THE FEEDBACK INTERCONNECTION OF LUMPED LINEAR TIME-INVARIANT SYSTEMS

by

C. A. Desoer and W. S. Chan

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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C. A. Desoer and W. S. Chan

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

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Abstract

This paper considers the feedback interconnection of two multi-input multi-output subsystems characterized by rational transfer functions $\hat{G}_1$ and $\hat{G}_2$. These transfer functions are not assumed to be proper nor exponentially stable. The effect of output disturbances on stability is taken into account. Ten examples are given to show that instabilities may appear anywhere around the loop. Next under a sequence of successively more restrictive assumptions, we prove four sets of necessary and sufficient conditions for the exponential stability of the system. Using coprime factorizations, we obtain four equivalent expressions for the system characteristic polynomial. Two stability tests are derived, the first one is based exclusively on transfer functions, the second is based on the characteristic polynomial. The paper ends by providing translation rules for reformulating all definitions and theorems for the discrete-time case (i.e. instead of Laplace transforms use Z-transforms, etc...).

I. Introduction

A general input-output theory of arbitrary interconnections of systems should clearly start with the feedback interconnection. It would also be very valuable to have a reasonably complete and systematic treatment of the lumped linear time-invariant case, firstly because of its importance in practice; secondly, because its study reduces to a purely algebraic problem which can be given an easily accessible as well as thorough treatment. Also because of the algebraic nature of the techniques, the continuous-time and the discrete-time treatments are essentially isomorphic as indicated in Sec. VII, below.

We consider the multi-input multi-output feedback system shown in Fig. I: we think of u_1 as the (vector) input and of u_2 as the effect of output disturbances. Our study starts, in Sec. II, by describing the system under consideration and imposing only one assumption, Eq. (3) below, without which the interconnection does not make sense. Section III presents ten examples whose purpose is to show that, unless special assumptions are made on the subsystems, instabilities may appear anywhere around the loop! As a preparation, we include two fundamental lemmas which will greatly simplify the analysis to follow. Section IV is devoted to a detailed study of necessary and sufficient conditions for the stability of the system under a sequence of successively more restricting assumptions on the two subsystems. Section V briefly describes the four forms of the characteristic polynomial of the system and shows how they are related to the transfer functions. Section VI describes algorithmically the stability tests: each test uses necessary and sufficient conditions, therefore failure of any part of the test implies instability. Section VII specifies the rule required to translate all the stability results to the discrete-time case.

Of course the present paper is based on an enormous amount of work done by many authors in many countries. Our contribution is to be found along several directions: first in the area of problem formulation: the ten examples alert the reader to the extent of the problem; for example it is not enough to restrict oneself to the transfer functions $u_1 \mapsto y_1$ and $u_1 \mapsto e_1$ as in [1, p.73], [2], [3].

Secondly, in previous work irrelevant assumptions are often made: typically, for certain theorems, $\hat{G}_1(s)$ and $\hat{G}_2(s)$ are required to be proper when it is not necessary [2]; certain conditions are stated as sufficient when, in fact, they are necessary and sufficient [2]; special assumptions are made on the feedback, e.g., it is a polynomial matrix [3], or a constant [1] or it is exponentially stable.

Thirdly, we discovered interesting conditions where the algebraic interrelation of the transfer functions involved can lead, in certain cases, to simplified conditions: see Theorem II, below.

Fourthly, the algebraic techniques developed in [3], [5], [6], [7], [8] and subsequent observations [4] are now so well polished that it is easy to obtain simple and elegant proofs. Many of the results of this paper were presented at the Allerton Conference, October 1974 [9].

Notations. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$, $\mathbb{R}[s]$ denote respectively the fields of real numbers, of complex numbers, of rational functions with real coefficients and the commutative ring of polynomials with real coefficients. The superscripts "n" and "n×n" (as in $\mathbb{R}^n$, $\mathbb{R}(s)^{n\times n}$) denote the corresponding ordered n-tuples and n×n arrays. $\bar{\mathbb{C}}$, $\mathbb{C}_+$, $\bar{\mathbb{C}}_+$ and $\mathring{\mathbb{C}}_+$ denote respectively the complex plane including the point at infinity, the closed right-half-plane, the closed right-half-plane including the point at infinity, and the open right-half plane. Laplace transforms are identified by a "^", e.g. $\hat{G}_1$. Matrix (scalar) transfer functions are denoted by upper case letters (e.g. $\hat{H}_e$), (lower case letters, (e.g. $\hat{g}_1$), resp.). MIMO, (SISO), denotes multiple-input multiple-output (single-input single-output, resp.).

II. System Description

We consider two n-input n-output linear time-invariant lumped subsystems. We assume that they are completely characterized by their rational transfer functions: $\hat{G}_i(s) \in \mathbb{R}(s)^{n\times n}$, $i = 1,2$. The feedback interconnection is shown in Fig. I. The inputs u_i, errors e_i and outputs y_i are functions mapping from $\mathbb{R}_+$ into $\mathbb{R}^n$. From Fig. I the interconnection equations are

$$\hat{e}_1 = \hat{u}_1 - \hat{y}_2 \quad , \quad \hat{e}_2 = \hat{u}_2 + \hat{y}_1 \tag{1}$$

The subsystems equations are, for k = 1,2,

$$\hat{y}_k(s) = \hat{G}_k(s)\hat{e}_k(s) \quad , \quad \hat{G}_k(s) \in \mathbb{R}(s)^{n\times n} \tag{2}$$

We assume once and for all that

$$\det[I + \hat{G}_2(s)\ \hat{G}_1(s)] \not\equiv 0 \tag{3}$$

As we shall see later, this assumption is necessary in order to be able to define closed-loop transfer functions.[†]

We may consider two closed-loop transfer functions: the first one, $\hat{H}_y$, takes $(\hat{u}_1,\hat{u}_2)$ into $(\hat{y}_1,\hat{y}_2)$, and the second one, $\hat{H}_e$, takes $(\hat{u}_1,\hat{u}_2)$ into $(\hat{e}_1,\hat{e}_2)$:

