OPEN-LOOP UNSTABLE CONVOLUTION FEEDBACK SYSTEMS WITH DYNAMICAL FEEDBACKS

by

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Open-loop Unstable Convolution Feedback Systems with Dynamical Feedbacks,[†]

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Abstract

This paper considers distributed multivariable convolution feedback systems characterized by $y_1 = G_1 * e_1$, $y_2 = G_2 * e_2$, $e_1 = u_1 - y_2$ and $e_2 = u_2 + y_1$ where the subsystem transfer functions $\hat{G}_1$ and $\hat{G}_2$ both admit a pseudo-coprime factorization in the subalgebra of absolutely summable distributions of order zero. The most general result, Theorem 1, gives necessary and sufficient conditions for stability of the system. This condition is specialized to the lumped case in Theorem 1L. Finally for distributed systems which have a finite number of open-loop unstable poles Theorem 1D gives an algorithmic test for stability. The graphical interpretation of both Theorem 1 and 1D is given in detail, and illustrated by examples.

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I. Introduction

This paper gives necessary and sufficient conditions for the stability of distributed n-input n-output convolution feedback systems that are open-loop unstable. An example of a system which this new theory can handle is described by (see Fig. 1 for notations)

$$\hat{G}_1(s) = \left[\begin{array}{c|c} \dfrac{e^{-\sqrt{2}\,s}}{s(s+1)} & \dfrac{2}{s+2+e^{-s}} \\ \hline \dfrac{1}{(s^2+1)^2} & \dfrac{s}{(s-2)^2} \end{array}\right] \quad ; \quad \hat{G}_2(s) = \left[\begin{array}{c|c} \dfrac{e^{-s}}{(s-1)^2} + e^{-s} & \dfrac{s}{s-2} \\ \hline 0 & \dfrac{1}{s+\sqrt{s^2+1}} \end{array}\right]$$

Note that the matrix transfer functions $\hat{G}_1$ and $\hat{G}_2$ are unstable and contain <u>multiple</u> poles in the <u>closed</u> right half-plane; they include also <u>delay terms</u> and <u>transcendental functions</u>. In fact the new theory presented below can handle any combination of polynomials in s and delay terms encountered in practice in transfer functions provided these contain a finite number of unstable poles.

The stability of distributed n-input n-output convolution feedback systems that are open-loop unstable has been discussed in [2,4,5,6,14]. In [6] and [4,5] the case of constant nonsingular feedback, respectively, stable feedback was considered. In these two cases the stability of the closed loop system was guaranteed by requiring that the transfer functions from u_1 to e_1 and to y_1 be stable (only u_1 was considered as input) and it can be shown that the presence of output disturbances represented by the "input" u_2 cannot cause instability. This is no longer the case when we allow both the feedforward and the feedback to be unstable, as considered in this paper and [14]: suppose that the transfer functions from u_1 to e_1 and to y_1 are stable, then stability is guaranteed only when inputs are applied at u_1, however output disturbances u_2 could excite unstable modes of $\hat{G}_2$ and cause closed loop instability! For this reason, as opposed to [14], we consider two inputs u_1 and u_2 and consequently two errors e_1 and e_2 and two outputs y_1 and y_2; therefore we have eight matrix transfer functions from u_i to e_j and y_j for $i,j = 1,2$. It is shown in Section III of this paper that a) for stability considerations it is sufficient to consider the four matrix transfer functions from u_i to e_j for $i,j = 1,2$ and that b) these can be aggregated to form the input-error transfer matrix of a unity feedback system giving insight to the problem. The necessity of considering each one of these four matrix transfer functions has been shown in a subsequent contribution [20]; any three of them may be stable while the fourth is unstable!

In [14], simple open-loop unstable poles were considered and the key tool was the decomposition lemma. Unfortunately that technique does not allow poles on the $j\omega$-axis and is cumbersome when multiple unstable poles are present. In the present paper $\hat{G}_1$ and $\hat{G}_2$ may have multiple unstable poles in the closed right half plane. To tackle this most general distributed case elegantly we use in section IV the recently developed pseudo-coprime factorization technique [4,5] and develop a "characteristic polynomial"--generalizing the characteristic polynomial familiar from the lumped case--to obtain the necessary and sufficient conditions for stability. These conditions are shown to be testable graphically.

In section V the specialization to the lumped case is easily carried out. Section VI considers in detail the practically important situation of distributed subsystems with a finite number of unstable poles; it describes two algorithms for testing stability and it includes two examples.

Section II collects once and for all useful definition and facts. All proofs are collected in the Appendix.

II. Useful Definitions and Facts.

Throughout the paper we shall use the convolution algebra $\mathcal{A}$ [1,2]: recall that f belongs to $\mathcal{A}$ iff, for $t < 0, f(t) = 0$, and, for $t > 0, f(t) = f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t-t_i)$ where $f_a(\cdot)$ belongs to $L^1[0,\infty)$, $f_i \in \mathbb{C}$ (the field of complex numbers) for all i, $\sum_{i=0}^{\infty} |f_i| < \infty$, $0 = t_o$, $t_i > 0$ for $i \geq 1$, and $\delta(\cdot)$ is the Dirac delta "function". Thus f is a complex-valued distribution of order zero with support on $\mathbb{R}_+$. An n-vector v, (n×n matrix A), is said to be in $\mathcal{A}^n$, ($\mathcal{A}^{n\times n}$ resp.), iff all its elements are in $\mathcal{A}$. Let $\hat{f}$ denote the Laplace transform of f: f belongs to the convolution algebra $\mathcal{A}$ if and only if $\hat{f}$ belongs to the algebra $\hat{\mathcal{A}}$ (with pointwise product). An element f of $\mathcal{A}$ is invertible in $\mathcal{A}$ iff $\inf_{\mathrm{Re}\, s \geq 0} |\hat{f}(s)| > 0$, [3] p. 150; an element A of $\mathcal{A}^{n\times n}$ is invertible in $\mathcal{A}^{n\times n}$ iff $\inf_{\mathrm{Re}\, s \geq 0} |\det \hat{A}(s)| > 0$, [5] Appendix D.3.3. In the sequel $\mathbb{C}_+$, $\mathring{\mathbb{C}}_+$, $\mathbb{C}^{n\times n}(s)$ will denote respectively the <u>closed</u> right-half of the complex plane, the <u>open</u> right-half of the complex plane and the noncommutative ring of n×n matrices whose entries are rational functions in s with complex coefficients, (if n = 1 we simply write $\mathbb{C}(s)$). An element of $\mathbb{C}^{n\times n}(s)$ is said to be (strictly) <u>proper</u> if it is (zero) bounded at infinity. We shall be concerned with n×n matrix-valued Laplace transformable distributions (L.t.d.) G with support on $\mathbb{R}_+$ whose Laplace transforms $\hat{G}$ admit so-called pseudo-coprime factorizations, [4], [5] Sec. IV.4. In order to alleviate the notation, elements in $\hat{\mathcal{A}}^{n\times n}$ will also be denoted by script letters.

Definition 1 [4,5]: A pair of elements $(\mathcal{N}_r, \mathcal{D}_r)$ where $\mathcal{N}_r$, $\mathcal{D}_r$ belong to $\hat{\mathcal{A}}^{n\times n}$ is said to be pseudo-right-coprime (p.r.c.) in $\hat{\mathcal{A}}^{n\times n}$ iff there exist elements $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ in $\hat{\mathcal{A}}^{n\times n}$ such that (i) $\det \mathcal{W}(s) \neq 0$ whenever $s \in \mathbb{C}_+$, and (ii)

$$\mathcal{U}(s)\mathcal{N}_r(s) + \mathcal{V}(s)\,\mathcal{D}_r(s) = \mathcal{W}(s) \text{ for all } s \in \mathbb{C}_+. \quad (1)$$

If in the above $\inf_{\mathrm{Re}\, s \geq 0} |\det \mathcal{W}(s)| > 0$, then the pair $(\mathcal{N}_r, \mathcal{D}_r)$ is said to be right-coprime (r.c.) in $\hat{\mathcal{A}}^{n\times n}$.

Definition 2 [4,5]: Given a L.t.d. G with support on $\mathbb{R}_+$, the ordered pair $(\mathcal{N}_r, \mathcal{D}_r)$ is said to be a pseudo-right-coprime factorization (p.r.c.f.) of $\hat{G}$ in $\hat{\mathcal{A}}^{n\times n}$ iff

(i) $\hat{G}(s) = \mathcal{N}_r(s)\,\mathcal{D}_r(s)^{-1}$ for all $s \in \mathbb{C}_+$, (2)#

(ii) the pair $(\mathcal{N}_r, \mathcal{D}_r)$ is p.r.c. in $\hat{\mathcal{A}}^{n\times n}$,

(iii) whenever $(s_i)_{i=1}^{\infty}$ is a sequence in $\mathbb{C}_+$ with $|s_i| \to \infty$, we have $\liminf_{i\to\infty} |\det \mathcal{D}_r(s_i)| > 0$.

The definition of a pseudo-left-coprime (p.ℓ.c.) pair $(\mathcal{N}_\ell, \mathcal{D}_\ell)$, a left-coprime (ℓ.c.) pair $(\mathcal{N}_\ell, \mathcal{D}_\ell)$, and a pseudo-left-coprime factorization (p.ℓ.c.f.) $(\mathcal{N}_\ell, \mathcal{D}_\ell)$ of $\hat{G}$ in $\hat{\mathcal{A}}^{n\times n}$ is completely similar: replace the subscript r by the subscript ℓ in the above definitions and interchange the order of the factors in (1) and (2). In our later work we will need a slightly more involved factorization, hence the definition:

The factorization (2) can always be performed algorithmically for the cases of section V and VI. The factorization (2) leads to formulae very close to the familiar ones encountered in the case of single-input single-output lumped systems.

