partial \mathcal{V}(\underset{\sim}{x})}{\partial \underset{\sim}{x}} \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{\xi}) > 0 \quad , \qquad \forall \ \|\underset{\sim}{x}\| > k_0 \quad , \quad \forall \ \|\underset{\sim}{\xi}\| \leq k_1 \tag{23}$$

then the solutions of (15) are eventually uniformly bounded. In particular, define $\hat{k}_0 \triangleq \sup_{\|\underset{\sim}{x}\| \leq k_0} \mathcal{V}(\underset{\sim}{x})$; then for any solution $\underset{\sim}{x}(\cdot)$ of (15) there exists $t_0 \in \mathbb{R}^1$ such that

$$\underset{\sim}{x}(t) \in \mathcal{K}_0 \triangleq \left\{\underset{\sim}{x}: \mathcal{V}(\underset{\sim}{x}) \leq \hat{k}_0\right\} \quad , \quad \forall \ t \geq t_0 \tag{24}$$

Finally, if in addition $\underset{\sim}{\xi}(\cdot)$ is periodic with period T > 0, then (15) has a solution $\underset{\sim}{x}^*(\cdot)$ which is periodic with period T.

Remarks: 1. The conclusion that the solutions of (15) are bounded or eventually uniformly bounded is proved in [16] and [22], and is discussed in [14]. The conclusion that (15) has a T-periodic solution $\underset{\sim}{x}^*(\cdot)$ if $\underset{\sim}{\xi}(\cdot)$ is T-periodic can be shown in the following way [22], [18]: We first show that there is a compact and convex set $\hat{\mathcal{K}}_0 \supseteq \mathcal{K}_0$ in $\mathbb{R}^n$ such that if $\underset{\sim}{x}(\cdot)$ is a solution such that $\underset{\sim}{x}(0) \in \hat{\mathcal{K}}_0$, then $x(t) \in \hat{\mathcal{K}}_0$ for all $t \geq 0$. This fact can be shown using the eventual uniform boundedness conclusion. Next, for every $\underset{\sim}{x}(0) \in \mathbb{R}^n$, and corresponding solution $\underset{\sim}{x}(\cdot)$, the mapping $\underset{\sim}{x}(0)$ into $\underset{\sim}{x}(T)$ is a continuous map from $\hat{\mathcal{K}}_0$ into $\hat{\mathcal{K}}_0$. Using Brower's Fixed Point Theorem [20] we see that there exists $\underset{\sim}{x}^*(0) \in \hat{\mathcal{K}}_0$ such that $\underset{\sim}{x}^*(T) = \underset{\sim}{x}^*(0)$. From this we conclude that the solution $\underset{\sim}{x}^*(\cdot)$ is periodic with period T.

2. If (15) is autonomous, i.e., $\underset{\sim}{\xi}(t) \equiv \underset{\sim}{\xi} \in \mathbb{R}^m$, and the solutions of (15) are eventually uniformly bounded, then (15) has an equilibrium point $\underset{\sim}{x}^* \in \mathcal{K}_0$. We can reach this conclusion by using the theorem to show that for every T > 0 (15) has a solution $\underset{\sim}{x}_T(\cdot)$ which is

periodic with period T. This is true in particular for arbitrarily small $T > 0$, and we can show that (15) has a constant solution in $\mathcal{K}_0$.

3. In application to the study of circuits, the difference between bounded and eventually uniformly bounded waveforms is nontrivial. For example, the linear network of Fig. 2(a) has no unbounded solutions (so long as $\omega \neq 1/\sqrt{LC}$), but the magnitude of the solutions can be arbitrarily large. On the other hand, the waveforms of the linear networks of Fig. 2(b) and Fig. 2(c) are eventually uniformly bounded.

The following theorem and corollaries are similar to Theorem 19.1 of [18].

Theorem A.2: Assume the solutions of (15) are eventually uniformly bounded. Let $D_x \supseteq \mathcal{K}_0$ (compact set $\mathcal{K}_0$ is defined in Def. 5, Eq. (16)) be open and bounded in $\mathbb{R}^n$, and let $D_\xi \supseteq \mathcal{R}(\underset{\sim}{\xi}(\cdot))$ be open and bounded in $\mathbb{R}^m$. Assume there exists a C^1-function $\mathcal{V}_\Delta$: $D_x \times D_x \to \mathbb{R}^1$ such that

$$\begin{aligned} \mathcal{V}_\Delta(\underset{\sim}{x},\underset{\sim}{x}) &= 0 \quad , \quad \forall\, \underset{\sim}{x} \in D_x \\ \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'') &> 0 \quad , \quad \forall\, \underset{\sim}{x}' \neq \underset{\sim}{x}'' \in D_x \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\frac{\partial \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}'} \underset{\sim}{f}(\underset{\sim}{x}',\underset{\sim}{\xi}) + \frac{\partial \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}''} \underset{\sim}{f}(\underset{\sim}{x}'',\underset{\sim}{\xi}) \\ &\qquad = 0, \qquad \forall\, \underset{\sim}{x}' = \underset{\sim}{x}'' \in D_x, \ \forall\, \underset{\sim}{\xi} \in D_\xi \\ &\frac{\partial \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}'} \underset{\sim}{f}(\underset{\sim}{x}',\underset{\sim}{\xi}) + \frac{\partial \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}''} \underset{\sim}{f}(\underset{\sim}{x}'',\underset{\sim}{\xi}) \\ &\qquad > 0, \qquad \forall\, \underset{\sim}{x}' \neq \underset{\sim}{x}'' \in D_x, \ \forall\, \underset{\sim}{\xi} \in D_\xi \end{aligned} \tag{26}$$

Then (15) has a unique steady-state solution.

Remarks: 1. This theorem and Corollaries A.1-A.4 are proved or discussed in the Appendix.

2. We call $\mathcal{V}_\Delta(\cdot,\cdot)$ an Incremental Lyapunov Function. It can be interpreted in the following way: Let $\underset{\sim}{x}'(\cdot)$ and $\underset{\sim}{x}''(\cdot)$ be two arbitrary solutions of (15). Then $\mathcal{V}_\Delta\left(\underset{\sim}{x}'(t),\underset{\sim}{x}''(t)\right)$ is a measure of the "incremental Lyapunov energy" between the two solutions at time t. We use (25) and (26) to show $\lim_{t\to\infty} \mathcal{V}_\Delta\left(\underset{\sim}{x}'(t),\underset{\sim}{x}''(t)\right) = 0$ which in turn implies (17). An alternate interpretation of $\mathcal{V}_\Delta(\cdot,\cdot)$ is contained in the following:

Corollary A.1: The condition in Theorem A.2 requiring that the solutions of (15) be eventually uniformly bounded may be replaced by the conditions (i) $\underset{\sim}{\xi}(\cdot)$ is bounded, and (15) has a bounded solution, (ii) Eqs. (25) and (26) are satisfied with $D_x = \mathbb{R}^n$ and $D_\xi = \mathbb{R}^m$, and (iii)

$$\lim_{\|\underset{\sim}{x}'-\underset{\sim}{x}''\|\to\infty} \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'') = +\infty \tag{27}$$

Remark: Here, if $\underset{\sim}{x}^*(\cdot)$ is the bounded solution of the hypothesis, then $\mathcal{V}_\Delta\left(\underset{\sim}{x}(t),\ \underset{\sim}{x}^*(t)\right)$ is the "Lyapunov energy" between any other solution $\underset{\sim}{x}(\cdot)$ and $\underset{\sim}{x}^*(\cdot)$ at time t. Equations (25), (26) and (27) are used to show that all solutions asymptotically converge to the solution $\underset{\sim}{x}^*(\cdot)$ which is therefore called the unique steady-state solution. In the special case where $\underset{\sim}{x}^*(t) \equiv \underset{\sim}{x}^* \in \mathbb{R}^n$; i.e., $\underset{\sim}{x}^*$ is an equilibrium point, then $\mathcal{V}_\Delta(\cdot,\underset{\sim}{x}^*)$ is the standard Lyapunov function.

In the following corollary, conditions are given such that, as in linear systems, the "transients" of (15) decay exponentially to the unique steady-state solution.

Corollary A.2: Assume further in Theorem A.2 or in Corollary A.1 that there exist constants $\gamma_2 \geq \gamma_1 > 0$, $\gamma_4 \geq \gamma_3 > 0$ and $\beta > 0$ such that for all $\underset{\sim}{x}', \underset{\sim}{x}'' \in D_x$ and for all $\underset{\sim}{\xi} \in D_\xi$,

$$\gamma_1 \|\underset{\sim}{x}'-\underset{\sim}{x}''\|^\beta \leq \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'') \leq \gamma_2 \|\underset{\sim}{x}'-\underset{\sim}{x}''\|^\beta \tag{28}$$

$$\gamma_3 \|\underset{\sim}{x}'-\underset{\sim}{x}''\|^\beta \leq \frac{\partial \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}'} f(\underset{\sim}{x}',\underset{\sim}{\xi}) + \frac{\partial \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}} \underset{\sim}{f}(\underset{\sim}{x}'',\underset{\sim}{\xi}) \leq \gamma_4 \|\underset{\sim}{x}'-\underset{\sim}{x}''\|^\beta \tag{29}$$

Then for any continuous $\underset{\sim}{\xi}(\cdot)$ such that $\mathcal{R}_0\left(\underset{\sim}{\xi}(\cdot)\right) \subseteq D_\xi$ and any two solutions $\underset{\sim}{x}'(\cdot)$ and $\underset{\sim}{x}''(\cdot)$ of (15) which lie in D_x for all $t \geq 0$, we have for all $t \geq 0$,

$$\left[\frac{\gamma_1}{\gamma_2}\right]^{1/2} e^{-t/\tau_{min}} \|\underset{\sim}{x}'(0)-\underset{\sim}{x}''(0)\| \leq \|\underset{\sim}{x}'(t)-\underset{\sim}{x}''(t)\| \leq \left[\frac{\gamma_2}{\gamma_1}\right]^{1/2} e^{-t/\tau_{max}} \|\underset{\sim}{x}'(0)-\underset{\sim}{x}''(0)\| \tag{30}$$

where

$$\tau_{min} \triangleq \frac{\beta\gamma_4}{\gamma_1} \quad \text{and} \quad \tau_{max} \triangleq \frac{\beta\gamma_2}{\gamma_3} \tag{31}$$

<u>Remark</u>: When the hypotheses of <u>Theorem A.2</u> are satisfied, the constants γ_4, γ_3, γ_2, and γ_1 in (29) and (30) exist if, and only if, for constant $\beta > 0$, the expressions

$$\lim_{\|\underset{\sim}{x}'-\underset{\sim}{x}''\|\to 0} \left[\frac{1}{\|\underset{\sim}{x}'-\underset{\sim}{x}''\|^\beta} \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')\right]$$
$$\lim_{\|\underset{\sim}{x}'-\underset{\sim}{x}''\|\to 0} \left[\frac{1}{\|\underset{\sim}{x}'-\underset{\sim}{x}''\|^\beta} \frac{\partial \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}'} f(\underset{\sim}{x}',\underset{\sim}{\xi}) + \frac{\partial \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}''} \underset{\sim}{f}(\underset{\sim}{x}'',\underset{\sim}{\xi})\right] \tag{32}$$

are well-defined, positive, and bounded away from zero, for all $\underset{\sim}{x}'' \in D_x$ and for all $\underset{\sim}{\xi} \in D_\xi$.

<u>Corollary A.3</u>: In <u>Theorem A.2</u> and in <u>Corollary A.1</u>, the hypothesis (26)

may be replaced by the conditions: (i) $\underset{\sim}{\xi}(\cdot)$ satisfies a global Lipschitz condition, (ii)

$$\frac{\partial \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}} \underset{\sim}{f}(\underset{\sim}{x}',\underset{\sim}{\xi}) + \frac{\partial \mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}} \underset{\sim}{f}(\underset{\sim}{x}'',\underset{\sim}{\xi})$$

$$\geq 0, \qquad \forall\ \underset{\sim}{x}',\ \underset{\sim}{x}'' \in D_x, \qquad \forall\ \underset{\sim}{\xi} \in D_\xi \tag{33}$$

and, (iii) for any continuous $\underset{\sim}{\xi}(\cdot)$ such that $\mathcal{R}_\infty\left(\underset{\sim}{\xi}(\cdot)\right) \subseteq D_\xi$ and for any two solutions $\underset{\sim}{x}'(\cdot)$ and $\underset{\sim}{x}''(\cdot)$, there exists a time $\hat{t}_0 \in \mathbb{R}^1$ such that for any compact time interval

$$I_t \triangleq [\tau_1,\tau_2] \qquad \hat{t}_0 \leq \tau_1 < \tau_2 \tag{34}$$

we have

$$\begin{aligned} &\left[\frac{d}{dt}\mathcal{V}_\Delta\left(\underset{\sim}{x}'(t),\underset{\sim}{x}''(t)\right) = 0, \qquad \forall\ t \in I_t\right] \\ &\Longleftrightarrow \left[\underset{\sim}{x}'(t) = \underset{\sim}{x}''(t), \qquad \forall\ t \in I_t\right] \end{aligned} \tag{35}$$

Remarks: 1. Note that if the definition of I_t in (34) is changed from $\tau_1 < \tau_2$, to $\tau_1 \leq \tau_2$, then this corollary is identical to Theorem A.2 or Corollary A.1. In other words, in this corollary the hypotheses of the previous theorem and corollary are relaxed to allow $\frac{d}{dt}\mathcal{V}_\Delta\left(\underset{\sim}{x}'(t),\underset{\sim}{x}''(t)\right) = 0$ at isolated times $t \in \mathbb{R}^1$.

2. There are a number of networks where it is possible to show that there is a unique steady-state solution using this corollary while it is not possible to do so using Theorem A.2 or Corollary A.1. For example, at the end of Sec. V we show that the network of Fig. 2(c) has a unique steady-state solution using Theorem A.2, but the same conclusion is reached for the network of Fig. 2(b) only by applying Corollary A.3. An autonomous version of this corollary is discussed and applied in [15].

The next corollary is used in Example 3.

Corollary A.4: In Theorem A.2 and in Corollary A.1 the hypothesis that $\mathcal{V}_\Delta$ is C^1 and that (26) is true may be replaced by the conditions: (i) $\mathcal{V}_\Delta$ is C^0, (ii) the right-hand derivative [20] $\frac{d^+}{dt}\mathcal{V}_\Delta\left(\underset{\sim}{x}'(\cdot),\underset{\sim}{x}''(\cdot)\right)$ of $\mathcal{V}_\Delta\left(\underset{\sim}{x}'(\cdot), \underset{\sim}{x}''(\cdot)\right)$ exists for every pair of solutions $\underset{\sim}{x}'(\cdot)$ and $\underset{\sim}{x}''(\cdot)$, and (iii)

$$\begin{aligned} &\frac{d^+}{dt}\mathcal{V}_\Delta\left(\underset{\sim}{x}'(t),\underset{\sim}{x}''(t)\right) \leq 0, && \forall\ t \in \mathbb{R}^1 \\ &\left[\underset{\sim}{x}'(\cdot) \neq \underset{\sim}{x}''(\cdot)\right] \Rightarrow \left[\frac{d^+}{dt}\mathcal{V}_\Delta\left(\underset{\sim}{x}'(t),\underset{\sim}{x}''(t)\right) < 0,\ \forall\ t \in \mathbb{R}^1\right] \end{aligned} \tag{36}$$

Remark: It is also possible to extend the hypothesis of Corollary A.2 in the same way. Specifically, Eq. (29) may be rewritten in the form of (36).

An important class of nonautonomous networks consists of networks containing periodic sources. Often, the periodic sources do not generate periodic waveforms in the network. For example, examine the network of Fig. 2(a). The capacitor voltage waveform $v_C(\cdot)$ has the form (when $\omega \neq 1/\sqrt{LC}$)

$$v_C(t) = A \sin t + B \sin \frac{1}{\sqrt{LC}}\, t \tag{37}$$

where A and B are appropriate constants. Now, $v_C(\cdot)$ is periodic if, and only if, $\sqrt{LC} \cdot \omega \in \mathbb{R}^1$ is rational. The probability of this is zero (i.e. the set of rational numbers has measure zero in $\mathbb{R}^1$). Thus we expect that $v_C(\cdot)$ is not periodic. However, for every L, C and ω, $v_C(\cdot)$ is almost periodic. We make explicit the definition of an almost periodic function in the theorem which follows. The theorem also states some important properties of almost periodic functions which are used in the proof of Theorem B.3. The following three theorems are stated without proof; references include the classic work by Favard [23] and the modern treatment by Yoshizawa [18].

Theorem B.1: Let $\underset{\sim}{\xi}(\cdot)$: $\mathbb{R}^1 \to \mathbb{R}^m$ be continuous. The following statements are equivalent:

(i) $\underset{\sim}{\xi}(\cdot)$ is almost periodic

(ii) for any $\varepsilon > 0$ there exists $\ell(\varepsilon) > 0$ such that every time interval of length $\ell(\varepsilon)$ in $\mathbb{R}^1$ contains a time τ such that

$$\|\underset{\sim}{\xi}(t+\tau) - \underset{\sim}{\xi}(t)\| < \varepsilon \quad , \quad \forall\, t \in \mathbb{R}^1 \tag{38}$$

Here, τ is called the ε-translation number of $\underset{\sim}{\xi}(\cdot)$.

(iii) there exists a countable set of real numbers $\{\omega_k\}$ called Fourier exponents and a corresponding countable set of vectors in $\mathbb{R}^m$ $\{\underset{\sim}{\xi}_k\}$ called Fourier coefficients such that

$$\underset{\sim}{\xi}(t) \sim \sum_k \underset{\sim}{\xi}_k e^{j\omega_k t} \tag{39}$$

where $j \triangleq \sqrt{-1}$. Let $\mathcal{S}_\xi$ denote the countable set of real numbers which are integer combinations of the ω_k. The set $\mathcal{S}_\xi$ is called the spectrum (also known as a σ-module) of $\underset{\sim}{\xi}(\cdot)$.