[†]Footnotes are listed at the end of the report.

$$\begin{bmatrix}\hat{y}_1(s)\\ \hat{y}_2(s)\end{bmatrix} = \begin{bmatrix}\hat{H}_y(s)\end{bmatrix}\begin{bmatrix}\hat{u}_1(s)\\ \hat{u}_2(s)\end{bmatrix} \; ; \; \begin{bmatrix}\hat{e}_1(s)\\ \hat{e}_2(s)\end{bmatrix} = \begin{bmatrix}\hat{H}_e(s)\end{bmatrix}\begin{bmatrix}\hat{u}_1(s)\\ \hat{u}_2(s)\end{bmatrix} \quad (4)$$

Note that $\hat{H}_e(s)$ and $\hat{H}_y(s) \in \mathbb{R}(s)^{2n\times 2n}$

In view of the fact that all transfer functions are rational functions, we adopt the following definition: a transfer function $\in \mathbb{R}(s)^{n\times n}$ is said to be exponentially stable (abbr. exp. st.) iff it is proper (i.e. bounded at infinity) and has all its poles in the open left-half plane (denoted by $\mathbb{C}_-$). Note that a) a rational transfer function is exp. st. if and only if it is analytic and bounded in $\bar{\mathbb{C}}_+$ ($\bar{\mathbb{C}}_+$ denotes the closed right half plane together including the point at infinity); b) a linear time-invariant system with transfer function $\hat{G}(s) \in \mathbb{R}(s)^{n\times n}$ is zero-state bounded-input bounded-output stable if and only if $\hat{G}$ is exp. st. Any transfer function which is not exp. st. will be called unstable.

The concept of exponential stability is very strict: let $\hat{H}(s) \in \mathbb{R}(s)^{n\times n}$ be exp. st. and let $\hat{y}(s) = \hat{H}(s)\hat{u}(s)$, then it is well-known that

(a) if $u \in L_p^n$ for $1 \leq p \leq \infty$, then $y \in L_p^n$ and $\|y\|_p \leq \|H\|_1\|u\|_p$;

(b) if $u \in L_\infty^n$ and as $t \to \infty$, $u(t) \to u_\infty$, a constant vector in $\mathbb{R}^n$, then $y(t) \to y_\infty = \hat{H}(0)u_\infty$ as $t \to \infty$;

(c) if u is periodic and applied at $t = 0$, then $y(t)$ approaches exponentially the periodic response whose K^{th} Fourier coefficient is $\hat{H}(\frac{jK2\pi}{T}) \times u_K$ where T is the period and u_K the K^{th} Fourier coefficient of the input.

III. Fundamental Lemmas and Examples

From (1), (2), (3) we see that

$$[\hat{H}_e(s)]^{-1} = \begin{bmatrix} I & \hat{G}_2(s) \\ -\hat{G}_1(s) & I \end{bmatrix} \tag{5}$$

By assumption (3), the matrix in (5) has an inverse in $\mathbb{R}(s)^{2n\times 2n}$. By direct computation from (5) we obtain

$$\hat{H}_e(s) = \begin{bmatrix} (I+\hat{G}_2(s)\hat{G}_1(s))^{-1} & -\hat{G}_2(s)(I+\hat{G}_1(s)\hat{G}_2(s))^{-1} \\ \hat{G}_1(s)(I+\hat{G}_2(s)\hat{G}_1(s))^{-1} & (I+\hat{G}_1(s)\hat{G}_2(s))^{-1} \end{bmatrix} \tag{6}$$

From (2) it follows that $\hat{H}_y$ is obtained from $\hat{H}_e$ by multiplying the first row by $\hat{G}_1$ on the left and by multiplying the second row by $\hat{G}_2$ on the left.

Introducing the $2n\times 2n$ matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

we easily obtain from (1) and (4), the important relations

$$\hat{H}_y = J(\hat{H}_e - I) \qquad ; \qquad \hat{H}_e = I - J\hat{H}_y \tag{7}$$

Fundamental Lemma I

For the MIMO system described by (1) (2) and (3), $\hat{H}_e$ is exp. st. if and only if $\hat{H}_y$ is exp. st.[§]

This lemma allows us to restrict our attention to only one of $\hat{H}_e$ and $\hat{H}_y$. Since the expressions describing $\hat{H}_e$ are simpler, we will work with $\hat{H}_e$ exclusively. For $i,j = 1,2$ we use $\hat{H}_{eij}$ and $\hat{H}_{yij}$ to denote the (i,j) $n\times n$ submatrix of $\hat{H}_e$ and $\hat{H}_y$ respectively.

The following well-established identities will be used repeatedly

[§]All proofs are to be found in the Appendix.

throughout this paper.

$$\det(I+\hat{G}_1\hat{G}_2) = \det(I+\hat{G}_2\hat{G}_1) \tag{8}$$

$$(I+\hat{G}_i\hat{G}_j)^{-1}\hat{G}_i = \hat{G}_i(I+\hat{G}_j\hat{G}_i)^{-1} \quad \forall i,j = 1,2 \tag{9}$$

and $$I - \hat{G}_i\hat{G}_j(I+\hat{G}_i\hat{G}_j)^{-1} = (I+\hat{G}_i\hat{G}_j)^{-1} \quad \forall i,j = 1,2 \tag{10}$$

The following lemma will be used in extending our simplifying theorems.

Fundamental Lemma II

Let (i,j) denote any one of the following ordered pairs (1,1), (1,2), (2,1), (2,2).

If (a) $\det(I+\hat{G}_2(s)\hat{G}_1(s)) \neq 0$, $\forall s \in \bar{\mathbb{C}}_+$

and (b) $\hat{H}_{eij}$ is analytic at every $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$ and of $\hat{G}_2$

then $\hat{H}_{eij}$ is exponentially stable.

Consider now $\hat{H}_e$ as shown in (6): $\hat{H}_e$ has four n×n submatrices where each of them may be exp. st. or unstable. This gives $16 = 2^4$ patterns of exponential stability and instability; the number is further reduced to 10 by interchanging subscripts 1 and 2. Table I shows ten examples, the corresponding submatrices $\hat{H}_{eij}$ and patterns of instability are given in Table II and III respectively.

