THE SPATIAL STRUCTURE OF PRODUCTION WITH A LEONTIEF TECHNOLOGY

by

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Abstract

The structure of the optimal spatial pattern of production is studied when there are interdependencies among production units which can be described by a Leontief technology, and when there is a single marketplace of final demand, the CBD. Transportation cost is proportional to distance. It is shown that the various goods are produced in rings which can be ranked by distance from the CBD independently of the levels of final demand. Furthermore shipment of goods for meeting intermediate and final demand can only be in the direction of the CBD and no shipment of goods towards the periphery can occur. A finite algorithm is given for the construction of the optimal pattern and for determining a system of f.o.b. prices and land rents which sustain it as a competitive equilibrium.

1. Introduction. In this paper we examine the spatial structure of production within the context of a model whose elements can be briefly described as follows.

a) Space. This is a featureless plane at the center of which is the central business district (CBD) of fixed radius $u_0 \geq 0$. Land outside the CBD, at distances $u \geq u_0$, is devoted exclusively to production.

b) Commodities. There is a finite number of goods indexed $j = 1,\ldots,n$ produced outside the CBD. There is one other commodity which is available throughout and serves as a numeraire; we shall call it money ($). Money can be regarded as a composite commodity.

c) Technology. Land is the only non-produced factor of production. To produce x_j units of j requires $\alpha_j x_j$ units of land and $a_{ij} x_j$ units of i, where $\alpha_j > 0$, $a_{ij} \geq 0$, $a_{jj} < 1$ are constants. Thus we have a linear technology with intermediate goods but without joint products and without substitution.

d) Transportation. It costs $\$t_j$ to transport each unit of good j over a unit distance where $t_j > 0$ is a constant.

Now suppose we are given a vector $Q = (Q_1,\ldots,Q_n)' \geqq 0$ of final demands at the CBD. Since we have assumed a Leontief technology, Q uniquely determines the total amounts of the various goods which must be produced in order to meet this demand. They are given by the vector $x = (x_1,\ldots,x_n)'$ which satisfies $(I-A)x=Q$ where I is the identity matrix and $A = (a_{ij})$. However the spatial organization of this production and the transportation flows, both between production units and between them and the CBD, are still undetermined. Among all possible ways of organizing these spatially we study the _optimal_ pattern, i.e., the one which minimizes total transportation cost. Specifically, we wish to know whether certain qualitative properties enjoyed by the optimal pattern in the two simple examples discussed next continue to hold in general.

In the first example there are no intermediate goods i.e., all the $a_{ij} = 0$. Then it is easy to show that the optimal pattern enjoys the following two properties (see e.g. Artle and Varaiya [1975]).

(A) If the different goods are ordered according to the rule "i precedes j if, in the optimum, production of i does not occur further away from the CBD than production of j." then this ordering is _independent_ of the final demand Q and depends only on the technology and transportation costs.

(B) It is possible to sustain the optimal pattern as a competitive equilibrium by means of a system of land rents r(u) and f.o.b. prices for the goods $p_j(u)$, $u \geq u_0$. Furthermore r(u) and p_j (u) descrease with u whereas $\frac{dr}{du}(u)$ and $\frac{dp_j}{du}(u)$ are non decreasing in u. The analytically trivial nature of this result stems from the fact that there are no intermediate goods, hence no locational interdependence between production units.

Recently, Mills [1970] and Hartwick [1974] analyzed the simplest possible example with intermediate goods. They consider a two-goods economy, n = 2. Furthermore $a_{21} > 0$, but $a_{12} = 0$, and $Q_1 > 0$ but $Q_2 = 0$. Thus good 2 is strictly an intermediate good. With these assumptions it is clear that, at the optimum, production of good 1 cannot be located further away from the CBD than production of good 2, since this would entail that some units of good 2 would have to be shipped outwards, incorporated into good 1, and then shipped inwards which is inefficient. Secondly, they show that property (B) holds also.

It is difficult to draw any conclusions about the general case from these examples. Furthermore the methods of analysis employed do not appear to be extendable to the general case for it is important there that the ordering of goods - in the optimum, by distance from the CBD - be established a priori and that no outwards shipment occur in the optimum. But in the general case it appears that *a priori* we can neither establish this ordering nor can we rule out outwards shipment.

Nevertheless, we shall show below that properties (A) and (B) always hold for the model outlined before. We show also that in the optimum no outward shipment of goods can occur. Furthermore, the optimum is unique. The method of proof is somewhat unusual. In Section 2 we formulate the determination of the optimal pattern of production as a problem of optimal control and we exhibit necessary and sufficient conditions for optimality. From these conditions we can conclude that the optimum pattern can be sustained as a competitive equilibrium. Next, in Section 3, we propose a particular pattern of production and transportation in which there is no outwards shipment and prove that it satisfies these optimality conditions. Hence the proposed pattern is optimal. The land rents and f.o.b. prices corresponding to this proposed pattern satisfy the qualitative properties listed under (B). It is also shown that the optimum is unique. The price system which sustains the optimum need not be unique (although the rent system is unique). We characterize precisely the set of equilibrium price systems. In Section 4, we study some properties of the entire region as an aggregate production function. Finally, in Section 5, we compare our results with those obtained by Goldstein and Moses [1975] for the two-goods case. In their analysis backwards shipment is possible and we show how this can occur. Throughout the analysis critical use is made of the assumption that there are no substitute production techniques and no joint production. We hope to remove these restrictions in subsequent papers.