(iv) For any infinite sequence of real numbers $\{t_k\}_{k=1}^{+\infty}$ there is a subsequence $\{t_{k_\ell}\}$ such that the sequence of functions $\{\underset{\sim}{\xi}(\cdot+t_{k_\ell})\}$ converges uniformly to a continuous function $\hat{\underset{\sim}{\xi}}$: $\mathbb{R}^1 \to \mathbb{R}^m$.

Remark: While (ii) is the formal definition of an almost periodic function, much insight can be gained by examining (iii): Eq. (39) states that the continuous $\underset{\sim}{\xi}(\cdot)$ can be uniformly approximated by the C^∞-summation on the right side of (39). Thus, for example, $v_C(\cdot)$ in (37) is almost periodic. The relationship between almost periodic and periodic functions is clear using statement (iii); an almost periodic function $\underset{\sim}{\xi}(\cdot)$ is periodic if, and only if, for any integers k and ℓ, ω_k/ω_ℓ is rational. Carrying this comparison further, we interpret the spectrum $\mathcal{S}_\xi$ as the set of harmonics generated by the "frequencies" ω_k. Note also that the continuous almost periodic function $\xi(\cdot)$ is uniformly continuous and bounded.

We know for the linear networks of Fig. 2(b) and Fig. 2(c) that all current and voltage waveforms converge to a (unique) periodic waveform. That is, the waveforms are "asymptotically" periodic. This concept may be extended to almost periodic waveforms and a formal definition is given in the following theorem due to Yoshitawa [18].

Theorem B.2: Let $\xi(\cdot)$: $\mathbb{R}^1 \to \mathbb{R}^m$ be continuous. The following statements are equivalent:

(i) $\xi(\cdot)$ is asymptotically almost periodic.

(ii) $\xi(\cdot)$ may be (uniquely) decomposed in the following way

$$\xi(t) = \xi_0(t) + \xi_T(t), \qquad \forall\ t \in \mathbb{R}^1 \tag{40}$$

where $\xi_0(\cdot)$ is continuous and almost periodic, $\xi_T(\cdot)$ is continuous and $\lim_{t\to+\infty} \xi_T(t) = 0$.

(iii) For every $\varepsilon > 0$ there exists $\ell(\varepsilon) > 0$ and $T_\varepsilon \in \mathbb{R}^1$ such that every time interval of length $\ell(\varepsilon)$ in $\mathbb{R}^1$ contains a time τ such that

$$\| \xi(t+\tau) - \xi(t)\| < \varepsilon, \qquad \forall\ t \geq T_\varepsilon \tag{41}$$

(iv) For any infinite sequence of real numbers $\{t_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} t_k = +\infty$ there exists an infinite subsequence $\{t_{k_\ell}\}$ such that the sequence of functions $\{\xi(\cdot+t_{k_\ell})\}$ converges uniformly to a continuous function $\hat{\xi}(\cdot)$: $\mathbb{R}^1 \to \mathbb{R}^m$.

We next examine the behavior of the solutions of (15) when $\xi(\cdot)$ is periodic, almost periodic, or asymptotically almost periodic, and (15) has a unique steady-state solution. We begin with the observation that if $\xi(\cdot)$ is T-periodic, and (15) has a unique steady-state solution, then every solution $x(\cdot)$ of (15) is asymptotically T-periodic. This conclusion comes from Theorem A.1. We can extend this observation to the case where $\xi(\cdot)$ is asymptotically almost periodic:

Theorem B.3: [12], [18] Assume $\xi(\cdot)$ is asymptotically almost periodic and is Lipschitz continuous. Assume (15) has a unique steady-state solution. Then every solution $x(\cdot)$ of (15) is asymptotically almost periodic, and in the steady state $\mathcal{S}_x \subseteq \mathcal{S}_\xi$.

Remarks: 1. This theorem is proved by Shaeffer [12] under the additional assumption that there exists an Incremental Lyapunov Function $\mathcal{V}_\Delta$ satisfying the conditions of Theorem A.2. In [18] Yoshizawa proves the theorem in the more general case, however he assumes that $\xi(\cdot)$ is almost periodic rather than asymptotically almost periodic. His proof uses the equivalence of (i) and (iv) of Theorem B.1 and by the similar equivalence of (i) and (iv) of Theorem B.2 we obtain the theorem as presented here. Note that since $\xi(\cdot)$ is Lipschitz continuous and is asymptotically almost periodic, it satisfies a global Lipschitz condition.

2. The conclusion that $\mathcal{S}_x \subseteq \mathcal{S}_\xi$ in the steady-state means that when we partition the asymptotically almost periodic functions $x(\cdot)$ and $\xi(\cdot)$ as in (ii) of Theorem B.2, then $\mathcal{S}_{x_0} \subseteq \mathcal{S}_{\xi_0}$. As in the periodic case, the conclusion may be interpreted as stating that the "harmonic content" of $x_0(\cdot)$ contains no component not found in the harmonic content of $\xi_0(\cdot)$.

The following mathematical theorem is discussed and proved in [15]:

Theorem C: Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-strictly-increasing diffeomorphic state function mapping $\mathbb{R}^n$ onto $\mathbb{R}^n$. Define $F: \mathbb{R}^n \to \mathbb{R}^1$ to be the unique C^2-function such that

$$\nabla F(\underset{\sim}{x}) = \underset{\sim}{f}(\underset{\sim}{x}), \qquad \forall\, \underset{\sim}{x} \in \mathbb{R}^n; \qquad F\left(\underset{\sim}{f}^{-1}(\underset{\sim}{0})\right) = 0 \tag{42}$$

Then the following properties hold:

A-1

F(·) is a strictly-convex function[6]

A-2

$$F(\underset{\sim}{x}) > 0 \qquad \forall\, \underset{\sim}{x} \neq \underset{\sim}{f}^{-1}(\underset{\sim}{0}) \tag{43}$$

A-3

$$\lim_{\|\underset{\sim}{x}\|\to\infty} \frac{1}{\|\underset{\sim}{x}\|} F(\underset{\sim}{x}) = +\infty \tag{44}$$

A-4

$$\lim_{\|\underset{\sim}{x}\|\to\infty} \frac{1}{\|\underset{\sim}{x}\|} \underset{\sim}{x}^T \underset{\sim}{f}(\underset{\sim}{x}) = +\infty \tag{45}$$

A-5 For each k > 0, the set

$$K \triangleq \{x \in \mathbb{R}^n:\ F(\underset{\sim}{x}) \leq k\} \tag{46}$$

is compact and convex[6] in $\mathbb{R}^n$.

IV. Networks with Eventually Uniformly Bounded Solutions

Theorem 1: Assume the nonlinear dynamic network $\mathcal{N}$ is described by the state equation (8). Assume the capacitor-inductor function $\underset{\sim}{h}_p$ is a C^1-state function,[7] and there exists a C^2-function $H_p:\ \mathbb{R}^{n_p} \to \mathbb{R}^1$ such

[6]A function F: $\mathbb{R}^n \to \mathbb{R}^1$ is strictly convex if, and only if, for each $\sigma \in (0,1)$, for each pair $\underset{\sim}{x}', \underset{\sim}{x}'' \in \mathbb{R}^n$,

$$F((1-\sigma)\underset{\sim}{x}'+\underset{\sim}{x}'') < (1-\sigma)F(\underset{\sim}{x}') + F(\underset{\sim}{x}'')$$

A set $S \subseteq \mathbb{R}^n$ is convex if, and only if, for each $\sigma \in (0,1)$, for each pair $\underset{\sim}{x}', \underset{\sim}{x}'' \in S$, $\underset{\sim}{x}_\sigma \triangleq (1-\sigma)\underset{\sim}{x}' + \sigma\underset{\sim}{x}'' \in S$.

[7]The condition that h_p is a state function is equivalent to requiring that the capacitors and inductors be reciprocal. This is a weak condition and is satisfied by most capacitors and inductors of practical interest. This assumption is made throughout this paper.

that $\nabla H_p(\underset{\sim}{z}_p) \equiv \underset{\sim}{h}_p(\underset{\sim}{z}_p)$. Let $\underset{\sim}{h}_p$ and $\underset{\sim}{H}_p$ satisfy

$$\lim_{\|\underset{\sim}{z}_p\|\to\infty} \underset{\sim}{h}_p(\underset{\sim}{z}_p) = +\infty$$
$$\lim_{\|\underset{\sim}{z}_p\|\to\infty} H_p(\underset{\sim}{z}_p) = +\infty \tag{47}$$

Let $k_1 > 0$ and assume $\underset{\sim}{u}_S(\cdot)$: $\mathbb{R}^1 \to \mathbb{R}^{n_S}$ is continuous, satisfying $\|\underset{\sim}{u}_S(t)\| \leq k_1$ for all $t \in \mathbb{R}^1$. Under these conditions,

1. If $\underset{\sim}{g}_p(\cdot,\underset{\sim}{u}_S)$ is eventually passive for all $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$ such that $\|\underset{\sim}{u}_S\| \leq k_1$, then every solution $\underset{\sim}{z}_p(\cdot)$ of (8) is bounded.

2. If $\underset{\sim}{g}_p(\cdot,\underset{\sim}{u}_S)$ is eventually <u>strictly</u> passive for all $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$ such that $\|\underset{\sim}{u}_S\| \leq k_1$, then the solutions of (8) are eventually uniformly bounded. In particular, let $k > 0$ be the constant such that

$$\underset{\sim}{x}_p^T\underset{\sim}{g}_p(\underset{\sim}{x}_p,\underset{\sim}{u}_S) > 0 \quad , \quad \forall\ \|\underset{\sim}{x}_p\| > k, \quad \forall\ \|\underset{\sim}{u}_S\| \leq k_1 \tag{48}$$

then there exists a positive constant k_0 such that

$$\left[\|\underset{\sim}{z}_p\| > k_0\right] \Rightarrow \left[\|\underset{\sim}{x}_p\| = \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)\| > k\right] \tag{49}$$

and, if we define the constant

$$\hat{k}_0 \triangleq \sup_{\|\underset{\sim}{z}_p\| \leq k_0} H_p(\underset{\sim}{z}_p) \tag{50}$$

then there exists a compact set

$$Z_p \triangleq \{\underset{\sim}{z}_p \in \mathbb{R}^{n_p}:\ H_p(\underset{\sim}{z}_p) \leq \hat{k}_0\} \tag{51}$$

such that for any solution $\underset{\sim}{z}_p(\cdot)$ of (8) there exists a time $t_0 \in \mathbb{R}^1$ such that

$$\underset{\sim}{z}_p(t) \in Z_p \quad , \quad \forall\, t \geq t_0 \tag{52}$$

Furthermore, if $\underset{\sim}{u}_S(\cdot)$ is periodic with period T, then (8) has a periodic solution with period T.

Proof: First, from (47) we conclude that for every $k > 0$ there exists a $k_0 > 0$ such that (49) is true, and also from (47) we conclude that $\hat{k}_0$ in (50) is well-defined and that Z_p in (51) is compact. The proof comes directly from Theorem A.1 where $\mathcal{V}(\cdot) = H_p(\cdot)$. ∎

Remark: From Theorem C we know that (47) is satisfied if $\underset{\sim}{h}_p$ is a C^1-strictly increasing diffeomorphism mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$.

We next examine conditions placed upon the resistors of $\mathcal{N}$ such that $\underset{\sim}{g}_p$ has the appropriate properties of Theorem 1. First, we note that even if each resistor function $\underset{\sim}{g}_{R_\alpha}(\cdot)$ of $\mathcal{N}$ is eventually (strictly) passive, the composite resistor function $\underset{\sim}{g}_R(\cdot)$ may not be eventually (strictly) passive. This fact is illustrated by the two resistor v - i curves of Fig. 3; assume $\mathcal{N}$ contains solely the two resistors of Fig. 3. Resistor R_1 is eventually strictly passive, while R_2 is strictly passive. Yet $\underset{\sim}{g}_R = \begin{pmatrix} g_{R_1} \\ g_{R_2} \end{pmatrix}$ is not eventually strictly passive. To show this, fix $v_{R_1} = 3/2$, then

$$\underset{\sim}{v}_R^T \underset{\sim}{g}_R(\underset{\sim}{v}_R) = \begin{cases} -9/4 + (v_{R_2})^2 & , \quad \forall\, |v_{R_2}| \leq 1 \\ -9/4 + \dfrac{1}{(v_{R_2})^2} & , \quad \forall\, |v_{R_2}| > 1 \end{cases}$$
$$< 0 \quad , \quad \forall\, v_{R_2} \in R^1 \tag{53}$$

The reason that $\underset{\sim}{g}_R$ is not eventually strictly passive is because while R_2 is strictly passive $|v_{R_2} i_{R_2}| \leq 1$ for all v_{R_2}. It is shown in [14] that if

$\lim_{\|\underset{\sim}{x}_{R_\alpha}\| \to \infty} [\underset{\sim}{x}_{R_\alpha}]^T [\underset{\sim}{g}_{R_\alpha}(\underset{\sim}{x}_{R_\alpha})] = +\infty$ for each $\alpha = 1,2,\ldots,m_R$, then indeed $\underset{\sim}{g}_R$ is eventually strictly passive. When the network contains constant independent sources, it is shown further in [14] that the requirement for $\underset{\sim}{g}_R$ to be eventually strictly passive is $\lim_{\|\underset{\sim}{x}_{R_\alpha}\| \to \infty} \frac{1}{\|\underset{\sim}{x}_{R_\alpha}\|} (\underset{\sim}{x}_{R_\alpha})^T \underset{\sim}{g}_{R_\alpha}(\underset{\sim}{x}_{R_\alpha}) = +\infty$. However, with conditions of this form, it is no longer possible to prescribe an eventually passive $\underset{\sim}{g}_R$ that is not eventually strictly passive. Hence, in the following theorem, we prove only that $\underset{\sim}{g}_p$ is eventually strictly passive as in (ii) of Theorem 1.

Theorem 2: Assume the dynamic nonlinear network $\mathcal{N}$ is described by the state equation (8). Assume the capacitor-inductor function $\underset{\sim}{h}_p$ is a C^1-state function, and there exists a C^2-function $H_p: \mathbb{R}^{n_p} \to \mathbb{R}^1$ such that $\nabla H_p(\underset{\sim}{z}_p) \equiv \underset{\sim}{h}_p(\underset{\sim}{z}_p)$. Let $\underset{\sim}{h}_p$ and H_p satisfy

$$\lim_{\|\underset{\sim}{z}_p\| \to \infty} \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)\| = +\infty$$

$$\lim_{\|\underset{\sim}{z}_p\| \to \infty} H_p(\underset{\sim}{z}_p) = +\infty \tag{54}$$

Assume further that $\mathcal{N}$ satisfies the Fundamental Topological Assumption. Under these conditions if each resistor function $\underset{\sim}{g}_{R_\alpha}$ is eventually strictly passive, satisfying

$$\lim_{\|\underset{\sim}{x}_{R_\alpha}\| \to \infty} \frac{1}{\|\underset{\sim}{x}_{R_\alpha}\|} (\underset{\sim}{x}_{R_\alpha})^T \underset{\sim}{g}_{R_\alpha}(\underset{\sim}{x}_{R_\alpha}) = +\infty \tag{55}$$

then all voltage and current waveforms of $\mathcal{N}$ are eventually uniformly bounded. Furthermore, if the sources of $\mathcal{N}$ are periodic with period T, then $\mathcal{N}$ has a T-periodic solution.

Proof: This follows from Theorem 1 and Theorems 8 and 9 of [14]. ∎

In special cases it is possible to relax the condition (55) on the resistor functions $\underset{\sim}{g}_{R_\alpha}$.

Corollary 1: Assume in Theorem 2 that $\mathcal{N}$ contains only voltage-controlled resistors (resp., current-controlled resistors), i.e., for each resistor R_α, $\underset{\sim}{x}_{R_\alpha} = \underset{\sim}{v}_{R_\alpha}$ (resp., $\underset{\sim}{x}_{R_\alpha} = \underset{\sim}{i}_{R_\alpha}$). Further assume that all independent sources of $\mathcal{N}$ are voltage-sources (resp., current-sources). Then condition (55) may be replaced by

$$\lim_{\|\underset{\sim}{x}_{R_\alpha}\|\to\infty} (\underset{\sim}{x}_{R_\alpha})^T \underset{\sim}{g}_{R_\alpha}(\underset{\sim}{x}_{R_\alpha}) = +\infty \tag{56}$$

Proof: This proof also follows from Theorem 1 and Theorems 8 and 9 of [14]. ∎

Theorem 2 and Corollary 1 are discussed (in their autonomous form) extensively in [15]. In this paper they are used in conjunction with theorems establishing the existence of a unique steady-state solution. We present below two examples which are re-examined in Sec. V.