These examples show that when one allows the transfer functions $\hat{G}_1$ and $\hat{G}_2$ to be unstable, then instabilities may appear anywhere around the loop. For example as in example 6, the transfer function from u_1 to y_1, $\hat{H}_{e21} = \hat{H}_{y11}$ may be exp. st. but $-\hat{H}_{e12} = \hat{H}_{y22}$ may be unstable. Since in practical systems there are output disturbances, the "input" u_2 of Fig. I may not be assumed to be identically zero. It is for this reason that we take as stability requirement for the system of Fig. I the condition that $\hat{H}_e$ and $\hat{H}_y$ be exp. st. By Lemma I above, we need only consider the stability of $\hat{H}_e$. Thus in principle we should check the exp. st. of the four n×n submatrices of $\hat{H}_e$. In the theorems that follow we exhibit various assumptions under which the necessary and sufficient condition for exp. st. simplify.

IV. Simplifying Theorems

We state below four sets of necessary and sufficient conditions for exponential stability under successively more restrictive assumptions.

Theorem I

Consider the MIMO system described by (1), (2) and (3). $\hat{H}_e$ and $\hat{H}_y$ are exp. st. if and only if (a) $\det(I+\hat{G}_2(s)\hat{G}_1(s)) \neq 0$, $\forall s \in \bar{\mathbb{C}}_+$ and (b) $\forall i,j = 1,2$, $\hat{H}_{eij}$ is analytic at every $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$ and $\hat{G}_2$.

Corollary I (single-input single-output)

Consider the SISO system described by (1), (2) and (3). $\hat{H}_e$ and $\hat{H}_y$ are exp. st. if and only if $\hat{H}_{e12} \triangleq \frac{-\hat{g}_2}{1+\hat{g}_1\hat{g}_2}$ and $\hat{H}_{e21} \triangleq \frac{\hat{g}_1}{1+\hat{g}_2\hat{g}_1}$ are exp. st.

Examples 7 and 9 show that for MIMO case, Corollary I does not hold. The underlying reason is that it is only for a scalar transfer function that a zero cannot coincide with a pole. In the MIMO case, a transfer function may have zeros which coincide with poles: to wit $\hat{G}(s) = \text{diag}[(s-1)/(s+1),\ (s+1)/(s-1)]$ has a pole as well as a zero at +1 and at -1 [12].

The following theorem gives the condition under which the exp. stability of $\hat{H}_{e12}$ and $\hat{H}_{e21}$ imply that of $\hat{H}_e$ for MIMO case.

Theorem II

Consider the MIMO system described by (1), (2) and (3). If $\hat{G}_1$, $\hat{G}_2$

have no common $\bar{\mathbb{C}}_+$-pole, then $\hat{H}_e$ and $\hat{H}_y$ are exp. st. if and only if $\hat{H}_{e12} \triangleq -\hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1}$ and $\hat{H}_{e21} \triangleq \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$ are exp. st.

Corollary II

Consider the MIMO system described by (1), (2) and (3). If $\hat{G}_1$, $\hat{G}_2$ have no common $\bar{\mathbb{C}}_+$-pole, then $\hat{H}_e$ and $\hat{H}_y$ are exp. st. if and only if (a) $\det(I+\hat{G}_2(s)\hat{G}_1(s)) \neq 0 \ \forall s \in \bar{\mathbb{C}}_+$ and (b) $\hat{H}_{e12} \triangleq -\hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1}$ and $\hat{H}_{e21} \triangleq \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$ are analytic at every $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$ and $\hat{G}_2$.

Theorem III

Consider the MIMO system described by (1), (2) and (3). If $\hat{G}_2$ is exp. st., then $\hat{H}_e$ and $\hat{H}_y$ are exp. st. if and only if $\hat{H}_{e21} \triangleq \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$ is exp. st.

Since $\hat{H}_{e21} = \hat{H}_{y11}$, this theorem says that if $\hat{G}_2$ is exp. st., then we need only test the exp. stability of $\hat{H}_{y_{11}} : \hat{u}_1 \mapsto \hat{y}_1$ to guarantee the exp. stability of $\hat{H}_e$ and $\hat{H}_y$. It justifies the elementary approaches to discussion of stability of MIMO systems.

We also observe that by symmetry, we have if $\hat{G}_1$ is exp. st., then $\hat{H}_e$ and $\hat{H}_y$ are exp. st. if and only if $\hat{H}_{e_{12}}$ is exp. st.

Corollary III

Consider the MIMO system described by (1), (2) and (3). If $\hat{G}_2$ is exp. st., then $\hat{H}_e$ and $\hat{H}_y$ are exp. st. if and only if (a) $\det(I+\hat{G}_2(s)\hat{G}_1(s)) \neq 0 \ \forall s \in \bar{\mathbb{C}}_+$ and (b) $\hat{H}_{e21} \triangleq \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$ is analytic at every $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$.

For completeness we state the well-known conditions when both $\hat{G}_1$ and $\hat{G}_2$ are exp. st. (see e.g. [2, p. 380]).

