DIFFERENTIATION FORMULAS FOR STOCHASTIC INTEGRALS IN THE PLANE

by

Eugene Wong and Moshe Zakai

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College of Engineering
University of California, Berkeley
94720

DIFFERENTIATION FORMULAS FOR STOCHASTIC INTEGRALS IN THE PLANE

Eugene Wong and Moshe Zakai†

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract

For a one-parameter process of the form

$$X_t = X_0 + \int_0^t \phi_s dW_s + \int_0^t \psi_s ds$$

where W is a Wiener process and $\int \phi dW$ is a stochastic integral, a twice continuously differentiable function $f(X_t)$ is again expressible as the sum of a stochastic integral and an ordinary integral via the Ito differentiation formula. In this paper we present a generalization for the stochastic integrals associated with two-parameter Wiener process.

Let $\{W_z, z \in R_t^2\}$ be a Wiener process with a two-dimensional parameter. Erstwhile, we have defined stochastic integrals $\int \phi dW$ and $\int \psi dWdW$, as well as mixed integrals $\int hdzdW$ and $\int gdWdz$. Now, let X_z be a two-parameter process defined by the sum of these four integrals and an ordinary Lebesgue integral. The objective of this paper is to represent a suitably differentiable function $f(X_z)$ as such a sum once again. In the process we will derive the (basically one-dimensional) differentiation formulas of $f(X_z)$ on increasing paths in R_t^2.

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†Presently at the Israel Technical Institute, Technion, Haifa, Israel.

1. Introduction.

Let R_+^2 denote the positive quadrant of the plane. For two points $a = (a_1, a_2)$, $b = (b_1, b_2)$ in R_+^2 we denote $a \prec b$ if $a_1 \leq b_1$ and $a_2 \leq b_2$. A family of σ-fields $\{\mathcal{F}_z, z \in R_+^2\}$ is said to be increasing if $a \prec b \Rightarrow \mathcal{F}_a \subseteq \mathcal{F}_b$. A two-parameter stochastic process $\{X_z, \mathcal{F}_z, z \in R_+^2\}$ is said to be a martingale if

$$E(X_b | \mathcal{F}_a) = X_a \quad \text{almost surely whenever } b \succ a. \tag{1.1}$$

One of the simplest examples of 2-parameter martingales is the Wiener process. We say $\{W_z, z \in R_+^2\}$ is a Wiener process if it is Gaussian, zero-mean, with

$$EW_aW_b = \min(a_1,b_1)\min(a_2,b_2) \qquad \forall a, b \in R_+^2 \tag{1.2}$$

Consider any increasing family of σ-fields $\{\mathcal{F}_z, z \in R_+^2\}$ such that, (1) W_z is $\mathcal{F}_z$-measurable for every z, and (2) for $b \succ a$ $\Delta W = W_b - W_{(a_1,b_2)} - W_{(b_1,a_2)} + W_a$ is $\mathcal{F}_a$-independent. It is easy to verify that $\{W_z, \mathcal{F}_z, z \in R_+^2\}$ is a martingale.

In view of the close connection between martingales and stochastic integrals in the one-parameter case, the possibility of defining stochastic integrals of the form

(1.3) $$Z_z = \int_{\zeta \prec z} \phi_\zeta \, dW_\zeta$$

as martingales suggests itself readily. This was done by Wong [3], and by Cairoli [1] who used it to study a class of stochastic differential equations. Wong and Zakai [4] noted that stochastic integrals of the form (3) were clearly incomplete for any reasonable calculus. In particular, unlike the one-parameter case, not every martingale defined on the sample space of a Wiener process can be represented in the form of (3). For such representations a second stochastic integral is needed and was introduced in [4]. In the process, a differentiation formula was derived for those transformations $f(W_z, z)$ which are themselves martingales. While this formula has already found some applications [5], it is inadequate for a general calculus.

The natural question is the following: Let X_z be defined as the sum of a Lebesgue integral and stochastic integrals of the first and second types, i.e.,

(1.4) $$X_z = \int_{\zeta \prec z} \theta_\zeta d\zeta + \int_{\zeta \prec z} \phi_\zeta dW_\zeta + \int_{\zeta,\zeta' \prec z} \psi_{\zeta,\zeta'} \, dW_\zeta dW_{\zeta'}$$

Let $f(x,z)$ be a suitably differentiable function. Can $f(X_z,z)$ again be expressed as a sum of three integrals as in (4)? The answer, interestingly, is no. For a complete generalization of the Ito lemma, we need the mixed area integrals introduced in [6]. The purpose of this paper is to derive the general differentiation formula and some related results.

2. Notations and Preliminaries.

Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be two points in the positive quadrant R_+^2. We denote $a \prec b$ if $a_1 \le b_1$ and $a_2 \le b_2$, $a \prec\prec b$ if $a_1 < b_1$ and $a_2 < b_2$, $a \curlywedge b$ if $a_1 \le b_1$ and $a_2 \ge b_2$, $a \not\curlywedge b$ if $a_1 < b_1$ and $a_2 > b_2$. Furthermore, we shall adopt the notations:

$$a \otimes b = (a_1, b_2)$$

$$a \wedge b = (\min(a_1, b_1),\ \min(a_2, b_2))$$

$$a \vee b = (\max(a_1, b_1),\ \max(a_2, b_2))$$

Note that if $a \curlywedge b$ then $a \otimes b = a \wedge b$, if $b \curlywedge a$ then $a \otimes b = a \vee b$. Note also that $a \otimes b \otimes c = a \otimes c$.

For a fixed point $a \in R_+^2$, R_a will denote the rectangle $\{z: z \in R_+^2, z \curlywedge a\}$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and let $\{\mathcal{F}_z, z \in R_a\}$ be a family of σ-subfields such that:

F_1) $z \curlywedge z'$ implies $\mathcal{F}_z \subset \mathcal{F}_{z'}$

F_2) $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$ (0 denotes the origin)

F_3) For every z, $\mathcal{F}_z = \bigcap_{z' \succ\succ z} \mathcal{F}_{z'}$

F_4) For each z, $\mathcal{F}_z^1 = \mathcal{F}_{z \otimes a}$ and $\mathcal{F}_z^2 = \mathcal{F}_{a \otimes z}$ are conditionally independent given $\mathcal{F}_z$.

The first three conditions are natural ones, and the fourth one was introduced in [2].

<u>Definition</u>: A stochastic process $\{M_z, z \in R_a\}$ is a <u>martingale</u> if: (1) for each z M_z is $\mathcal{F}_z$-measurable, (2) for each z $E|M_z| < \infty$, (3) $z \prec z'$ implies $E(M_{z'} | \mathcal{F}_z) = M_z$ almost surely.

Let $z' \succ\succ z$. Then $(z, z']$ will denote the rectangle $\{\zeta: \zeta \succ\succ z$ and $\zeta \prec z'\}$. If $\{X_z, z \in R_a\}$ is a stochastic process then we will denote

$$X(z, z'] = X_{z'} - X_{z\otimes z'} - Z_{z'\otimes z} + X_z$$

Several martingale related concepts were defined by Cairoli and Walsh [2] in terms of $X(z, z']$. These were slightly modified in [6]. In the following definitions $X = \{X_z, z \in R_a\}$ is assumed to be $\mathcal{F}_z$-adapted and integrable for each z, and the defining condition is to hold for all $z \prec\prec z'$:

<u>Definitions</u>: (a) X is a <u>weak martingale</u> if $E[X(z, z'] \mid \mathcal{F}_z] = 0$

(b) X is a <u>strong martingale</u> if it vanishes at the axis and $E[X(z, z'] \mid \mathcal{F}_z^1 \vee \mathcal{F}_z^2] = 0$.