Example 1: Examine the transistor circuit of Fig. 4. The transistor may be modeled as a grounded two-port resistor using the Ebers-Moll equation [8]. Let i_E and v_E be the current and voltage respectively of the emitter-base junction, and let i_C and v_C be the current and voltage respectively of the collector-base junction. The resistive two-port is described by its constitutive relation:

$$\begin{pmatrix} i_E \\ i_C \end{pmatrix} = \underset{\sim}{g}_{tr}\begin{pmatrix} v_E \\ v_C \end{pmatrix} \triangleq \begin{bmatrix} 1 & -\alpha_R \\ -\alpha_F & 1 \end{bmatrix}\begin{pmatrix} I_{ES}(e^{v_E/V_T}-1) \\ I_{CS}(e^{v_C/V_T}-1) \end{pmatrix} \tag{57}$$

where the subscript "tr" denotes transistor. In (57), I_{ES}, I_{CS}, α_R, V_T, and α_F are positive constants, and furthermore $\alpha_R < 1$, $\alpha_F < 1$, and $\alpha_R I_{CS} = \alpha_F I_{ES}$. Now, it can easily be shown that

$$\begin{pmatrix} v_E \\ v_C \end{pmatrix}^T \underset{\sim}{g}_{tr} \begin{pmatrix} v_E \\ v_C \end{pmatrix} = (1-\alpha_F) I_{ES} v_E (e^{v_E/V_T} - 1) + (1-\alpha_R) I_{CS} v_C (e^{v_C/V_T} - 1)$$
$$+ \alpha_R I_{CS} (v_E - v_C)(e^{v_E/V_T} - e^{v_C/V_T}) \tag{58}$$

Hence, we conclude (see [15]) that $\underset{\sim}{g}_{tr}$ is strictly passive and satisfies (56). The state equation of the network can be written in the form (8), where

$$\begin{pmatrix} v_{C_1} \\ v_{C_2} \\ v_{C_3} \end{pmatrix} = \underset{\sim}{h}_p \begin{pmatrix} q_{C_1} \\ q_{C_2} \\ q_{C_3} \end{pmatrix} = \begin{bmatrix} \frac{1}{C_1} & 0 & 0 \\ 0 & \frac{1}{C_2} & 0 \\ 0 & 0 & \frac{1}{C_3} \end{bmatrix} \begin{pmatrix} q_{C_1} \\ q_{C_2} \\ q_{C_3} \end{pmatrix} \tag{59}$$

and

$$\begin{pmatrix} i_{C_1} \\ i_{C_2} \\ i_{C_3} \end{pmatrix} = -\underset{\sim}{g}_p \left(\begin{pmatrix} v_{C_1} \\ v_{C_2} \\ v_{C_3} \end{pmatrix}, \begin{pmatrix} E_S(t) \\ E_1 \\ E_2 \end{pmatrix} \right)$$

$$= - \begin{bmatrix} G_1+G_2+G_3+G_5+G_6 & -G_5-G_6 & G_6 \\ -G_5-G_6 & G_4+G_5+G_6 & -G_6 \\ G_6 & -G_6 & G_6+G_7 \end{bmatrix} \begin{pmatrix} v_{C_1} \\ v_{C_2} \\ v_{C_3} \end{pmatrix}$$
$$- \begin{bmatrix} 1 & -\alpha_r \\ -\alpha_F & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} I_{ES}(e^{v_{C_1}/V_T} - 1) \\ I_{CS}(e^{v_{C_2}/V_T} - 1) \end{pmatrix} - \begin{bmatrix} G_1 & G_2 & G_5 \\ 0 & 0 & -G_5 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} E_S(t) \\ E_1 \\ E_2 \end{pmatrix} \tag{60}$$

We apply <u>Corollary 1</u> and conclude that if $E_S(\cdot)$ is continuous and bounded, then the voltage and current waveforms of the network of Fig. 4 are eventually uniformly bounded. Furthermore, if $E_S(\cdot)$ is periodic with period T, then there is a T-periodic solution.

Example 2: The network of Fig. 5. is a straight-forward illustration of Theorem 2. Since each network element is described by a one-dimensional C^∞-strictly-increasing bijective function, the hypotheses (54) and (55) of Theorem 2 follow immediately. Thus, the voltage and current waveforms of the network are eventually uniformly bounded. Furthermore, there is a periodic solution with frequency $\omega = 1$. In Sec. V we display a periodic solution. We also present an "almost subharmonic" solution.

In the next two theorems and corollary, the methods used in Theorems 1 and 2 are extended to show that under certain conditions a "small signal" $\underset{\sim}{u}_S(\cdot)$ yields a "small signal" $\underset{\sim}{z}_p(\cdot)$. The main result is:

Theorem 3: Assume the nonlinear dynamic network $\mathcal{N}$ is described by the state equation (8). Let the capacitor-inductor function $\underset{\sim}{h}_p$ be a C^1-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$. Let the function $\underset{\sim}{g}_p(\cdot,\underset{\sim}{u}_S)$ be a strictly increasing eventually strictly passive C^1-diffeomorphism mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$ for all $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$.

Under these conditions, for any $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$, and any $\varepsilon > 0$ there exists $\delta > 0$ and a unique $\underset{\sim}{z}_p^* \in \mathbb{R}^{n_S}$ such that $\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p^*),\underset{\sim}{u}_S^*\right) = \underset{\sim}{0}$. Moreover, for any continuous and bounded $\underset{\sim}{u}_S(\cdot)$ satisfying[8]

$$\left\|\mathcal{R}_\infty\left(\underset{\sim}{u}_S(\cdot)\right) - \underset{\sim}{u}_S^*\right\| < \delta \tag{61}$$

the corresponding solution $\underset{\sim}{z}_p(\cdot)$ satisfies

[8] $\mathcal{R}_\infty\left(\underset{\sim}{u}_S(\cdot)\right)$ is the ultimate range of $\underset{\sim}{u}_S(\cdot)$ (Def. 7). Equation (61) has the following interpretation: for any $u_S \in \mathcal{R}_\infty\left(u_S(\cdot)\right)$, we have $\|u_S-u_S^*\| < \delta$.

$$\| \mathcal{R}_\infty\left(z_p(\cdot)\right) - z_p^*\| < \varepsilon \tag{62}$$

regardless of the initial conditions.

Before we present the lengthy proof of this theorem, it is instructive to present first a corollary and some remarks.

Corollary 2: Assume the functions h_p and g_p satisfy the hypotheses of Theorem 3. Then for every $u_S^* \in \mathbb{R}^{n_S}$ there exists a unique $z_p \in \mathbb{R}^{n_S}$ such that $g_p\left(h_p(z_p^*), u_S^*\right) = 0$. Moreover, for any continuous $u_S(\cdot)$ such that

$$\lim_{t\to\infty} u_S(t) = u_S^* \tag{63}$$

the corresponding solution $z_p(\cdot)$ satisfies

$$\lim_{t\to\infty} z_p(t) = z_p^* \tag{64}$$

regardless of the initial conditions.

Remarks: 1. The C^1-strictly increasing diffeomorphism $g_p(\cdot, u_S)$ mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$ is eventually strictly passive if it is a state function (Theorem C) or if it is a uniformly increasing function (see [14]).

2. Corollary 2 is an extension of Theorem 5 of [15] where the same conclusion is found assuming $u_S(t) \equiv u_S^*$, and without assuming that $g_p(\cdot, u_S)$ is eventually strictly passive for all $u_S \in \mathbb{R}^{n_S}$. Note that z_p^* is not an equilibrium point of (8) unless $u_S(t) \equiv u_S^*$. That is, $z_p(t) \equiv z_p^*$ is not a solution of (8) which is driven by a time-dependent input $u_S(t)$.

Proof of Theorem 3: First, we conclude that there is a unique $z_p^* \in \mathbb{R}^{n_p}$ such that $g_p\left(h_p(z_p^*), u_S^*\right) = 0$. This is because the function $g_p\left(h_p(\cdot), u_S^*\right)$ is a composite of two bijections and is therefore itself a bijection. Next we see that the hypotheses of Theorem 1 are satisfied and thus the solutions of (8) are eventually uniformly bounded. That is, there exists a compact set $Z_p \subseteq \mathbb{R}^{n_p}$ such that for every solution $\underset{\sim}{z}_p(\cdot)$ of (8) there is a time $t_1 \in \mathbb{R}^1$ such that $\underset{\sim}{z}_p(t) \in Z_p$ for all $t \geq t_1$.

In order to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that (61) implies (62) we first derive the constants $\gamma_1 > 0$, $\gamma_2 > 0$ and $\varepsilon' > 0$, time $t_0 \in \mathbb{R}^1$ and the compact set $Z_p(\varepsilon') \subseteq Z_p$: First assume $\delta > 0$ is known and (61) is true. Since $\mathcal{R}_\infty\left(\underset{\sim}{u}_S(\cdot)\right)$ is closed, there exists time $t_0 \in \mathbb{R}^1$ such that

$$\|\underset{\sim}{u}_S(\cdot) - \underset{\sim}{u}_S^*\| < \delta, \qquad \forall\, t \geq t_0 \tag{65}$$

Furthermore, assume for any particular solution $\underset{\sim}{z}_p(\cdot)$ in this proof that $t_0 \geq t_1$; i.e., for any particular solution $\underset{\sim}{z}_p(\cdot)$, we can assume also that $\underset{\sim}{z}_p(t) \in Z_p$ for all $t \geq t_0$. Next, using <u>Theorem C</u> we conclude that Z_p (in (51) of <u>Theorem 1</u>) is also convex. Thus the strictly increasing function $\underset{\sim}{h}_p(\cdot)$ and the strictly increasing function $\underset{\sim}{g}_p(\cdot,\underset{\sim}{u}_S)$ are <u>uniformly increasing</u> in Z_p[14]. More specifically, there exists constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that [14];

$$(\underset{\sim}{x}_p'-\underset{\sim}{x}_p'')^T\left(\underset{\sim}{g}_p(\underset{\sim}{x}_p',\underset{\sim}{u}_S)-\underset{\sim}{g}_p(\underset{\sim}{x}_p'',\underset{\sim}{u}_S)\right) \geq \gamma_1\|\underset{\sim}{x}_p'-\underset{\sim}{x}_p''\|^2$$

$$\forall\, \underset{\sim}{h}_p^{-1}(\underset{\sim}{x}_p'),\ \underset{\sim}{h}_p^{-1}(\underset{\sim}{x}_p'') \in Z_p \tag{66}$$

$$\forall\, \|\underset{\sim}{u}_S-\underset{\sim}{u}_S^*\| < \delta$$

and

$$(\underset{\sim}{z}_p'-\underset{\sim}{z}_p'')^T\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p')-\underset{\sim}{h}_p(\underset{\sim}{z}_p'')\right) \geq \gamma_2\|\underset{\sim}{z}_p'-\underset{\sim}{z}_p''\|^2$$

$$\forall\, \underset{\sim}{z}_p',\ \underset{\sim}{z}_p'' \in Z_p \tag{67a}$$

from which we obtain [15]

$$\|\underset{\sim}{h}_p(\underset{\sim}{z}_p')-\underset{\sim}{h}_p(\underset{\sim}{z}_p'')\| \geq \gamma_2\|\underset{\sim}{z}_p'-\underset{\sim}{z}_p''\| \quad , \quad \forall\ \underset{\sim}{z}_p',\underset{\sim}{z}_p'' \in Z_p \tag{67b}$$

Because $\underset{\sim}{g}_p$ is continuous, for every $\varepsilon' > 0$ there exists $\delta > 0$ such that for all $\|\underset{\sim}{u}_S-\underset{\sim}{u}_S^*\| < \delta$,

$$\left\|\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p^*),\underset{\sim}{u}_S\right) - \underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p^*),\underset{\sim}{u}_S^*\right)\right\| = \left\|\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p^*),\underset{\sim}{u}_S\right)\right\| < \varepsilon' \tag{68}$$

We now reapply Theorem A.1; the function $\underset{\sim}{h}_p(\cdot) - \underset{\sim}{h}_p(\underset{\sim}{z}_p^*)$ is a C^1-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$, hence using Theorem C we can define the C^2-function $\mathcal{V}$: $\mathbb{R}^{n_p} \to \mathbb{R}^1$ such that $\nabla\mathcal{V}(\underset{\sim}{z}_p) \equiv \underset{\sim}{h}_p(\underset{\sim}{z}_p) - \underset{\sim}{h}_p(\underset{\sim}{z}_p^*)$. From Theorem C we conclude that (20) of Theorem A.1 is satisfied. We have only to find a constant $k_0 \geq 0$ such that (23) is true.

Using Eqs. (66), (67) and (68),

$$\begin{aligned}
\frac{\partial\mathcal{V}(\underset{\sim}{z}_p)}{\partial \underset{\sim}{z}_p}\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p),\underset{\sim}{u}_S\right) &= \left[\underset{\sim}{h}_p(\underset{\sim}{z}_p)-\underset{\sim}{h}_p(\underset{\sim}{z}_p^*)\right]^T\left[\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p),\underset{\sim}{u}_S\right)\right] \\
&= \left[\underset{\sim}{h}_p(\underset{\sim}{z}_p)-\underset{\sim}{h}_p(\underset{\sim}{z}_p^*)\right]^T\left[\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p),\underset{\sim}{u}_S\right) - \underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p^*),\underset{\sim}{u}_S\right)\right] \\
&\quad + \left[\underset{\sim}{h}_p(\underset{\sim}{z}_p)-\underset{\sim}{h}_p(\underset{\sim}{z}_p^*)\right]^T\left[\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p^*),\underset{\sim}{u}_S\right)\right] \\
&\geq \gamma_1\|\underset{\sim}{h}_p(\underset{\sim}{z}_p)-\underset{\sim}{h}_p(\underset{\sim}{z}_p^*)\|^2 - \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)-\underset{\sim}{h}_p(\underset{\sim}{z}_p^*)\|\varepsilon' \\
&\geq \|\underset{\sim}{h}_p(\underset{\sim}{z}_p)-\underset{\sim}{h}_p(\underset{\sim}{z}_p^*)\|\left[\gamma_1\gamma_2\|\underset{\sim}{z}_p-\underset{\sim}{z}_p^*\| - \varepsilon'\right] \\
&> 0 , \qquad \forall\ \|\underset{\sim}{z}_p-\underset{\sim}{z}_p^*\| > \frac{\varepsilon'}{\gamma_1\gamma_2},\ \underset{\sim}{z}_p \in Z_p \\
&\qquad\qquad \forall\ \|\underset{\sim}{u}_S-\underset{\sim}{u}_S^*\| < \delta
\end{aligned} \tag{69}$$

Define the positive constant

$$k(\varepsilon') \triangleq \max_{\|\underset{\sim}{z}_p-\underset{\sim}{z}_p^*\| \leq \frac{\varepsilon'}{\gamma_1\gamma_2}} \mathcal{V}(\underset{\sim}{z}_p) \tag{70}$$

and the compact set in $\mathbb{R}^{n_p}$

$$Z_p(\varepsilon') \triangleq \{\underset{\sim}{z}_p: \mathcal{V}(\underset{\sim}{z}_p) \leq k(\varepsilon')\} \tag{71}$$

It follows from Theorem A.1 that

$$\mathcal{R}_\infty\left(\underset{\sim}{z}_p(\cdot)\right) \subseteq Z_p(\varepsilon') \tag{72}$$

We now prescribe $Z_p(\varepsilon')$ such that (61) implies (62): For every $\varepsilon > 0$ define the positive constant

$$k(\varepsilon) \triangleq \min_{\|\underset{\sim}{z}_p - \underset{\sim}{z}_p^*\| \leq \varepsilon} \mathcal{V}(\underset{\sim}{z}_p)$$

with γ_1 and γ_2 defined in (66) and (67). Pick $\varepsilon' > 0$ such that for $k(\varepsilon')$ defined in (70), $k(\varepsilon') \leq k(\varepsilon)$. With ε' so chosen, find $\delta > 0$ such that (68) is true. Thus, from (69) and (72), we conclude that (62) is true. ■

The following theorem is similar to Theorem 3 and Corollary 2 but with the mathematical conditions on $\underset{\sim}{g}_p(\cdot, \underset{\sim}{u}_S)$ replaced by more circuit-theoretic conditions involving the constitutive relations of the internal resistors and topological constraints.

Theorem 4: Assume in the dynamic nonlinear network $\mathcal{N}$ that the capacitor-inductor function $\underset{\sim}{h}_p$ is a C^1-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$. Assume $\mathcal{N}$ satisfies the Fundamental Topological Assumption and let each resistor function $\underset{\sim}{g}_{R_\alpha}$ be a C^1-strictly increasing homeomorphism mapping $\mathbb{R}^{n_\alpha}$ onto $\mathbb{R}^{n_\alpha}$, satisfying

$$\lim_{\|\underset{\sim}{x}_{R_\alpha}\| \to \infty} \frac{1}{\|\underset{\sim}{x}_{R_\alpha}\|} \left[(\underset{\sim}{x}_{R_\alpha})^T \underset{\sim}{g}_{R_\alpha}(\underset{\sim}{x}_{R_\alpha})\right] = +\infty \tag{74}$$

Under these conditions, the state equation (8) describing $\mathcal{N}$ exists, and

1. For any $\underset{\sim}{u}_S^* \in \mathbb{R}^{n_S}$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that

for any continuous bounded $\underset{\sim}{u}_S(\cdot)$ satisfying

$$\| \mathcal{R}_\infty\left(\underset{\sim}{u}_S(\cdot)\right) - \underset{\sim}{u}_S^* \| < \delta \tag{75}$$

there is a unique $\underset{\sim}{z}_p^* \in \mathbb{R}^{n_S}$ such that $\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p^*), \underset{\sim}{u}_S^*\right) = \underset{\sim}{0}$. Moreover, the corresponding solution $\underset{\sim}{z}_p(\cdot)$ satisfies

$$\| \mathcal{R}_\infty\left(\underset{\sim}{z}_p(\cdot)\right) - \underset{\sim}{z}_p^* \| < \varepsilon \tag{76}$$

regardless of the initial conditions.

2. If, in addition $\lim_{t\to\infty} \underset{\sim}{u}_S(t) = \underset{\sim}{u}_S^*$, then the corresponding solution $\underset{\sim}{z}_p(\cdot)$ satisfies $\lim_{t\to\infty} \underset{\sim}{z}_p(t) = \underset{\sim}{z}_p^*$ regardless of the initial conditions. Furthermore, every voltage and current waveform of $\mathcal{N}$ asymptotically converges to a unique constant.

Proof: This theorem follows directly from Theorem 3, Corollary 1, and Theorems 8 and 9 in [14]. ∎

Remarks: 1. As we have previously noted, (74) is true if the function $\underset{\sim}{g}_{R_\alpha}$ is (in addition to the other conditions of Theorem 4) either a uniformly increasing function or a state function. This remark also applies to Theorems 6, 8 and 10 which follow.