Theorem IV

Consider the MIMO system described by (1), (2) and (3). If $\hat{G}_1$, $\hat{G}_2$

are exp. st., then $\hat{H}_e$ and $\hat{H}_y$ are exp. st. if and only if $\det(I+\hat{G}_2(s)\ \hat{G}_1(s)) \neq 0\ \forall s \in \bar{\mathbb{C}}_+$.††

V. Characteristic Polynomial

Since the two subsystems constituting the feedback system of Fig. I are lumped, it is clear that the necessary and sufficient conditions for exponential stability for $\hat{H}_e$ and $\hat{H}_y$ can be obtained by requiring that all the zeros of some characteristic polynomial be in $\mathring{\mathbb{C}}_-$, the open left half plane. In the scalar case where, for $i = 1,2$, $\hat{g}_i(s) = n_i(s)/d_i(s)$, (the polynomials n_i and d_i are coprime, i.e. have no common factor which is a polynomial of positive degree), it is easy to see that the characteristic polynomial is

$$n_1(s)n_2(s) + d_1(s)d_2(s)$$

This fact becomes clear by examining the four scalar transfer functions appearing in $\hat{H}_e$ and noting that neither a (possible) common factor of n_1 and d_2 nor a (possible) common factor of n_2 and d_1 may cause a cancellation in all four transfer functions. Thus for the single-input single-output case:

$$p \in \mathbb{C} \text{ is a pole of } \hat{H}_e \Leftrightarrow n_1(p)n_2(p) + d_1(p)d_2(p) = 0 \qquad (11)$$

To tackle the multi-input multi-output case, we need to use the polynomial factorization of rational matrices [1], [3], [6], [7]. Let $\mathbb{R}[s]^{n\times n}$ denote the non-commutative ring of polynomials with coefficients in $\mathbb{R}^{n\times n}$, [10]. For $k = 1,2$, consider the left-coprime and right-coprime factorization of $\hat{G}_k$:

$$\hat{G}_k = N_{kr}D_{kr}^{-1} = D_{k\ell}^{-1}N_{k\ell} \tag{12}$$

(where the second subscript is r or ℓ according to whether the factorization is right or left coprime) we have for $k = 1,2$

$$N_{k\ell},\ N_{kr},\ D_{k\ell},\ D_{kr} \in \mathbb{R}[s]^{n\times n} \tag{13}$$

and the coprime condition,

$$\text{rank}\begin{bmatrix} D_{kr}(s) \\ N_{kr}(s) \end{bmatrix} = \text{rank}\ [N_{k\ell}(s) \quad D_{k\ell}(s)] = n \quad \forall s \in \mathbb{C} \tag{14}$$

In order to simplify later expressions, we shall assume from now on, that the polynomial matrices in the factorizations (12) are multiplied by a suitable nonzero constant so that the polynomials $\det[D_{kr}(s)]$ and $\det[D_{k\ell}(s)]$ are monic. It is well known that [1], [3], [6], [7] $p \in \mathbb{C}$ is a pole of $\hat{G}_k \Leftrightarrow \det D_{k\ell}(p) = 0 \Leftrightarrow \det D_{kr}(p) = 0$.

Let us now calculate the factorization of $\hat{H}_e$

$$\hat{H}_e = \begin{bmatrix} I & \hat{G}_2 \\ -\hat{G}_1 & I \end{bmatrix}^{-1} \tag{15}$$

$$= \left[\begin{array}{c|c} I & N_{2r}D_{2r}^{-1} \\ \hline -N_{1r}D_{1r}^{-1} & I \end{array}\right]^{-1} = \begin{bmatrix} D_{1r} & 0 \\ 0 & D_{2r} \end{bmatrix}\begin{bmatrix} D_{1r} & N_{2r} \\ -N_{1r} & D_{2r} \end{bmatrix}^{-1} \tag{16}$$

$$= \left[\begin{array}{c|c} I & D_{2\ell}^{-1}N_{2\ell} \\ \hline -D_{1\ell}^{-1}N_{1\ell} & I \end{array}\right]^{-1} = \begin{bmatrix} D_{2\ell} & N_{2\ell} \\ -N_{1\ell} & D_{1\ell} \end{bmatrix}^{-1}\begin{bmatrix} D_{2\ell} & 0 \\ 0 & D_{1\ell} \end{bmatrix} \tag{17}$$

It is easily shown that the two factorizations of $\hat{H}_e$ given in (16) and (17) are right-coprime and left-coprime, resp.

We also have two other possible factorizations

$$\hat{H}_e = \left[\begin{array}{c|c} I & N_{2r}D_{2r}^{-1} \\ \hline -D_{1\ell}^{-1}N_{1\ell} & I \end{array}\right]^{-1} = \begin{bmatrix} I & 0 \\ 0 & D_{2r} \end{bmatrix} \begin{bmatrix} I & N_{2r} \\ -N_{1\ell} & D_{1\ell}D_{2r} \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & D_{1\ell} \end{bmatrix} \quad (18)$$

$$= \left[\begin{array}{c|c} I & D_{2\ell}^{-1}N_{2\ell} \\ \hline -N_{1r}D_{1r}^{-1} & I \end{array}\right]^{-1} = \begin{bmatrix} D_{1r} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{2\ell}D_{1r} & N_{2\ell} \\ -N_{1r} & I \end{bmatrix}^{-1} \begin{bmatrix} D_{2\ell} & 0 \\ 0 & I \end{bmatrix} \quad (19)$$

Using (14) it is easy to show that the right-hand sides of (18) and of (19) are such that their first two factors are right-coprime and their last two factors are left coprime.

Theorem V

Given the notations above, for the MIMO system described by (1), (2) and (3) we have

$$p \in \mathbb{C} \text{ is a pole of } \hat{H}_e \Leftrightarrow \det[D_{1\ell}D_{2r} + N_{1\ell}N_{2r}](p) = 0 \quad (20)$$

$$\Leftrightarrow \det[D_{2\ell}D_{1r} + N_{2\ell}N_{1r}](p) = 0 \quad (21)$$

$$\Leftrightarrow \det\begin{bmatrix} D_{1r} & N_{2r} \\ -N_{1r} & D_{2r} \end{bmatrix}(p) = 0 \quad (22)$$

$$\Leftrightarrow \det\begin{bmatrix} D_{2\ell} & N_{2\ell} \\ -N_{1\ell} & D_{1\ell} \end{bmatrix}(p) = 0 \quad (23)$$

Note that Theorem V does not consider the behavior of $\hat{H}_e$ at infinity.

Theorem V implies that the four polynomials in (20)-(23) differ at most by a nonzero constant factor. Let us normalize these expressions to their common monic polynomial form, say $\chi(s)$. The polynomial $\chi(s)$ will be called the characteristic polynomial of the system described by (1), (2) and (3).