(c) X is an <u>i-martingale</u> (i = 1, 2) if $E[X(z, z'] \mid \mathcal{F}_z^i] = 0$ and $X_{z\otimes 0}$ $(X_{0\otimes z})$ is a one-parameter martingale for i = 1 (i = 2).

With these definitions, a strong martingale is also a martingale, a process is a martingale if and only if it is both a 1-martingale and a 2-martingale (see [2]), and either a 1-martingale or a 2-martingale is also a weak martingale.

3. Stochastic Integrals.

Let M be a continuous square integrable strong martingale. Then, four types of stochastic integrals have been defined: ([6])

$$\int \phi_\zeta \, dM_\zeta$$

$$\int \psi_{\zeta,\zeta'} \, dM_\zeta \, dM_{\zeta'}$$

$$\int \psi_{\zeta,\zeta'} \, d\zeta \, dM_{\zeta'}$$

$$\int \psi_{\zeta,\zeta'} \, dM_\zeta \, d\zeta'$$

In this paper we shall consider only the special case where $M = W$ is a two-parameter Wiener process, which can be defined as a continuous strong martingale such that $X_z = W_z^2 - \text{Area}\ (R_z)$ is a martingale. Next, we shall summarize the principal properties of stochastic integrals with respect to W.

Let $\{W_z, \mathcal{F}_z, z \in R_a\}$ be a Wiener process. Let $\{\phi_z, z \in R_a\}$ be a process such that:

(3.1) (a) ϕ is bimeasurable function of (ω,z).

(b) $\int_{R_a} E\phi_\zeta^2 \, d\zeta < \infty$

and for each z

either (c_0) ϕ_z is $\mathcal{F}_z$-measurable

or (c_1) ϕ_z is $\mathcal{F}_z^1$-measurable or (c_2) ϕ_z is $\mathcal{F}_z^2$-measurable

Let $\mathcal{H}_i$ denote the space of ϕ satisfying (a), (b) and (c_i). For $\phi \in \mathcal{H}_i$, $i = 0, 1, 2$, the stochastic integral $\int_{R_a} \phi_\zeta \, dW_\zeta$ is well-defined. If we define

$$(3.2) \qquad (\phi \circ W)_z = \int_{R_z} \phi_\zeta \, dW_\zeta = \int_{R_a} I_{\zeta \prec z} \phi_\zeta \, dW_\zeta, \quad z \in R_a$$

then the process $\phi \circ W$ is a strong martingale if $\phi \in \mathcal{H}_0$, a 1-martingale if $\phi \in \mathcal{H}_1$ and a 2-martingale if $\phi \in \mathcal{H}_2$. Furthermore, define

$$(3.3) \qquad X_z = (\phi \circ W)_z \, (\psi \circ W)_z - \int_{R_z} \phi_\zeta \, \psi_\zeta \, d\zeta$$

Then X is a martingale if $\phi, \psi \in \mathcal{H}_0$, a 1-martingale if $\phi, \psi \in \mathcal{H}_1$, and a 2-martingale if $\phi, \psi \in \mathcal{H}_2$. In all cases continuous versions can be chosen.

Proposition 3.1. Let $\{X_z, z \in R_a\}$ be a process defined by

$$X_z = X_0 + \int_{R_z} f(z, \zeta) \, dW_\zeta$$

where X_0 is $\mathcal{F}_0$-measurable and f satisfies the conditions

(3.4) (a) $f(z,\zeta) = 0$ unless $\zeta \prec z$

(b) $f(z,\zeta) = f(\zeta \otimes z, \zeta)$

((b') $f(z, \zeta) = f(z \otimes \zeta, \zeta)$)

(c) For each $z \in R_a$, $f(z, \cdot) \in \mathcal{H}_1$

((c') $f(z, \cdot) \in \mathcal{H}_2$)

Then, X_z is a 1-martingale (respectively, a 2-martingale).

proof: Consider the first case. Let $z' \succ z$. Then

$$E(X_{z'} \mid \mathcal{F}^1_z) = \int_{R_{z \otimes z'}} f(z',\zeta)\, dW_\zeta + X_0$$

$$= \int_{R_{z \otimes z'}} f(\zeta \otimes z',\zeta)\, dW_\zeta + Z_0$$

$$= \int_{R_{z \otimes z'}} f(\zeta \otimes z \otimes z',\zeta)\, dW_\zeta + X_0$$

$$= X_{z \times z'}$$

Therefore,

$$E\{X(z,z'] \mid \mathcal{F}^1_z\} = E\{X_{z'} - X_{z \otimes z'} - X_{z' \otimes z} + X_z \mid \mathcal{F}^1_z\}$$

$$= -\, E\{X_{z' \otimes z} - X_z \mid \mathcal{F}^1_z\}$$

$$= -\, \{X_{z \otimes z' \otimes z} - X_z \mid \mathcal{F}^1_z\}$$

$$= 0$$

The proof is identical for the 2-martingale case. ¤

Remark: Except for notational differences and an explicit display of the dependence of the integrand on limit of integration, proposition 3.1 is a restatement of proposition 2.3 of Cairoli and Walsh [2].

Next, consider functions $\psi(\omega,\zeta,\zeta')$, ζ, $\zeta' \in R_a$, such that

(3.5) (a) ψ is a measurable process and for each (ζ, ζ') $\psi_{\zeta,\zeta'}$

is $\mathcal{F}_{\zeta \vee \zeta'}$-measurable.

(b) $\int_{R_a \times R_a} E\psi^2_{\zeta,\zeta'} \, d\zeta d\zeta' < \infty$

(c) $\psi_{\zeta,\zeta'} = 0$ unless $\zeta \curlywedge \zeta'$

Consider a function satisfying (3.5) and of the form

(3.6) $$\psi_{\zeta,\zeta'} = \psi \quad \text{for } \zeta \in A \text{ and } \zeta' \in B$$
$$= 0 \quad \text{otherwise}$$

where A and B are rectangles. We define

$$\int_{R_a \times R_a} \psi_{\zeta,\zeta'} \, dW_\zeta \, dW_{\zeta'} = \psi \, W(A) \, W(B)$$

$$\int_{R_a \times R_a} \psi_{\zeta,\zeta'} \, d\zeta \, dW_{\zeta'} = \psi \, \mathrm{Area}(A) \, W(B)$$

$$\int_{R_a \times R_a} \psi_{\zeta,\zeta'} \, dW_\zeta \, d\zeta' = \psi \, W(A) \, \mathrm{Area}(B)$$

For ψ which is a sum of such functions, the integrals are defined by linearity. For a general ψ satisfying (3.5) the integrals are defined by approximations and passage to quadratic-mean limit. Finally, for ψ satisfying conditions (a) and (b) of (3.5) but not (c) we define the integrals as being the same as those with $\psi_{\zeta,\zeta'}$ replaced by $I(\zeta \curlywedge \zeta')\psi_{\zeta,\zeta}$ where $I(\zeta \curlywedge \zeta') = 1$ or 0 according as $\zeta \curlywedge \zeta'$ or not. We shall denote by $\mathcal{H}$ the space of functions satisfying (3.5) (a) and (b).