Definition 3: Given a L.t.d. G with support on $\mathbb{R}_+$, the ordered triple $(\mathcal{D}_1, \mathcal{N}, \mathcal{D}_2)$, where $\mathcal{D}_1$, $\mathcal{N}$ and $\mathcal{D}_2$ are elements in $\hat{\mathcal{A}}^{n\times n}$, is said to be a pseudo-left-right-coprime factorization (p.ℓ.r.c.f.) of $\hat{G}$ in $\hat{\mathcal{A}}^{n\times n}$ iff

(i) $\hat{G}(s) = \mathcal{D}_1(s)^{-1}\mathcal{N}(s)\,\mathcal{D}_2(s)^{-1}$ for all $s \in \mathbb{C}_+$, (3)

(ii) the pair $(\mathcal{D}_1, \mathcal{N})$ is p.ℓ.c. and the pair $(\mathcal{N}, \mathcal{D}_2)$ is p.r.c.

(iii) whenever $(s_i)_{i=1}^{\infty}$ is a sequence in $\mathbb{C}_+$ with $|s_i| \to \infty$, we have $\liminf_{i\to\infty} |\det \mathcal{D}_j(s_i)| > 0$ for $j = 1, 2$.

Comment: If a given Laplace transform $\hat{G}$ admits a pseudo-coprime factorization, then $\hat{G}$ is meromorphic in $\mathring{\mathbb{C}}_+$ and all $\mathring{\mathbb{C}}_+$-singularities of $\hat{G}$ are contained in a bounded subset of $\mathbb{C}_+$. Thus $\hat{G}$ can only have poles as singularities in $\mathring{\mathbb{C}}_+$ and the $j\omega$ axis may contain singularities, which, however, cannot be poles.

Fact 1: Let G be a L.t.d. with support on $\mathbb{R}_+$ and let $\hat{G}$ admit a p.r.c.f. $(\mathcal{N}_r, \mathcal{D}_r)$ and p.ℓ.c.f. $(\mathcal{N}_\ell, \mathcal{D}_\ell)$ in $\hat{\mathcal{A}}^{n\times n}$. Let $s_o \in \mathbb{C}_+$ and let $\mathcal{B}(s_o)$ denote any sufficiently small disk centered on s_o in $\mathbb{C}$. Under these conditions

$$\hat{G} \text{ is unbounded in } \mathcal{B}(s_o) \cap \mathbb{C}_+ \tag{4}$$

if and only if any one of the following equivalent conditions hold

either $\quad \det \mathcal{D}_r(s_o) = 0,$ (5)

or $\quad \det \mathcal{D}_\ell(s_o) = 0.$ (6)

The proof is in the Appendix. We are now ready to start our system description.

III. System Description and Definition of Stability.

We consider the feedback system S shown on Fig. 1; u_1, u_2 are inputs; e_1, e_2 are errors; y_1, y_2 are outputs. The u_i, e_i, y_i for $i = 1, 2$ are functions from $\mathbb{R}_+$, (defined as $[0,\infty)$), into $\mathbb{C}^n$, (the complex n-vectors).[†] Both subsystems denoted by G_1* and G_2* are causal, possibly unstable, convolution subsystems. Thus we have the basic equations

$$e_1 = u_1 - y_2 \tag{7}$$

$$e_2 = u_2 + y_1 \tag{8}$$

$$y_1 = G_1 * e_1 \tag{9}$$

$$y_2 = G_2 * e_2 \tag{10}$$

where G_1, G_2 are the complex-valued $n \times n$ matrix impulse responses of the subsystems. We assume that, for $i = 1, 2$,

(i) G_i is a L.t.d. with support on $\mathbb{R}_+$,[§] (11)

[†]In certain manipulations (expressing cosines in exponential form) and in certain models (as a result of band-pass to low-pass transformations), the expressions become complex-valued. So for simplicity, we allow them to be complex valued throughout. Also to simplify the description of the results we assume that the u_i's, e_i's, y_i's, $(i = 1, 2)$, have n-components. If that is not the case one may insert appropriate modifications or simply add to the matrices G_i and the vectors u_i, y_i, e_i elements that are identically zero or one, at appropriate places.

[§]If for some $c \in \mathbb{C}$, $\exp(-ct)G(t)$ is a distribution of slow growth, then, for $\text{Re } s > c$, the Laplace transform of G is well defined and is an analytic function of s, [22,p.310, 23,p.213].

(ii) G_i is such that the $\hat{G}_i$ admit a p.r.c.f. $(\mathcal{N}_{ir}, \mathcal{D}_{ir})$ and a p.ℓ.c.f. $(\mathcal{N}_{i\ell}, \mathcal{D}_{i\ell})$. (12)

To avoid trivial singular cases, we assume that

$$\det[I + \hat{G}_2(s)\,\hat{G}_1(s)] = \det\,[I + \hat{G}_1(s)\,\hat{G}_2(s)] \neq 0 \text{ in } \mathbb{C}_+. \tag{13}$$

This ensures us that the system S viewed as a map from $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ into both $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is specified by a well-defined matrix transfer function. Observe that (7)-(10) can be rewritten in the form

$$\begin{bmatrix} I & \hat{G}_2(s) \\ -\hat{G}_1(s) & I \end{bmatrix} \begin{bmatrix} \hat{e}_1(s) \\ \hat{e}_2(s) \end{bmatrix} = \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \end{bmatrix} \quad \text{for all } s \in \mathbb{C}_+$$

i.e. considering the input-error system: $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$, the input-error $\hat{H}_e$ satisfies

$$\hat{H}_e(s)^{-1} = \begin{bmatrix} I & \hat{G}_2(s) \\ -\hat{G}_1(s) & I \end{bmatrix} \quad \text{for all } s \text{ in } \mathbb{C}_+. \tag{14}$$

Furthermore considering the input-output system: $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ the input-output transfer function $\hat{H}_y$ satisfies

$$\hat{H}_y(s) = \begin{bmatrix} \hat{G}_1(s) & 0 \\ 0 & \hat{G}_2(s) \end{bmatrix} \hat{H}_e(s) \quad \text{for all } s \text{ in } \mathbb{C}_+ \tag{15}$$

Introducing the symplectic matrix J, [17],

$$J \triangleq \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{15a}$$

the relation between H_e and H_y becomes

$$\hat{H}_e = I - J\hat{H}_y \tag{16}$$

or

$$\hat{H}_y = -J + J\hat{H}_e, \tag{17}$$

where we used $J^{-1} = -J$.

Hence

$$\hat{H}_y \in \hat{\mathcal{A}}^{2n\times 2n} \Leftrightarrow \hat{H}_e \in \hat{\mathcal{A}}^{2n\times 2n}. \tag{18}$$

Definition 4 (definition of stability): The system S described by (7)-(10) is said to be $\mathcal{A}$-stable iff

$$\hat{H}_y \in \hat{\mathcal{A}}^{2n\times 2n} \tag{19}$$

and

$$\hat{H}_e \in \hat{\mathcal{A}}^{2n\times 2n}. \tag{20}$$

Comments: (i) $\mathcal{A}$-stability implies that, for any $p \in [1,\infty]$, the system S viewed as taking the input $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ into the error $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ and the output $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is L^p-stable, [2]. Furthermore S takes continuous and bounded inputs (almost periodic inputs, periodic inputs, resp.) into outputs and errors belonging to the same classes, resp. [18].

(ii) Because of (18) for $\mathcal{A}$-stability to hold it is necessary and sufficient that (20) holds. Let us relate the system under consideration with systems with unity feedback [19]. Simple calculations establish the following.

Fact 2. Consider the system S described by (7-11) and (13); let $u = (u_1^T, u_2^T)^T$, $e = (e_1^T, e_2^T)^T$, $y = (y_1^T, y_2^T)^T$; let

$$\hat{G}(s) = \begin{bmatrix} 0 & \hat{G}_2(s) \\ -\hat{G}_1(s) & 0 \end{bmatrix} \tag{21}$$

then the system S (see Fig. 2) is described by

$$e = u - G * e \tag{22}$$

$$y = J^{-1} \; G * e \tag{23}$$

$$\det[I + \hat{G}(s)] \neq 0 \qquad \text{for } s \in \mathbb{C}_+. \tag{24}$$

Comment. Such unity feedback systems have been studied in detail in [4], [5], [6]. However direct application of these results does not take advantage of the fact that $\hat{G}$ defined in (21) has a very special form. Thus one of the thrusts of this paper is to take full advantage of this special structure and obtain the corresponding physical interpretations.

IV. Main Result.

The main result is Theorem I below. To clarify the mathematical structure of Theorem I we state a preliminary lemma.

Lemma 1. Consider the system S described by (7)-(11) and (13); let $\hat{G}$, defined by (21), have p.ℓ.r.c.f. $(\mathcal{D}_1, \mathcal{N}, \mathcal{D}_2)$. Under these conditions,

$$S \text{ is } \mathcal{A}\text{-stable} \tag{25}$$

if and only if

$$\inf_{\mathrm{Re}\ s \geq 0} \left| \det[\mathcal{D}_1(s)\ \mathcal{D}_2(s) + \mathcal{N}(s)] \right| > 0. \tag{26}$$

Proof: see Appendix.

Lemma 1 is an extension to distributed systems of the theorem in [21].

Remarks. (i) If $(\mathcal{N}_r, \mathcal{D}_r)$ is a p.r.c.f. of $\hat{G}$, then $(I, \mathcal{N}_r, \mathcal{D}_r)$ is a p.ℓ.r.c.f. of $\hat{G}$; similarly, if $(\mathcal{N}_\ell, \mathcal{D}_\ell)$ is a p.ℓ.c.f. of $\hat{G}$ then $(\mathcal{D}_\ell, \mathcal{N}_\ell, I)$ is a p.ℓ.r.c.f. of $\hat{G}$. The corresponding necessary and sufficient conditions for $\mathcal{A}$-stability become, respectively,

$$\inf_{\mathrm{Re}\ s \geq 0} \left| \det[\mathcal{D}_r(s) + \mathcal{N}_r(s)] \right| > 0\ , \tag{27}$$

$$\inf_{\mathrm{Re}\ s \geq 0} \left| \det[\mathcal{D}_\ell(s) + \mathcal{N}_\ell(s)] \right| > 0\ . \tag{28}$$

(ii) The proof of Lemma 1 actually shows that if the determinant in (26) is zero at some $s_o \in \mathbb{C}_+$, then $\hat{G}$ has a pole at s_o. (Compare with Fact 1).