2. Using Theorem 11 of [14] we may relax the condition that $\mathcal{N}$ satisfies the Fundamental Topological Assumption to allow loops of capacitors and constant voltage sources, and cutset of inductors and constant current sources. Indeed, this remark also applies to Theorems 6, 8 and 10 which follow. For a complete discussion of this extension of these theorems, see [14] and [15].

V. Networks with Unique Steady-State Solutions

In this section we apply Theorem A.2 and Corollaries A.1-A.4 to establish that a variety of nonlinear dynamic networks have a unique steady-state solution. In each of these theorems it is required that the resistor functions g_{R_α} are strictly increasing. We shall see that this condition leads to an intuitive application of Theorem A.2 and its corollaries. Before we derive these theorems, however, it is instructive to point out that Theorem A.2 may be applied to networks whose resistors are not strictly monotone. For example, Sandberg [9] (this article is also reprinted in [5]) uses a form of Corollary A.4 to show that networks containing transistors, linear resistor and voltage sources have unique steady-state solutions (the network dynamics are due to nonlinear capacitors which are intrinsic parts of the transistor model) when certain conditions are satisfied. Sandberg's method can be easily extended to allow external capacitors and inductors which are not intrinsic in the device's circuit model. The following example is a case in point.

Example 3: Examine the network of Fig. 4 where the transistor is described by the Ebers-Moll equation (57), and the network state equation is given (implicitly) by (59) and (60).

Claim: If

$$G_6 < \min[G_1+G_2+G_3,\ G_4,\ G_7] \tag{77}$$

then for any bounded and continuous $E_S(\cdot)$, the network has a unique steady-state solution. That is, each solution $\left[q_{C_1}(\cdot),\ q_{C_2}(\cdot),\ q_{C_3}(\cdot)\right]^T \triangleq \underset{\sim}{z}_p(\cdot)$ of (59) and (60) converges to a unique steady-state solution. Furthermore, if $E_S(\cdot)$ is Lipschitz continuous and asymptotically almost periodic, then each solution $\underset{\sim}{z}_p(\cdot)$ is asymptotically almost periodic, and in the steady-state, $\mathcal{S}_{z_p} \subseteq \mathcal{S}_{E_S}$.

Remark: In this claim as well as in the following theorems, we will show that the solutions of the state equations converge to a unique steady-state solution. From the assumptions discussed in the Introduction (e.g., the Fundamental Topological Condition) this leads us to conclude that each voltage and current waveform converges to a unique steady-state. This conclusion is abbreviated by the phrase "the network has a unique steady-state solution." In the same way, the conclusion that $\mathcal{S}_{z_p} \subseteq \mathcal{S}_{E_S}$ in the steady-state implies that for each waveform of the network we have the same conclusion.

Proof: We have shown in Example 1 that the solutions of this network are eventually uniformly bounded. We apply Corollary A.4: Define for each $\underset{\sim}{z}'_p = \left[q'_{C_1}, q'_{C_2}, q'_{C_3}\right]^T$ and $\underset{\sim}{z}''_p = \left[q''_{C_1}, q''_{C_2}, q''_{C_3}\right]^T$ the Incremental Lyapunov Function $\mathcal{V}_\Delta: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^1$

$$\mathcal{V}_\Delta(\underset{\sim}{z}'_p, \underset{\sim}{z}''_p) \triangleq |q'_{C_1} - q''_{C_1}| + |q'_{C_2} - q''_{C_2}| + |q'_{C_3} - q''_{C_3}| \tag{78}$$

For any differential function $x(\cdot): \mathbb{R}^1 \to \mathbb{R}^1$, define at each time $t \in \mathbb{R}^1$

$$\text{sgn}\big(x(t)\big) \triangleq \begin{cases} 1, & \text{if } x(t) > 0, \text{ or} \\ & \text{if } x(t) = 0, \text{ and } \dot{x}(t) > 0 \\ -1, & \text{if } x(t) < 0, \text{ or} \\ & \text{if } x(t) = 0, \text{ and } \dot{x}(t) < 0 \\ 0, & \text{if } x(t) = 0, \text{ and } \dot{x}(t) = 0 \end{cases} \tag{79}$$

It is easy to show that for any pair of solutions $\underset{\sim}{z}'_p(\cdot) \neq \underset{\sim}{z}''_p(\cdot)$, the right-hand derivative of $\mathcal{V}_\Delta\big(\underset{\sim}{z}'_p(\cdot), \underset{\sim}{z}''_p(\cdot)\big)$ takes the following form:

$$\frac{d^+}{dt}\mathcal{V}_\Delta\left(z_p'(t),z_p''(t)\right) =$$

$$- \begin{bmatrix} \text{sgn}(q_{C_1}'-q_{C_1}'') \\ \text{sgn}(q_{C_2}'-q_{C_2}'') \\ \text{sgn}(q_{C_3}'-q_{C_3}'') \end{bmatrix}^T \begin{bmatrix} G_1+G_2+G_3+G_5+G_6 & -G_5-G_6 & G_6 \\ -G_5-G_6 & G_4+G_5+G_6 & -G_6 \\ G_6 & -G_6 & G_6+G_7 \end{bmatrix} \begin{bmatrix} \frac{1}{C_1}[q_{C_1}'-q_{C_1}''] \\ \frac{1}{C_2}[q_{C_2}'-q_{C_2}''] \\ \frac{1}{C_3}[q_{C_3}'-q_{C_3}''] \end{bmatrix}$$

$$- \begin{bmatrix} \text{sgn}(q_{C_1}'-q_{C_1}'') \\ \text{sgn}(q_{C_2}'-q_{C_2}'') \end{bmatrix}^T \begin{bmatrix} 1 & -\alpha_r \\ -\alpha_f & 1 \end{bmatrix} \begin{bmatrix} I_{ES}\left(e^{q_{C_1}'/V_TC_1} - e^{q_{C_1}''/V_TC_1}\right) \\ I_{CS}\left(e^{q_{C_2}'/V_TC_2} - e^{q_{C_2}''/V_TC_T}\right) \end{bmatrix}$$

$$\leq -\frac{1}{C_1}\left|q_{C_1}'-q_{C_1}''\right|\left[G_1+G_2+G_3-G_6\right] - \frac{1}{C_2}\left|q_{C_2}'-q_{C_2}''\right|\left[G_4-G_6\right] - \frac{1}{C_3}\left|q_{C_3}'-q_{C_3}''\right|\left[G_7-G_6\right]$$

$$- \left|q_{C_1}'-q_{C_1}''\right| I_{ES}(1-\alpha_f)\left(e^{q_{C_1}'/V_TC_1} - e^{q_{C_1}''/V_TC_1}\right)/(q_{C_1}'-q_{C_1}'')$$

$$- \left|q_{C_2}'-q_{C_2}''\right| I_{CS}(1-\alpha_r)\left(e^{q_{C_2}'/V_TC_2} - e^{q_{C_2}''/V_TC_2}\right)/(q_{C_2}'-q_{C_2}'') \qquad (80)$$

The first three terms on the right side of (80) are negative because of (77). The last two terms are negative because e^x is a strictly increasing function of x. The remaining conclusions come from Theorem B.3. ∎

Networks with Linear Capacitors and Inductors

Theorem 5: Assume the nonlinear dynamic network $\mathcal{N}$ is described by the state equation (8), where the capacitor-inductor function h_p is linear; i.e.,

$$h_p(z_p) = \Gamma_p z_p \qquad (81)$$

where $\Gamma_p \in \mathbb{R}^{n_p \times n_p}$ is positive definite symmetric. Assume further that $g_p(\cdot,u_S)$ is a C^1-strictly increasing, eventually strictly passive function for all $u_S \in \mathbb{R}^{n_S}$. Under these conditions, for any continuous and bounded

$u_S(\cdot)$, (8) has a unique steady-state solution. Furthermore, if $u_S(\cdot)$ is asymptotically almost periodic and Lipschitz continuous, then each solution $z_p(\cdot)$ of (8) is asymptotically almost periodic, and in the steady state, $\mathcal{S}_{z_p} \subseteq \mathcal{S}_{u_S}$.

Proof: The hypotheses of Theorem 1 are satisfied and the solutions of (8) are eventually uniformly bounded. We apply Theorem A.2; define the following Incremental Lyapunov function $\mathcal{V}_\Delta$: $\mathbb{R}^{n_p} \times \mathbb{R}^{n_p} \to \mathbb{R}^1$:

$$\mathcal{V}_\Delta(z_p', z_p'') \triangleq (z_p' - z_p'')^T \Gamma_p (z_p' - z_p'') \tag{82a}$$

The function $\mathcal{V}_\Delta$ is C^∞, and (25) of Theorem A.2 is satisfied. We next show (26) using the strictly increasing property of $g_p(\cdot, u_S)$;

$$\begin{aligned}
&\frac{\partial \mathcal{V}_\Delta(z_p', z_p'')}{\partial z_p'} g_p(\Gamma_p z_p', u_S) + \frac{\partial \mathcal{V}_\Delta(z_p', z_p'')}{\partial z_p''} g_p(\Gamma_p z_p'', u_S) \\
&= 2 \cdot (z_p' - z_p'')^T \Gamma_p \left[g_p(\Gamma_p z_p', u_S) - g_p(\Gamma_p z_p'', u_S) \right] \\
&= 2 \cdot (\Gamma_p z_p' - \Gamma_p z_p'')^T \left[g_p(\Gamma_p z_p', u_S) - g_p(\Gamma_p z_p'', u_S) \right] \\
&> 0, \qquad \forall\, z_p' \neq z_p'', \qquad \forall\, u_S \in \mathbb{R}^{n_S}
\end{aligned} \tag{82b}$$

Hence, (8) has a unique steady-state solution. The periodic nature of the solutions comes from Theorem B.3. ∎

The condition that $g_p(\cdot, u_S)$ is strictly increasing means that $g_p(\cdot, u_S)$ is a homeomorphism in $\mathbb{R}^{n_p}$. By strengthening this condition slightly, we can apply Corollary A.2 to obtain the following result:

Corollary 3: Assume in addition to the hypotheses of Theorem 5 that $g_p(\cdot, u_S)$ is a C^1-diffeomorphism[9] mapping $\mathbb{R}^{n_p}$ into $\mathbb{R}^{n_p}$ for all $u_S \in \mathbb{R}^{n_S}$.

---

[9] Since $g_p(\cdot, u_S)$ is a homeomorphism, it is in addition a C^1-diffeomorphism if, and only if, it is a local diffeomorphism everywhere in $\mathbb{R}^{n_p}$; if, and only if, the determinant of its Jacobian is nonzero for all $z_p \in \mathbb{R}^{n_p}$, for all $u_S \in \mathbb{R}^{n_p}$.

Then for every continuous and bounded $\underset{\sim}{u}_S(\cdot)$ and for every pair of solutions $\underset{\sim}{z}'_p(\cdot)$ and $\underset{\sim}{z}''_p(\cdot)$ of (8), there exists a constant γ satisfying $1 \geq \gamma > 0$ and times τ_{max} and τ_{min} satisfying $\tau_{max} \geq \tau_{min} > 0$ such that

$$[\gamma]^{1/2} e^{-t/\tau_{min}} \| \underset{\sim}{z}'_p(0) - \underset{\sim}{z}''_p(0) \| \leq \| \underset{\sim}{z}'_p(t) - \underset{\sim}{z}''_p(t) \|$$
$$\leq [\frac{1}{\gamma}]^{1/2} e^{-t/\tau_{max}} \| \underset{\sim}{z}'_p(0) - \underset{\sim}{z}''_p(0) \| \quad , \quad \forall\, t \geq 0 \tag{83}$$

Furthermore, if $\bar{\lambda} \triangleq$ max. eigenvalue $[\underset{\sim}{\Gamma}_p]$ and $\underline{\lambda} \triangleq$ min. eigenvalue $[\underset{\sim}{\Gamma}_p]$, then $\gamma = \frac{\underline{\lambda}}{\bar{\lambda}}$.

Proof: We apply Corollary A.2. First, we see that from (82), Eq. (28) is satisfied for all $\underset{\sim}{z}'_p$, $\underset{\sim}{z}''_p \in \mathbb{R}^{n_p}$, where $\gamma_1 = \underline{\lambda}$, $\gamma_2 = \bar{\lambda}$, and $\beta = 2$. Next, since the solutions of (8) are eventually uniformly bounded, let $Z_p \subseteq \mathbb{R}^{n_p}$ be any compact and convex set which contains $\underset{\sim}{z}'_p(\cdot)$ and $\underset{\sim}{z}''_p(\cdot)$ for all $t \geq 0$. Let $D_{u_S} \subseteq \mathbb{R}^{n_S}$ be any compact set containing the bounded $\underset{\sim}{u}_S(\cdot)$ for all $t \geq 0$. Then, the strictly increasing C^1-diffeomorphism $\underset{\sim}{g}_p(\cdot, \underset{\sim}{u}_S)$ is strongly uniformly increasing [14] on the compact set Z_p, uniformly for all $\underset{\sim}{u}_S \in D_{u_S}$. We can find constant $\bar{\gamma} \geq \underline{\gamma} > 0$ and form an equation similar to (14) which holds for all $\underset{\sim}{u}_S \in D_{u_S}$. The verification of Eq. (29) of Corollary A.2 follows directly from this. ∎

Theorem 6: Assume in the dynamic nonlinear network $\mathcal{N}$ that the capacitor-inductor function $\underset{\sim}{h}_p$ is linear; i.e., $\underset{\sim}{h}_p(\underset{\sim}{z}_p) = \underset{\sim}{\Gamma}_p \underset{\sim}{z}_p$ where $\underset{\sim}{\Gamma}_p$ $\mathbb{R}^{n_p \times n_p}$ is positive definite symmetric. Assume $\mathcal{N}$ satisfies the Fundamental Topological Assumption and let each resistor function $\underset{\sim}{g}_{R_\alpha}$ be a strictly increasing homeomorphism mapping $\mathbb{R}^{n_\alpha}$ onto $\mathbb{R}^{n_\alpha}$ and satisfies

$$\lim_{\|\underset{\sim}{x}_{R_\alpha}\| \to \infty} \frac{1}{\|\underset{\sim}{x}_{R_\alpha}\|} \left[(\underset{\sim}{x}_R)^T \underset{\sim}{g}_{R_\alpha}(\underset{\sim}{x}_{R_\alpha}) \right] = +\infty \tag{84}$$

Under these conditions, the state equation (8) describing $\mathcal{N}$ exists, and for any continuous and bounded $u_S(\cdot)$, $\mathcal{N}$ has a unique steady-state solution. Furthermore, if $u_S(\cdot)$ is Lipschitz continuous and asymptotically almost periodic, then every solution of $\mathcal{N}$ is asymptotically almost periodic and, in the steady state, $\mathcal{S}_{z_p} \subseteq \mathcal{S}_{u_S}$. Moreover, if each g_{R_α} is a C^1-diffeomorphism mapping $\mathbb{R}^{n_\alpha}$ onto $\mathbb{R}^{n_\alpha}$, then for any continuous and bounded $u_S(\cdot)$ and for any two solutions $z_p'(\cdot)$ and $z_p''(\cdot)$ of (8), there exists a constant γ satisfying $1 \geq \gamma > 0$ and times τ_{max} and τ_{min} satisfying $\tau_{max} \geq \tau_{min} > 0$ such that (83) is true.

Remarks: 1. This theorem is proved using the previous theorem and corollary, and the results in [14].

2. In [15], an algorithm is presented which computes γ and τ_{max} of (83). The algorithm does not require forming the state equation (8). Rather, the algorithm requires knowledge only of the individual network constitutive relations, and their interconnections.

Example 4: We apply Theorem 6 to the network of Fig. 6, and conclude that the network has a unique steady-state solution which, in the steady-state, is periodic with frequency $\omega = 1/2$. We apply the algorithm in [15] to conclude that for any pair of solutions $z_p'(\cdot) \triangleq \left[\phi_L'(\cdot), q_{C_1}'(\cdot), q_{C_2}'(\cdot)\right]^T$ and $z_p''(\cdot) = \left[\phi_L''(\cdot), q_{C_1}(\cdot), q_{C_2}''(\cdot)\right]^T$, we have for all $t \geq 0$

$$\|z_p'(t)-z_p''(t)\| \leq \sqrt{2}\|z_p'(0)-z_p''(0)\| e^{-t/8} \tag{85}$$

RC and RL Networks with Linear Resistors

Theorem 7: Assume the nonlinear dynamic network $\mathcal{N}$ is described by the state equation (8). Let the capacitor-inductor function h_p be a C^1-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$. Assume g_p is linear; i.e.,

$$g_p(x_p,u_S) = G_p x_p + G_S u_S \tag{86}$$

where $G_p \in \mathbb{R}^{n_p \times n_p}$ is positive definite symmetric, and $G_S \in \mathbb{R}^{n_p \times n_S}$. Under these conditions, for any continuous and bounded $u_S(\cdot)$, (8) has a unique steady-state solution. Moreover, for any pair of solutions $z_p'(\cdot)$ and $z_p''(\cdot)$ there exists a constant γ satisfying $1 \geq \gamma > 0$ and times τ_{max} and τ_{min} satisfying $\tau_{max} \geq \tau_{min} > 0$ such that

$$[\gamma]^{1/2} e^{-t/\tau_{min}} \| z_p'(0) - z_p''(0) \| \leq \| z_p'(t) - z_p''(t) \|$$
$$\leq [\tfrac{1}{\gamma}]^{1/2} e^{-t/\tau_{max}} \| z_p'(0) - z_p''(0) \|, \qquad \forall\, t \geq 0 \tag{87}$$

Moreover, if u_S is Lipschitz continuous and asymptotically almost periodic, then each solution $z_p(t)$ of (8) is asymptotically almost periodic and, in the steady-state, $\mathcal{S}_{z_p} \subseteq \mathcal{S}_{u_S}$.