Proposition 3.2. Let $\psi \in \mathcal{H}$ and define

(3.7)
$$X_z = \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \, dW_\zeta dW_{\zeta'}$$

$$Y_{1z} = \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \, d\zeta dW_{\zeta'}$$

$$Y_{2z} = \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \, dW_\zeta d\zeta,$$

Then, X, Y_1, Y_2 are respectively a martingale, a 1-martingale, and a 2-martingale for which almost surely sample continuous versions can be chosen. Furthermore, let

(3.8)
$$f_1(z, \zeta') = \int_{R_z} I(\zeta \, \zeta')\psi_{\zeta,\zeta'} \, dW_\zeta$$

$$f_2(z,\zeta) = \int_{R_z} I(\zeta \curlywedge \zeta')\psi_{\zeta,\zeta'} \, dW_{\zeta'}$$

$$g_1(z, \zeta') = \int_{R_z} I(\zeta \curlywedge \zeta')\psi_{\zeta,\zeta'} \, d\zeta$$

$$g_2(z, \zeta) = \int_{R_z} I(\zeta \curlywedge \zeta')\psi_{\zeta,\zeta'} \, d\zeta'$$

Then,

(3.9)
$$X_z = \int_{R_z} f_1(z,\zeta') \, dW_{\zeta'}$$

$$= \int_{R_z} f_2(z,\zeta) dW_\zeta$$

(3.10) $$Y_{1z} = \int_{R_z} g_1(z, \zeta')dW_{\zeta'}$$

$$= \int_{R_z} f_2(z, \zeta)d\zeta$$

(3.11) $$Y_{2z} = \int_{R_z} g_2(z, \zeta)dW_\zeta$$

$$= \int_{R_z} f_1(z, \zeta')d\zeta'$$

Proof: Let $\tilde{\mathcal{H}}$ denote the space of all functions ψ which are sums of functions satisfying both (3.5) and (3.6). The conclusions of the propositions are obvious for $\psi \in \tilde{\mathcal{H}}$. For ψ satisfying $\mathcal{H}$ let $\{\psi_n\}$ be a sequence in $\tilde{\mathcal{H}}$ such that

$$\|\psi_n - \psi\|^2 = \int_{R_a} E(\psi_{n,\zeta,\zeta'} - \psi_{\zeta,\zeta'})^2 \, d\zeta d\zeta' \xrightarrow[n\to\infty]{} 0$$

and define f_{in} and g_{in} by using ψ_n in (3.8). Then

$$\int_{R_z} E[f_{in}(z,\zeta) - f_i(z,\zeta)]^2 \, d\zeta \leq \|\psi_n - \psi\|^2 \xrightarrow[n\to\infty]{} 0$$

and

$$\int_{R_z} E[g_{in}(z,\zeta) - g_i(z,\zeta)]^2 \, d\zeta \leq \text{Area}(R_z) \, \|\psi_n - \psi\|^2 \xrightarrow[n\to\infty]{} 0$$

Hence, if we denote $X_{nz} = \int_{R_z \times R_z} \psi_{n\zeta,\zeta'} \, dW_\zeta dW_{\zeta'}$, then

$$E[X_z - \int_{R_z} f_1(z,\zeta')dW_{\zeta'}]^2 \leq 2 \, E(X_z - X_{nz})^2 + 2\|\psi_n - \psi\|^2 \xrightarrow[n\to\infty]{} 0$$

Similarly,

$$E[Y_{1z} - \int_{R_z} f_2(z,\zeta)d\zeta]^2 \leq 2\ E(Y_{1z} - Y_{1nz})^2 + 2\ \text{Area}(R_z)\|\psi_n - \psi\|^2 \underset{n\to\infty}{\longrightarrow} 0$$

These two cases are prototypical of all the others.

The martingale-properties can be proved using approximations, but they also follow directly from the iterated integrals by using proposition 3.1. Continuity is proved by showing that a subsequence of $\{\psi_n\}$ can be so chosen that the resulting approximations of X and Y_i converge uniformly almost surely. ¤

Remark: Proposition 3.2 might be viewed as stochastic Fubini's theorems.

As in the one-dimensional parameter case, we would like to extend the stochastic integrals to integrands which are square-integrable almost surely. This can be done and will be given in a forthcoming paper, but we have no proof that the resulting processes defined by the four types of stochastic integrals are then sample continuous. For the derivation of the differentiation formulas, we shall extend the stochastic integrals as follows: Instead of conditions (3.1b) and (3.5b), assume

(3.1b') $\quad \sup_{\zeta \in R_a} |\phi_\zeta| < \infty \quad$ almost surely

(3.5b') $\quad \sup_{\zeta,\, \zeta' \in R_a} |\psi_{\zeta,\,\zeta'}| < \infty \quad$ almost surely

For stochastic integrals of the first type, choose an increasing sequence K_n such that

$$\mathcal{P}(\sup_{\zeta}|\phi_\zeta|>K_n) \leq 1/n^2$$

and

$$\zeta_{n\zeta} = \begin{cases} \phi_\zeta & \text{if} \quad |\phi_\zeta| \le K_n \\ K_n & \text{if} \quad \phi_\zeta > K_n \\ -K_n & \text{if} \quad \phi_\zeta < -K_n \end{cases}$$

Note that $\int_{R_a} E\phi_{n\zeta}^2 \, d\zeta < \infty$ and

$$\mathcal{P}\left(\sup_{m>0} \sup_{z} \left| \int_{R_z} \phi_{n+m,\zeta} dW_\zeta - \int_{R_z} \phi_{n\zeta} \, dW_\zeta \right| > 0\right)$$

$$\le \mathcal{P}(\sup_{\zeta} |\phi_{n\zeta}| > K_n) \le 1/n^2$$

Therefore, Borel-Cantilli lemma implies that the sequence

$$\int_{R_z} \phi_{n\zeta} \, dW_\zeta$$

converges uniformly with probability 1. We now define $\int_{R_z} \phi_\zeta d\zeta$ as the limit, which being the uniform limit of sample-continuous process is itself sample continuous.

Stochastic integrals of the second type and mixed integrals can be defined under condition (3.5b') in a similar way, and the resulting processes are again sample continuous. In all these cases martingales properties must be replaced by the corresponding "local" martingale properties in a way similar to the one-dimensional parameter case.

4. Formulas on Partial Differentiation

In [6] we have shown that under suitable differentiability conditions, every weak martingale can be represented as the sum of stochastic integrals of the four types. If we call processes of the form X_z = (weak martingale) + $\int_{R_z} u_\zeta d\zeta$ <u>weak semi-martingales</u>, then our principal result (section 5) will be a representation of sufficiently smooth functions $F(X_z)$ as weak semi-martingales once again, via a differentiation formula.

Suppose that $\{X_z,\ z \in R_a\}$ is a process of the form

$$(4.1) \qquad X_z = X_o + \int_{R_z} f(z,\zeta)\, dW_\zeta + \int_{R_z} u(z,\zeta)\, d\zeta$$

where f satisfies the conditions of proposition of 3.1 to make the stochastic integral $\int_{R_z} f(z,\zeta) dW_\zeta$ a 1-martingale and u satisfies $u(z,\ \zeta) = u(\zeta \otimes z, \zeta)$.