2. Formulation as an optimal control problem. For any vector z, $z \geq 0$ means that all its components are non-negative; $z > 0$ means $z \geq 0$ and $z \neq 0$; $z \gg 0$ means that all its components are positive. We need the following notations.

$A^j = (a_{1j}, \ldots, a_{nj})'$ is the (column) vector of inputs necessary to produce one unit of j. $A^j \geq 0$ all j.

$A = (A^1 \ldots A^n)$, $D = I - A$ is the input-output matrix with columns $D^1, .., D^n$. D is the ith activity vector.

$\alpha = (\alpha_1, \ldots, \alpha_n)'$ is the vector of land inputs; $\alpha \gg 0$

$t = (t_1, \ldots, t_n)$ is the (row) vector of unit transportation costs; $t \gg 0$

$Q = (Q_1, \ldots, Q_n)'$ is the vector of final demands at the CBD; $Q \geq 0$

$\theta(u)$ = amount of land at $u \geq u_0$ available for production; $\theta(u) > 0$. (For e.g., in a circular region $\theta(u) = 2\pi u$)

We impose the following assumption.

Assumption: A is productive i.e., there exists a vector $x \geq 0$ such that $Dx >> 0$.

Definition 2.1: An allocation with final output Q is a 4-tuple $\omega = (\bar{u}, x(\cdot), f(\cdot), \phi(\cdot))$ where (i) $\bar{u} \geq u_0$ is the maximum distance at which production occurs, (ii) $x(u) = (x_1(u), \ldots, x_n(u))' \geq 0$ is the vector of activity levels at u, $u_0 \leq u \leq \bar{u}$, (iii) $f(u)$, respectively $\phi(u)$, is the amount of (net) local production at u which is shipped inwards to the CBD, respectively outwards from the CBD; and such that the following feasibility conditions are satisfied.

a) $0 \leq \alpha' x(u) \leq \theta(u), \quad u_0 \leq u \leq \bar{u}$

b) $y(u) = DX(u) = f(u) + \phi(u), \quad u_0 \leq u \leq \bar{u}$

c) If $s(u)$ is obtained from the differential equation

(2.1) $$\dot{s}(u) = \frac{ds}{du}(u) = -f(u), \quad s(\bar{u}) = 0,$$

then $s(u) \geq 0$, $u_0 \leq u \leq \bar{u}$ and $s(u_0) = Q$.

d) If $\sigma(u)$ is obtained from the differential equation

(2.2) $$\dot{\sigma}(u) = \phi(u), \quad \sigma(u_0) = 0,$$

then $\sigma(u) \geq 0$, $u_0 \leq u \leq \bar{u}$.

The transport cost corresponding to the allocation is

(2.3) $$T(\omega) = \int_{u_0}^{\bar{u}} t[s(u) + \sigma(u)]\, du$$

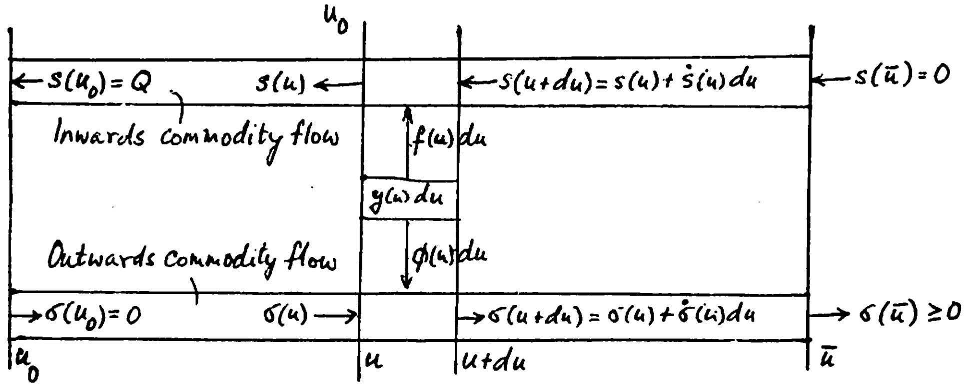

Fig 2.1. Commodity flows in an allocation.

Condition a) states that land devoted to production at u cannot exceed land available at u, whereas b) states that commodity flows originating at u equal net production at u. Conditions c) and d) can be deduced from Figure 1. They are essentially material balance equations with the additional restriction that the total flows must be non-negative.

Definition 2.2: An allocation with final output Q, $\omega^* = (\bar{u}^*, x^*(\cdot), f^*(\cdot), \phi^*(\cdot))$, is optimal if $T(\omega^*) \leq T(\omega)$ for all allocations ω with final output Q.