Proof: We will apply Corollary A.2 to show (87). The remaining conclusions follow from Theorem B.3. First note that the hypotheses of Theorem 1 are satisfied, and the solutions of (8) are eventually uniformly bounded. Define the following Incremental Lyapunov function $\mathcal{V}_\Delta$: $\mathbb{R}^{n_p} \times \mathbb{R}^{n_p} \to \mathbb{R}^1$:

$$\mathcal{V}_\Delta(z_p', z_p'') \triangleq (z_p' - z_p'')^T G_p^{-1} (z_p' - z_p'') \tag{88}$$

The function $\mathcal{V}_\Delta$ is C^∞, and (28) of Corollary A.2 is satisfied, where $\beta = 2$, γ_2 = min. eigenvalue of $[G_p]$ and γ_1 = max. eigenvalue of $[G_p]$. Next,

$$\frac{\partial \mathcal{V}_\Delta(z_p', z_p'')}{\partial z_p'} \left[G_p h_p(z_p') + G_S u_S \right] + \frac{\partial \mathcal{V}_\Delta(z_p', z_p'')}{\partial z_p''} \left[G_p h_p(z_p'') + G_S u_S \right]$$
$$= 2(z_p' - z_p'')^T G_p^{-1} \left[G_p h_p(z_p') - G_p h_p(z_p'') \right] = 2(z_p' - z_p'') \left[h_p(z_p') - h_p(z_p'') \right] \tag{89}$$

Since the hypotheses of Theorem 1 are satisfied, we can assume that the right side of Eq. (89) is evaluated for $z'_p, z''_p \in Z_p$ where Z_p is compact and convex. Then there exists a constant $\gamma_2 > 0$ such that (67a) is satisfied, and this equation together with (89) shows that (29) of Corollary A.2 is satisfied. ∎

Remark: The key condition in this theorem is that G_p is symmetric. That is, we know [14] that if $\mathcal{N}$ contains strictly passive linear resistors such that the Fundamental Topological Assumption is satisfied, then g_p in (86) exists and G_p is positive definite. Matrix G_p may be symmetric if $\mathcal{N}$ contains linear controlled sources, but when no such sources exist, matrix G_p is not symmetric if $\mathcal{N}$ contains both capacitors and inductors [7]. Hence this theorem in general may be applied only to RC and RL networks. The following theorem is stated for RC networks, but its dual applies immediately to RL networks.

Theorem 8: Assume the dynamic nonlinear network $\mathcal{N}$ contains no inductors and let the capacitor function h_p be a C^1-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$. Assume $\mathcal{N}$ satisfies the Fundamental Topological Assumption and let each resistor function g_{R_α} be linear; i.e.,

$$g_{R_\alpha}(x_{R_\alpha}) = G_\alpha x_{R_\alpha} \tag{90}$$

where $G_\alpha \in \mathbb{R}^{n_\alpha \times n_\alpha}$ is positive definite symmetric. Under these conditions, the state equation (8) describing $\mathcal{N}$ exists, where g_p is given by (86), and for any continuous and bounded $u_S(\cdot)$, $\mathcal{N}$ has a unique steady-state solution. Moreover, for any pair of solutions $z'_p(\cdot)$ and $z''_p(\ )$ of (8) (where $z_p = q_C$) there exists constant $\gamma > 0$ and times $\tau_{max} \geq \tau_{min} > 0$ such that (87) is true. Furthermore, if $u_S(\cdot)$ is Lipschitz continuous and asymptotically almost periodic, then every solution of $\mathcal{N}$ is asymptotically almost periodic and, in the steady state, $\mathcal{S}_{z_p} \subseteq \mathcal{S}_{u_S}$.

Proof: The important part of this theorem is the conclusion that if each G_α is positive definite symmetric, then the resulting $\underset{\sim}{G}_p$ in (86) is positive definite symmetric. This fact comes as a corollary of theorems in [21]. The remainder of the proof follows from Theorem 7 and results in [14]. ∎

Networks with "Small Signal" Inputs

Theorem 9: [12] Assume the nonlinear dynamic network $\mathcal{N}$ is described by the state equation (8). Let the capacitor-inductor function $\underset{\sim}{h}_p$ be a C^1-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$, and let $\frac{\partial \underset{\sim}{h}_p(\cdot)}{\partial \underset{\sim}{z}_p}$ be (locally) Lipschitz[10] continuous everywhere in $\mathbb{R}^{n_p}$. Assume $\underset{\sim}{g}_p(\cdot,\underset{\sim}{u}_S)$ is a C^1-strictly increasing, eventually strictly passive diffeomorphism mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$ for all $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$. Under these conditions, for every $\underset{\sim}{u}_S^* \in \mathbb{R}^{n_S}$ there exists $\delta > 0$ such that for every continuous and bounded $\underset{\sim}{u}_S(\cdot)$ satisfying

$$\left\| \mathcal{R}_\infty\left(\underset{\sim}{u}_S(\cdot)\right) - \underset{\sim}{u}_S^* \right\| < \delta \tag{91}$$

where $\mathcal{R}_\infty \underset{\sim}{u}_S(\cdot)$ is the eventual range of $\underset{\sim}{u}_S(\cdot)$, Eq. (8) has a unique steady-state solution. Furthermore, if $\underset{\sim}{u}_S(\cdot)$ is Lipschitz continuous and asymptotically almost periodic, then every solution $\underset{\sim}{z}_p(\cdot)$ is asymptotically almost periodic, and, in the steady state, $\mathcal{S}_{z_p} \subseteq \mathcal{S}_{u_S}$.

---

[10] That is, for every $\underset{\sim}{z}_p'' \in \mathbb{R}^{n_p}$ there exists constants $\delta > 0$ and $\ell_z > 0$ such that

$$\left\| \frac{\partial \underset{\sim}{h}_p(\underset{\sim}{z}_p')}{\partial \underset{\sim}{z}_p} - \frac{\partial \underset{\sim}{h}_p(\underset{\sim}{z}_p'')}{\partial \underset{\sim}{z}_p} \right\| \le \ell_z \, \| \underset{\sim}{z}_p' - \underset{\sim}{z}_p'' \| \;, \qquad \forall \; \| \underset{\sim}{z}_p' - \underset{\sim}{z}_p'' \| < \delta$$

Proof: The hypotheses of Theorem 3 are satisfied by this theorem, and we shall use the conclusions and the details of the proof of Theorem 3. Specifically, we will define constants γ_1, γ_2, γ_z and ε_0 such that (94) is satisfied, and then Theorem A.2 will be applied.

First, let $\underset{\sim}{u}_S^* \in \mathbb{R}^{n_S}$ be fixed and let $\hat{D}_{u_S}$ be any bounded open set containing $\underset{\sim}{u}_S^*$ in $\mathbb{R}^{n_S}$ which is large enough so that if $\|\underset{\sim}{u}_S-\underset{\sim}{u}_S^*\| < \delta$, then $\underset{\sim}{u}_S \in D_{u_S}$. For every continuous and bounded $\underset{\sim}{u}_S(\cdot)$ such that $\mathcal{R}_\infty\left(\underset{\sim}{u}_S(\cdot)\right) \subseteq \hat{D}_{u_S}$ Theorem 1 is applicable, and there exists $Z_p \subseteq \mathbb{R}^{n_p}$ which is compact and convex (and is independent of the choice of $\underset{\sim}{u}_S(\cdot)$) such that every solution $\underset{\sim}{z}_p(\cdot)$ of (8) eventually lies in Z_p. Also, the functions $\underset{\sim}{h}_p$ and $\underset{\sim}{g}_p(\cdot,\underset{\sim}{u}_S)$ are uniformly increasing in Z_p and there exists constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that (66) and (67) are true (where in (66) we have $\underset{\sim}{u}_S \in \hat{D}_{u_S}$ instead of $\|\underset{\sim}{u}_S-\underset{\sim}{u}_S^*\| < \delta$). In the same way, $\dfrac{\partial \underset{\sim}{h}_p(\cdot)}{\partial \underset{\sim}{z}_p}$ satisfies a global Lipschitz condition with a Lipschitz constant $\ell_z > 0$ in Z_p; namely,

$$\left\|\frac{\partial \underset{\sim}{h}_p(\underset{\sim}{z}_p')}{\partial \underset{\sim}{z}_p} - \frac{\partial \underset{\sim}{h}_p(\underset{\sim}{z}_p'')}{\partial \underset{\sim}{z}_p}\right\| \le \ell_z\|\underset{\sim}{z}_p'-\underset{\sim}{z}_p''\| \quad , \quad \forall\ \underset{\sim}{z}_p',\underset{\sim}{z}_p'' \in Z_p \tag{92}$$

Next, as in Theorem 3 let $\underset{\sim}{z}_p^* \in \mathbb{R}^{n_p}$ be the unique vector such that $\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p^*),\underset{\sim}{u}_p^*\right) = \underset{\sim}{0}$. From the continuity of $\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\cdot),\cdot\right)$, for every $\varepsilon_0 > 0$ there exists $\varepsilon > 0$ and $\delta > 0$ such that

$$\begin{aligned}\left\|\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p),\underset{\sim}{u}_S\right)\right\| &= \left\|\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p),\underset{\sim}{u}_S\right)-\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p^*),\underset{\sim}{u}_p^*\right)\right\| \\ &< \varepsilon_0 \quad , \quad \forall\ \|\underset{\sim}{z}_p-\underset{\sim}{z}_p^*\| < \varepsilon \\ &\qquad\qquad\quad \forall\ \|\underset{\sim}{u}_S-\underset{\sim}{u}_S^*\| < \delta\end{aligned} \tag{93}$$

From Theorem 3 we further conclude that for every $\varepsilon > 0$ there exists $\delta > 0$ such that (91) (which is (61)) implies (62); namely $\|\mathcal{R}_\infty \underset{\sim}{z}_p(\cdot)-\underset{\sim}{z}_p^*\| < \varepsilon$ for every solution $\underset{\sim}{z}_p(\cdot)$. Combining these facts together, we reach the following conclusion: For every $\varepsilon_0 > 0$ there exists $\varepsilon > 0$ and $\delta > 0$ such that (61) implies (62), such that (93) is true, and

$$\gamma_1(\gamma_2)^2 - \frac{1}{2}\ell_z\varepsilon_0 > 0 \tag{94}$$

This equation is true because $\varepsilon_0 \to 0$ as $\delta \to 0$, while γ_1, γ_2 and ℓ_z do not depend on δ. We can now apply Theorem A.2, where $D_{z_p} \overset{\Delta}{=} \{z_p: \|\underset{\sim}{z}_p - \underset{\sim}{z}_p^*\| < \varepsilon\}$, and $D_{u_S} \overset{\Delta}{=} \{\underset{\sim}{u}_S: \|\underset{\sim}{u}_S - \underset{\sim}{u}_S^*\| < \delta\}$. Define the Incremental Lyapunov function

$$\mathcal{V}_\Delta(\underset{\sim}{z}_p', \underset{\sim}{z}_p'') \triangleq \frac{1}{2}(\underset{\sim}{z}_p' - \underset{\sim}{z}_p'')^T\left[\underset{\sim}{h}_p(\underset{\sim}{z}_p') - \underset{\sim}{h}_p(\underset{\sim}{z}_p'')\right] \tag{95}$$

and (25) follows. We have only to show (26); let us denote as in (8)

$$\begin{aligned}\dot{\underset{\sim}{z}}_p' &\overset{\Delta}{=} -\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p'), \underset{\sim}{u}_S\right)\\ \dot{\underset{\sim}{z}}_p'' &\overset{\Delta}{=} -\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p''), \underset{\sim}{u}_S\right)\end{aligned} \tag{96}$$

then for all $\underset{\sim}{z}_p', \underset{\sim}{z}_p'' \in D_{z_p}$, for all $\underset{\sim}{u}_S \in D_{u_S}$,

$$\begin{aligned}&\frac{\partial\mathcal{V}_\Delta(\underset{\sim}{z}_p', \underset{\sim}{z}_p'')}{\partial\underset{\sim}{z}_p'}\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p'), \underset{\sim}{u}_S\right) + \frac{\partial\mathcal{V}_\Delta(\underset{\sim}{z}_p', \underset{\sim}{z}_p'')}{\partial\underset{\sim}{z}_p''}\underset{\sim}{g}_p\left(\underset{\sim}{h}_p(\underset{\sim}{z}_p''), \underset{\sim}{u}_S\right)\\ &= -\frac{1}{2}\left[\underset{\sim}{h}_p(\underset{\sim}{z}_p') - \underset{\sim}{h}_p(\underset{\sim}{z}_p'')\right]^T(\dot{\underset{\sim}{z}}_p' - \dot{\underset{\sim}{z}}_p'') - \frac{1}{2}(\underset{\sim}{z}_p' - \underset{\sim}{z}_p'')^T\left[\frac{\partial\underset{\sim}{h}_p(\underset{\sim}{z}_p')}{\partial\underset{\sim}{z}_p}\dot{\underset{\sim}{z}}_p' - \frac{\partial\underset{\sim}{h}_p(\underset{\sim}{z}_p'')}{\partial\underset{\sim}{z}_p}\dot{\underset{\sim}{z}}_p''\right]\\ &= -\left[\underset{\sim}{h}_p(\underset{\sim}{z}_p') - \underset{\sim}{h}_p(\underset{\sim}{z}_p'')\right]^T(\dot{\underset{\sim}{z}}_p' - \dot{\underset{\sim}{z}}_p'') + \frac{1}{2}\left[\underset{\sim}{h}_p(\underset{\sim}{z}_p') - \underset{\sim}{h}_p(\underset{\sim}{z}_p'')\right]^T(\dot{\underset{\sim}{z}}_p' - \dot{\underset{\sim}{z}}_p'')\\ &\quad - \frac{1}{2}(\underset{\sim}{z}_p' - \underset{\sim}{z}_p'')^T\left[\frac{\partial\underset{\sim}{h}_p(\underset{\sim}{z}_p')}{\partial\underset{\sim}{z}_p}\dot{\underset{\sim}{z}}_p' - \frac{\partial\underset{\sim}{h}_p(\underset{\sim}{z}_p'')}{\partial\underset{\sim}{z}_p}\dot{\underset{\sim}{z}}_p''\right]\end{aligned} \tag{97}$$

Now, applying the Mean Value Theorem [20],

$$(\dot{\underset{\sim}{z}}_p' - \dot{\underset{\sim}{z}}_p'')^T\left[\underset{\sim}{h}_p(\underset{\sim}{z}_p') - \underset{\sim}{h}_p(\underset{\sim}{z}_p'')\right] = (\dot{\underset{\sim}{z}}_p' - \dot{\underset{\sim}{z}}_p'')^T\left[\frac{\partial\underset{\sim}{h}_p(\tilde{\underset{\sim}{z}}_p)}{\partial\underset{\sim}{z}_p}(\underset{\sim}{z}_p' - \underset{\sim}{z}_p'')\right] \tag{98}$$

where for some $\lambda \in [0,1]$,

$$\tilde{z}_p \overset{\Delta}{=} z''_p + \lambda[z'_p - z''_p] \tag{99}$$

Hence

$$\frac{\partial \mathcal{V}_\Delta(z'_p, z''_p)}{\partial z'_p} g_p\left(h_p(z'_p), u_S\right) + \frac{\partial \mathcal{V}_\Delta(z'_p, z''_p)}{\partial z''_p} g_p\left(h_p(z''_p), u_S\right)$$

$$= -\left[h_p(z'_p) - h_p(z''_p)\right]^T (\dot{z}'_p - \dot{z}''_p) - \frac{1}{2}(z'_p - z''_p)^T \left[\left(\frac{\partial h_p(z'_p)}{\partial z_p} - \frac{\partial h_p(\tilde{z}_p)}{\partial z_p}\right)\dot{z}'_p + \left(\frac{\partial h_p(\tilde{z}_p)}{\partial z_p} - \frac{\partial h_p(z''_p)}{\partial z_p}\right)\dot{z}''_p\right] \tag{100}$$