Fact 2 has connected the given system to a unity feedback system characterized by $\hat{G}$ (see (21) to (23)). Lemma 1 above, gives stability conditions in terms of factorization of $\hat{G}$. It remains to exploit the particular structure of $\hat{G}$ to obtain the required factorizations of $\hat{G}$ in terms of those of the subsystems $\hat{G}_1$ and $\hat{G}_2$.

Lemma 2: Let $\hat{G}$ be as in (21) and let the $\hat{G}_i$ for $i = 1, 2$ admit pseudo-coprime factorizations as in (12).

Under these conditions

$$\text{(i)}\quad \left\{\begin{bmatrix} 0 & \mathcal{N}_{2r} \\ -\mathcal{N}_{1r} & 0 \end{bmatrix}, \begin{bmatrix} \mathcal{D}_{1r} & 0 \\ 0 & \mathcal{D}_{2r} \end{bmatrix}\right\} \text{ is a p.r.c.f. of } \hat{G}, \tag{29}$$

$$\text{(ii)}\quad \left\{\begin{bmatrix} 0 & \mathcal{N}_{2\ell} \\ -\mathcal{N}_{1\ell} & 0 \end{bmatrix}, \begin{bmatrix} \mathcal{D}_{2\ell} & 0 \\ 0 & \mathcal{D}_{1\ell} \end{bmatrix}\right\} \text{ is a p.}\ell\text{.c.f. of } \hat{G}, \tag{30}$$

$$\text{(iii)}\quad \left\{\begin{bmatrix} \mathcal{D}_{2\ell} & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{N}_{2\ell} \\ -\mathcal{N}_{1r} & 0 \end{bmatrix}, \begin{bmatrix} \mathcal{D}_{1r} & 0 \\ 0 & I \end{bmatrix}\right\} \text{ is a p.}\ell\text{.r.c.f. of } \hat{G}, \tag{31}$$

$$\text{(iv)}\quad \left\{\begin{bmatrix} I & 0 \\ 0 & \mathcal{D}_{1\ell} \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{N}_{2r} \\ -\mathcal{N}_{1\ell} & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & \mathcal{D}_{2r} \end{bmatrix}\right\} \text{ is a p.}\ell\text{.r.c.f. of } \hat{G}. \tag{32}$$

The proof is in the Appendix.

We are now in a position to apply directly the mathematical results of Lemmas 1 and 2 to the problem at hand. The result of the substitutions is immediate, therefore the steps of the proof are left to the reader.

<u>Theorem 1</u>. Consider the system S described by (7)-(13). Under these conditions, the system S is $\mathcal{a}$-stable if and only if any one of the four following equivalent conditions is satisfied:

$$\text{(i)}\quad \inf_{\text{Re } s \geq 0} \left| \det \begin{bmatrix} \mathcal{D}_{1r}(s) & \mathcal{N}_{2r}(s) \\ -\mathcal{N}_{1r}(s) & \mathcal{D}_{2r}(s) \end{bmatrix} \right| > 0, \tag{33}$$

(ii) $$\inf_{\text{Re } s \geq 0} \left| \det \begin{bmatrix} \mathcal{D}_{2\ell}(s) & \mathcal{N}_{2\ell}(s) \\ -\mathcal{N}_{1\ell}(s) & \mathcal{D}_{1\ell}(s) \end{bmatrix} \right| > 0, \tag{34}$$

(iii) $$\inf_{\text{Re } s \geq 0} \left| \det[\mathcal{D}_{2\ell}(s)\ \mathcal{D}_{1r}(s) + \mathcal{N}_{2\ell}(s)\ \mathcal{N}_{1r}(s)] \right| > 0, \tag{35}$$

(iv) $$\inf_{\text{Re } s \geq 0} \left| \det[\mathcal{D}_{1\ell}(s)\ \mathcal{D}_{2r}(s) + \mathcal{N}_{2\ell}(s)\ \mathcal{N}_{2r}(s)] \right| > 0. \tag{36}$$

Comments. (i) The four expressions (33)-(36) give necessary and sufficient conditions for $\mathcal{Q}$-stability in terms of the factorizations (12) of the two subsystems. In a very rough way, we may think of them as "characteristic polynomials" (valid for the closed right-half plane only) of the closed loop system S. This interpretation will be made precise and rigorous in the discussion of the lumped case below. In view of Fact 1 and Lemma 1, we see that if any one of these expressions goes to zero in $\mathbb{C}_+$, so must the other three.

(ii) The conditions above require the $\hat{G}_i$'s to be factorized. This is easily done in the lumped case, ($\hat{G}_i$ is a proper element of $\mathbb{C}^{n\times n}(s)$), [7,9,10]; also in the distributed case when $\hat{G}_i$ has a finite number of unstable poles [4,5].

Remark. It is important to note that the $\mathcal{Q}$-stability conditions (33)-(36) are closely related to the return difference; more precisely.

$$\det \begin{bmatrix} \mathcal{D}_{1r}(s) & \mathcal{N}_{2r}(s) \\ -\mathcal{N}_{1r}(s) & \mathcal{D}_{2r}(s) \end{bmatrix} = \det\left(I + \hat{G}_2(s)\hat{G}_1(s)\right)\ \det \mathcal{D}_{1r}(s)\ \det \mathcal{D}_{2r}(s), \tag{37}$$

$$\det\begin{bmatrix} \mathcal{D}_{2\ell}(s) & \mathcal{N}_{2\ell}(s) \\ -\mathcal{N}_{1\ell}(s) & \mathcal{D}_{1\ell}(s) \end{bmatrix} = \det\left(I + \hat{G}_2(s)\hat{G}_1(s)\right) \det \mathcal{D}_{1\ell}(s) \det \mathcal{D}_{2\ell}(s), \tag{38}$$

$$\det[\mathcal{D}_{2\ell}(s)\ \mathcal{D}_{1r}(s) + \mathcal{N}_{2\ell}(s)\ \mathcal{N}_{1r}(s)] = \det\left(I + \hat{G}_2(s)\ \hat{G}_1(s)\right)$$

$$\det \mathcal{D}_{1r}(s) \det \mathcal{D}_{2\ell}(s), \tag{39}$$

$$\det[\mathcal{D}_{1\ell}(s)\ \mathcal{D}_{2r}(s) + \mathcal{N}_{2\ell}(s)\ \mathcal{N}_{2r}(s)] = \det\left(I + \hat{G}_2(s)\ \hat{G}_1(s)\right)$$

$$\det \mathcal{D}_{1\ell}(s) \det \mathcal{D}_{2r}(s). \tag{40}$$

In the distributed case, the expressions in the left-hand side of (33)-(36) are transcendental functions and there is no general algorithm for testing analytically these conditions. They can, however, be tested graphically by using Theorem 2 of [13]. For simplicity we shall state the result for (33) only. Let

$$\Delta(s) \triangleq \det\begin{bmatrix} \mathcal{D}_{1r} & \mathcal{N}_{2r} \\ -\mathcal{N}_{1r} & \mathcal{D}_{2r} \end{bmatrix}, \quad \Delta_{ap}(s) \triangleq \text{almost periodic part of } \Delta \tag{41}$$

The meaning of Δ_{ap} is clear: $\Delta \in \hat{\mathcal{A}}$, hence Δ_{ap} is the sum of all the terms of Δ of the form $\alpha_i e^{-st_i}$. As $|s| \to \infty$ in $\mathbb{C}_+$, $\Delta(s) \to \Delta_{ap}(s)$.

<u>Theorem 1'</u>. Consider the system S defined by (7)-(13). Let $s = \sigma + j\omega$ and define Δ and Δ_{ap} as in (41). Under these conditions, S is $\mathcal{A}$-stable if and only if

$$\text{(i)} \quad \lim_{\sigma \to +\infty} \Delta(\sigma) \neq 0, \tag{42}$$

(ii) $\inf_{\omega \in \mathbb{R}} |\Delta(j\omega)| > 0,$ (43)

(iii) the mean angular velocity of $\omega \mapsto \Delta_{ap}(j\omega)$ is zero, (44)

(iv) the Nyquist curve $\omega \mapsto \Delta(j\omega)/\Delta_{ap}(j\omega)$, with ω increasing from $-\infty$ to ∞, does not encircle the origin of the complex plane. (45)

An example is given below.