Thus the problem of finding an optimal allocation is an optimal control problem: we must find a "control function" x(u), so as to minimize the functional (2.3), subject to the "dynamical equations" (2.1) and (2.2) and the "state constraints" $s(u) \geq 0$, $\sigma(u) \geq 0$. The presence of these state constraints makes the problem quite difficult since, as we shall see below, the resulting optimality conditions do not permit us to deduce any of the interesting properties of the optimal allocation that we are seeking.

Theorem 2.1. (Optimality conditions) An allocation with final output Q, $\omega^* = (\bar{u}^*, x^*(\cdot), f^*(\cdot), \phi^*(\cdot))$ is optimal if and only if there exists an absolutely continuous price system $p^*(u) \geq 0$, $u \geq u_0$, such that the following conditions hold.

a) For $u_0 \leq u < \bar{u}^*$

(2.4) $$p^*(u)\, Dx^*(u) = \text{Max}\ \{p^*(u)\, Dx \mid x \geq 0,\ \alpha' x \leq \theta(u)\}$$

b) If $s^*(u)$, $\sigma^*(u)$ are solutions of (2.1), (2.2) corresponding to $f^*(u)$, $\phi^*(u)$ respectively, then

$$(2.5) \qquad (s^*(u))' \sigma^*(u) = 0 , \; u_0 \leq u \leq \bar{u}^*$$

$$(2.6) \qquad p^*(u) = 0, \; u \geq \bar{u}^*$$

$$(2.7) \qquad \dot{p}_j^*(u) = \begin{cases} -t_j \;, \text{ if } s_j^*(u) > 0 \\ t_j \;, \text{ if } \sigma_j^*(u) > 0 \\ \in [-t_j, t_j], \text{ if } s_j^*(u) = 0 \text{ and } \sigma_j^*(u) = 0 \end{cases}$$

for $j=1,\ldots,n$ and $u_0 \leq u \leq \bar{u}^*$

(Note: Here and elsewhere the price vector p^* is always considered a row vector)

Proof: See Appendix.

The conditions can be easily interpreted. (2.4) says that $x^*(u)$ is the maximum profit activity vector at u when $p^*(u)$ is the f.o.b. price at u. Since $s_j^*(u) > 0$, $\sigma_j^*(u) > 0$, (2.5) says merely that we cannot have simultaneously $s_j^*(u) > 0$ and $\sigma_j^*(u) > 0$ which would clearly be inefficient. (2.6) limits the total amount of land devoted to production. (2.7) is a version of the condition first observed by Samuelson [1952]. It asserts that if it is optimal at u to ship the ith good inwards, respectively outwards, then the price of i must decrease, respectively increase, at the rate of the transport cost; on the other hand price cannot change faster than transport cost to prevent arbitrage. From these conditions it is not at all evident that it is not optimal to have outwards shipment of goods, nor that prices must decline with distance from CBD. As an immediate consequence however we can obtain the following result.

Corollary 2.1. The system of prices $p^*(u)$, $u_0 \leq u \leq \bar{u}^*$ and land rents $r^*(u)$ given by

$$(2.8) \qquad r^*(u) = [\alpha' x^*(u)]^{-1} [p^*(u) \, Dx^*(u)]$$

sustains the optimal allocation ω^* as a competitive equilibrium.

Subsequently we shall need the next result.

Corollary 2.2. Let $p^*(u)$ be a price system which sustains ω^* as in Theorem 2.1. Let ω be another optimal allocation with final output Q. Then p^* also sustains ω .

Proof: See Appendix.

We will show in the next section that for every $Q \geq 0$ there exists an optimal allocation with final output Q. We can then define the equilibrium price vector $P(Q) = p^*(u_0)$ at the CBD corresponding to the final output Q. We may call P(Q) the region's supply function. Some of its properties will be studied in Section 4.

3.1 An optimal allocation. We first display a particular allocation $\omega^* = (\bar{u}^*, x^*(\cdot), f^*(\cdot), \phi^*(\cdot))$ with final output Q and then show that it is optimum. The allocation ω^* is determined in two steps. First, in Lemma 3.2, we establish an ordering among the goods according to the distance from the CBD at which they are produced. This ordering is independent of Q. Secondly, we specify the actual activity levels x^*, f^* and ϕ^* such that the final output is indeed Q.

As a preliminary we need this well-known result.

Lemma 3.1 The following conditions are equivalent:

(i) A is productive

(ii) A' is productive

(iii) For every $y \in R^n$, $y \geq 0$ there exists $x \in R^n$, $x \geq 0$ such that $Dx = y$

(iv) D is non-singular and $D^{-1} \geq 0$

(v) All principal minors of D are positive.

Proof: See for example Nikaido [1968].

We now give the first step. The ordering of the goods is given by the next lemma. It should be noted that the proof is constructive.