Let $z = (s,t)$ and $\zeta = (\sigma,\tau)$. Then $\zeta \otimes z = (\sigma,t)$ and by setting $f((\sigma,t),(\sigma,\tau)) = \tilde{f}(t;\sigma,\tau)$ and $u((\sigma,t),(\sigma,\tau)) = u(t;\sigma,\tau)$, we can reexpress X_z as

$$(4.2) \qquad X_{s,t} = X_0 + \int_{R_{s,t}} \tilde{f}(t,\zeta) dW_\zeta + \int_{R_{s,t}} \tilde{u}(t,\zeta)\, d\zeta$$

$X_{s,t}$ is a one-parameter semimartingale in s for each t. Rewriting it as

$$(4.3) \qquad X_{s,t} = X_0 + M_s^t + \int_0^s [\int_0^t \tilde{u}(t,\sigma,\tau)\, d\tau]\, d\sigma$$

we get the one-parameter formula

$$(4.4) \qquad F(X_{s,t}) = F(X_0) + \int_0^s F'(X_{\sigma,t})\{dM_\sigma^t + \int_0^t \tilde{u}(t,\sigma,\tau)\, d\tau]\, d\sigma\}$$

$$+ \frac{1}{2}\int_0^s F''(X_{\sigma,t})\, d\langle M^t, M^t\rangle_\sigma$$

for any twice continously differentiable F. Equation (4.4) can be rewritten as

$$F(X_{s,t}) = F(X_0) + \int_0^s \int_0^t F'(X_{\sigma,t})\{\tilde{f}(t;\sigma,\tau)\, dW_{\sigma,\tau} + \tilde{u}(t;\sigma,\tau)\, d\sigma d\tau\}$$

$$+ \frac{1}{2}\int_0^s \int_0^t F''(X_{\sigma,t})\, \tilde{f}^2(t;\sigma,\tau)\, d\sigma d\tau$$

or

(4.5) $$F(X_z) = F(X_0) + \int_{R_z} F'(X_{\zeta\otimes z})\{f(\zeta\otimes x,\zeta)dW_\zeta + u(\zeta\otimes z,\zeta)d\zeta\}$$

$$+ \frac{1}{2}\int_{R_z} F''(X_{\zeta\otimes z})f^2(\zeta\otimes z,\zeta)d\zeta$$

$$= F(X_0) + \int_{R_z} F'(X_{\zeta\otimes z})\{f(z,\zeta)dW_\zeta + u(z,\zeta)d\zeta\}$$

$$+ \frac{1}{2}\int_{R_z} F''(X_{\zeta\otimes z})f^2(z,\zeta)d\zeta$$

Proposition 4.1. Let X_{kz}, $z \in R_a$, $k = 1, 2, \cdots, n$, be processes defined by

(4.6) $$X_{kz} = X_{k0} + \int_{R_z} f_k(z,\zeta)dW_\zeta + \int_{R_z} u_k(z,\zeta)d\zeta$$

Suppose that for each k f satisfies the conditions of proposition 3.1 to make the stochastic integral a 1-martingale and $u_k(z,\zeta) = u_k(\zeta\otimes z,\zeta)$. Let $X = (X_1, X_2, \cdots, X_n)$ and F(X) be a function with continuous partials up to the second order. Then,

(4.7)
$$F(X_z) = F(X_0) + \sum_k \int_{R_z} F_k(X_{\zeta\otimes z})[f_k(z,\zeta)dW_\zeta + u_k(z,\zeta)d\zeta]$$
$$+ \frac{1}{2} \sum_{k,\ell} \int_{R_z} F_{k\ell}(X_{\zeta\otimes z}) f_k(z,\zeta) f_\ell(z,\zeta) d\zeta$$

where F_k and $F_{k\ell}$ denote partial derivatives. Alternatively, if f_k satisfy the conditions of proposition 3.1 to make the stochastic integral a 2-martingale and $u_k(z,\zeta) = u_k(z\otimes\zeta,\zeta)$ then

(4.7')
$$F(X_z) = F(X_0) + \sum_k \int_{R_z} F_k(X_{z\otimes\zeta})[f_k(z,\zeta)dW_\zeta + u_k(z,\zeta)d\zeta]$$
$$+ \frac{1}{2} \sum_{k,\ell} \int_{R_z} F_{k\ell}(X_{z\otimes\zeta}) f_k(z,\zeta) f_\ell(z,\zeta) d\zeta$$

An important special case of a process X which is of the form (4.6) is given by

(4.8)
$$X_z = \int_{R_z} \theta_\zeta \, d\zeta + \int_{R_z} \phi_\zeta dW_\zeta + \int_{R_z\times R_z} \psi_{\zeta,\zeta'} \, dW_\zeta dW_{\zeta'}$$
$$+ \int_{R_z\times R_z} g_{\zeta,\zeta'} \, d\zeta dW_{\zeta'} + \int_{R_z\times R_z} h_{\zeta,\zeta'} dW_\zeta d\zeta'$$

which can be written in the form of (4.6) in two ways, with either

$$f(z,\zeta) = \phi_\zeta + \int_{R_z} I(\zeta' \curlywedge \zeta)[\psi_{\zeta'\zeta} dW_{\zeta'} + g_{\zeta',\zeta} d\zeta']$$

(4.9)

$$u(z,\zeta) = \theta_\zeta + \int_{R_z} I(\zeta' \curlywedge \zeta) h_{\zeta',\zeta} dW_{\zeta'}$$

or

$$f(z,\zeta) = \phi_\zeta + \int_{R_z} I(\zeta \curlywedge \zeta')[\psi_{\zeta,\zeta'} dW_{\zeta'} + h_{\zeta,\zeta'} d\zeta']$$

(4.10)

$$u(z,\zeta) = \theta_\zeta + \int_{R_z} I(\zeta \curlywedge \zeta')\, g_{\zeta,\zeta'} dW_{\zeta'}$$

It is easy to verify that in the first case because of the term $I(\zeta' \curlywedge \zeta)$, $f(z, \zeta) = f(\zeta \otimes z,\zeta)$ and $u(z,\zeta) = u(\zeta \otimes z,\zeta)$ and for the second case $f(z,\zeta) = f(z \otimes \zeta,\zeta)$ and $u(z,\zeta) = u(z \otimes \zeta,\zeta)$. (See illustration.)

We note that for a fixed ζ, $f(z,\zeta)$ and $u(z,\zeta)$ as given by (4.9) and (4.10) are 1 and 2 semi-martingales, and differentiation rules apply once again.

5. The Ito Lemma for Stochastic Integrals in the Plane.

Let Z_{kz}, $z \in R_a$, $k = 1,2,\ldots,m$, be processes defined by

(5.1) $$X_{kz} = Z_{k0} + \int_{R_z} \theta_{k\zeta} d\zeta + \int_{R_z} \phi_{k\zeta} dW_\zeta + \int_{R_z \times R_z} \psi_{k,\zeta,\zeta'}\, dW_\zeta dW_{\zeta'}$$

$$+ \int_{R_z \times R_z} f_{k,\zeta,\zeta'} d\zeta dW_{\zeta'} + \int_{R_z \times R_z} g_{k,\zeta,\zeta'} dW_\zeta d\zeta'$$

If we set

(5.2) $$u_k(z,\zeta') = \phi_{k\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta')\psi_{k,\zeta,\zeta'} dW_\zeta + \int_{R_z} I(\zeta \curlywedge \zeta') f_{k,\zeta,\zeta'} d\zeta$$

and

(5.3) $$v_k(z,\zeta') = \theta_{k\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta') g_{k,\zeta,\zeta'} dW_\zeta$$

then (5.1) can be rewritten as

$$(5.4)\quad X_{kz} = X_{k0} + \int_{R_z} u_k(z,\zeta')dW_{\zeta'} + \int_{R_z} v_k(z,\zeta')d\zeta'$$

which is of the same form as (4.6), and u_k and v_k satisfy the conditions for (4.7). Therefore, we have

$$(5.5)\quad F(X_z) = F(Z_0) + \sum_k \int_{R_z} F_k(X_{\zeta'\otimes z})[u_k(z,\zeta')dW_{\zeta'} + v_k(z,\zeta')d\zeta']$$

$$+ \frac{1}{2}\sum_{k,\ell} \int_{R_z} F_{k\ell}(Z_{\zeta'\otimes z})\ u_k(z,\zeta')u_\ell(z,\zeta')\ d\zeta'$$