V. The lumped system case.

In the lumped case, the transfer functions $\hat{G}_i$ are proper (bounded at infinity) elements of $\mathbb{C}^{n\times n}(s)$, the noncommutative ring of n×n matrices whose elements are rational functions with coefficients in $\mathbb{C}$. Let $\mathbb{C}^{n\times n}[s]$ be the ring of n×n polynomial matrices with coefficients in $\mathbb{C}$. There exist well known procedures to factor any proper $\hat{G}_i(s) \in \mathbb{C}^{n\times n}(s)$ as a ratio of polynomial matrices

$$\hat{G}_i(s) = N_{ir}(s)\, D_{ir}(s)^{-1} = D_{i\ell}(s)^{-1}\, N_{i\ell}(s) \tag{46}$$

where (N_{ir}, D_{ir}) are right coprime and $(N_{i\ell}, D_{i\ell})$ are left coprime; Furthermore D_{ir}, $(D_{i\ell})$, can be made column proper, (row proper), and the polynomials det $D_{i\ell}(s)$ and det $D_{i\ell}(s)$ can be made monic. [7,8,9,10]. The lumped system case is covered by Theorem 1 by the following device: it is easy to find polynomial matrices $M_{ir} \in \mathbb{C}^{n\times n}[s]$ such that, [4,5],

$$N_{ir}\, M_{ir}^{-1} \triangleq \mathcal{N}_{ir} \in \hat{\mathcal{A}}^{n\times n} \text{ and } D_{ir}\, M_{ir}^{-1} \triangleq \mathcal{D}_{ir} \in \hat{\mathcal{A}}^{n\times n} \tag{46a}$$

and $\hat{G}_i = \mathcal{N}_{ir}\, \mathcal{D}_{ir}^{-1}$ is a p.r.c.f.. The recipe is as follows:

Choose

$$M_{ir}(s) = \text{diag}[(s+1)^{-\delta_1}, (s+1)^{-\delta_2}, \ldots, (s+1)^{-\delta_n}] \tag{46b}$$

where δ_k is the highest power of s in the k^{th}-column of D_{ir}, k = 1, 2, ..., n. The same procedure can be applied to the left factorizations. Consequently, for the lumped case, we have procedures to obtain the factorization postulated in (12). If we apply Theorem 1 to these factorizations, we obtain after a few manipulations the following result:

Theorem 1L. Consider the system S described by (7) to (10), and where the $\hat{G}_i(s) \in \mathbb{C}^{n\times n}(s)$, are proper and satisfy (13). Let the $\hat{G}_i$ be factored as in (46). Under these conditions, the lumped system S is $\mathcal{Q}$-stable if and only if any one of the four polynomials below satisfy the stated conditions:

$$\text{(i)}\quad \det\begin{bmatrix} D_{1r}(s) & N_{2r}(s) \\ -N_{1r}(s) & D_{2r}(s) \end{bmatrix} \neq 0 \quad \text{for all } s \in \mathbb{C}_+ , \tag{47}$$

$$\text{(ii)}\quad \det\begin{bmatrix} D_{2\ell}(s) & N_{2\ell}(s) \\ -N_{1\ell}(s) & D_{1\ell}(s) \end{bmatrix} \neq 0 \quad \text{for all } s \in \mathbb{C}_+ , \tag{48}$$

$$\text{(iii)}\quad \det[D_{2\ell}(s)\, D_{1r}(s) + N_{2\ell}(s)\, N_{1r}(s)] \neq 0 \quad \text{for all } s \in \mathbb{C}_+ , \tag{49}$$

$$\text{(iv)}\quad \det[D_{1\ell}(s)\, D_{2r}(s) + N_{1\ell}(s)\, N_{2r}(s)] \neq 0 \quad \text{for all } s \in \mathbb{C}_+ . \tag{50}$$

Remarks. (i) The four polynomial conditions can of course be expressed also in terms of the return difference as in (37)-(40).

(ii) Under the assumptions of Theorem 1L, we may think of the lumped subsystems S_i in terms of one of their minimal representations $[A_i, B_i, C_i, D_i]$. Call x_1 and x_2 the corresponding states. With these data, one may calculate for the feedback system S a representation [A, B, C, D] with input $u = (u_1^T, u_2^T)^T$, output $y = (y_1^T, y_2^T)^T$ and state $x = (x_1^T, x_2^T)^T$, [12]. For such a choice of state, input and output, the representation [A, B, C, D] is completely observable and completely controllable, hence minimal. The polynomials in (47)-(50) are, modulo a non-zero constant factor, equal to the characteristic polynomial of the matrix A.

(iii) It should be stressed that neither $\hat{G}_1$ nor $\hat{G}_2$ is assumed to be stable, hence by [20] we know that any of the four submatrices in $\hat{H}_e$ may be unstable while the other three submatrices are stable. Therefore the coprime factorization and characteristic polynomial approach developed here may be more efficient than the calculation of the four submatrices of $\hat{H}_e$ and testing each one of them for stability. The reader should keep in mind that the multiplication and inversion of matrices of rational functions is a very costly operation: indeed when two rational functions are added one has to find the least common multiple of the denominators in order to obtain the result as a ratio of coprime polynomials; similarly when two rational functions n_1/d_1 and n_2/d_2 are multiplied one has to cancel all common factors between n_1 and d_2, and n_2 and d_1. In contrast manipulations with matrices with polynomial elements are quite easy: each polynomial is stored and manipulated as the ordered sequence of its coefficients: addition of polynomials corresponds to term by term addition of the sequences, and multiplication corresponds to the convolution of the sequences.

VI. The Distributed Case with a finite number of open-loop unstable poles.

An important case to consider is that where each subsystem $\hat{G}_i$ has a finite number of unstable poles, more precisely, poles in $\mathbb{C}_+$. For this case, the general conditions of Theorem 1 above, can be made explicit and the tests be performed graphically.

Consider the system S described by (7)-(13) and assume that for $i = 1, 2$

$$\hat{G}_i(s) = R^{(i)}(s) + \hat{G}_\rho^{(i)}(s) \quad \text{for all } s \text{ in } \mathbb{C}_+ \tag{51}$$

where, for $i = 1, 2$,

a) the $R^{(i)}$ are strictly proper elements of $\mathbb{C}^{n\times n}(s)$; (52)

b) $\{p_{ik}\}_{k=1}^{\ell_i}$ is the family of pairwise-distinct poles of $R^{(i)}$; (53)

c) $\mathrm{Re}\, p_{ik} \geq 0$ for $k = 1, 2, \ldots, \ell_i$; (54)

d) $\hat{G}_\rho^{(i)}$ belong to $\hat{\mathcal{A}}^{n\times n}$. (55)

In section V, it was shown how to construct a p.r.c.f. $(\overline{\mathcal{N}}_{ir}, \mathcal{D}_{ir})$ and a p.ℓ.c.f. $(\overline{\mathcal{N}}_{i\ell}, \mathcal{D}_{i\ell})$ for the $R^{(i)}$, $i = 1, 2$. Then it is easy to see that if we define

$$\mathcal{N}_{i\ell} \triangleq \overline{\mathcal{N}}_{i\ell} + \mathcal{D}_{i\ell}\, \hat{G}_\rho^{(i)} \tag{56}$$

$$\mathcal{N}_{ir} \triangleq \overline{\mathcal{N}}_{ir} + \hat{G}_\rho^{(i)}\, \mathcal{D}_{ir} \tag{57}$$

the pairs $(\mathcal{N}_{i\ell}, \mathcal{D}_{i\ell})$ and $(\mathcal{N}_{ir}, \mathcal{D}_{ir})$ are a p.ℓ.c.f., resp. a p.r.c.f., of the $\hat{G}_i$. ([4], [5], sec. IV.4). We are now in a position to apply Theorem 1 to the system described above. The conclusion is stated as

<u>Theorem 1D</u>. Consider the system S described by (7)-(11), (51)-(55) and satisfying (13). Let (N_{ir}, D_{ir}) and $(N_{i\ell}, D_{i\ell})$ be any right-coprime, and left-coprime, resp., <u>polynomial</u> matrix factorizations of the unstable parts $R^{(i)}$ of the transfer functions $\hat{G}_i$, i = 1, 2.

Under these **conditions**

S is $\mathcal{a}$-stable

if and only if

i) $$\inf_{\text{Re } s \geq 0} |\det[I + \hat{G}_2(s)\, \hat{G}_1(s)]| > 0 \quad , \tag{58}$$

ii) any one of the four equivalent conditions is satisfied: for $k = 1, 2, \ldots, \ell_i$, and $i = 1, 2$

$$\det\left[\begin{array}{c|c} D_{1r}(p_{ik}) & N_{2r}(p_{ik}) + \hat{G}_\rho^{(2)}(p_{ik})\, D_{2r}(p_{ik}) \\ \hline -N_{1r}(p_{ik}) - \hat{G}_\rho^{(1)}(p_{ik})\, D_{1r}(p_{ik}) & D_{2r}(p_{ik}) \end{array}\right] \neq 0 \ , \tag{59}$$

$$\det\left[\begin{array}{c|c} D_{2\ell}(p_{ik}) & N_{2\ell}(p_{ik}) + D_{2\ell}(p_{ik})\, \hat{G}_\ell^{(2)}(p_{ik}) \\ \hline -N_{1\ell}(p_{ik}) - D_{1\ell}(p_{ik})\, \hat{G}_\rho^{(1)}(p_{ik}) & D_{1\ell}(p_{ik}) \end{array}\right] \neq 0 \ , \tag{60}$$

$$\det[D_{2\ell}(p_{ik})\, D_{1r}(p_{ik}) + (N_{2\ell}(p_{ik}) + D_{2\ell}(p_{ik})\, \hat{G}_\rho^{(2)}(p_{ik}))\cdot (N_{1r}(p_{ik}) + \hat{G}_\rho^{(1)}(p_{ik})\, D_{1r}(p_{ik}))] \neq 0 \ , \tag{61}$$

$$\det[D_{1\ell}(p_{ik})\, D_{2r}(p_{ik}) + (N_{1\ell}(p_{ik}) + D_{1\ell}(p_{ik})\, \hat{G}_\rho^{(1)}(p_{ik}))\cdot$$
$$(N_{2r}(p_{ik}) + \hat{G}_\rho^{(2)}(p_{ik})\, D_{2r}(p_{ik}))] \neq 0\,. \tag{62}$$

The Proof is in the Appendix.

Remarks. (i) The conditions (59) to (62) have a very natural interpretation: they prevent $\hat{H}_e$ (and, hence, $\hat{H}_y$) becoming unbounded in a small neighborhood of the p_{ik}'s. This is clear from (46 a-b) and the necessity proof of Lemma 1.