Lemma 3.2 There is an integer $m \leq n$, m (row) vectors $t^i = (t_1^i, \ldots, t_n^i)$ and m subsets J_i of $\{1,\ldots,n\}$ such that

(3.1) $$t = t^1 > t^2 > \ldots > t^m \gg 0$$

(3.2) $$J_1 \subset J_2 \ldots \subset J_m = \{1,\ldots,n\}$$

(3.3) $$t_j^i = t_j \text{ if and only if } j \notin J_{i-1}$$

(3.4) $$\alpha_j^{-1} t^i d^j \begin{cases} = \alpha_k^{-1} t^i d^k & \text{if } j \in J_i,\ k \in J_i \\ > \alpha_k^{-1} t^i d^k & \text{if } j \in J_i,\ k \notin J_i \end{cases}$$

Proof: Set $t^1 = t$ and define

$$J_1 = \{ j \mid \alpha_j^{-1} t^1 d^j = \max_{k=1,\ldots,n} \alpha_k^{-1} t^1 d^k \}$$

We now proceed by induction. Suppose t^i and J_i have been constructed. Suppose $J_i \neq \{1,\ldots,n\}$. By Lemma 3.1 there is a row vector $\Delta^i = (\Delta_1^i, \ldots, \Delta_n^i) > 0$ such that

(3.5) $$\Delta_j^i > 0 \text{ if and only if } j \in J_i ,$$

(3.6) $$\Delta^i d^j = \alpha_j \text{ if } j \in J_i$$

We claim that for $\lambda > 0$

(3.7) $$\alpha_j^{-1}(t^i - \lambda\Delta^i) d^j = \alpha_j^{-1} t^i d^j - \lambda \text{ if } j \in J_i ,$$

(3.8) $$\alpha_k^{-1}(t^i - \lambda\Delta^i) d^k \text{ is non-decreasing in } \lambda \text{ for } k \notin J_i .$$

To see (3.7) we simply substitute from (3.6); for future use we also note that by (3.4) the right hand side of (3.7) is independent of j for $j \in J_i$. To see (3.8) we observe that $\Delta^i_k = 0$ if $k \notin J_i$ by (3.5), whereas $d^k_j \leq 0$ if $k \neq j$; hence $\Delta^i d^k \leq 0$ for $k \notin J_i$.

Since (3.7) is strictly decreasing in λ and (3.8) is non-decreasing, therefore there exists $\lambda_i > 0$ such that

$$(3.9) \qquad \lambda_i = \min \{ \lambda > 0 \mid \text{there exists } k \notin J_i \text{ with } \alpha_k^{-1}(t^i - \lambda\Delta^i)d^k = \alpha_j^{-1} t^i d^j - \lambda \text{ for } j \in J_i \}$$

Set

$$(3.10) \qquad t^{i+1} = t^i - \lambda_i \Delta^i ,$$

$$(3.11) \qquad J_{i+1} = \{ k \mid \alpha_k^{-1} t^{i+1} d^k = \alpha_j^{-1} t^{i+1} d^j \text{ for } j \in J_i \}$$

We check that t^{i+1}, J_{i+1} satisfy (3.1)-(3.4). First of all since $\lambda_i > 0$ and $\Delta^i > 0$ therefore $t^i > t^{i+1}$ by (3.10). Secondly, from (3.9) and (3.11) we know that $J_i \subset J_{i+1}$; in fact $J_i \subsetneq J_{i+1}$ so that if m is the smallest integer such that $J_m = \{1,\ldots,n\}$, then $m \leq n$. Thirdly, $t^{i+1}_j = t^i_j$ if and only if $\Delta^i_j = 0$, but $\Delta^i_j = 0$ if and only if $j \notin J_i$ and so (3.3) is verified by induction. Fourthly, from (3.9) and (3.11) it follows that $\alpha_k^{-1} t^{i+1} d^k < \alpha_j^{-1} t^{i+1} d^j$ for $j \in J_{i+1}, k \notin J_{i+1}$, thus (3.4) is verified. To complete the proof of the lemma it only remains to show that $t^{i+1} \gg 0$ if $t^i \gg 0$. Now $t^{i+1}_j = t_j > 0$ if $j \notin J_i$, whereas for $j \in J_i$ (in fact for $j \in J_{i+1}$)

$$t^{i+1} d^j = \rho \alpha_j$$

where $\rho > 0$ by (3.11). By Lemma 3.1 these relations can hold only if $t^{i+1}_j > 0$ for $j \in J_i$. The lemma is proved.

From the proof of the previous lemma we can deduce the next result.

Corollary 3.1 For i = 1,...,m

$$(3.12) \qquad \rho_i = \alpha_j^{-1} t^i d^j , \quad j \in J_i$$

is independent of $j \in J_i$. Furthermore $\rho_1 > \rho_2 \ldots > \rho_m$.

Let $Q \geq 0$ be a fixed output vector for the remainder of this section.