Now, (5.1) can also be reexpressed as

$$(5.6)\quad X_{kz} = X_{k0} + \int_{R_z} [\tilde{u}_k(z,\zeta)dW_\zeta + \tilde{v}_k(z,\zeta)d\zeta]$$

with $\tilde{u}_k$ and $\tilde{v}_k$ given by

$$(5.7)\quad \tilde{u}_k(z,\zeta) = \phi_{k\zeta} + \int_{R_z} I(\zeta \curlywedge \zeta')[\psi_{k,\zeta,\zeta'}dW_{\zeta'} + g_{k,\zeta,\zeta'}d\zeta'$$

$$(5.8)\quad \tilde{v}_k(z,\zeta) = \theta_{k\zeta} + \int_{R_z} I(\zeta \curlywedge \zeta')f_{k,\zeta,\zeta'}dW_{\zeta'}$$

Observe that because of the term $I(\zeta \curlywedge \zeta')$ in the integrals $\tilde{u}_k(z,\zeta) = \tilde{u}_k(z\otimes\zeta,\zeta)$ and $\tilde{v}_k(z,\zeta) = v_k(z\otimes\zeta,\zeta)$. Therefore, for any fixed point ζ'

$$(5.9)\quad X_{k\zeta'\otimes z} - Z_{k\zeta'} = \int_{R_{\zeta'\otimes z}-R_{\zeta'}} [\tilde{u}_k(\zeta'\otimes\zeta,\zeta)dW_\zeta + \tilde{v}_k(\zeta'\otimes\zeta,\zeta)d\zeta]$$

$$= \int_{R_z} I(\zeta \curlywedge \zeta')[\tilde{u}_k(\zeta'\otimes\zeta,\zeta)dW_\zeta + v_k(\zeta'\otimes\zeta,\zeta)d\zeta]$$

The three equations (5.2), (5.3) and (5.9) are all of the same form, viz.,

$$(5.10)\quad Y(z,\zeta') = \alpha_{\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta')[\beta_{\zeta,\zeta'} dW_\zeta + \gamma_{\zeta,\zeta'} d\zeta]$$

which is a 2-semimartingale for each fixed ζ'. Therefore, we can reexpress the integrands of (5.5) using (4.7), the differentiation formula for 2-semimartingales, e.g.,

$$F_k(X_{\zeta' \otimes z})\, u_k(z,\zeta') = F_k(X_{\zeta'})\phi_{k\zeta'}$$

$$+ \int_{R_z} I(\zeta \curlywedge \zeta') F_k(X_{\zeta' \otimes \zeta})[\psi_{k,\zeta,\zeta'} dW_\zeta + f_{k,\zeta,\zeta'} d\zeta]$$

$$+ \int_{R_z} I(\zeta \curlywedge \zeta') u_k(\zeta' \otimes \zeta,\zeta') \sum_\ell F_{k\ell}(X_{\zeta' \otimes \zeta})[\tilde{u}_\ell(\zeta' \otimes \zeta,\zeta) dW + \tilde{v}_k(\zeta' \otimes \zeta,\zeta) d\zeta]$$

$$+ \int_{R_z} I(\zeta \curlywedge \zeta')[\sum_\ell F_{k\ell}(X_{\zeta' \otimes \zeta})\psi_{k,\zeta,\zeta'}\, \tilde{u}_\ell(\zeta' \otimes \zeta,\zeta)] d\zeta$$

$$+ \frac{1}{2}\int_{R_z} I(\zeta \curlywedge \zeta') u_k(\zeta' \otimes \zeta,\zeta')[\sum_{\ell,m} F_{k\ell m}(X_{\zeta' \otimes \zeta})\tilde{u}_\ell(\zeta' \otimes \zeta,\zeta)\tilde{u}_m(\zeta' \otimes \zeta,\zeta)] d\zeta$$

If this tedious but straightforward procedure is applied to every term of the integrand in (5.5), we get the following:

<u>Proposition 5.1</u>. Let X_{kz}, $z \in R_a$, $k = 1,2,\ldots,n$, be process defined by (5.1), where the integrands are almost surely bounded. Let $F(x)$, $x \in R^n$, be a function with continuous mixed partials through the fourth order. Then,

$$(5.11) \qquad F(X_z) = F(X_0) + \int_{R_z} F_k(X_\zeta)[\phi_k \, dW_\zeta + \theta_k \, d\zeta]$$

$$+ \frac{1}{2}\int_{R_z} F_{k\ell}(X_\zeta)\phi_{k\zeta}\phi_{\ell\zeta}d\zeta$$

$$+ \int_{R_z \times R_z} [F_{k\ell}(X_{\zeta\vee\zeta'})u_k\tilde{u}_\ell + F_k(X_{\zeta\vee\zeta'})\psi_k]dW_\zeta dW_{\zeta'}$$

$$+\int_{R_z \times R_z} [F_k(X_{\zeta\vee\zeta'})f_k + F_{k\ell}(X_{\zeta\vee\zeta'})(u_k\tilde{v}_\ell + \psi_k\tilde{u}_\ell)$$

$$+ \frac{1}{2} F_{k\ell m}(X_{\zeta\vee\zeta'})u_k\tilde{u}_\ell\tilde{u}_m]d\zeta dW_{\zeta'}$$

$$+ \int_{R_z \times R_z} [F_k(X_{\zeta\vee\zeta'})g_k + F_{k\ell}(X_{\zeta\vee\zeta'})(\tilde{u}_k v_\ell + \psi_k u_\ell)$$

$$+ \frac{1}{2} F_{k\ell m}(X_{\zeta\vee\zeta'})\tilde{u}_k u_\ell u_m]dW_\zeta d\zeta'$$

$$+ \int_{R_z \times R_z} I(\zeta\vee\zeta')\{F_{k\ell}(X_{\zeta\vee\zeta'})(v_k\tilde{v}_\ell + g_k\tilde{u}_\ell + f_k u_\ell + \frac{1}{2}\psi_k\psi_\ell)$$

$$+ F_{k\ell m}(X_{\zeta\vee\zeta'})(u_k\tilde{u}_\ell\psi_m + \frac{1}{2} v_k\tilde{u}_\ell\tilde{u}_m + \frac{1}{2}\tilde{v}_k u_\ell u_m)$$

$$+ \frac{1}{4} F_{k\ell mp}(X_{\zeta\vee\zeta'})u_k u_\ell\tilde{u}_m\tilde{u}_p\}d\zeta d\zeta'$$

when u and v have arguments $(\zeta\vee\zeta',\zeta')$, $\tilde{u}$ and $\tilde{v}$ have arguments $(\zeta\vee\zeta',\zeta)$, ψ, f and g have arguments $(\zeta.\zeta')$ and all repeated indices are summed from 1 to n. Observe that we have made use of the relationship $\zeta\vee\zeta' = \zeta'\otimes\zeta$ if $\zeta\curlywedge\zeta'$.