(ii) For the subsystems considered in Theorem 1D, we have proposed a specific method for obtaining the required pseudo coprime factorizations; this leads to the question: what is the effect on the four equivalent expressions (33) to (36) when we choose other pseudo coprime factorizations? Some simple manipulations based on the fact below establish that as one goes from one factorization to the other, each of these expressions is multiplied by a factor k(s) analytic in $\mathring{\mathbb{C}}_+$, bounded in $\mathbb{C}_+$ and bounded away from zero in $\mathbb{C}_+$. The fact in question is

Fact 3. Let G be an n×n matrix-valued L.t.d. with support on $\mathbb{R}_+$. Let $(\mathcal{N}_\ell, \mathcal{D}_\ell)$, $((\mathcal{N}_r, \mathcal{D}_r)$, resp.), be any p.ℓ.c.f., (p.r.c.f., resp.) of $\hat{G}$. Let $\hat{G}$ have a finite number of poles in $\mathbb{C}_+$. Then

$$\det \mathcal{D}_r(s) = k(s) \det \mathcal{D}_\ell(s) \qquad \text{for all } s \in \mathbb{C}_+$$

where $k(\cdot)$ is analytic in $\mathring{\mathbb{C}}_+$, bounded on $\mathbb{C}_+$ and bounded away from zero on $\mathbb{C}_+$.

The proof of Fact 3 is in the appendix. The necessary and sufficient conditions of Theorem 1D can be tested graphically. Again the basic tool is Theorem 2 of [13] and we use the concepts of [13]. To simplify notations, let

$$r(s) \triangleq \det[I + \hat{G}_2(s)\hat{G}_1(s)], \quad r_{ap}(s) \triangleq \text{almost periodic part of } r(s). \tag{63}$$

Let

$$n_p \triangleq \text{number of } \mathring{\mathbb{C}}_+\text{-zeros of } \det(D_{1r}\ D_{2r}), \text{ counting multiplicities.} \tag{64}$$

Let

$j\omega_i$, $i = 1, 2, \ldots, m$, denote the $j\omega$-axis poles of $\hat{G}_1$ and of $\hat{G}_2$.

The reason for treating the $j\omega$-axis poles separately is that, in the distributed case, it is not in general possible to continue analytically the transfer functions $\hat{G}_1$ and $\hat{G}_2$ into the open left half plane. Consequently, in tracing the Nyquist contour we are forced to perform indentations on the right at each $j\omega$-axis pole of $\hat{G}_1$ and $\hat{G}_2$. With these notations, we can state the following

Theorem 1D'. Consider the system S described in Theorem 1D and use the notation defined above. Under these conditions, S is $\mathcal{A}$-stable if and only if

(i) $\lim_{\sigma \to +\infty} r(\sigma) \neq 0$; (65)

(ii) $\inf_{\omega \in \mathbb{R}} |r(j\omega)| > 0$; (66)

(iii) the mean angular velocity of $\omega \mapsto r_{ap}(j\omega)$ is zero; (67)

(iv) the Nyquist curve of $\omega \mapsto r(j\omega)/r_{ap}(j\omega)$ as s moves up along the $j\omega$-axis from $-\infty$ to $+\infty$, (with right indentations at each $j\omega$-axis pole), encircles the origin of the complex plane n_p-times in the counterclockwise sense; (68)

(v)

$$\det\left[\begin{array}{c|c} D_{1r}(j\omega_i) & N_{2r}(j\omega_i) + \hat{G}_\rho^{(2)}(j\omega_i)\, D_{2r}(j\omega_i) \\ \hline -N_{1r}(j\omega_i)-\hat{G}_\rho^{(1)}(j\omega_i)\, D_{1r}(j\omega_i) & D_{2r}(j\omega_i) \end{array}\right] \neq 0 \tag{69}$$

for $i = 1, 2, \ldots, m$.

The proof is given in the Appendix.

Since in most cases graphical tests have major computational advantages over analytic tests, we give below abbreviated versions of algorithms for stability tests.

Algorithm I.

Step 1: Obtain the subsystem transfer matrices $\hat{G}_i$ in the form given by (51).

Step 2: Obtain (N_{ir}, D_{ir}), a right coprime factorization of the unstable parts $R^{(i)}$ of the $\hat{G}_i$, by the procedure described in [7, chapter 1] or [5, p. 64-65].

Step 3: Determine n_p as given by (64).

Step 4: Calculate $r(s)$ and $r_{ap}(s)$ as given by (63).

Step 5: Apply the graphical test as given by Theorem 1D'.

Comments: i) step 1 can be achieved for transfer functions of lumped differential delay systems with a finite number of delays by multiple applications of the decomposition lemma of [14]; ii) in most cases one knows the time-lags t_i of $r_{ap}(s) \triangleq \sum_{i=0}^{\infty} \beta_i e^{-st_i}, 0 = t_o < t_i, i > 0$: $r_{ap}(s)$ can then be identified by inspection or if need be by Fourier analysis of the almost periodic asymptote of $\omega \mapsto r(j\omega)$, [24, pp. 23 et seq.]; iii) (67) can be tested by looking at the map $\omega \mapsto \arg r_{ap}(j\omega)$:

given (66), (67) is satisfied iff this map is bounded, [13]; iv) if there are many $j\omega$-axis poles, checking (69) might in some cases be very tedious: in that case one can make use of the graphical test of Theorem 1'.

Algorithm II.

Step 1: Obtain the subsystem transfer matrices $\hat{G}_i$ in the form given by (51).

Step 2: Obtain $(\mathcal{N}_{ir}, \mathcal{D}_{ir})$, a pseudo right coprime factorization of the $\hat{G}_i$, as described in the introduction of section VI.

Step 3: Determine $\Delta(s)$ and $\Delta_{ap}(s)$ as given by (41).

Step 4: Apply the graphical test as given by Theorem 1'.

Example I. Consider the system described as follows:

$$\hat{G}_1(s) = \left[\begin{array}{c|c} 5/(s+3) & e^{-s} \\ \hline e^{-s} & 1/s \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1/s \end{array}\right] + \left[\begin{array}{c|c} 5/(s+3) & e^{-s} \\ \hline e^{-s} & 0 \end{array}\right] \tag{70}$$

$$N_{1r}(s) = \text{diag}(0,\ 1), \quad D_{1r}(s) = \text{diag}(1,\ s) \tag{71}$$

$$\hat{G}_2(s) = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 10(s+1)/(s-1) \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 20/(s-1) \end{array}\right] + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 10 \end{array}\right] \tag{72}$$

$$N_{2r}(s) = \text{diag}(0,\ 20), \quad D_{2r} = \text{diag}(1,\ s-1) \tag{73}$$

There is one simple $\mathcal{C}_+$-zero of $\det(D_{1r}\ D_{2r})$, namely, + 1 and one $j\omega$-axis pole of $\hat{G}_1$ and $\hat{G}_2$ at 0. Hence $n_p = 1$. Now

$$\det \left[\begin{array}{c|c} D_{1r} & N_{2r} + G_\rho^{(2)}\ D_{2r} \\ \hline -N_{1r}\ -G_\rho^{(1)}\ D_{1r} & D_{2r} \end{array}\right] = (s+10)\ (s-1) + 20 \qquad (74)$$

We apply Theorem 1D'. Using (74), condition (69) is obviously satisfied.

$$r(s) = \det[I + \hat{G}_2(s)\ \hat{G}_1(s)] = (s^2 + 9s + 10)\ (s-1)^{-1}\ s^{-1} \qquad (75)$$

Hence conditions (65) and (66) are satisfied. The Nyquist diagram required by condition (68) is shown on Fig. 3. Clearly (68) is satisfied. Since $r_{ap}(s) = 1$ for this case, (67) is satisfied. Therefore the closed loop system S is $\mathcal{Q}$-stable.

Example II. Consider the same system as above but apply the graphical test of Theorem 1'. We use the same factorization (46 a-b), (56) and (57):

$\mathcal{D}_{1r} = \text{diag}(1, s/(s+1))$; $\mathcal{N}_{1r} = \text{diag}(0, 1/(s+1))$;

$$\mathcal{N}_{1r} = \left[\begin{array}{c|c} 5/(s+3) & se^{-s}/(s+1) \\ \hline e^{-s} & 1/(s+1) \end{array}\right] ;$$

$\mathcal{D}_{2r} = \text{diag}[1, (s-1)/(s+1)]$; $\mathcal{N}_{2r} = \text{diag}[0, 20/(s+1)]$; $\mathcal{N}_{2r} = \text{diag}[0, 10]$

In the present case (see (41))

$$\Delta(s) = (s^2 + 9s + 10)\ (s+1)^{-2}, \quad \Delta_{ap}(s) = 1.$$

Immediately, conditions (42), (43) and (44) are satisfied. To check (45), the Nyquist diagram $\omega \mapsto \Delta(j\omega)/\Delta_{ap}(j\omega)$ is plotted on Fig. 4: it shows that condition (45) holds. In conclusion the system S is $\mathcal{Q}$-stable.

Conclusions

This paper derived necessary and sufficient conditions for the $\mathcal{Q}$-stability of a distributed continuous-time, multivariable, linear, time-invariant feedback system made of unstable subsystems. In the most general theorem, (Theorem 1), the only data required was the pseudo coprime factorization of the subsystems G_1 and G_2. Theorem 1' showed how these conditions could be tested graphically. Next it was shown how Theorem 1 specialized for lumped systems: the conditions were given in Theorem 1L. Next, for systems whose transfer functions have only a finite number of unstable poles, an algorithm for obtaining the pseudo coprime factorization was described and used to derive the stability conditions: see Theorem 1D. Theorem 1D' showed how these conditions can be tested graphically. Finally, examples illustrated the techniques.

It should be stressed that, for ease of exposition, we assumed that the transfer functions $\hat{G}_1$ and $\hat{G}_2$ were square. If $\hat{G}_1$ and $\hat{G}_2$ are rectangular, all the formulae given are still valid, the only change required is the dimensions of the matrices involved.

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Appendix

Proof of Fact 1.