We use (66), (67) and (96) to analyze the first term on the right side of (100);

$$-\left[h_p(z'_p) - h_p(z''_p)\right]^T (\dot{z}'_p - \dot{z}''_p) = \left[h_p(z'_p) - h_p(z''_p)\right]^T \left[g_p\left(h_p(z'_p), u_S\right) - g_p\left(h_p(z''_p), u_S\right)\right]$$

$$\geq \gamma_1 \| h_p(z'_p) - h_p(z''_p) \|^2$$

$$\geq \gamma_1 (\gamma_2)^2 \| z'_p - z''_p \|^2 \tag{101}$$

We use (92), (93) and (99) to analyze the second term on the right side of (100);

$$\frac{1}{2}(z'_p - z''_p)^T \left[\left(\frac{\partial h_p(z'_p)}{\partial z_p} - \frac{\partial h_p(\tilde{z}_p)}{\partial z_p}\right)\dot{z}'_p - \left(\frac{\partial h_p(\tilde{z}_p)}{\partial z_p} - \frac{\partial h_p(z''_p)}{\partial z_p}\right)\dot{z}''_p\right]$$

$$\geq -\frac{1}{2}\| z'_p - z''_p \| \cdot \left[\left\| \frac{\partial h_p(z'_p)}{\partial z_p} - \frac{\partial h_p(\tilde{z}_p)}{\partial z_p} \right\| \cdot \| \dot{z}'_p \| + \left\| \frac{\partial h_p(\tilde{z}_p)}{\partial z_p} - \frac{\partial h_p(z''_p)}{\partial z_p} \right\| \cdot \| \dot{z}''_p \| \right]$$

$$\geq -\frac{1}{2}\| z'_p - z''_p \| \left[\ell_z \| z'_p - \tilde{z}_p \| \cdot \| \dot{z}'_p \| + \ell_z \| \tilde{z}_p - z''_p \| \cdot \| \dot{z}''_p \| \right]$$

$$= -\frac{1}{2}\ell_z \| z'_p - z''_p \| \left[\| z'_p - z''_p \| (1-\lambda) \| \dot{z}'_p \| + \| z'_p - z''_p \| \lambda \| \dot{z}''_p \| \right]$$

$$\geq -\frac{1}{2}\ell_z\|z_p'-z_p''\|^2[(1-\lambda)\varepsilon_0+\lambda\varepsilon_0]$$

$$= -\frac{1}{2}\ell_z\varepsilon_0\|z_p'-z_p''\|^2 \qquad (102)$$

Hence, for all $z_p', z_p'' \in D_{z_p}$ for all $u_S \in D_{u_S}$, using (94), (100), (101) and (102), we obtain

$$\frac{\partial \mathcal{V}_\Delta(z_p', z_p'')}{\partial z_p'} g_p\left(h_p(z_p'), u_S\right) + \frac{\partial \mathcal{V}_\Delta(z_p', z_p'')}{\partial z_p''} g_p\left(h_p(z_p''), u_S\right)$$

$$\geq \|z_p'-z_p''\|^2[\gamma_1(\gamma_2)^2 - \frac{1}{2}\ell_z\varepsilon_0]$$

$$> 0 \qquad \forall\, z_p' \neq z_p'' \qquad ∎ \qquad (103)$$

Remark: The Lipschitz constant ℓ_z in (92) is a measure of the nonlinearity of the capacitors and inductors. Indeed, when the capacitors and inductors are linear, then $\ell_z = 0$, (94) is automatically satisfied, and Theorem 5 is therefore a corollary of Theorem 9. Equation (94) has the following interpretation: Assuming $\gamma_1 > 0$ and $\gamma_2 > 0$ are fixed, for every measure of the nonlinearity of the capacitors and inductors — ℓ_z — we must fix the "small signal" component of $u_S(\cdot)$ — δ — such that for the corresponding "small signal" component of $\dot{z}_p(\cdot)$ — ε_0 — Eq. (94) is satisfied. The relationship between ε_0 and δ is examined more closely following the next theorem which follows from Theorem 9 and the results in [14];

Theorem 10: Assume in the dynamic nonlinear network $\mathcal{N}$ that the capacitor-inductor function h_p is a C^1-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$, and its Jacobian is Lipschitz continuous. Assume $\mathcal{N}$ satisfies the Fundamental Topological Assumption and let each resistor function g_{R_α} be a C^1-strictly increasing diffeomorphism mapping $\mathbb{R}^{n_\alpha}$ onto $\mathbb{R}^{n_\alpha}$, satisfying

$$\lim_{\|\tilde{x}_{R_\alpha}\|\to\infty} \frac{1}{\|\tilde{x}_{R_\alpha}\|}\left[(\tilde{x}_{R_\alpha})^T \tilde{g}_{R_\alpha}(\tilde{x}_{R_\alpha})\right] = +\infty \tag{104}$$

Under these conditions, the state equation (8) describing $\mathcal{N}$ exists, and for every $\tilde{u}_S^* \in \mathbb{R}^{n_S}$ there exists $\delta > 0$ such that for any continuous and bounded $\tilde{u}_S(\cdot)$ satisfying

$$\|\mathcal{R}_\infty \tilde{u}_S(\cdot) - \tilde{u}_S^*\| < \delta \tag{105}$$

$\mathcal{N}$ has a unique steady-state solution. Furthermore, if $\tilde{u}_S(\cdot)$ is asymptotically almost periodic, every solution of $\mathcal{N}$ is asymptotically almost periodic and, in the steady-state, $\mathcal{S}_{z_p} \subseteq \mathcal{S}_{u_S}$. ∎

<u>Theorem 9</u> and <u>Theorem 10</u> are not constructive theorems in that there is no explicit method given to calculate the constant $\delta > 0$ such that if (105) is satisfied, then $\mathcal{N}$ has a unique steady-state solution. The problem with specifying δ is that the key condition is given by Eq. (94) which involves $\varepsilon_0 > 0$ and not δ. It is true that the continuity of $\tilde{g}_p(\tilde{h}_p(\cdot))$ guarantees that for every $\varepsilon_0 > 0$ there exists $\delta > 0$ such that (93) is true. However, in practice it is difficult to determine δ from ε_0. The difficulty is illustrated in the following example:

<u>Example 5</u>: Examine the network of Fig. 7. The state equation takes the form (8) where

$$\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \tilde{h}_p\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} \triangleq \begin{pmatrix} 3/2\ q_C + 1/8 \sin 4q_C \\ \phi_L \end{pmatrix} \tag{106}$$

and

$$\begin{pmatrix} i_C \\ v_L \end{pmatrix} = g_p \begin{pmatrix} v_C \\ i_L \\ \delta \sin t \end{pmatrix} \triangleq - \begin{pmatrix} v_C + i_L - 1 - \delta \sin t \\ i_L + 1/3(i_L)^3 - v_C \end{pmatrix} \tag{107}$$

Here, we compute directly that

$$\gamma_1 = 1 \; ; \; \gamma_2 = \min[1, \tfrac{3}{2} + \inf(\tfrac{1}{2} \cos 4q_C)] = 1 \tag{108}$$

and

$$\ell_z = \sup |-2 \sin 4q_C| = 2 \tag{109}$$

Hence to satisfy (94) we have to show

$$0 < \varepsilon_0 < \frac{\gamma_1 (\gamma_2)^2}{1/2 \; \ell_z} = 1 \tag{110}$$

Here $\underset{\sim}{u}_S^* = 1$. Upon setting $\delta = 0$ in (107), we see that

$$\underset{\sim}{x}_p^* = \begin{pmatrix} v_C^* \\ i_L^* \end{pmatrix} \cong \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}; \quad \underset{\sim}{z}_p^* = \begin{pmatrix} q_C^* \\ \phi_L^* \end{pmatrix} \cong \begin{pmatrix} .27 \\ .50 \end{pmatrix} \tag{111}$$

<u>Claim</u>: When $\delta \leq \frac{1}{4}$, the network has a unique steady-state solution.

To prove this claim we will show that ε_0 satisfies (110) when $\delta = 1/4$. We first find the equivalence of $\varepsilon > 0$ such that $\| \mathcal{R}_\infty \left(z_p(\cdot) \right) - z_p^* \| < \varepsilon$. Looking at the proof of <u>Theorem 3</u>;

$$\begin{pmatrix} v_C - 1/2 \\ i_L - 1/2 \end{pmatrix}^T \begin{pmatrix} v_C + i_L - 1 - 1/4 \sin t \\ i_L + 1/3(i_L)^3 - v_C \end{pmatrix} = (v_C - 1/2)^2 + (i_L - 1/2)^2$$
$$+ (i_L - 1/2) 1/3 (i_L)^3 - 1/4 (v_C - 1/2) \sin t \tag{112}$$

and the right side of (112) is positive for all $|v_C - 1/2| > 1/4$, and

$|i_L - 1/2| > 1/24$. That is, it is positive for all

$$1/8 < q_C < 2/5 \quad ; \quad 11/24 < \phi_L < 13/24 \tag{113}$$

Define

$$\mathcal{V}(\underset{\sim}{z}_p) \triangleq \frac{(\phi_L - 1/2)^2}{2} + 3/4\, q_C^2 - 3/4\, q_C^2 - 1/2\, q_C - 1/32 \cos 4\, q_C + .095 \tag{114}$$

such that we have $\nabla\mathcal{V}(\underset{\sim}{z}_p) \equiv \underset{\sim}{h}_p(\underset{\sim}{z}_p) - \underset{\sim}{h}_p(\underset{\sim}{z}_p^*)$, and $\mathcal{V}(\underset{\sim}{z}_p^*) = 0$. After some computation we see that if q_C and ϕ_L take on values given by (113), then

$$\mathcal{V}(\underset{\sim}{z}_p) \leq .02 \tag{115}$$

Hence, we further conclude that all solutions $\underset{\sim}{z}_p(\cdot)$ eventually lie in the compact and convex set

$$Z_p \triangleq \{\underset{\sim}{z}_p : \; 1/8 \leq q_C \leq 2/5, \; .3 \leq \phi_L \leq .7\} \tag{116}$$

and for $\underset{\sim}{z}_p$ in Z_p, we have from (106) $|v_C - 1/2| < 1/4$ and $|i_L - 1/2| < 1/5$. We are now ready to compute ε_0:

$$\begin{aligned}
(\varepsilon_0)^2 &\leq \sup_{\substack{|v_C - 1/2| < 1/4 \\ |i_L - 1/2| < 1/5}} \left\| \underset{\sim}{g}_p \begin{pmatrix} v_C \\ i_L \\ 1/4 \sin t \end{pmatrix} \right\|^2 \\
&\leq \sup_{\substack{|v_C - 1/2| < 1/4 \\ |i_L - 1/2| < 1/5}} \Big(2(i_L - 1/2)^2 + 2(v_C - 1/2)^2 + 1/16 \sin^2(t) \\
&\qquad - 1/2(i_L + v_C - 1)\sin t + 1/9(i_L)^6 \\
&\qquad + 2/3(i_L)^3(i_L - v_C) \Big) \\
&< .62
\end{aligned} \tag{117}$$

Hence $\varepsilon_0 < .78$ and the <u>claim</u> has been proved. ∎

Let us return to Theorems 6 and 8 and the Fundamental Topological Assumption. There are a number of networks which have a unique steady-state solution but do not satisfy this assumption; examine the networks of Fig. 2(b) and Fig. 2(c) where for simplicity we replace the voltage sources by short circuits. In both cases we know that the networks have a unique steady-state solution which is the globally asymptotically stable equilibrium point $\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This conclusion follows from Theorem 4 for the network of Fig. 2(c). But there is a loop formed by the capacitor and inductor in Fig. 2(b) and Theorem 4 is therefore not applicable. One way to show $\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the globally asymptotically stable equilibrium point of this network is to use the Lyapunov function $\mathcal{V} \triangleq C/2(v_C)^2 + L/2(i_L)^2$. Then, for any solution $\begin{pmatrix} v_C(\cdot) \\ i_L(\cdot) \end{pmatrix}$ of the network,

$$\frac{d}{dt}\mathcal{V}\begin{pmatrix} v_C(t) \\ i_L(t) \end{pmatrix} = -\frac{1}{R_2}(v_C(t)) \quad , \quad \forall\, t \in \mathbb{R}^1 \tag{118}$$

and

$$\left[\frac{d}{dt}\mathcal{V}\begin{pmatrix} v_C(t) \\ i_L(t) \end{pmatrix} \equiv 0\right] \Rightarrow [v_C(t) \equiv 0] \Rightarrow [i_C(t) \equiv i_R(t) \equiv 0] \Rightarrow [i_L(t) \equiv 0] \tag{119}$$

Now, by a well-known extension of Lyapunov's Theorem [22] (or by Corollary A.2) we conclude that $\begin{pmatrix} v_C \\ i_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is globally asymptotically stable. Of course, there are many networks where the Fundamental Topological Assumption is not satisfied, and there is no unique steady-state solution. For example, in the network of Fig. 2(d), we see that

$$\begin{pmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{pmatrix} = \begin{pmatrix} \beta \sin \omega t \\ -\beta \sin \omega t \end{pmatrix} \tag{120}$$

is a solution, for any $\beta \in \mathbb{R}^1$, where $\omega = 1/\sqrt{LC}$. We present below a condition which is weaker than the Fundamental Topological Assumption and will allow loops and cutsets such as in Fig. 2(b), and prohibit those as in Fig. 2(d). This hypothesis has been discussed extensively in [14] and [15]:

Inductor-Capacitor Loop-Cutset Hypothesis (L.C. Hypothesis)

Let the dynamic nonlinear network $\mathcal{N}$ contain capacitors, inductors, resistors and constant sources. The capacitors and inductors are described by $\underset{\sim}{h}_p$ in (4), where $\underset{\sim}{h}_p$ is a C^1-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$.

Let $\mathcal{S}$ be any set of capacitors and inductors such that any capacitor or inductor in $\mathcal{S}$ forms a loop and/or cutset exclusively with any combination of independent voltage and current sources, and other capacitors and inductors of $\mathcal{S}$. Let one of the following conditions be satisfied:

(a) There is a capacitor C_j in $\mathcal{S}$ which is in a loop formed exclusively with any combination of independent sources and other elements of $\mathcal{S}$, but not in a cutset formed exclusively with any combination of currents sources and elements of $\mathcal{S}$. This capacitor is not coupled[11] to any other capacitor of $\mathcal{S}$.

(b) There is an inductor L_j in $\mathcal{S}$ which is in a cutset formed exclusively with any combination of independent sources and other elements of $\mathcal{S}$ but not in a loop formed exclusively with any combination of voltage sources and elements of $\mathcal{S}$. This inductor is not coupled to any other inductor of $\mathcal{S}$.

Remark: The statement of the L.C. Hypothesis given here differs from that in [14]. Specifically, the interconnection conditions placed on the

[11]That is, for any other capacitor C_k in $\mathcal{S}$, $\frac{dv_{C_j}}{dq_{C_k}} \equiv \frac{dv_{C_k}}{dq_{C_j}} \equiv 0$.

independent voltage and current sources is slightly weaker in its present form. Theorem 12 of [14] remains valid using this form of the L.C. Hypothesis and its proof remains essentially the same as the earlier version.

Theorem 11: In Theorems 6 and 8 the hypothesis that $\mathcal{N}$ satisfies the Fundamental Topological Assumption may be replaced by the following conditions; (i) there is no loop (resp., cutset) formed exclusively by capacitors and voltage sources (resp., inductors and current sources), (ii) $\mathcal{N}$ satisfies the L.C. Hypothesis, (iii) the function $\underset{\sim}{u}_S(\cdot)$ satisfies a global Lipschitz condition, and the state equation (8) (which exists as part of the conclusion) has at least one bounded solution, and (iv) each resistor function $\underset{\sim}{g}_{R_\alpha}$ and the capacitor-inductor function $\underset{\sim}{h}_p$ are C^3-functions.

Proof: We will apply Corollary A.3. First, we apply results in [2] to conclude that state equation (8) describing $\mathcal{N}$ exists and all voltage and current waveforms of $\mathcal{N}$ are C^1-functions of time. This latter conclusion comes directly from the condition (iv), and is necessary in applying Theorem 12 of [14].

We will show that the conclusions of Theorem 6 are valid under these hypotheses. The application of these hypotheses to Theorem 8 is similar and need not be shown. Examine the Incremental Lyapunov function $\mathcal{V}_\Delta$ given by (82). Then, using Tellegen's Theorem (see [14] for the details)

$$\frac{\partial \mathcal{V}_\Delta(\underset{\sim}{z}_p', \underset{\sim}{z}_p'')}{\partial \underset{\sim}{z}_p'} \underset{\sim}{g}_p\left(\underset{\sim}{\Gamma}_p \underset{\sim}{z}_p', \underset{\sim}{u}_S\right) + \frac{\partial \mathcal{V}_\Delta(\underset{\sim}{z}_p', \underset{\sim}{z}_p'')}{\partial \underset{\sim}{z}_p''} \underset{\sim}{g}_p\left(\underset{\sim}{\Gamma}_p \underset{\sim}{z}_p'', \underset{\sim}{u}_S\right)$$

$$= 2(\underset{\sim}{x}_p' - \underset{\sim}{x}_p'')^T \left[\underset{\sim}{g}_p(\underset{\sim}{x}_p', \underset{\sim}{u}_S) - \underset{\sim}{g}_p(\underset{\sim}{x}_p'', \underset{\sim}{u}_S)\right]$$

$$= 2(\underset{\sim}{v}_R' - \underset{\sim}{v}_R'')^T (\underset{\sim}{i}_R' - \underset{\sim}{i}_R'') \qquad (121)$$

and since each resistor is strictly increasing, the right side of (121) is non-negative. Hence Eq. (33) of Corollary A.3 is satisfied. To show that Eq. (35) is satisfied, as an extension of Theorem 12 of [14] we see that for any time interval $I_T = [\tau_1, \tau_2]$, $\tau_1 < \tau_2$, $\underset{\sim}{v}_R'(t) - \underset{\sim}{v}_R''(t) = \underset{\sim}{i}_R'(t) - \underset{\sim}{i}_R''(t) = 0$ for all $t \in I_r$ if, and only if, $\underset{\sim}{z}_p'(t) - \underset{\sim}{z}_p''(t) = 0$ for all $t \in I_t$. ∎

A Network with More than One Steady-State Solution

In Theorems 3-10 we have shown that $\mathcal{N}$ has a unique steady-state solution if

(I) $\underset{\sim}{g}_p(\cdot, \underset{\sim}{u}_S)$ is a C^1-strictly increasing eventually strictly passive diffeomorphism mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$ for all $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$,

(II) $\underset{\sim}{h}_p(\cdot)$ is a C^1-strictly increasing diffeomorphic state function mapping $\mathbb{R}^{n_p}$ onto $\mathbb{R}^{n_p}$,

and if other conditions in each theorem are satisfied. A natural extension of these theorems is the following:

Conjecture: Assume $\mathcal{N}$ satisfies (I) and (II) above. Then for any continuous and bounded $\underset{\sim}{u}_S(\cdot)$, (8) has a unique steady-state solution.