Lemma 3.3 There is a unique set of activity vectors $x^i = (x^i_1, \ldots, x^i_n)' \geq 0$, $i = 1, \ldots, m$ such that

(3.13) $$Q = \sum_{i=1}^{m} Dx^i$$

(3.14) $$x^i_j = 0 \quad \text{if } j \notin J_i$$

(3.15) $$\sum_{k=1}^{n} d_{jk} x^i_k = \begin{cases} 0 & \text{if } j \in J_{i-1} \\ Q_j + \sum_{\ell=1}^{i-1} \sum_{k=1}^{n} a_{jk} x^\ell_k & \text{if } j \in J_i - J_{i-1} \end{cases}$$

Proof: Immediate from Lemma 3.1.

We now propose an allocation ω^* which meets final output Q. The amount of land needed for the activity vector x^i is $\alpha' x^i$. By induction we define the distances $u_0 \leq u^*_1 \leq u^*_2 < \ldots \leq u^*_m$ such that

(3.16) $$\int_{u_0}^{u^*_1} \theta(u)\, du = \alpha' x^1$$

(3.17) $$\int_{u^*_{i-1}}^{u^*_i} \theta(u)\, du = \alpha' x^i \quad \text{for } i = 2, \ldots, m$$

(We are implicitly assuming that $\theta(u)$ is so large that there is enough land to produce Q.)

$\omega^* = (\bar{u}^*, x^*(\cdot), f^*(\cdot), \phi^*(\cdot))$ is given by

(3.18a) $$\bar{u}^* = u^*_m$$

(3.18b) $$x^*(u) = \theta(u)[\alpha' x^i]^{-1} x^i \quad \text{for } u \in [u^*_{i-1}, u^*_i]$$
$$i = 1, \ldots, m \left(\text{hence } \int_{u^*_{i-1}}^{u^*_i} x^*(u)du = x^i\right)$$

(3.18c) $$f^*(u) = Dx^*(u) \quad \text{for } u \in [u_0, \bar{u}^*]$$

(3.18d) $$\phi^*(u) = 0 \quad \text{for } u \in [u_0, \bar{u}^*]$$

In light of (3.13) - (3.18) the proposed allocation can be described as follows. In the ith ring, $[u^*_{i-1}, u^*_i]$, there is no production, gross or net, of goods $j \notin J_i$ which are produced in subsequent rings (see (3.14)); there is no net production of goods $j \in J_{i-1}$ which are produced in preceding rings (see (3.15)); there is net production only of goods $j \in J_i - J_{i-1}$ in such amounts as to meet the final demand for these goods at the CBD as well as to meet the demands in preceding rings as intermediate inputs (see (3.15)). Finally, the sizes of the rings are such as to provide just enough land to sustain these activities in the requisite order (see (3.16),(3.17)). Note that in the proposed allocation there is no backwards shipment of goods ((3.18d)).

<u>Theorem 3.1</u> (Existence of an optimum) ω^* is an optimal allocation with final output Q.

<u>Proof</u>: First of all it is immediate from (3.13)-(3.18) that ω^* is indeed an allocation with final output Q i.e., it satisfies the various feasibility conditions of Definition 2.1. Let $s^*(u)$, $\sigma^*(u)$ be the solutions of (2.1), (2.2) corresponding to ω^*. It is easy to check that

(3.19) $\sigma^*(u) = 0 \quad , u_0 \le u \le \bar{u}^*$

(3.20) $s_j^*(u) = 0 \quad$ for $u \ge u_i^*$ and $j \in J_i$

We prove optimality of ω^* by using Theorem 2.1. Define the price system $p^*(u) = (p_1^*(u),\ldots,p_n^*(u))$, $u_0 \le u \le \bar{u}^*$ as follows:

(3.21) $$p^*(u) = \begin{cases} 0, & \text{if } u \ge \bar{u}^* = u_m^* \\ p^*(u_i^*) + (u_i^*-u)t^i, & \text{if } u \in [u^*_{i-1}, u^*_i], \end{cases}$$

$i = 1,\ldots,m,$

where the vectors $t^1, \ldots, t^m$ are specified in Lemma 3.2. So for $u \in [u^*_{i-1}, u^*_i]$

$$-\dot{p}^*_j = t^i_j \begin{cases} < t_j & \text{if } j \in J_{i-1} \\ = t_j & \text{if } j \notin J_{i-1} \end{cases}$$

by (3.3), whereas by (3.20) $s^*_j(u) > 0$ only if $j \notin J_{i-1}$; hence (2.7) holds. It only remains to verify (2.4). Define the profit per unit of land at u made by a producer of good j as

$$(3.22) \qquad r_j(u) = \alpha_j^{-1} p^*(u) D^j$$

and let $r^*(u) = \max\{r_1(u), \ldots, r_n(u)\}$. From (3.21), Lemma 3.1 and Corollary 3.1 it follows that

$$(3.23) \qquad \dot{r}^*(u) = -\rho_i \quad \text{for } u \in [u^*_{i-1}, u^*_i],$$

$$(3.24) \qquad r^*(u) \begin{cases} = r_j(u) & \text{for } j \in J_i, \ u \in [u^*_{i-1}, u^*_i] \\ > r_j(u) & \text{for } j \notin J_i, \ u \in [u^*_{i-1}, u^*_i] \end{cases}$$