$\Leftarrow$. By assumption, (5) holds. For a proof by contradiction, suppose that (4) is false, i.e. $\hat{G}$ is bounded in $\mathcal{B}(s_o) \cap \mathbb{C}_+$. Now by (1) and (2)

$$\mathcal{U}(s)\ \hat{G}(s) + \mathcal{V}(s) = \mathcal{W}(s)\ \mathcal{D}_r(s)^{-1} \qquad \forall s \in \mathbb{C}_+. \tag{A.1}$$

Consider any sequence $(s_i)_1^\infty$ in $\mathbb{C}_+$ with $s_i \to s_o$. Since (4) is false, the left hand side of (A.1) is bounded on the sequence, while the right hand side is unbounded because det $\mathcal{W}(s_i) \neq 0$ for i = 0, 1, 2, ... and (5). Consequently (5) implies (4).

$\Rightarrow$. By assumption, (4) holds. Hence $\hat{G}(s_i) = \mathcal{N}_i(s_i) \cdot \mathcal{M}(s_i)/\det[\mathcal{D}_r(s_i)]$, where $\mathcal{M}(s_i)$ is the matrix of cofactors of $\mathcal{D}_r(s_i)$, is unbounded as $s_i \to s_o$. Since $\mathcal{N}_r(s)$ and $\mathcal{M}(s)$ are bounded on $\mathbb{C}_+$, this requires that (5) be true.//

Proof of Lemma 1.

$\Leftarrow$. We have $\hat{G} = \mathcal{D}_1^{-1} \mathcal{N} \mathcal{D}_2^{-1}$ and (26). From Fact 2,

$$\hat{H}_e = (I + \hat{G})^{-1} = \mathcal{D}_2(\mathcal{D}_1 \mathcal{D}_2 + \mathcal{N})^{-1}\ \mathcal{D}_1 \tag{A2}$$

By (26), $(\mathcal{D}_1 \mathcal{D}_2 + \mathcal{N})^{-1} \in \hat{\mathcal{A}}^{2n\times 2n}$, [2,5]. Hence $\hat{H}_e \in \hat{\mathcal{A}}^{2n\times 2n}$ as a product of three elements in the algebra.

$\Rightarrow$. By assumption (25) holds, equivalently $\hat{H}_e \in \hat{\mathcal{A}}^{2n\times 2n}$. Since $(\mathcal{D}_1, \mathcal{N}, \mathcal{D}_2)$ is a p.ℓ.r.c.f. of $\hat{G}$, it is easy to show that $(\mathcal{D}_1, \mathcal{D}_1\mathcal{D}_2 + \mathcal{N})$ and $(\mathcal{D}_1\mathcal{D}_2 + \mathcal{N}, \mathcal{D}_2)$ are p.ℓ.c. and p.r.c. respectively. Hence, for i = 1, 2, there exist elements $\mathcal{U}_i$, $\mathcal{V}_i$, $\mathcal{W}_i$ in $\hat{\mathcal{A}}^{2n\times 2n}$ with det $\mathcal{W}_i(s) \neq 0$ in $\mathbb{C}_+$

such that

$$(\mathcal{D}_1\mathcal{D}_2 + \mathcal{N})\mathcal{U}_1 + \mathcal{D}_1\mathcal{V}_1 = \mathcal{W}_1 \tag{A3}$$

$$\mathcal{U}_2(\mathcal{D}_1\mathcal{D}_2 + \mathcal{N}) + \mathcal{V}_2\mathcal{D}_2 = \mathcal{W}_2 \tag{A4}$$

For the purpose of a proof by contradiction assume that (26) is false. Therefore, either

there is a sequence $(s_i)_1^\infty$ in $\mathbb{C}_+$ with $|s_i| \to \infty$ such that

$$\lim_{i \to \infty} \left| \det[\mathcal{D}_1(s_i)\mathcal{D}_2(s_i) + \mathcal{N}(s_i)] \right| = 0 \tag{A5}$$

or

there is an $s_o \in \mathbb{C}_+$ such that $\det[\mathcal{D}_1(s_o)\mathcal{D}_2(s_o) + \mathcal{N}(s_o)] = 0$. (A6)

We show first that (A5) leads to a contradiction. From (A2)

$$\det \hat{H}_e(s_i) = \det \mathcal{D}_1(s_i) \det \mathcal{D}_2(s_i)/\det[\mathcal{D}_1(s_i)\mathcal{D}_2(s_i) + \mathcal{N}(s_i)] \tag{A7}$$

By definition 3 , the first two factors are bounded away from zero on the tail of the sequence; by (A5), the denominator goes to zero. Hence (A7) implies that $i \mapsto \hat{H}_e(s_i)$ is an unbounded sequence. This contradicts the assumption that $\hat{H}_e$ belongs to the algebra $\hat{\mathcal{A}}^{2n\times 2n}$. Hence (A5) is false. Next we show that (A6) leads to a contradiction. From (A3) and (A4) we obtain

$$\mathcal{U}_1 + (\mathcal{D}_1\mathcal{D}_2 + \mathcal{N})^{-1}\mathcal{D}_1\mathcal{V}_1 = (\mathcal{D}_1\mathcal{D}_2 + \mathcal{N})^{-1}\mathcal{W}_1 \tag{A8}$$

Recalling that for i = 1, 2, $\det \mathcal{W}_i(s_o) \neq 0$, (A6) implies that the right hand side of (A8) becomes unbounded as $s \to s_o$. Hence $(\mathcal{D}_1\mathcal{D}_2 + \mathcal{N})^{-1}\mathcal{D}_1$

becomes unbounded. Now use (A2) to write (A4) as follows

$$\mathcal{U}_2 \mathcal{D}_1 + \mathcal{V}_2 \hat{H}_e = \mathcal{W}_2 (\mathcal{D}_1 \mathcal{D}_2 + \mathcal{N})^{-1} \mathcal{D}_1 \tag{A9}$$

Again the right hand side becomes unbounded hence $\hat{H}_e$ also becomes unbounded as $s \to s_o$. This contradiction implies that (A6) is false.

Proof of Lemma 2.

To prove the lemma, requirements (i), (ii) and (iii) of the Definition 2 (or 3, as required) need be established. In all cases, (i) follows by calculation from (21) and the definition of pseudo-coprime factorization; (iii) follows from the requirements on the $\mathcal{D}_{ir}$'s and $\mathcal{D}_{i\ell}$'s. We now prove (ii) for the pair (29); more precisely, we prove that the pair

$$\left(\begin{bmatrix} 0 & \mathcal{N}_{2r} \\ -\mathcal{N}_{1r} & 0 \end{bmatrix} , \begin{bmatrix} \mathcal{D}_{1r} & 0 \\ 0 & \mathcal{D}_{2r} \end{bmatrix} \right) \quad \text{is p.r.c. in } \hat{\mathcal{A}}^{2n\times 2n} . \tag{A10}$$

By condition (ii) of Definition 2 , for $i = 1, 2$, there are elements $\mathcal{U}_i, \mathcal{V}_i, \mathcal{W}_i$ in $\hat{\mathcal{A}}^{n\times n}$ with $\det \mathcal{W}_i(s) \neq 0$ in $\mathbb{C}_+$ such that

$$\mathcal{U}_i \mathcal{N}_{ir} + \mathcal{V}_i \mathcal{D}_{ir} = \mathcal{W}_i .$$

Hence

$$\begin{bmatrix} \mathcal{V}_1 & 0 \\ 0 & \mathcal{V}_2 \end{bmatrix} \begin{bmatrix} \mathcal{D}_{1r} & 0 \\ 0 & \mathcal{D}_{2r} \end{bmatrix} + \begin{bmatrix} 0 & -\mathcal{U}_1 \\ \mathcal{U}_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathcal{N}_{2r} \\ -\mathcal{N}_{1r} & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{W}_1 & 0 \\ 0 & \mathcal{W}_2 \end{bmatrix} \tag{A11}$$

from which (A10) follows. The proof of the left coprimeness of (30) is entirely similar. To prove (31), we must prove that

$$\text{the pair } \left(\begin{bmatrix} 0 & \mathcal{N}_{2\ell} \\ -\mathcal{N}_{1r} & 0 \end{bmatrix}, \begin{bmatrix} \mathcal{D}_{1r} & 0 \\ 0 & I \end{bmatrix} \right) \text{ is p.r.c. in } \hat{\mathcal{A}}^{2n\times 2n}. \tag{A12}$$

Since $(\mathcal{N}_{1r}, \mathcal{D}_{1r})$ constitute a p.r.c. factorization there are elements $\mathcal{U}, \mathcal{V}, \mathcal{W}_1$ in $\hat{\mathcal{A}}^{n\times n}$ with $\det \mathcal{W}_1(s) \neq 0$ in $\mathbb{C}_+$ such that

$$\mathcal{U}\mathcal{N}_{1r} + \mathcal{V}\mathcal{D}_{1r} = \mathcal{W}_1; \tag{A13}$$

hence

$$\begin{bmatrix} \mathcal{V} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{D}_{1r} & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & -\mathcal{U} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathcal{N}_{2\ell} \\ -\mathcal{N}_{1r} & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{W}_1 & 0 \\ 0 & I \end{bmatrix} \tag{A14}$$

hence (A12) is established. The other pseudo coprime factorizations in (31) and (32) are established in the same way.//

Proof of Theorem 1D.