Because of its complexity, the final expression for the differentiation formula may not be as useful as the partial differentiation formulas which give rise to it. Specifically, we are referring to (5.5) and the three Eqs. (5.2), (5.3) and (5.9). Note that (5.5) is a representation of $F(X_z)$ as a 1-semimartingale, and (5.2), (5.3) and (5.9) provide a representation of the integrands as 2-semimartingales. An alternative form with the roles of 1 and 2 semimartingales reversed also exists. It is useful to summarize these results as follows.

$$(5.12)\quad F(X_z) = F(X_0) + \int_{R_z} F_k(X_{\zeta'\otimes z})[u_k(z,\zeta')dW_{\zeta'} + v_k(z,\zeta')d\zeta']$$

$$+ \frac{1}{2}\int_{R_z} F_{k\ell}(X_{\zeta'\otimes z})u_k(z,\zeta')u_\ell(z,\zeta')d\zeta'$$

$$= F(X_0) + \int_{R_z} F_k(X_{z\otimes\zeta})[\tilde{u}_k(z,\zeta)dW_\zeta + \tilde{v}_k(z,\zeta)d\zeta]$$

$$+ \frac{1}{2}\int_{R_z} F_{k\ell}(X_{z\otimes\zeta})\tilde{u}_k(z,\zeta)\tilde{u}_\ell(z,\zeta)d\zeta$$

$$(5.13)\quad X_{k\zeta'\otimes z} = X_{k\zeta'} + \int_{R_z} I(\zeta\curlywedge\zeta')[\tilde{u}_k(\zeta'\otimes\zeta,\zeta)dW_\zeta + \tilde{v}_k(\zeta'\otimes\zeta,\zeta)d\zeta]$$

$$X_{kz\otimes\zeta} = X_{k\zeta} + \int_{R_z} I(\zeta\curlywedge\zeta')[u_k(\zeta'\otimes\zeta,\zeta')dW_{\zeta'} + v_k(\zeta'\otimes\zeta,\zeta')d\zeta']$$

$$(5.14)\quad u_k(z,\zeta') = \phi_{k\zeta'} + \int_{R_z} I(\zeta\curlywedge\zeta')\psi_{k,\zeta,\zeta'}dW_\zeta + \int_{R_z} I(\zeta\curlywedge\zeta')f_{k,\zeta,\zeta'}d\zeta$$

$$\tilde{u}_k(z,\zeta) = \phi_{k\zeta} + \int_{R_z} I(\zeta\curlywedge\zeta')\psi_{k,\zeta,\zeta'}dW_{\zeta'} + \int_{R_z} I(\zeta\curlywedge\zeta')g_{k,\zeta,\zeta'}d\zeta'$$

$$(5.15)\quad v_k(z,\zeta') = \theta_{k\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta') g_{k,\zeta,\zeta'} dW_\zeta$$

$$\tilde{v}_k(z,\zeta) = \theta_{k\zeta} + \int_{R_z} I(\zeta \curlywedge \zeta') f_{k,\zeta,\zeta'} dW_{\zeta'}$$

As an application consider the problem of characterizing a positive square-integrable martingale M_z on the sample space of a Wiener process. From [4] we know that M has a representation of the form

$$(5.16)\quad M_z = M_0 + \int_{R_z} \phi_\zeta dW_\zeta + \int_{R_z \times R_z} \psi_{\zeta,\zeta'} dW_\zeta dW_{\zeta'}$$

without less of generality we can assume $M_0 = 1$. Now, suppose ϕ and ψ are almost surely bounded. Then, write

$$(5.17)\quad M_z = 1 + \int_{R_z} u(z,\zeta') dW_{\zeta'}$$

$$= 1 + \int_{R_z} \tilde{u}(z,\zeta) dW_\zeta$$

where

$$(5.18)\quad u(z,\zeta') = \phi_{\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta') \psi_{\zeta;\zeta'} dW_\zeta$$

$$\tilde{u}(z,\zeta) = \phi_\zeta + \int_{R_z} I(\zeta \curlywedge \zeta') \psi_{\zeta,\zeta'} dW_{\zeta'}$$

Equation (5.12) now yields

$$(5.19)\quad \ell n\, M_z = \int_{R_z} [u(z,\zeta')/M_{\zeta' \otimes z}] dW_{\zeta'} - \frac{1}{2} \int_{R_z} [u(z,\zeta')/M_{\zeta' \otimes z}]^2 d\zeta'$$

The second equation in (5.17) yields

$$(5.20) \quad M_{\zeta'\otimes z} = M_{\zeta'} + \int_{R_z} I(\zeta\curlywedge\zeta')\tilde{u}(\zeta'\otimes\zeta,\zeta)dW_\zeta$$

The first equation in (5.18) can now be used with (5.20) to yield

$$h(z,\zeta') = [u(z,\zeta')/M_{\zeta'\otimes z}]$$

$$= \alpha_{\zeta'} + \int_{R_z} \beta_{\zeta,\zeta'}[dW_\zeta - \tilde{h}(\zeta\vee\zeta',\zeta)d\zeta]$$

where $\alpha_{\zeta'} = (\phi_{\zeta'}/M_{\zeta'})$

$$\tilde{h}(z,\zeta) = \tilde{u}(z,\zeta)/M_{z\otimes\zeta}$$

and

$$\beta_{\zeta,\zeta'} = [(\psi_{\zeta,\zeta'}/M_{\zeta\vee\zeta'}) - h(\zeta\vee\zeta',\zeta')\tilde{h}(\zeta\vee\zeta',\zeta)]I(\zeta\vee\zeta')$$

We now have the following alternative representations for M_z:

$$M_z = \exp\left\{\int_{R_z} h(z,\zeta')dW_{\zeta'} - \frac{1}{2}\int_{R_z} h^2(z,\zeta')d\zeta'\right\}$$

$$M_z = \exp\left\{\int_{R_z} \tilde{h}(z,\zeta)dW_\zeta - \frac{1}{2}\int_{R_z} \tilde{h}^2(z,\zeta)d\zeta\right\}$$

$$M_z = \exp\left\{\int_{R_z} \alpha_\zeta dW_\zeta + \int_{R_z\times R_z} \beta_{\zeta,\zeta'}dW_\zeta dW_{\zeta'}\right.$$

$$- \frac{1}{2}\int_{R_z} \alpha_\zeta^2 d\zeta - \frac{1}{2}\int_{R_z\times R_z} \beta_{\zeta,\zeta'}^2 d\zeta d\zeta'$$

$$- \int_{R_z\times R_z} \beta_{\zeta,\zeta'}[h(\zeta\vee\zeta',\zeta')dW_\zeta d\zeta' + \tilde{h}(\zeta\vee\zeta',\zeta)d\zeta dW_{\zeta'}$$

$$\left. - h(\zeta\vee\zeta',\zeta')\tilde{h}(\zeta\vee\zeta',\zeta)d\zeta d\zeta']\right\}$$

$$M_z = 1 + \int_{R_z} \alpha_\zeta M_\zeta dW_\zeta + \int_{R_z \times R_z} M_{\zeta \vee \zeta'} [\beta_{\zeta,\zeta'} + h(\zeta \vee \zeta', \zeta')\tilde{h}(\zeta \vee \zeta', \zeta)] dW_\zeta dW_{\zeta'}$$

The function h, $\tilde{h}$ are related to α and β by the equations

$$h(z,\zeta') = \alpha_{\zeta'} + \int_{R_z} \beta_{\zeta,\zeta'} [dW_\zeta - \tilde{h}(\zeta \vee \zeta', \zeta) d\zeta]$$

$$\tilde{h}(z,\zeta) = \alpha_\zeta + \int_{R_z} \beta_{\zeta,\zeta'} [dW_{\zeta'} - h(\zeta \vee \zeta', \zeta') d\zeta']$$

The application of these results to transformation of probability measures will be considered in a separate paper.