Now fix $u \in [u^*_{i-1}, u^*_i]$ and let $x \geq 0$ be any activity vector with $\alpha' x = \theta(u)$. Then $p^*(u)Dx = \sum_j x_j \, p^*(u)D^j = \sum_j \alpha_j x_j r_j(u)$ by (3.22). Hence by (3.24)

$$(3.25) \qquad p^*(u)Dx \begin{cases} < \theta(u)r^*(u) & \text{if } x_j > 0 \text{ for some } j \notin J_i \\ = \theta(u)r^*(u) & \text{if } x_j = 0 \text{ for all } j \notin J_i, \end{cases}$$

and so, in particular, since $x^*_j(u) = 0$ for $j \notin J_i$ by (3.14),(3.18b), therefore

$$p^*(u)Dx \leq \theta(u)r^*(u) = p^*(u)Dx^*(u) \text{ and so (2.4)}$$

is verified. The theorem is proved.

The price system $p^*(u)$ defined in (3.21) sustains ω^* as a competitive equilibrium. Under this price system producers of good j make a bid rent offer of $r_j(u)$ and the equilibrium rent is $r^*(u)$. From Corollary 3.1 and (3.22) we observe that $r^*(u)$ is decreasing and $\dot{r}^*(u)$ is nondecreasing in u.

As an example, suppose $n = 3$ and $J_1 = \{1\}$, $J_2 = \{1,2\}$, $J_3 = \{1,2,3\}$. Then the price system and rent functions have the pattern shown in Figure 3.1.

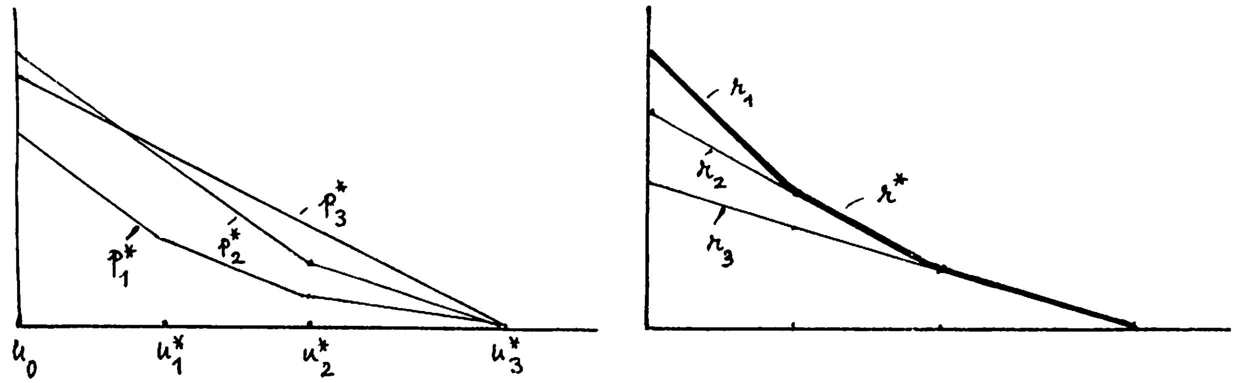

Figure 3.1. Equilibrium prices and rents of example.

The next result shows that the optimal allocation ω^* proposed above is essentially unique.

Theorem 3.2. (Uniqueness of the optimum) Let $\omega = (\bar{u}, x(\cdot), f(\cdot), \phi(\cdot))$ be any allocation with final output Q. Then ω is optimal if and only if

(3.26) $$\bar{u} = \bar{u}^*$$

(3.27) $$\int_{u_{i-1}^*}^{u_i^*} x(u)\, du = \int_{u_{i-1}^*}^{u_i^*} x^*(u)\, du = x^i, \quad i = 1,\ldots,m$$

(3.28) $$f(u) = Dx(u)$$

(3.29) $$\phi(u) = 0$$

Proof: The sufficiency is easy to prove since it is readily verified that whenever ω satisfies (3.26)-(3.29) then ω also satisfies the conditions of Theorem 2.1 with $p^*(u)$ given by (3.21) so that ω is indeed optimal. It remains to prove necessity. Suppose ω is optimal. Then by corollary 2.2 ω satisfies the conditions of Theorem 2.1 with $p^*(u)$ given by (3.21). Since by construction $\dot{p}^* \leq 0$ therefore by (2.7) there can be no backwards shipment in ω which implies (3.29) and (3.28). Next, the maximum profit condition (2.4), together with (3.25), implies that for $u \in [u^*_{i-1}, u^*_i]$ and $j \notin J_i$ we must have $x_j(u) = 0$. Define

$$z^i = \int_{u^*_{i-1}}^{u^*_i} x(u)\, du \quad , \; i = 1,\ldots,m$$

Then $z^i_j = 0$ for $j \notin J_i$ and $Q = \sum_{i=1}^{m} Dz^i$ since ω meets final demand Q. Finally, if $j \in J_i$ then by (3.21) $p^*_j(u) > -t_j$ and so by (2.7) good j cannot be shipped and hence there is no net production of this good in $[u^*_{i-1}, u^*_i]$ i.e. $\sum_{k=1}^{n} d_{jk} z^i_k = 0$ for $j \in J_{i-1}$. Then, since ω meets final demand Q, therefore $\sum_k d_{jk} z^i_k = Q_j + \sum_{l=1}^{i-1} \sum_k a_{jk} z^l_k$ for $j \in J_i - J_{i-1}$. By Lemma 3.3 $z^i = x^i$ and so (3.27) holds. (3.26) is then immediate. The theorem is proved.