Without loss of generality, we may assume that the D_{ir}, ($D_{i\ell}$, resp.), are column-proper, (row-proper, resp.) and that the polynomial $\det D_{ir}(s)$, ($\det D_{i\ell}(s)$, resp.) is monic. Consequently $\det \mathcal{D}_{ir}(s)$ and $\det \mathcal{D}_{i\ell}(s)$ tend to 1 as $s \to \infty$. Since we have the pseudo-coprime factorizations of the $\hat{G}_i$'s, we may apply Theorem 1. For the $\mathcal{A}$ stability of S it is necessary and sufficient that (33)-(36) hold, or better still, that the right-hand sides of (37)-(40) be bounded away from zero in $\mathbb{C}_+$. Now by construction, $\det \mathcal{D}_{i\ell}(s) \neq 0$ and $\det \mathcal{D}_{ir}(s) \neq 0$ for all $s \in \mathbb{C}_+$ except at the p_{ik}'s, where they are both zero, for $k = 1, 2, \ldots, \ell_i$, and $i = 1, 2$. Therefore S is $\mathcal{A}$-stable if and only if (58) holds as well as one of the four equivalent conditions that follow:

for $k = 1, 2, \ldots, \ell_i$, and $i = 1, 2$

$$\det \begin{bmatrix} \mathcal{D}_{1r}(p_{ik}) & \mathcal{N}_{2r}(p_{ik}) \\ -\mathcal{N}_{1r}(p_{ik}) & \mathcal{D}_{2r}(p_{ik}) \end{bmatrix} \neq 0 \quad , \tag{A15}$$

$$\det \begin{bmatrix} \mathcal{D}_{2\ell}(p_{ik}) & \mathcal{N}_{2\ell}(p_{ik}) \\ -\mathcal{N}_{1\ell}(p_{ik}) & \mathcal{D}_{1\ell}(p_{ik}) \end{bmatrix} \neq 0 \quad , \tag{A16}$$

$$\det[\mathcal{D}_{2\ell}(p_{ik})\, \mathcal{D}_{1r}(p_{ik}) + \mathcal{N}_{2\ell}(p_{ik})\, \mathcal{N}_{1r}(p_{ik})] \neq 0 \ , \tag{A17}$$

$$\det[\mathcal{D}_{1\ell}(p_{ik})\, \mathcal{D}_{2r}(p_{ik}) + \mathcal{N}_{1\ell}(p_{ik})\, \mathcal{N}_{2r}(p_{ik})] \neq 0. \tag{A18}$$

Now in each of these determinantal expressions we can factor out the "multiplier" matrices M_{ir} and $M_{i\ell}$ (see (46a)) and since det $M_{ir}(s)$ and det $M_{i\ell}(s)$ are bounded away from zero in $\mathbb{C}_+$, the four conditions (59)-(62) are equivalent to the four conditions above.//

<u>Proof of Theorem 1D'</u>

Theorem 1' gives a graphical test for guaranteeing that $\Delta(s)$, the left hand side of (37), is bounded away from zero on $\mathbb{C}_+$. Now from (37) and (63) we have

$$\Delta(s) = r(s) \det[\mathcal{D}_{1r}(s)\mathcal{D}_{2r}(s)]; \tag{A19a}$$

furthermore, $r(s)$ is meromorphic in $\mathring{\mathbb{C}}_+$ since by (12), $\Delta(s)$ and $\det[\mathcal{D}_{1r}(s)\mathcal{D}_{2r}(s)]$ are a) analytic in $\mathring{\mathbb{C}}_+$, b) bounded and continuous in $\mathbb{C}_+$, and $\det[\mathcal{D}_{1r}(s)\mathcal{D}_{2r}(s)]$ is bounded away from zero as $|s| \to \infty$. It follows also that a) for some Ω, $r(j\omega)$ is continuous and bounded on $(-\infty,-\Omega]$ and $[\Omega,\infty)$; b) on $[-\Omega,\Omega]$, $r(j\omega)$ is continuous and bounded except in some intervals centered on the $j\omega_i$'s, which are $j\omega$-singularities of $r(s)$, where $r(s)$ is unbounded, and which, by (A19a), occur necessarily at zeros of $\det[\mathcal{D}_{1r}\mathcal{D}_{2r}]$.

Therefore $\Delta(s)$ is bounded away from zero in $\mathbb{C}_+$ if and only if a) $r(s)$ is bounded away from zero in $\mathbb{C}_+$ and b) each $\mathbb{C}_+$-zero of $\det(\mathcal{D}_{1r}\mathcal{D}_{2r})$ is cancelled by an unbounded singularity of $r(s)$ so that their product is different from zero. (If the zero is in $\mathring{\mathbb{C}}_+$, then the singularity of $r(s)$ has to be a pole of the same order). Now, by (A19a) and (64), the number of $\mathring{\mathbb{C}}_+$-poles of $r(s)$ is at most n_p, (counting multiplicities). Consequently, $\Delta(s)$ is bounded away from zero in $\mathbb{C}_+$ if and only if (i) exact (unbounded singularity)-zero cancellation occurs on the $j\omega$-axis, i.e. (69) holds; (ii) $r(s)$ is bounded away from zero in $\mathbb{C}_+$ and each of its $\mathring{\mathbb{C}}_+$-poles is cancelled by a $\mathring{\mathbb{C}}_+$-zero of $\det(\mathcal{D}_{1r}\mathcal{D}_{2r})$. Condition (ii) is tested graphically by (65), (66), (67) and (68), ([13],Theorem 2).

Proof of Fact 3.

Assertion I: If $s_o \in \mathbb{C}_+$ is a zero of order ℓ of $\det \mathcal{D}_r(s)$, ($\mathcal{D}_\ell(s)$, resp.) then ℓ is the maximal order of s_o as a pole of any minor of $\hat{G}(s)$.

Since $(\mathcal{N}_r, \mathcal{D}_r)$ is a p.r.c.f. of $\hat{G}(s)$,

$$\operatorname{rank}\begin{bmatrix}\mathcal{D}_r(s)\\ \mathcal{N}_r(s)\end{bmatrix} = n \quad \text{for all } s \in \mathbb{C}_+ \tag{A19}$$

Let us express any minor of order p of $\hat{G} = \mathcal{N}_r \mathcal{D}_r^{-1}$ in terms of minors of order p of $\mathcal{N}_r$ and minors of order n-p of $\mathcal{D}_r$. By well known methods and notations [15, p. 21-22], we consider the minor of $\hat{G}$ made of the intersection of rows $i_1, i_2, \ldots, i_p$ and columns $k_1, k_2, \ldots, k_p$, denoted by $\hat{G}\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix}$, and we obtain

$$\hat{G}\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix} = \sum_{1 \le \ell_1 < \ell_2 < \cdots < \ell_p \le n} \mathcal{N}_r\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ \ell_1 & \ell_2 & \cdots & \ell_p \end{pmatrix} \mathcal{D}_r^{-1}\begin{pmatrix} \ell_1 & \ell_2 & \cdots & \ell_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix}$$

$$= \frac{\displaystyle\sum_{1 \le \ell_1 < \ell_2 < \cdots < \ell_p \le n} \mathcal{N}_r\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ \ell_1 & \ell_2 & \cdots & \ell_p \end{pmatrix} (-1)^{\sum_{\nu=1}^{p} \ell_\nu + k_\nu} \mathcal{D}_r\begin{pmatrix} k'_1 & k'_2 & \cdots & k'_{n-p} \\ \ell'_1 & \ell'_2 & \cdots & \ell'_{n-p} \end{pmatrix}}{\det \mathcal{D}_r} \quad \text{(A.20)}$$

where $\ell_1 < \ell_2 < \ldots < \ell_p$ and $\ell'_1 < \ell'_2 < \ldots < \ell'_{n-p}$, $k_1 < k_2 < \ldots < k_p$ and $k'_1 < k'_2 < \ldots < k'_{n-p}$ form a complete system of indices 1, 2, ..., n. Observe that the numerator of the above expression is proportional to the Laplace expansion, [16] Exercise 7.23, of the minor of order n of $\begin{bmatrix} \mathcal{D}_r \\ \mathcal{N}_r \end{bmatrix}$ obtained by adjoining rows $i_1, i_2, \ldots, i_p$ of $\mathcal{N}_r$ to rows $k'_1, k'_2, \ldots, k'_{n-p}$, of $\mathcal{D}_r$. For all s in $\mathbb{C}_+$, (A.19) implies that at least one such minor of order n is nonzero, hence there exists at least one minor of some order p of $\hat{G}$ whose numerator, as given by (A.20), is nonzero. Hence the assertion I follows. The proof for $\hat{G} = \mathcal{D}_\ell^{-1}\mathcal{N}_\ell$ is entirely similar.

II. Consider

$$k(s) = \det \mathcal{D}_r(s)/\det \mathcal{D}_\ell(s) \ . \quad \text{(A21)}$$

By assumption det $\mathcal{D}_r(s)$ and det $\mathcal{D}_\ell(s) \in \hat{\mathcal{A}}$, hence k is analytic in $\mathring{\mathbb{C}}_+$ except, maybe, at zeros of det $\mathcal{D}_\ell(s)$. By assumption $\hat{G}$ has a finite number of $\mathbb{C}_+$-poles, hence by Fact 1, det $\mathcal{D}_r$ and det $\mathcal{D}_\ell$ can have zeros only at these poles. Furthermore at each pole, say p_o, det $\mathcal{D}_r$ and det $\mathcal{D}_\ell$ have zeros of the <u>same</u> order because that order is equal, by assetion I, to the maximal order of p_o as a pole of all minors of $\hat{G}$. Hence by (A21), in any sufficiently small neighborhood of p_o, k(s) is bounded and bounded away from zero. Hence k(s) is bounded and bounded away from zero on compact subsets of $\mathbb{C}_+$. The same holds at infinity, because along any sequence $(s_i)_1^\infty \subset \mathbb{C}_+$ with $|s_i| \to \infty$, det $\mathcal{D}_r(s_i)$ and det $\mathcal{D}_\ell(s_i)$ are both bounded, (because they belong to $\hat{\mathcal{A}}$) and both bounded away from zero because of the pseudo coprime factorizations.