Example:

Let us calculate $\chi(s)$ for Example 6 of Table I.

$$\hat{G}_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 0 & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} 1 & s+1 \\ 0 & s \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s(s+1) \end{bmatrix}^{-1} = N_{1r} \times D_{1r}^{-1} \tag{24}$$

$$\hat{G}_2(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s+1} \\ 0 & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s+1 \end{bmatrix}^{-1} = N_{2r} \times D_{2r}^{-1} \tag{25}$$

By (22)

$$\chi(s) = \det\begin{bmatrix} D_{1r} & N_{2r} \\ -N_{1r} & D_{2r} \end{bmatrix}(s)$$

$$= s(s^2+s+1)(s^2+2s+2)$$

Remark. Theorems I to IV and the examples of Sec. III show that it is only when both $\hat{G}_1$ and $\hat{G}_2$ are exp. st. that $\det[I + \hat{G}_2(s)\hat{G}_1(s)] \neq 0$ in $\bar{\mathbb{C}}_+$ is a valid stability criterion. Some insight in its failure to be a valid criterion in more general cases is obtained if we use the factorizations (16), (17) to relate the four expressions of the characteristic polynomial to $\det(I+\hat{G}_2\hat{G}_1)$. The result is

$$\det[I + \hat{G}_2(s)\hat{G}_1(s)] = \det[I + \hat{G}_1(s)\hat{G}_2(s)] = \frac{\chi(s)}{\det[D_{1r}(s)]\det[D_{2r}(s)]} \tag{26}$$

and three other similar expressions obtained by replacing the subscripts r by ℓ.

In general cancellations of common factors involving $\mathbb{C}_+$-zeros in the right hand side of (26) may occur; in that case there is a $p \in \mathbb{C}_+$ which is a zero of $\chi(s)$ and not a zero of $\det[I + \hat{G}_2(s)\hat{G}_1(s)]$.

Example:

Consider Example 6 of Table I. By direct calculation, we have

$$\det(I+\hat{G}_2\hat{G}_1) = \frac{(s^2+s+1)(s^2+2s+2)}{s(s+1)^3} .$$

From (24) and (25), we have

$\det D_{1r} = s(s+1)^2$ and

$\det D_{2r} = s(s+1)$.

Formula (26), which is obtained strictly on the basis of transfer function descriptions of the sybsystems 1 and 2, should be compared to the result of Hsu and Chen [11], [1], namely, that

$$\det(I+\hat{G}_2(s)\hat{G}_1(s)) = k\,\frac{\det(sI-A)}{\det(sI-A_1)\det(sI-A_2)} \tag{27}$$

where k is a nonzero constant, A_1 and A_2 are the A-matrices of minimal representations of $\hat{G}_1$ and $\hat{G}_2$, and A is the A-matrix of the state representation of the feedback system which uses as state space the product of the state spaces of minimal representations of $\hat{G}_1$ and $\hat{G}_2$. The connection between (26) and (27) is further illuminated if we note that $\det(sI-A_i) = \det[D_{ir}(s)] = \det[D_{i\ell}(s)]$, for i = 1,2.([7, p. 22], [5, p. 160]).

VI. Stability Tests

Consider the MIMO systems described by (1), (2) and (3). In the following we use the necessary and sufficient conditions proved above to test for the exp. stability of $\hat{H}_e$ and $\hat{H}_y$ under various conditions. Consequently, if any part of a test fails, the system is unstable.

Test using transfer functions:

Case 1:

If $\hat{G}_1$, $\hat{G}_2$ are exp. st., test for $\det(I+\hat{G}_2(s)\hat{G}_1(s)) \neq 0 \ \forall s \in \bar{\mathbb{C}}_+$.†††

Case 2:

If $\hat{G}_1$ is unstable and $\hat{G}_2$ is exp. st., test for (1) $\det(I+\hat{G}_2(s)\hat{G}_1(s)) \neq 0$ $\forall s \in \bar{\mathbb{C}}_+$ and (2) $\hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$ analytic at every $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$.

Case 3:

If $\hat{G}_1, \hat{G}_2$ are unstable with no common $\bar{\mathbb{C}}_+$-pole, test for (1) $\det(I+\hat{G}_2(s)\hat{G}_1(s)) \neq 0 \ \forall s \in \bar{\mathbb{C}}_+$ and (2) $\hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$ and $(I+\hat{G}_2\hat{G}_1)^{-1}\hat{G}_2$ analytic at every $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$ and $\hat{G}_2$.

Case 4:

If $\hat{G}_1$, $\hat{G}_2$ are unstable, test for (1) $\det(I+\hat{G}_2(s)\hat{G}_1(s)) \neq 0 \ \forall s \in \bar{\mathbb{C}}_+$ and (2) $\hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$, $(I+\hat{G}_2\hat{G}_1)^{-1}\hat{G}_2$, $\hat{G}_2\hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$, $\hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}\hat{G}_2$ analytic at every $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$ and $\hat{G}_2$.

<u>Test using the characteristic polynomial</u>

Step I: Check that all four submatrices of $\hat{H}_e$ are proper. (Note: if $\hat{G}_1$, $\hat{G}_2$ are proper and if $\det[I+\hat{G}_2(\infty)\hat{G}_1(\infty)] \neq 0$, then $\hat{H}_e$ is proper.)

Step II: Factor $\hat{G}_1$ and $\hat{G}_2$ using standard algorithms. (See [1], [3], [6], [7].)

Calculate one of the polynomials in Theorem V.

Step III: Apply to the characteristic polynomial $\chi(s)$, either Liénard-Chipart Test or the graphical test.

VII. <u>The Discrete-Time Case</u>

All the results above are stated for the continuous-time case. Since all the proofs are purely algebraic and are based on simple properties of rational functions, determinants and matrices, all the results above apply equally well to the discrete-time case with modifications indicated in Table IV where B(0,1) and $B(0,1)^c$ denote the open unit ball centered at 0 in $\mathbb{C}$ and its complement in $\bar{\mathbb{C}}$, respectively.