From this result we can see that the only way we can have two different optimal allocations is when there is more than one production pattern x(u) which satisfies (3.27). In turn this can happen only when for some i the set $J_i - J_{i-1}$ contains at least two goods, say j and k, so that then $r_j(u) = r_k(u) = r^*(u)$ in the ring $[u^*_{i-1}, u^*_i]$. Suppose further that production of either good does not require the other as an intermediate input. Then it is clearly immaterial how the production of j and k in the ring is arranged as long as the aggregate production requirement in the ring, (3.27), is met. Thus except for these "singular" cases when $J_i - J_{i-1}$ contains two or more goods (equivalently, when $m < n$) the optimal allocation is unique.

In the "regular" case, $m = n$, we may relabel the goods so that $J_i = \{1,\ldots,i\}$. Then, in the optimum, net production of good i always occurs in the ith ring. Furthermore, if production of i requires some (or all) of the goods $1,\ldots,i-1$ as inputs then these intermediate goods must be produced "locally" i.e., in ring i, whereas if some of the goods i + 1,...,n are needed as inputs then these must be "imported" from the outer rings. Thus as we move closer to the CBD production is "specialized": in the outermost ring all inputs are produced locally, in the innermost ring all the inputs are imported.

In the regular case we can readily see which particular combinations of the activities $A^1,\dots,A^n$ are adopted in each ring. Suppose the activities are relabeled so that $J_i = \{1,\dots,i\}$. Let $\Psi = (\Psi^i_j)$ be a n x n nonnegative upper triangular matrix ($\Psi^i_j = 0$, $i < j$) such that the matrix $C = A\Psi$ is lower triangular ($C^i_j = 0$, $i > j$), and $C^i_i = +1$, all i. Then in the ith ring $[u^*_{i-1}, u^*_i]$ techniques $A^1,\dots,A^i$ are adopted in the proportions $\Psi^i_1,\dots,\Psi^i_i$, and the resulting net input-output technique in this ring is given by the vector $C^i = (0,\dots,0,C^i_i,\dots,C^i_n)$. The land used to operate C^i at a unit level is $\alpha_1\Psi^i_1+\dots+\alpha_i\Psi^i_i$. We see that the matrix C of net input-output techniques, when its columns are ordered according to the rings from the CBD in which they are adopted have a triangular structure. This is possible because there is no indivisibility in production
(cf. Andersson and Marksjo [1972, p.135]).

Next we investigate the extent to which the price and rent profiles which sustain an optimal allocation as a competitive equilibrium are unique. It is easy to see that the price profile will not in general be unique. For suppose that in the optimum allocation there is no production or shipment of good j at some distance $u \in [u_0, \bar{u}^*]$. Then the equilibirum price system must be such that these three conditions hold: (i) producers find it unprofitable to supply j at u, (ii) producers find it unprofitable to demand j at u, and (iii) producers find it unprofitable to ship j to or from u. Hence at u the supply and demand curves cannot intersect at positive quantity levels so they must have a behavior as portrayed in Figure 3.2. But then (given all other prices) any price p_j of

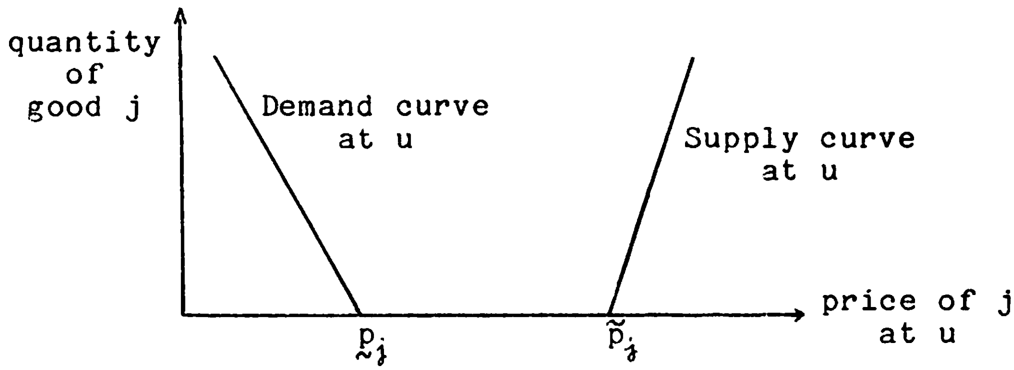

Figure 3.2. Non-uniqueness of equilibrium prices

good j at u which satisfies $\underset{\sim}{p}_j \leq p_j \leq \tilde{p}_j$ will satisfy the equilibrium conditions (i) and (ii) above. Of course, not all such prices are permissible since condition (iii) induces some relations between the equilibrium prices at u and at other locations. Theorem 3.3 shows that with the exception of the situation depicted above the equilibrium price and rent profiles are indeed unique. The next lemma is preliminary to this result.