The conjecture is true when $n_p = 1$, i.e., when $\mathcal{N}$ is a one-dimensional system. For then, $\hat{g}_p(\cdot, \underset{\sim}{u}_S) \triangleq g_p\left(h_p(\cdot), \underset{\sim}{u}_S\right)$ is a C^1-strictly increasing, eventually strictly passive mapping $\mathbb{R}^1$ onto $\mathbb{R}^1$, for all $\underset{\sim}{u}_S \in \mathbb{R}^{n_S}$, and the conjecture follows from Theorem 5. For $n_p \geq 2$, however, this conjecture is false. The counter example is given in the following:

Example 6: Examine the network of Fig. 5. It satisfies the hypotheses of Theorem 2, hence (Example 2) its solutions are eventually uniformly bounded, and it has a periodic solution with period 2π. Furthermore, this network is a simple extension of the results presented previously in that (i) the only nonlinear element is the inductor; otherwise Theorem 4 is directly applicable (ii) if there is no capacitor the dual of Theorem 6 is directly applicable (iii) $n_p = 2$, and the Conjecture is true for $n_p = 1$, (iv) Theorem 10 is applicable and if the voltage source $E(t) = \sin t$ is replaced with a source $E(t) = \delta \sin t$ then we know that for some $\delta > 0$ the network has a unique steady-state solution.

We have analyzed this network using computer simulation in a computer-graphics system CSMP [24]. The network has (at least) two steady-state solutions. The waveforms of these solutions are shown in Fig. 8. Both are "local" steady-state solutions in the sense that all other solutions "nearby" converge to them. The periodic solution which is guaranteed by Theorem 1 is shown in Fig. 8(a). The top waveform is $\phi_L(t)$, the lower waveform is $q_C(t)$, and the frequency is $\omega = 1$. The second solution is shown in Fig. 8(b). It is an "almost subharmonic" waveform. That is, it is an almost periodic waveform with "frequency" $\omega \cong 1/10$ (its almost periodic nature is easily observed in the lower waveform). Again, the upper waveform is $\phi_L(t)$ and the lower waveform is $q_C(t)$. In Fig. 8(c) the upper waveform is the "almost subharmonic" $\phi_L(t)$ and the lower waveform is the periodic $\phi_L(t)$.

Networks of this type have been analyzed in the past (see, for example [10]) because the inductor is a model of a non-hysteretic iron core inductor. It is well known that such networks generate subharmonics.

We have shown using this example that the conjecture is false and that Theorems 3-10 are the best possible for networks where $\underset{\sim}{h}_p(\cdot)$ and $\underset{\sim}{g}_p(\cdot,\underset{\sim}{u}_S)$ satisfy respectively (I) and (II) above.

VI. Conclusions:

A number of results concerning the qualitative behavior of nonlinear dynamic networks are presented. The hypotheses of these results are of two types: First, very general and practical conditions on the network state equation, and second, conditions upon the individual element constitutive relations and their interconnection. In the latter form, the hypotheses include (in general) the Fundamental Topological Assumption and the L.C. Hypothesis. These conditions are simple, easy to verify, and therefore quite practical. These are the best possible conditions for the class of networks such that $\underset{\sim}{h}_p(\cdot)$ is a C^1-strictly increasing diffeomorphic state function, and $\underset{\sim}{g}_p(\cdot,\underset{\sim}{u}_S)$ is a C^1-strictly increasing eventually strictly passive diffeomorphism for all $\underset{\sim}{u}_S$.

The results developed in this paper and in [15] may be applied in a useful way to the study of the structural sensitivity of nonlinear dynamic networks. For example, assume we are building a circuit whose capacitors, inductors and resistors are presumed linear. We analyze the behavior of the linear model so to predict the behavior of the real circuit. Of course, every real electrical element is nonlinear, but so long as the elements retain the strongly uniformly increasing nature of their idealized linear models, then the behavior of the real network is the same as the behavior of the linear model of the network. In particular, using the results presented here we conclude that the boundedness of solutions, the almost periodic nature of solutions, the existence of an equilibrium point, and the existence of a unique steady-state solution are properties inherent in the network and not dependent upon the linearity. This is a comforting though expected conclusion.

We conclude with a comment concerning Incremental Lyapunov Functions whose properties are illustrated in Theorem A.2 and Corollaries A.1-A.4. We believe that the establishment of a unique steady-state solution is important in the study of nonlinear dynamic networks, and therefore the application of Incremental Lyapunov Functions will be of use to future researchers.

APPENDIX

Proof of Theorem A.2: Since the solutions of (15) are eventually uniformly bounded, using (25) it suffices to show that for any two solutions $\underset{\sim}{x}'(\cdot)$ and $\underset{\sim}{x}''(\cdot)$,

$$\lim_{t\to+\infty} \mathcal{V}_\Delta\left(\underset{\sim}{x}'(t),\underset{\sim}{x}''(t)\right) = 0 \tag{A-1}$$

For purposes of contradiction, assume there are two solutions $\underset{\sim}{x}'(\cdot)$ and $\underset{\sim}{x}''(\cdot)$ such that (A-1) is not true. More specifically, let $t_0 \in \mathbb{R}^1$ be the time such that $\underset{\sim}{x}'(\cdot)$ and $\underset{\sim}{x}''(\cdot)$ lie in $\mathcal{K}_0$ for all $t \geq t_0$. Then, (notationally, we set $\mathcal{V}_\Delta(t)$ to denote $\mathcal{V}_\Delta\left(\underset{\sim}{x}'(t),\underset{\sim}{x}''(t)\right)$) Eqs. (25) and (26) imply $\mathcal{V}_\Delta(t) \geq 0$ and $\frac{d}{dt}\mathcal{V}_\Delta(t) \leq 0$ for all $t \geq t_0$, and (A-1) is not satisfied if, and only if, there exists some $\varepsilon > 0$ such that

$$\mathcal{V}_\Delta(t) \geq \varepsilon \qquad , \qquad \forall\, t \geq t_0 \tag{A-2}$$

By hypothesis $D_\xi \supseteq \mathcal{R}_\infty\left(\underset{\sim}{\xi}(\cdot)\right)$ is open, and let time $t_1 \in \mathbb{R}^1$ be such that $\underset{\sim}{\xi}(t) \in D_\xi$ for all $t \geq t_1$. Define $\hat{t}_0 \triangleq \max\{t_0,t_1\}$. We will show that (A-2) contradicts (26). First, we find constants $\varepsilon' > 0$ and $\delta > 0$:

The continuous function $\mathcal{V}_\Delta(\cdot,\cdot)$ is uniformly continuous on the compact set $\mathcal{K}_0 \times \mathcal{K}_0$. Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $(\underset{\sim}{x}',\underset{\sim}{x}'') \in \mathcal{K}_0 \times \mathcal{K}_0$,

$$[\|\underset{\sim}{x}'-\underset{\sim}{x}''\| < \delta] \Rightarrow [\mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'') < \varepsilon] \tag{A.3a}$$

Observe that Eq. (A.3a) is equivalent to

$$[\mathcal{V}_\Delta(\underset{\sim}{x}',\underset{\sim}{x}'') \geq \varepsilon] \Rightarrow [\|\underset{\sim}{x}'-\underset{\sim}{x}''\| \geq \delta] \tag{A.3b}$$

Next, since D_ξ and D_x are bounded and $\mathcal{V}_\Delta$ is C^1, for every $\delta > 0$ there exists $\varepsilon' > 0$ such that (26) implies

$$\inf_{\underset{\sim}{\xi}\in D_{\xi}} \inf_{\substack{(\underset{\sim}{x}',\underset{\sim}{x}'')\in D_x \times D_x \\ \|\underset{\sim}{x}'-\underset{\sim}{x}''\|\geq\delta}} \left[\frac{\partial \mathcal{V}_{\Delta}(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}'} \underset{\sim}{f}(\underset{\sim}{x}',\underset{\sim}{\xi}) + \frac{\partial \mathcal{V}_{\Delta}(\underset{\sim}{x}',\underset{\sim}{x}'')}{\partial \underset{\sim}{x}''} \underset{\sim}{f}(\underset{\sim}{x}'',\underset{\sim}{\xi})\right] \geq \varepsilon' \tag{A.4}$$

Thus, using Eqs. (15), (A.2), (A.3b) and (A.4), we conclude

$$\frac{d}{dt}\mathcal{V}_{\Delta}(t) \leq -\varepsilon' \quad , \quad \forall\ t \geq \hat{t}_0 \tag{A.5}$$

Define time $t_2 > \hat{t}_0$;

$$t_2 \triangleq \frac{\mathcal{V}_{\Delta}(\hat{t}_0)-\varepsilon}{\varepsilon'} + \hat{t}_0 + 1 \tag{A.6}$$

Then

$$\begin{aligned} \mathcal{V}_{\Delta}(t_2) &= \mathcal{V}_{\Delta}(\hat{t}_0) + \int_{\hat{t}_0}^{t_2} \frac{d}{dt}\mathcal{V}_{\Delta}(t)dt \\ &\leq \mathcal{V}_{\Delta}(\hat{t}_0) + \int_{\hat{t}_0}^{t_2} (-\varepsilon')dt \\ &= \mathcal{V}_{\Delta}(\hat{t}_0) - \varepsilon'\left[\frac{\mathcal{V}_{\Delta}(\hat{t}_0)-\varepsilon}{\varepsilon'} + 1\right] \\ &= \varepsilon - \varepsilon' \end{aligned} \tag{A.7}$$

which contradicts (A.2). ■

The proof of Corollary A.1 is similar to the proof of Theorem A.2 and need only be outlined: Let $\underset{\sim}{x}'(\cdot)$ be the *a priori* bounded solution of (15) and let $\underset{\sim}{x}''(\cdot)$ be any other solution. Since $D_{\xi} = \mathbb{R}^m$ and $D_x = \mathbb{R}^n$ $\mathcal{V}_{\Delta}(t) \triangleq \mathcal{V}_{\Delta}\left(x'(t),x''(t)\right)$ is defined for all $t \in \mathbb{R}^1$. From (25) and (26) we conclude as in Theorem A.2 that $\mathcal{V}_{\Delta}(t) \geq 0$ and $\frac{d}{dt}\mathcal{V}_{\Delta}(t) \leq 0$ for all

$t \in \mathbb{R}^1$. This means $\mathcal{V}_\Delta(\cdot)$ is bounded and from (27) we conclude that $\underset{\sim}{x}''(\cdot)$ is also bounded. Hence there exists open and bounded sets $\tilde{D}_\xi \subseteq \mathbb{R}^m$ and $\tilde{D}_x \subseteq \mathbb{R}^n$ such that $\underset{\sim}{\xi}(t) \in \tilde{D}_\xi$ for all $t \geq 0$, and $\underset{\sim}{x}'(t), \underset{\sim}{x}''(t) \in \tilde{D}_x$ for all $t \geq 0$. At this point Corollary A.1 follows directly from Theorem A.2.

The proof of Corollary A.2 follows from (28) and (29) by noting that for any two solutions $\underset{\sim}{x}'(\cdot)$ and $\underset{\sim}{x}''(\cdot)$ which lie in $\mathcal{K}_0$ for all $t \geq t_0$,

$$-\frac{\gamma_4}{\gamma_1}\mathcal{V}_\Delta(t) \leq \frac{d}{dt}\mathcal{V}_\Delta(t) \leq -\frac{\gamma_3}{\gamma_2}\mathcal{V}_\Delta(t) \qquad \forall\, t \geq t_0 \tag{A.8}$$

and (30) follows from this.

Proof of Corollary A.3: As in the proof of Theorem A.2, we have only to show (A.1). For purposes of contradiction, suppose there exists a $\underset{\sim}{\xi}(\cdot)$ which satisfies a global Lipschitz condition such that $\mathcal{R}_\infty\left(\underset{\sim}{\xi}(\cdot)\right) \subseteq D_\xi$, and there are two solutions $\underset{\sim}{x}'(\cdot)$ and $\underset{\sim}{x}''(\cdot)$ such that (A.2) is true. Let $\hat{t}_0 \in \mathbb{R}^1$ be the time such that $\underset{\sim}{x}'(t), \underset{\sim}{x}''(t) \in D_x$ for all $t \geq \hat{t}_0$, $\underset{\sim}{\xi}(t) \in D_\xi$ for all $t \geq \hat{t}_0$, and (this is condition (ii) of Corollary A.2) for any time interval $I_t \triangleq [\tau_1, \tau_2]$, $\hat{t}_0 \leq \tau_1 < \tau_2$ (35) is true. Let us examine (A.2) more closely; since $\frac{d}{dt}\mathcal{V}_\Delta(t) \leq 0$ for all $t \geq \hat{t}_0$, $\mathcal{V}_\Delta(\cdot)$ is a decreasing function of time for $t \in [\hat{t}_0, \infty)$ with values in the compact set $[\varepsilon, \mathcal{V}_\Delta(\hat{t}_0)] \subseteq \mathbb{R}^1$. This means that there exists $\varepsilon_0 \in [\varepsilon, \mathcal{V}_\Delta(\hat{t}_0)]$ such that

$$\lim_{t\to\infty} \mathcal{V}_\Delta(t) = \varepsilon_0 \tag{A.9}$$

which implies

$$\lim_{t\to\infty} \frac{d}{dt}\mathcal{V}_\Delta(t) = 0 \tag{A.10}$$

We will show that this contradicts (35). We define the following sequences:

(i) The sequence of <u>times</u> $\{T_j\}_{j=1}^{\infty}$ such that $T_{j+1} > T_j$, $T_1 \geq \hat{t}_0$, and

$$0 \geq \frac{d}{dt}\mathcal{V}_\Delta(t) \geq -\frac{1}{j} \tag{A.11}$$

The existence of such a sequence $\{T_j\}$ is guaranteed by (A.10).

(ii) The sequence of <u>functions</u> $\{\underset{\sim}{\xi}_j(\cdot)\}_{j=1}^{\infty}$ defined by

$$\underset{\sim}{\xi}_j(t) \triangleq \underset{\sim}{\xi}(t+T_j) \tag{A.12}$$

Since $\underset{\sim}{\xi}(\cdot)$ satisfies a global Lipschitz condition, the sequence is equicontinuous and there exists a subsequence which converges uniformly on the time interval $I_t = [\tau_1, \tau_2]$, $\hat{t}_0 \leq \tau_1 < \tau_2$ to a continuous function which we denote by $\underset{\sim}{\xi}_\infty(\cdot)$ [17]. We extend $\underset{\sim}{\xi}_\infty(\cdot)$ so that it is defined continuously for all $t \in \mathbb{R}^1$, and $\underset{\sim}{\xi}_\infty(t) \in D$ for all $t \in \mathbb{R}^1$.

(iii) The sequences of solutions $\{\underset{\sim}{x}_j'(\cdot)\}_{j=1}^{\infty}$ and $\{\underset{\sim}{x}_j''(\cdot)\}_{j=1}^{\infty}$ defined by

$$\begin{aligned} \underset{\sim}{x}_j'(t) &\triangleq \underset{\sim}{x}'(t+T_j) \\ \underset{\sim}{x}_j''(t) &\triangleq \underset{\sim}{x}''(t+T_j) \end{aligned} \tag{A.13}$$

These sequences also are equicontinuous and corresponding to the subsequence of $\{\underset{\sim}{\xi}_j(\cdot)\}$ which converges to $\underset{\sim}{\xi}_\infty(\cdot)$, the subsequences of $\{\underset{\sim}{x}_j'(\cdot)\}$ and $\{\underset{\sim}{x}_j''(\cdot)\}$ converge to $\underset{\sim}{x}_\infty'(\cdot)$ and $\underset{\sim}{x}_\infty''(\cdot)$ on I_t [17]. These are furthermore solutions of (15) when $\underset{\sim}{\xi}(\cdot) = \underset{\sim}{\xi}_\infty(\cdot)$. Corresponding to the extension of $\underset{\sim}{\xi}_\infty(\cdot)$, we extend the solutions $\underset{\sim}{x}_\infty'(\cdot)$ and $\underset{\sim}{x}_\infty''(\cdot)$ such that (possibly redefining $\underset{\sim}{\xi}_\infty(\cdot)$ if necessary for $t \notin I_t$) $\underset{\sim}{x}_\infty'(t)$, $\underset{\sim}{x}_\infty''(t) \in D_x$ for all $t \geq \hat{t}_0$. By construction, from (A.10),

$$0 > \frac{d}{dt}\mathcal{V}_\Delta\left(\underset{\sim}{x}_j'(t), \underset{\sim}{x}_j''(t)\right) \geq -\frac{1}{j} \quad , \quad \forall\ t \in I_t \tag{A.16}$$

Hence $\frac{d}{dt}\mathcal{V}_{\Delta}\left(\underset{\sim}{x}'_{\infty}(t),\underset{\sim}{x}''_{\infty}(t)\right) = 0$ for all $t \in I_t$. This contradicts (35). ∎

Remark: The condition that $\underset{\sim}{\xi}(\cdot)$ satisfies a global Lipschitz condition is necessary so that $\underset{\sim}{\xi}_{\infty}(\cdot)$ is "continuous in the same way as $\underset{\sim}{\xi}(\cdot)$." For example, $\xi(t) \stackrel{\Delta}{=} \cos(t)^2$ is not Lipschitz continuous at "$t = \infty$."

The proof of Corollary A.4 is identical to the proof of Theorem A.2 and Corollary A.1 since the behavior of $\mathcal{V}_{\Delta}(t)$ is the same in all instances.

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FIGURE CAPTIONS

Fig. 1 The Dynamic Nonlinear Network $\mathcal{N}$.

Fig. 2 (a) A Network whose Waveforms are Bounded and (Unless $\omega\sqrt{LC}$ is a Rational Number) Almost Periodic. (b) and (c) Networks with Eventually Uniformly Bounded Solutions and Unique Steady-State Solutions, (d) A Network which Oscillates.

Fig. 3 (a) the v-i Curve of Resistor R_1; the Function g_{R_1} is Eventually Strictly Passive, (b) The v-i Curve of Resistor R_2; the Function g_{R_2} is Strictly Passive. The Composite Function $g_R = (g_{R_1}, g_{R_2})^T$ Is Not Eventually Strictly Passive.