6. Integration with Respect to Paths

The formulas on partial differentiation given in section 4 can be interpreted as formulas on horizontal and vertical paths, relating path integrals to stochastic (area) integrals. So interpreted, they are not unlike the Green's formulas of Cairoli and Walsh [2].

Let Γ be an increasing path ($\Gamma : \{z(t),\ 0 \le t \le 1;\ t > s \Rightarrow z(t) \succ z(s)\}$) connecting points z_0 and z_f ($z_f \succ z_0$). Let D_1 be the area below Γ, and D_2 the area to the left of Γ. It is clear that D_1^Γ and D_2^Γ intersect only on Γ and their union is $R_{z_f} - R_{z_0}$. Let ϕ be a measurable process such that

$$(6.1) \quad \int_{R_{z_f} - R_{z_0}} \phi_\zeta^2 d\zeta < \infty \qquad \text{almost surely}$$

For each point ζ in $R_{z_f} - R_{z_0}$ let ζ_Γ denote the smallest point on Γ such that $\zeta_\Gamma \succ \zeta$. We say ϕ is Γ-adapted if ϕ_ζ is $\mathcal{F}_{\zeta_\Gamma}$ measurable for each $\zeta \in R_{z_f} - R_{z_0}$. For such a ϕ define

$$(6.2) \quad \phi^{\Gamma}_{i\zeta} = \phi_{\zeta} \quad \text{if} \quad \zeta \in D_i$$

$$= 0 \quad \text{otherwise}$$

Then ϕ^{Γ}_{i} is adapted to $\mathcal{F}^{i}$ and

$$(6.3) \quad M^{\Gamma}_{iz} = \int_{R_z} \phi^{\Gamma}_{i\zeta}\, dW_{\zeta}$$

defines a local i-martingale, which is a one-parameter continuous local martingale for $z \in \Gamma$, with

$$(6.4) \quad \langle M^{\Gamma}_{i}, M^{\Gamma}_{j} \rangle_z = \int_{R_z} \phi^{\Gamma}_{i\zeta}\phi^{\Gamma}_{j\zeta} d\zeta \quad , \quad z \in \Gamma$$

Hence,

$$(6.5) \quad M^{\Gamma}_{z} = M^{\Gamma}_{1z} + M^{\Gamma}_{2z}$$

is a continuous local martingale on Γ and

$$(6.6) \quad \langle M^{\Gamma}, M^{\Gamma} \rangle_z = \int_{R_z - R_{z_0}} \phi^{2}_{\zeta} d\zeta, \quad z \in \Gamma$$

If z_0 is the origin then $\phi^{\Gamma}_{1\zeta} + \phi^{\Gamma}_{2\zeta} = \phi_{\zeta}$ for all ζ in R_{z_f}. Hence, it is tempting to write

$$(6.7) \quad \int_{R_z} \phi_{\zeta} dW_{\zeta} = \int_{R_z} \phi^{\Gamma}_{1\zeta}\, dW_{\zeta} + \int_{R_z} \phi^{\Gamma}_{2\zeta}\, dW_{\zeta}$$

and use the right hand side to define the stochastic integral $\phi \circ W$. However, for this to be justified we would have to show that the right hand side is independent of Γ. Specifically, we need to show the following:

<u>Lemma</u> Let Γ and Γ' be two increasing paths, both starting from the origin and passing through z, such that ϕ is adapted to both Γ and Γ'. Then,

$$(6.8)\quad \int_{R_z} \phi^{\Gamma}_{1\zeta} dW_\zeta + \int_{R_z} \phi^{\Gamma}_{2\zeta} dW_\zeta = \int_{R_z} \phi^{\Gamma'}_{1\zeta} dW_\zeta + \int_{R_z} \phi^{\Gamma'}_{2\zeta} dW_\zeta$$

<u>proof</u>: with no loss of generality we can assume that both Γ and Γ' end at z. Then ϕ^{Γ}_i and $\phi^{\Gamma'}_i$ differ only on the sets $(D^{\Gamma}_1 \cap D^{\Gamma'}_2)$ and $(D^{\Gamma}_2 \cap D^{\Gamma'}_1)$. Observe that for every point ζ in these sets $\zeta_{\Gamma_1} \wedge \zeta_{\Gamma_2} = \zeta$. Since ϕ is adapted to both paths, for every ζ in these sets ϕ_ζ is measurable with respect to $\mathcal{F}_\zeta = \mathcal{F}_{\zeta_{\Gamma_1}} \cap \mathcal{F}_{\zeta_{\Gamma_2}}$. Hence,

$$\int_{D^{\Gamma}_i \cap D^{\Gamma'}_j} \phi^{\Gamma}_{i\zeta} dW_\zeta = \int_{D^{\Gamma}_i \cap D^{\Gamma'}_j} \phi^{\Gamma'}_{j\zeta} dW_\zeta = \int_{D^{\Gamma}_i \cap D^{\Gamma'}_j} \phi_\zeta dW_\zeta$$

for $i \neq j$. This completes the proof. ∎

Let Γ be an increasing path starting from the origin and let ϕ be Γ adapted. Let M be a continuous martingale on Γ defined by

$$(6.9)\quad M_z = \int_{R_z} \phi_\zeta dW_\zeta , \qquad z \in \Gamma$$

Let f be a process defined on Γ, adapted to $\{\mathcal{F}_z, z \in \Gamma\}$, and satisfying

$$(6.10)\quad \int_{R_z} f^2_{\zeta_\Gamma} \phi^2_\zeta d\zeta < \infty \qquad \text{a.s.}$$

for each $z \in \Gamma$. Then, the path integral $f \circ \partial M^{T}$ is well-defined as a continuous local martingale on Γ, and is equal to

$$(6.11) \quad (f\circ\partial M^{\Gamma})_z = \int_{R_z} f_{\zeta_\Gamma}\phi_\zeta dW_\zeta \quad , \quad z \in \Gamma$$

with

$$(6.12) \quad \langle f_1\circ\partial M^{\Gamma}, f_2\circ\partial M^{\Gamma}\rangle_z = \int_{R_z} f_{1\zeta_\Gamma} f_{2\zeta_\Gamma}\phi_\zeta^2 d\zeta \quad , \quad z \in \Gamma$$

For a point z let H_z and V_z denote the horizontal and vertical lines connecting z to the axes. Note that for $\Gamma = H_z$ ζ_Γ is $\zeta\otimes z$ and for $\Gamma = V_z$, ζ_Γ is $z\otimes\zeta$. Hence,

$$(6.13) \quad (f\circ\partial M^{\Gamma})_z = \int_{R_z} f_{\zeta\otimes z}\phi_\zeta dW_\zeta \qquad \text{for } \Gamma = H_z$$

$$= \int_{R_z} f_{z\otimes\zeta}\phi_\zeta dW_\zeta \qquad \text{for } \Gamma = V_z$$

We can now generalize proposition 4.1 as follows:

<u>Proposition 6.1</u>. Let Γ be an increasing path starting from the origin. Let X_{kz}, $z \in \Gamma$, $k = 1,2,\ldots,n$, be continuous local semimartingales defined by

$$(6.14) \quad X_{kz} = X_{k0} + \int_{R_z} \phi_{k\zeta} dW_\zeta + \int_{R_z} u_{k\zeta} d\zeta$$

where ϕ_k are Γ adapted. Let X denote $(X_1, X_2, \ldots, X_n)$ and let F(X) be a function with continuous mixed partial derivative up to second order.

$$(6.15) \quad F(X_z) = F(X_0) + \int_{R_z} F_k(X_{\zeta_\Gamma})[\phi_{k\zeta} dW_\zeta + u_{k\zeta} d\zeta]$$

$$+ \frac{1}{2}\int_{R_z} F_{k\ell}(X_{\zeta_\Gamma})\ \phi_{k\zeta}\phi_{\ell\zeta} d\zeta \qquad z \in \Gamma$$

where F_k and F_k denote partial derivatives, and summation over all repeated indices is implied.