TABLE IV

Continuous-time		Discrete-time
Laplace transform	→	Z-transform
$\mathring{\mathbb{C}}_-$	→	B(0,1)
$\bar{\mathbb{C}}_+$	→	$B(0,1)^c$
$\mathbb{R}(s)^{n\times n}$	→	$\mathbb{R}(z)^{n\times n}$
$s \to \infty$	→	$z \to \infty$

CONCLUSION

This paper has given a complete discussion of the stability of the feedback interconnection of linear, time-invariant, lumped systems. The results above can be extended to certain classes of linear, time-invariant, distributed systems [13]. The cost of the extension is that the required methods are quite different: they demand from the reader a considerable sophistication in mathematical analysis. In contrast, the above treatment of lumped systems is transparent and easily accessible because it is purely algebraic.

FOOTNOTES

†It should be recalled that equ. (1) and (2) have an n-port interpretation [1, p. 37]: $\hat{G}_1$, $(\hat{G}_2)$ is the impedance, (admittance) of an n-port; e_1, (e_2) is the input current, (input voltage) to $\hat{G}_1$, $(\hat{G}_2)$; y_1, (y_2) is the output voltage, (output current) of $\hat{G}_1$, $(\hat{G}_2)$. The n-ports $\hat{G}_1$ and $\hat{G}_2$ are plugged into each other with the input current-sources u_1, in parallel, and the input voltage-sources u_2, in series.

††There are many papers where authors fail to require that $\hat{G}_1$ and $\hat{G}_2$ be exp. st. and that $\det[I+\hat{G}_2(\infty)\hat{G}_1(\infty)] \neq 0$. They merely assert that $\det[I+\hat{G}_2(s)\hat{G}_1(s)] \neq 0$, $\forall s \in \mathbb{C}_+$, (or in the SISO case $1 + \hat{g}_2(s)\hat{g}_1(s) \neq 0$ $\forall s \in \mathbb{C}_+$) is a "sufficient condition" for "system stability." The best cure for such misconception is a <u>counter</u> <u>example</u>: consider $\hat{g}_1(s) = 1/(s-1)$, $\hat{g}_2(s) = (s-1)/(s+1)$, hence $1 + \hat{g}_2(s)\hat{g}_1(s) = (s+2)/(s+1)$. However the (1,1) element of $\hat{H}_y$, i.e. the transfer function from u_1 to y_1, is unstable : $\hat{H}_{y11} \triangleq \hat{g}_1(1+\hat{g}_2\hat{g}_1)^{-1} = (s+1)/[(s-1)(s+2)]$. An easy way to perceive why the "sufficient condition" quoted above is inadequate is to study equ. (26) or (27) below.

†††There are two ways of applying the test, (I): Verify that $\det[I+\hat{G}_2(\infty)\hat{G}_1(\infty)] \neq 0$ and apply the Liénard-Chipart Test to the numerator polynomial of $\det[I+\hat{G}_2(s)\hat{G}_1(s)]$; or (II): Apply the graphical test to $\omega \mapsto \det[I+\hat{G}_2(j\omega)\hat{G}_1(j\omega)]$, $\omega \in \mathbb{R}_+$.

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APPENDIX: PROOFS

Proof of Fundamental Lemma I

The result follows immediately from (7): indeed J is a constant nonsingular matrix, therefore $p \in \bar{\mathbb{C}}$ is a pole of $\hat{H}_e$, (resp. $\hat{H}_y$) if and only if p is a pole of $J\hat{H}_e$, (resp. $J\hat{H}_y$).//

Proof of Fundamental Lemma II

Case 1: i = 1, j = 1

$$\therefore \hat{H}_{eij} = \hat{H}_{e11} = {}^{-}(I+\hat{G}_2\hat{G}_1)^{-1}$$

$$= \frac{\operatorname{adj}(I+\hat{G}_2\hat{G}_1)}{\det(I+\hat{G}_2\hat{G}_1)} \qquad \text{by Cramer's Rule}$$

(a) implies that every $\bar{\mathbb{C}}_+$-pole of $\hat{H}_{e11}$ must be a $\bar{\mathbb{C}}_+$-pole of $\operatorname{adj}(I+\hat{G}_2\hat{G}_1)$, and hence a $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$ or $\hat{G}_2$.

Therefore, by (b), $\hat{H}_{e11}$ does not have any $\bar{\mathbb{C}}_+$-pole.

Case 2: i = 1, j = 2.

$$\therefore \hat{H}_{eij} = \hat{H}_{e12} = -\hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1}$$

$$= -\hat{G}_2 \times \frac{\operatorname{adj}(I+\hat{G}_1\hat{G}_2)}{\det(I+\hat{G}_2\hat{G}_1)} \qquad \text{by Cramer's Rule.}$$

(a) implies that every $\bar{\mathbb{C}}_+$-pole of $\hat{H}_{e12}$ must be a $\bar{\mathbb{C}}_+$-pole of

$\hat{G}_2 \times \operatorname{adj}(I+\hat{G}_1\hat{G}_2)$ and hence a $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$ or $\hat{G}_2$.

Therefore, by (b), $\hat{H}_{e12}$ does not have any $\bar{\mathbb{C}}_+$-pole.

The proof of the remaining two cases follows in a similar manner.//

Proof of Theorem I

$\Leftarrow$ By Fundamental Lemma II, all four submatrices of $\hat{H}_e$ are exp. st. and hence so is $\hat{H}_e$.

$\Rightarrow$ Note that

$$\det \hat{H}_{e11} = \det[(I+\hat{G}_2\hat{G}_1)^{-1}]$$

$$= \frac{1}{\det(I+\hat{G}_2\hat{G}_1)}.$$

Since $\hat{H}_{e11}$ is exp. st., i.e. analytic and bounded in $\bar{\mathbb{C}}_+$, (a) follows. (b) follows immediately from the exp. stability of $\hat{H}_e$. //

<u>Proof of Corollary I</u>

$\Rightarrow$ Immediate

$\Leftarrow$ First, we note that for SISO case,

$$(1+\hat{g}_1\hat{g}_2)^{-1} = (1+\hat{g}_2\hat{g}_1)^{-1}$$

Suppose, for the sake of contradiction, $(1+\hat{g}_1\hat{g}_2)^{-1}$ has a $\bar{\mathbb{C}}_+$-pole, say, p.