Footnote

†In certain manipulations (expressing cosines in exponential form) and in certain models (as a result of band-pass to low-pass transformations), the expressions become complex-valued. So for simplicity, we allow them to be complex valued throughout. Also to simplify the description of the results we assume that the u_i's, e_i's, y_i's ($i = 1, 2$), have n-components. If that is not the case one may insert appropriate modifications or simply add to the matrices G_i and the vectors u_i, y_i, e_i elements that are identically zero or one, at appropriate places.

§If for some $c \in \mathbb{C}$, $\exp(-ct)G(t)$ is a distribution of slow growth, then, for $\mathrm{Re}\, s > c$, the Laplace transform of G is well defined and is an analytic function of s, [22,p.310, 23,p.213].

#The factorization (2) can always be performed algorithmically for the cases of section V and VI. The factorization (2) leads to formulas very close to the familiar one encountered in the case of single-input single-output lumped systems.

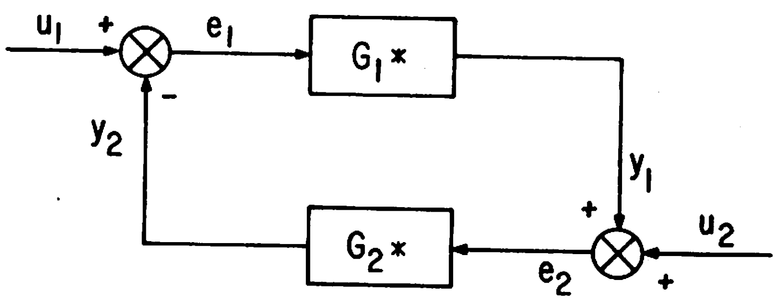
Figure Captions

Fig. 1. The System S.

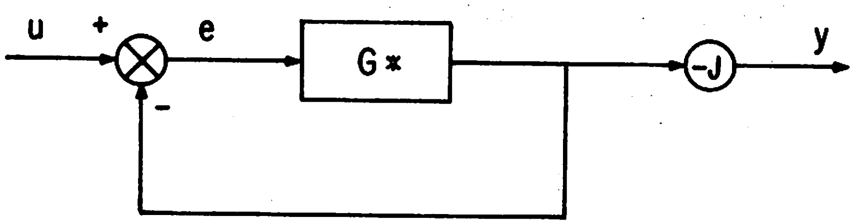
Fig. 2. The Convolution Feedback System S.

Fig. 3. Nyquist plot for example 1.

Fig. 4. Nyquist plot for example 2.



Callier Desoer Fig 1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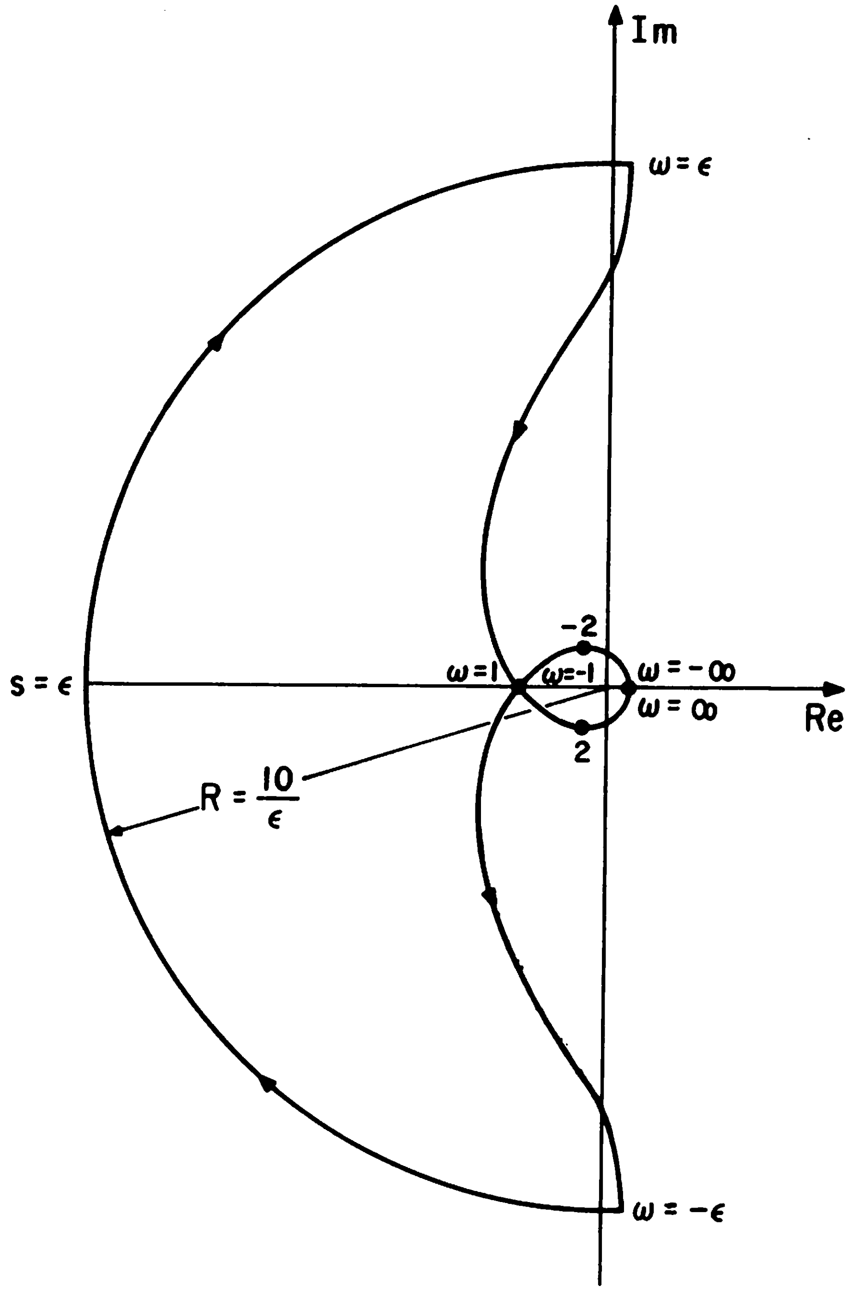
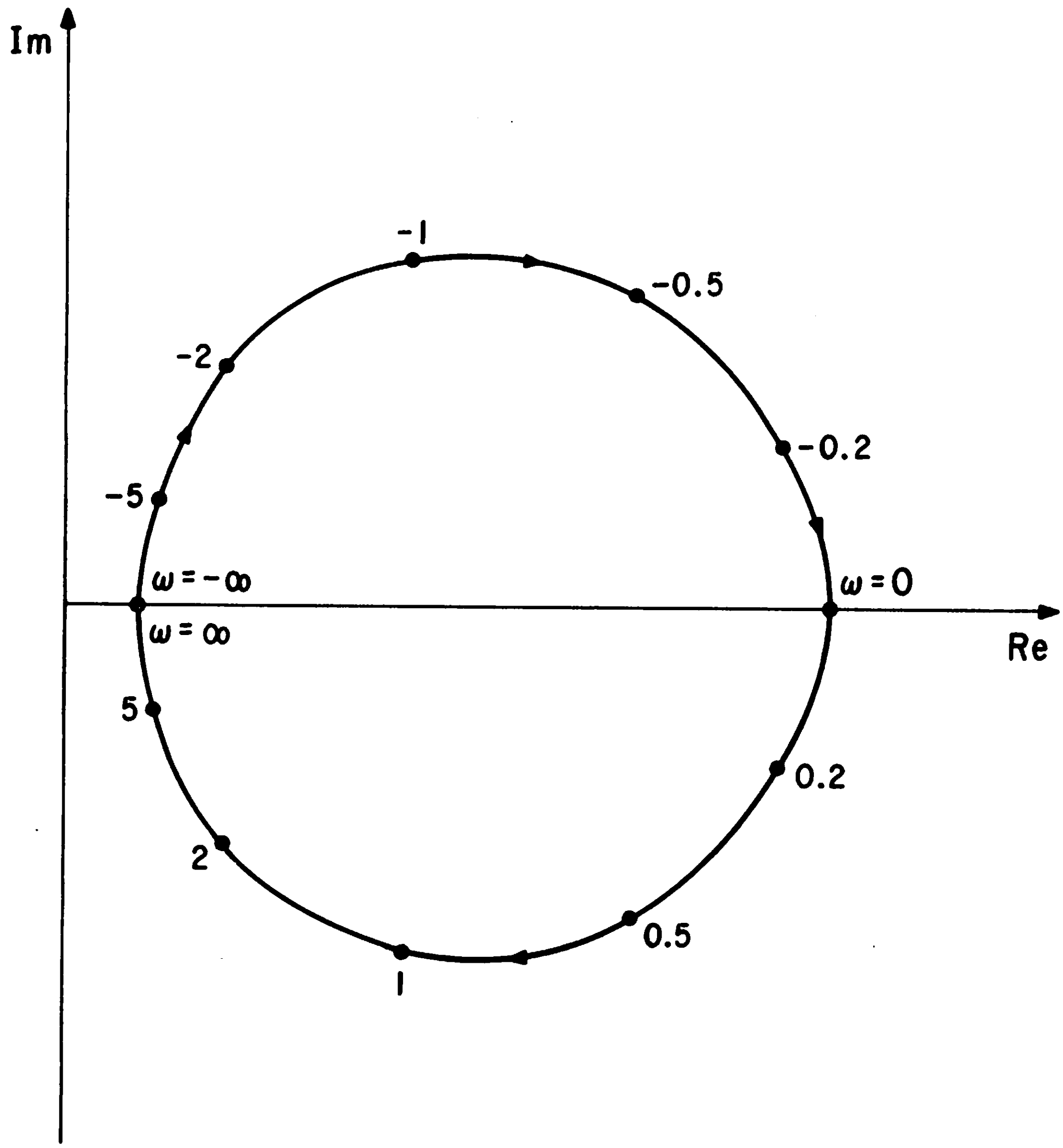


Callier Desoer Fig 3



Callier Desoer Fig 4