We note that a stochastic integral of the second type

$$(6.16) \qquad M_z = \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \, dW_\zeta dW_{\zeta'}$$

can be reexpressed in the form

$$(6.17) \qquad M_z = \int_{R_z} \phi_\zeta \, dW_\zeta$$

in a multitude of ways. Take any increasing path Γ from the origin to z and define

$$(6.18) \qquad \phi_\zeta = \int_{\zeta \vee \zeta' \in D_1^\Gamma} I_{\zeta' \curlywedge \zeta} \, \psi_{\zeta',\zeta} \, dW_{\zeta'} + \int_{\zeta \vee \zeta' \in D_2^\Gamma} I_{\zeta \curlywedge \zeta'} \, \psi_{\zeta,\zeta'} \, dW_{\zeta'}$$

Then, ϕ is Γ adapted and

$$(6.19) \qquad \int_{R_z} \phi_\zeta \, dW_\zeta = \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \, dW_\zeta dW_{\zeta'}$$

It follows that on any increasing path

$$(6.20) \qquad \langle M,M \rangle_z^\Gamma = \int_{R_z} \left[\int_{\zeta \vee \zeta' \in D_1} I_{\zeta' \curlywedge \zeta} \psi_{\zeta',\zeta} dW_{\zeta'} + \int_{\zeta \vee \zeta' \in D_2} I_{\zeta \curlywedge \zeta'} \psi_{\zeta,\zeta'} dW_{\zeta'} \right]^2 d\zeta$$

Cairoli and Walsh [2] considered path integrals of the form $(f \circ \partial M)_\Gamma$ where f has certain stochastic partial derivatives and obtained a Green's formula. Our development of the path integral as a stochastic (area) integral makes the nature of the Green's formula, (at least in the special case and modified form which we treat) rather transparent. Let M_z be a strong martingale of the form

$$(6.21)\quad M_z = \int_{R_z} \phi_\zeta dW_\zeta \qquad z \in R_a .$$

we shall write dM_z for $\phi_z dW_z$. Let f_z be a function which has a representation

$$(6.22)\quad f_z = f_{z\otimes 0} + \int_{R_z} [u_{z\otimes\zeta} dM_\zeta + v_{z\otimes\zeta} d\zeta]$$

$$= f_{0\otimes z} + \int_{R_z} [\hat{u}_{\zeta\otimes z} dM_\zeta + \hat{v}_{\zeta\otimes z} d\zeta] \quad , \quad \forall z \in R_a$$

Observe that (6.22) implies f can be represented as a path integral with respect to ∂M and ∂s (= path length) on V_z and H_z.

Next, we consider $\int f\partial M$ on horizontal and vertical paths. On a horizontal path we have

$$(6.23)\quad (f\circ\partial M)_{H_z} = \int_{R_z} f_{\zeta'\otimes z}\, dM_{\zeta'}$$

Using the first equation of (6.22), we can write

$$(6.24)\quad f_{\zeta'\otimes z} - f_{\zeta'} = \int_{R_{\zeta'\otimes z} - R_{\zeta'}} [u_{\zeta'\otimes\zeta} dM_\zeta + v_{\zeta'\otimes\zeta} d\zeta]$$

For any $\zeta' \in R_z$, the set $R_{\zeta'\otimes z} - R_{\zeta'}$ is identical to the set $\{\zeta : \zeta \in R_z, \zeta \curlywedge \zeta'\}$. Hence

$$(6.25)\quad f_{\zeta'\otimes z} - f_{\zeta'} = \int_{R_z} I(\zeta\lambda\zeta')[u_{\zeta'\otimes\zeta} dM_\zeta + v_{\zeta'\otimes\zeta} d\zeta]$$

$$= \int_{R_z} I(\zeta\lambda\zeta')[u_{\zeta'\vee\zeta} dM_\zeta + v_{\zeta'\vee\zeta} d\zeta]$$

and (6.23) becomes

$$(6.26)\quad (f\circ\partial M)_{H_z} = \int_{R_z} f_\zeta dM_\zeta + \int_{R_z\times R_z} u_{\zeta\vee\zeta'} dM_\zeta dM_{\zeta'} + \int_{R_z\times R_z} v_{\zeta\vee\zeta'} d\zeta dM_{\zeta'}$$

Similarly, the corresponding expression for $(f\circ\partial M)_{V_z}$ is given by

$$(6.27)\quad (f\circ\partial M)_{V_z} = \int_{R_z} f_\zeta dM_\zeta + \int_{R_z\times R_z} \hat{u}_{\zeta\vee\zeta'} dM_\zeta dM_{\zeta'} + \int_{R_z\times R_z} \hat{v}_{\zeta\vee\zeta'} dM_\zeta d\zeta'$$

If we define for a decreasing path Γ

$$(f\circ\partial M)_\Gamma = -\ (f\circ\partial M)_{\hat{\Gamma}}$$

where $\hat{\Gamma}$ denotes Γ in the opposite direction, then (6.26) and (6.27) suffice to show that for a rectangle D

$$(6.28)\quad (f\circ\partial M)_{\partial D} = \int_{D\times D} (u_{\zeta\vee\zeta'} - \hat{u}_{\zeta\vee\zeta'})\ dM_\zeta dM_{\zeta'}$$

$$+ \int_{D\times D} v_{\zeta\vee\zeta'}\ d\zeta dM_{\zeta'} - \int_{D\times D} \hat{v}_{\zeta\vee\zeta'}\ dM_\zeta d\zeta'$$

where D is taken in the clockwise direction. Finally, for a region D whose boundary is piecewise pure (i.e., a parametric representation of the boundary $z(t) = (x(t), y(t))$, $0 \leq t \leq 1$, has piecewise monotonic components), (6.28) follows by approximating ∂D by stepped paths as is done in [2]. Equation (6.28) is the Green's theorem of Cairoli and Walsh.

REFERENCES

1. R. Cairoli: Sur une équation differtielle stochastique. Compte Rendus Acad. Sc., Paris 274 (June 12, 1972) Ser A, 1739-1742.
2. R. Cairoli and J. B. Walsh: Stochastic integrals in the plane. To be published in Acta Matematica (1975).
3. E. Wong: The dynamics of random fields. Proc. U.S.-Japan Joint Seminar on Stochastic Methods in Dynamical Problems, Kyoto, 1971.
4. E. Wong and M. Zakai: Martingales and stochastic integrals for processes with a multidimensional parameter. Z. für Wahrscheinlichkeitstheorie verw. Gebiete, 29, 109-122 (1974).
5. E. Wong: A likelihood ratio formula for two-dimensional random fields. IEEE Trans. Information Theory, IT-20, 418-422 (1974).
6. E. Wong and M. Zakai: Weak martingales and stochastic integrals in the plane. Electronics Research Laboratory, Univ. of California, Berkeley, Tech. Memorandum 496, 1975.