Since $\hat{H}_{e12} \stackrel{\Delta}{=} -\hat{g}_2(1+\hat{g}_1\hat{g}_2)^{-1}$ and $\hat{H}_{e21} \stackrel{\Delta}{=} \hat{g}_1(1+\hat{g}_2\hat{g}_1)^{-1}$ are scalar and exp. st., we have $\hat{g}_2(p) = 0 = \hat{g}_1(p)$.

$$\therefore\ [1 + \hat{g}_1(p)\hat{g}_2(p)]^{-1} = 1$$

which contradicts the assumption that p is a pole of $(1+\hat{g}_1\hat{g}_2)^{-1}$.
Hence $(1+\hat{g}_1\hat{g}_2)^{-1}$ is exp. st.
Since every submatrix of $\hat{H}_e$ is exp. st., so is $\hat{H}_e$.//

<u>Proof of Theorem II</u>

$\Rightarrow$ Immediate

$$\Leftarrow\ I - \hat{G}_2 \cdot \hat{H}_{e21} = I - \hat{G}_2\hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1} = (I+\hat{G}_2\hat{G}_1)^{-1} \quad \text{(A1)}$$

$$= I - \hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1}\hat{G}_1 = I + \hat{H}_{e12} \cdot \hat{G}_1 \quad \text{(A2)}$$

In view of (A1) and the exp. stability of $\hat{H}_{e21}$, every $\bar{\mathbb{C}}_+$-pole of $(I+\hat{G}_2\hat{G}_1)^{-1}$ must be a $\bar{\mathbb{C}}_+$-pole of $\hat{G}_2$.

Similarly, in view of (A2) and the exp. stability of $\hat{H}_{e12}$, every

$\bar{\mathbb{C}}_+$-pole of $(I+\hat{G}_2\hat{G}_1)^{-1}$ must be a $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$.

But $\hat{G}_1$ and $\hat{G}_2$ have no common $\bar{\mathbb{C}}_+$-pole.

$\therefore$ $(I+\hat{G}_2\hat{G}_1)^{-1}$ is exp. st.

Repeating the above derivation with subscripts 1 and 2 interchanged, we see that $(I+\hat{G}_1\hat{G}_2)^{-1}$ is exp. st.

Since every submatrix of $\hat{H}_e$ is exp. st., so is $\hat{H}_e$.//

<u>Proof of Corollary II</u>

$\Rightarrow$ This implication follows from Theorem I.

$\Leftarrow$ This implication follows from Fundamental Lemma II and Theorem II.//

<u>Proof of Theorem III</u>

$\Rightarrow$ Immediate

$\Leftarrow$
$$
\begin{aligned}
I - \hat{H}_{e21}\cdot\hat{G}_2 &= I - \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}\cdot\hat{G}_2 \\
&= I - \hat{G}_1\hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1} \quad \text{(using (9))} \\
&= (I+\hat{G}_1\hat{G}_2)^{-1} \qquad\qquad \text{(using (10))} \qquad\qquad \text{(A3)}
\end{aligned}
$$

In view of (A1), (A3) and the exp. stability of $\hat{G}_2$ and $\hat{H}_{e21}$ $(I+\hat{G}_2\hat{G}_1)^{-1}$, $(I+\hat{G}_1\hat{G}_2)^{-1}$ are exp. st.

The exp. stability of $\hat{G}_2$ and $(I+\hat{G}_1\hat{G}_2)^{-1}$ imply that of $\hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1}$.

Since every submatrix of $\hat{H}_e$ is exp. st., so is $\hat{H}_e$.//

<u>Proof of Corollary III</u>

$\Rightarrow$ This implication follows from Theorem I.

$\Leftarrow$ Since $\hat{G}_2$ is exp. st., condition (b) of Fundamental Lemma II becomes $\hat{H}_{eij}$ is analytic at every $\bar{\mathbb{C}}_+$-pole of $\hat{G}_1$. Hence, by assumption $\hat{H}_{e21} = \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$ satisfies the conditions of Fundamental Lemma II and so it is exp. st. Therefore, by Theorem III, $\hat{H}_e$ is exp. st.//

Proof of Theorem IV

⇒ This implication follows from Theorem I.

⇐ Since $\hat{G}_1$ and $\hat{G}_2$ are exp. st., condition (b) of Fundamental Lemma II is fulfilled for all $\hat{H}_{eij}$, i,j = 1,2. By assumption condition (a) of Fundamental Lemma II is satisfied. Hence, every submatrix $\hat{H}_{eij}$ of $\hat{H}_e$ is exp. st. and so does $\hat{H}_e$.//

Proof of Theorem V

The first two assertions follow from the coprimeness of the factors in (18), (19) and a theorem of [12]. The last two are consequences of the coprimeness of the factorizations (16) and (17).//

TABLE I

EXAMPLE NO.	$\hat{G}_1$	$\hat{G}_2$
1	$1/(s+1)$	$-1/(s+1)$
2	$\begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s-1} \\ 0 & \frac{1}{s} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{s-1} & \frac{1}{s} \\ 0 & \frac{s+2}{s+1} \end{bmatrix}$
3	$1/s$	$s/(s-1)$
4	$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s-2} \\ 0 & \frac{1}{s+2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{s-2}{s+2} \end{bmatrix}$
5	$s/(s-1)$	$(s-1)/s$
6	$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 0 & \frac{1}{s+1} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{s} & \frac{1}{s+1} \\ 0 & \frac{1}{s+1} \end{bmatrix}$
7	$\begin{bmatrix} \frac{2}{s+1} & \frac{1}{s-1} \\ 0 & \frac{2}{s+1} \end{bmatrix}$	SAME AS $\hat{G}_1$
8	$1/s$	$s/(s+1)$
9	$\begin{bmatrix} \frac{s}{s+1} & \frac{1}{s} \\ 0 & \frac{1}{s} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 0 & \frac{1}{s+1} \end{bmatrix}$
10	$2/s$	$(s+1)/(s-1)$