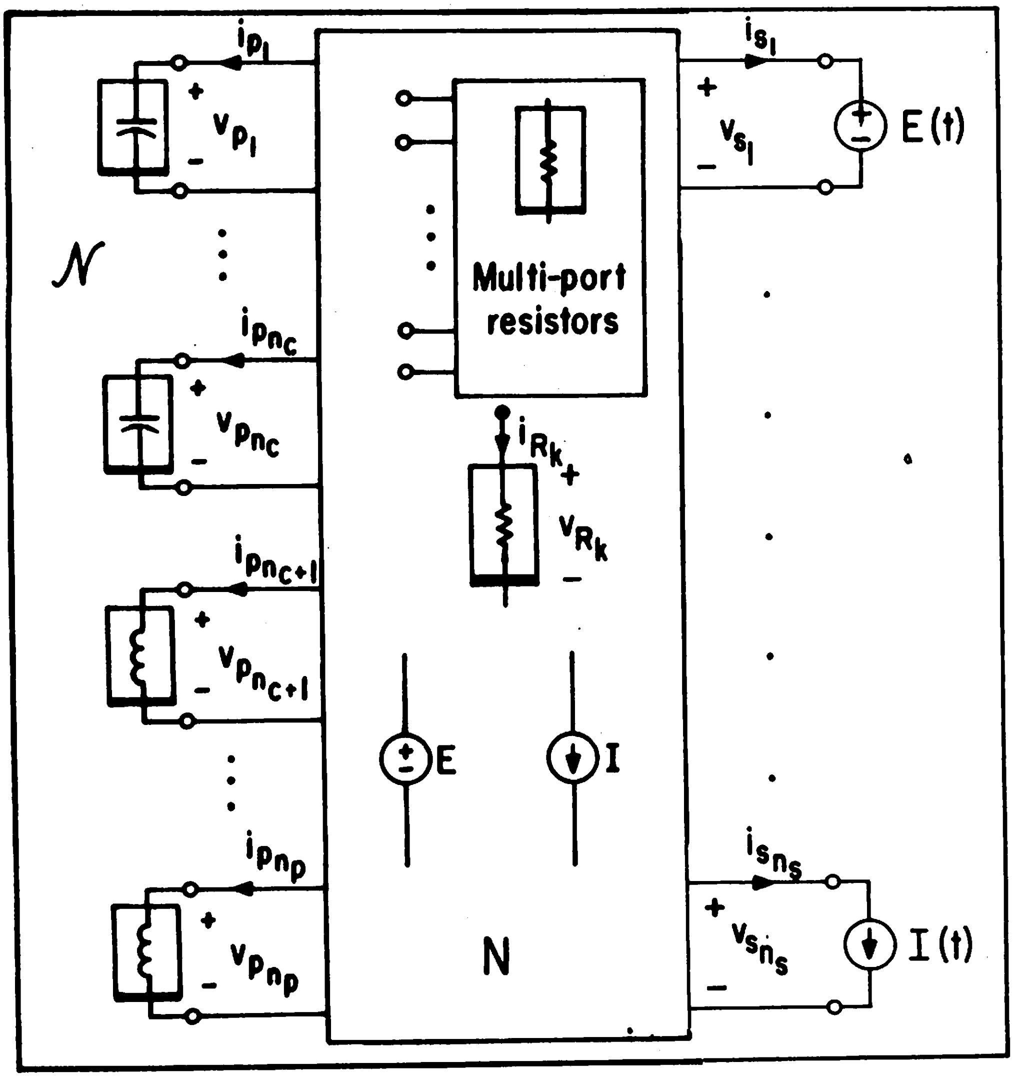
Fig. 4 A Transistor Network whose Solutions are Eventually Uniformly Bounded (Example 1) and which has a Unique Steady-State Solution (Example 3). The Linear Resistors are Described by Their Conductances.

Fig. 5 A Network whose Solutions are Eventually Uniformly Bounded (Example 2) and has More than One Steady-State Solution (Example 6).

Fig. 6 A Network with a Unique Steady-State Solution (Example 4).

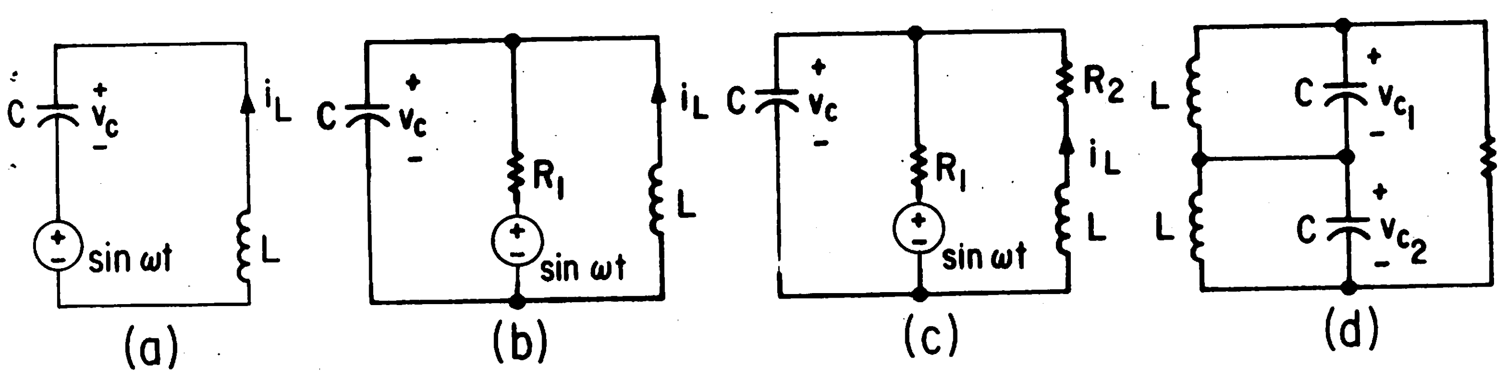
Fig. 7 A Network with a "Small Signal" Input and a Unique Steady-State Solution (Example 5)

Fig. 8 Two Local Steady-State Waveforms of Fig. 5. (a) A Periodic Waveform (ω=1); $\phi_L(t)$ is the Upper Waveform and $q_C(t)$ is the Lower Waveform, (b) An "Almost Subharmonic" Waveform; $\phi_L(t)$ is the Upper Waveform and $q_C(t)$ is the Lower Waveform, (c) $\phi_L(t)$ of Both Waveforms.



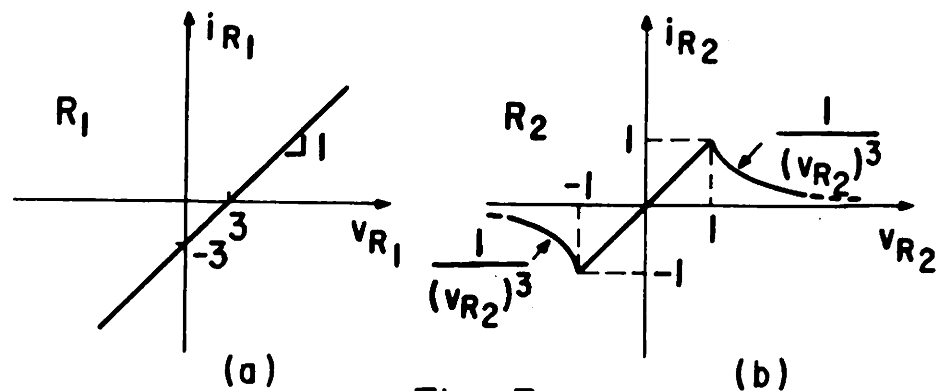
i_{p_1}
v_{p_1}
$\mathcal{N}$
$i_{p_{n_c}}$
$v_{p_{n_c}}$
$i_{p_{n_c+1}}$
$v_{p_{n_c+1}}$
$i_{p_{n_p}}$
$v_{p_{n_p}}$
Multi-port resistors
i_{R_k}
v_{R_k}
E
I
N
i_{s_1}
v_{s_1}
E(t)
$i_{s_{n_s}}$
$v_{s_{n_s}}$
I(t)

Fig. 1



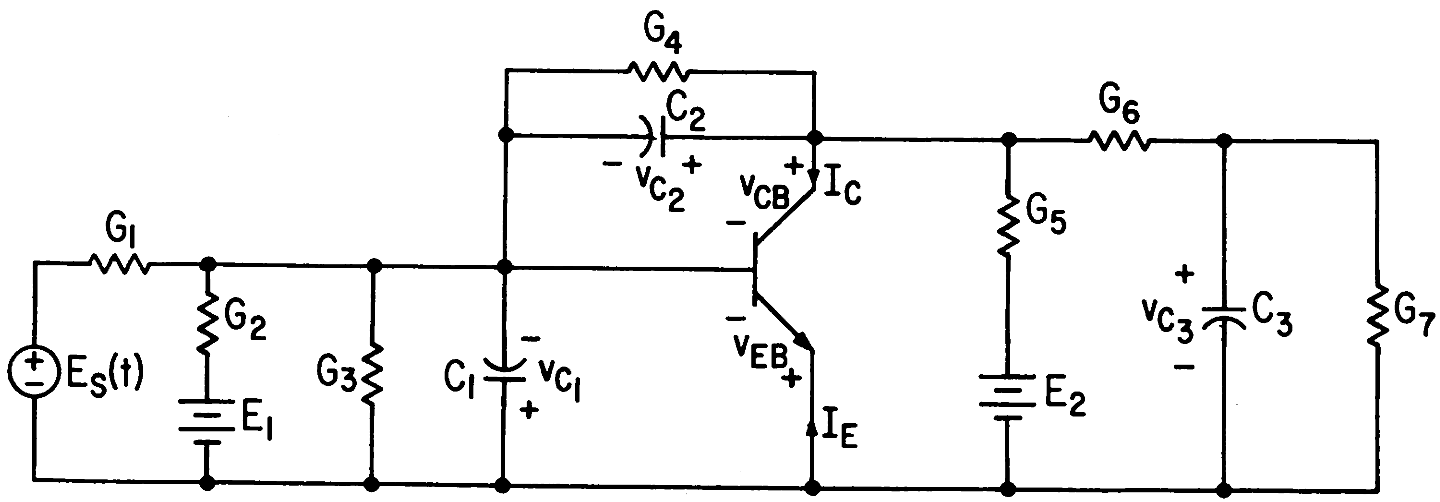
C
v_c
i_L
sin ωt
L
(a)
C
v_c
R_1
i_L
L
sin ωt
(b)
C
v_c
R_1
R_2
i_L
L
sin ωt
(c)
L
C
v_{c_1}
R
L
C
v_{c_2}
(d)

Fig. 2



R1
iR1
vR1
1
1
3
-3
(a)
R2
iR2
vR2
1
-1
1
-1
1/(vR2)^3
1/(vR2)^3
(b)

Fig. 3



G1
G2
G3
G4
G5
G6
G7
C1
C2
C3
E1
E2
Es(t)
vC1
vC2
vC3
vCB
vEB
IC
IE

Fig. 4

$$i_L = 0.46(\phi_L)^5 + 0.005\,\phi_L$$

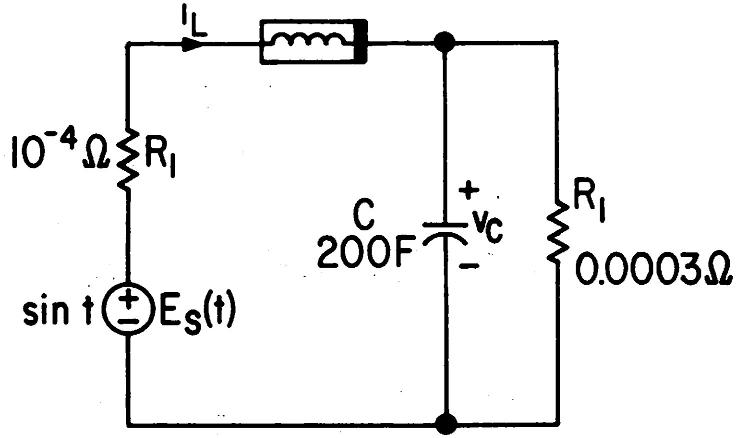
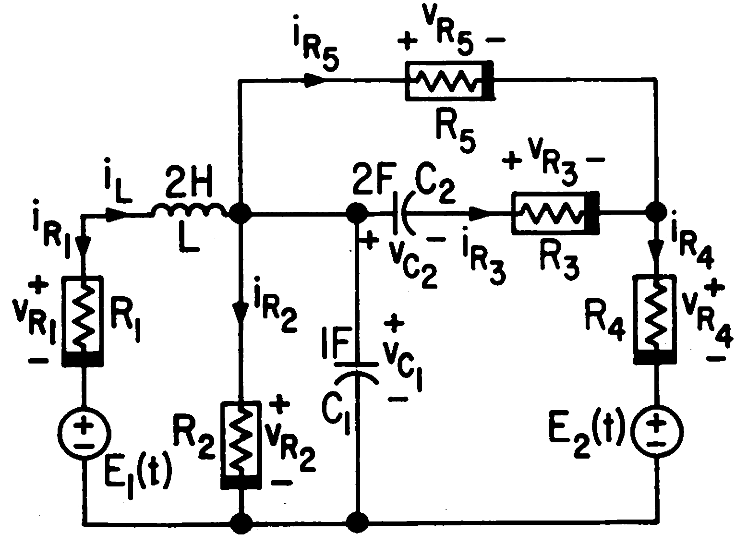


Fig. 5



$$v_{R_1} = i_{R_1} + (i_{R_1})^3 + 2$$

$$i_{R_2} = 2v_{R_2} + \sin(v_{R_2})$$

$$i_{R_3} = 2v_{R_3} + e^{v_{R_3}}$$

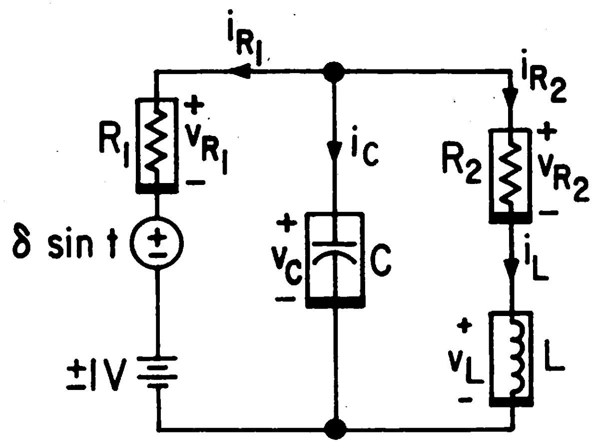
$$i_{R_4} = 3v_{R_4}$$

$$i_{R_5} = v_{R_5} + (v_{R_5})^2 + (v_{R_5})^3$$

$$E_1(t) = \sin t + e^{-t}$$

$$E_2(t) = \cos\left(\frac{1}{2}t\right)$$

Fig. 6



$$v_C = \frac{3}{2}q_C + \frac{1}{8}\sin(4q_C)$$

$$i_L = \phi_L$$

$$i_{R_1} = v_{R_1}$$

$$v_{R_2} = i_{R_2} + \frac{1}{3}(i_{R_2})^3$$

Fig. 7

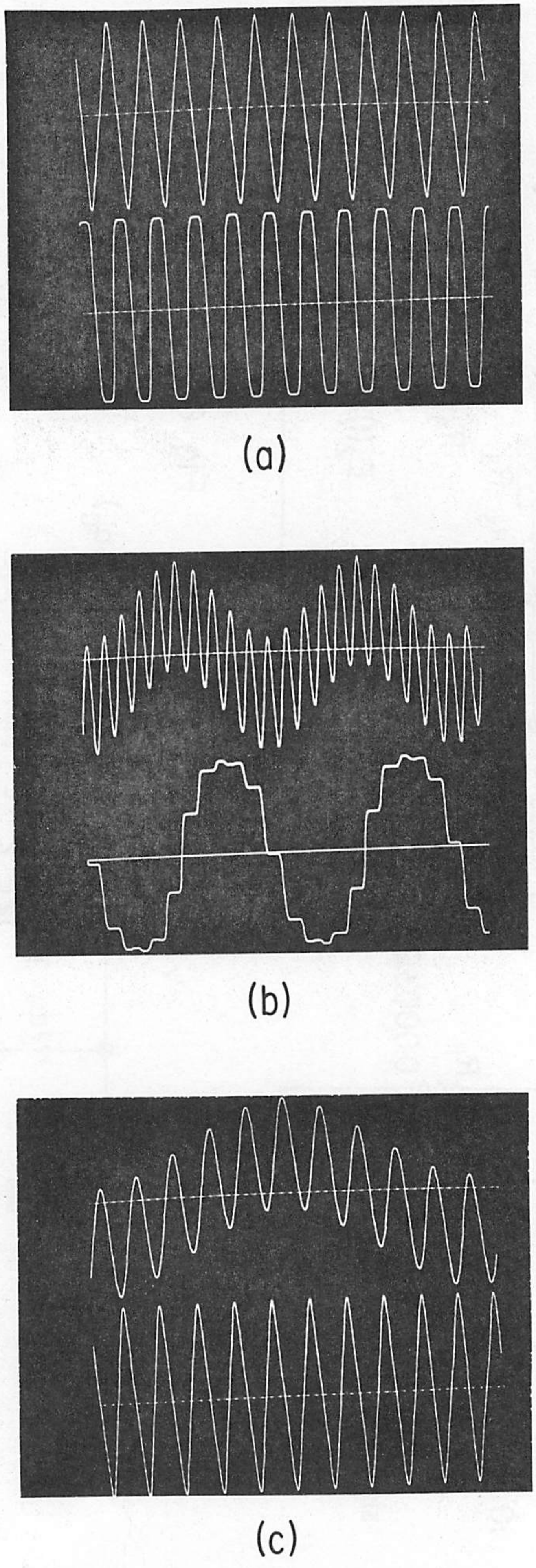


Fig 8