TABLE II

EXAMPLE NO.	$\hat{H}_{e22} \triangleq (I+\hat{G}_1\hat{G}_2)^{-1}$	$-\hat{H}_{e12} \triangleq \hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1}$	$\hat{H}_{e11} \triangleq (I+\hat{G}_2\hat{G}_1)^{-1}$	$\hat{H}_{e21} \triangleq \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$
1	$\frac{(s+1)^2}{s(s+2)}$	$\frac{-(s+1)}{s(s+2)}$	$\frac{(s+1)^2}{s(s+2)}$	$\frac{(s+1)}{s(s+2)}$
2	Let $a = s^2+s+1$, $b = s^2+2s+1$ $\begin{bmatrix} \frac{s^2-1}{a} & \frac{-(s+1)(s+2)(2s-1)}{ab} \\ 0 & \frac{s(s+1)}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{s+1}{a} & \frac{s^3-2s^2-3s+1}{(s-1)ab} \\ 0 & \frac{s(s+2)}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{s^2-1}{a} & \frac{-(s+1)^2(2s^2-2s+1)}{s(s-1)ab} \\ 0 & \frac{s(s+1)}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{(s+2)(s-1)}{a} & \frac{(s+1)(s^3-s+2)}{sab} \\ 0 & \frac{s+1}{b} \end{bmatrix}$
3	$\frac{s-1}{s}$	1	$\frac{s-1}{s}$	$\frac{s-1}{s^2}$
4	Let $a = s^2+2s+2$, $b = s^2+5s+2$ $\begin{bmatrix} \frac{(s+1)^2}{a} & -\frac{(s+1)^2(s+2)}{ab} \\ 0 & \frac{(s+2)^2}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{s+1}{a} & -\frac{(s+1)(s+2)}{ab} \\ 0 & \frac{(s+2)(s-2)}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{(s+1)^2}{a} & -\frac{(s+1)(s+2)^2}{(s-2)ab} \\ 0 & \frac{(s+2)^2}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{s+1}{a} & \frac{(s+1)^2(s+2)^2}{(s-2)ab} \\ 0 & \frac{s+2}{b} \end{bmatrix}$
5	$\frac{1}{2}$	$\frac{(s-1)}{2s}$	$\frac{1}{2}$	$\frac{s}{2(s-1)}$
6	Let $a = s^2+s+1$, $b = s^2+2s+2$ $\begin{bmatrix} \frac{s(s+1)}{a} & \frac{-(s+1)(2s+1)}{ab} \\ 0 & \frac{(s+1)^2}{a} \end{bmatrix}$	$\begin{bmatrix} \frac{s+1}{a} & \frac{(s+1)^3(s-1)}{sab} \\ 0 & \frac{s+1}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{s(s+1)}{a} & -\frac{(2s^2+2s+1)(s+1)}{sab} \\ 0 & \frac{(s+1)^2}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{s}{a} & \frac{s^3+3s^2+2s+1}{ab} \\ 0 & \frac{s+1}{b} \end{bmatrix}$
7	Let $a = s^2+2s+5$ $\begin{bmatrix} \frac{(s+1)^2}{a} & \frac{-4(s+1)^3}{(s-1)a^2} \\ 0 & \frac{(s+1)^2}{a} \end{bmatrix}$	$\begin{bmatrix} \frac{2(s+1)}{a} & \frac{(s+1)^2(s+3)}{a^2} \\ 0 & \frac{2(s+1)}{a} \end{bmatrix}$	same as $\hat{H}_{e22}$	same as $-\hat{H}_{e12}$
8	$\frac{s+1}{s+2}$	$\frac{s}{s+2}$	$\frac{s+1}{s+2}$	$\frac{s+1}{s(s+2)}$
9	Let $a = s^2+3s+1$, $b = s^2+s+1$ $\begin{bmatrix} \frac{(s+1)^2}{a} & \frac{(s+1)^3}{ab} \\ 0 & \frac{s(s+1)}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{s(s+1)}{a} & \frac{s(s+1)(s+2)}{ab} \\ 0 & \frac{s+1}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{(s+1)^2}{a} & \frac{-(2s+1)(s+1)^2}{sab} \\ 0 & \frac{s(s+1)}{b} \end{bmatrix}$	$\begin{bmatrix} \frac{s(s+1)}{a} & \frac{s(s+1)^2}{ab} \\ 0 & \frac{s+1}{b} \end{bmatrix}$
10	Let $a = s^2+s+2$ $\frac{s(s-1)}{a}$	$\frac{s(s+1)}{a}$	$\frac{s(s-1)}{a}$	$\frac{2(s-1)}{a}$

TABLE III

SUBMATRICES OF $\hat{H}_e$ \ EXAMPLE NO.	1	2	3	4	5	6	7	8	9	10
$\hat{H}_{e22} = (I+\hat{G}_1\hat{G}_2)^{-1}$	U	S	U	S	S	S	U	S	S	S
$\hat{H}_{e12} = -\hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1}$	U	U	S	S	U	U	S	S	S	S
$\hat{H}_{e11} = (I+\hat{G}_2\hat{G}_1)^{-1}$	U	U	U	U	S	U	U	S	U	S
$\hat{H}_{e21} = \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$	U	U	U	U	U	S	S	U	S	S

S, (U), indicates that the corresponding submatrices is exponentially stable, (unstable, respectively).

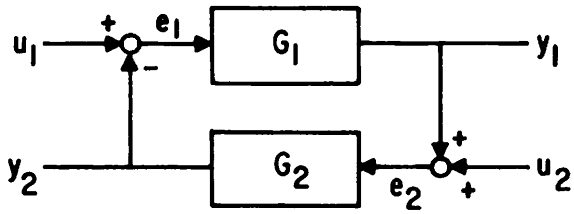

Fig. 1. Multivariable feedback system under consideration: the inputs are u_1 and u_2, the outputs are y_1, y_2 and e_1, e_2 are the errors.