<u>Lemma 3.4</u>. Let $J = \{1,\ldots,n\}$. Let F, G, H be subsets of J so that $J = F \cup G \cup H$. Let $\bar{p}_f$, $f \in F$, and $\bar{r}$ be fixed numbers. Let $p \in R^n$ be a vector such that

$$p_f = \bar{p}_f \quad , f \in F \tag{3.30}$$

$$pD^g = \alpha_g \bar{r} \quad , g \in G \tag{3.31}$$

(3.32) $\quad D_h^g = 0 \quad , h \in H$ and $g \in G$

Then p_g , $g \in G$, is uniquely determined by $\{\bar{p}_f\}$ and $\bar{r}$. Moreover, if $F \cap G$ is non-empty then $\bar{r}$ is uniquely determined by $\{\bar{p}_f\}$.

<u>Proof</u>: We can find subsets $F' \subset F$, $H' \subset H$ so that F', G, H' are disjoint and $F' \cup G \cup H' = J$. Corresponding to this partition of indices we can define the vector $\bar{p}^{F'}$ with components $\bar{p}_f$, $f \in F'$, and we can partition the vector p and the matrix D as

$$p = (p^{F'}, p^G, p^{H'}),$$

and

$$D = \begin{bmatrix} D_{F'}^{F'} & D_{F'}^{G} & D_{F'}^{H'} \\ D_{G}^{F'} & D_{G}^{G} & D_{G}^{H'} \\ D_{H'}^{F'} & D_{H'}^{G} & D_{H'}^{H'} \end{bmatrix}$$

By (3.32) $D_{H'}^G = 0$ and so by (3.31)

(3.33) $$p^{F'} D_{F'}^G + p^G D_G^G = \alpha^G \bar{r}$$

where α^G is the subvector of α corresponding to the indices $g \in G$. From (3.30) and (3.32) we obtain

(3.34) $$p^G = [\alpha^G \bar{r} - \bar{p}^{F'} D_{F'}^G][D_G^G]^{-1}$$

so that p^G is determined by $\{\bar{p}_f\}$ and $\bar{r}$. Moreover, if there exists a j in $F \cap G$ so that from (3.30) $p_j = \bar{p}_j$, then from (3.34) we get

$$\bar{p}_j = [\alpha^G \bar{r} - \bar{p}^{F'} D_{F'}^G] \delta^j$$

where δ^j is the column of $[D_G^G]^{-1}$ corresponding to index j. It follows that $\bar{r}$ is determined by the $\{\bar{p}_f\}$ and the lemma is proved.

Now let $Q = (Q_1, \dots, Q_n)' \geq 0$ be fixed. Let $J_1 \subset J_2 \dots \subset J_m = \{1,..,n\} = J$ be as before. Let x^i , $i = 1,..,m$ be the vectors defined in Lemma 3.1 and let $u_1^* \leq .. < u_m^*$ be as defined in (3.16) and (3.17). Define

$$J_i^Q = \{j \in J \mid x_j^i > 0\} \quad , \quad S_i^Q = \{j \in J \mid s_j^*(u) > 0 \, , \, u \in [u_{i-1}^*, u_i^*]\}$$

so that J_i^Q is the set of all activities which are operated in the ith ring $[u^*_{i-1}, u^*_i]$, and S_i^Q is the set of all goods which are shipped through this ring in positive amounts (cf. (3.18c)).

Theorem 3.3 (Uniqueness of equilibrium prices and rents.) Let p(u) and $r(u) = \max_j r_j(u)$, $u \geq u_0$, be non-negative, absolutely continuous functions, with $r_j(u) = \alpha_j^{-1} p(u) D^j$ the bid rent offer of producers of good j.

(i) Then p and r form equilibrium price and rent profiles which sustain an optimal allocation with final demand Q if and only if the following conditions hold for each i=1,...m and each $u \in [u^*_{i-1}, u^*_i]$.

(3.35) $$p(u)D^j = \alpha_j r(u) \quad , j \in J_i^Q$$

(3.36) $$\dot{p}_j(u) = -t_j \quad , j \in S_i^Q$$

(3.37) $$p(u)D^j \leq \alpha_j r(u) \quad , j \in J - J_i^Q$$

(3.38) $$\dot{p}_j(u) \in [-t_j, t_j] \quad , j \in J - S_i^Q$$

(3.39) $$p(u) = 0 \quad , u \geq u^*_m$$

(ii) r(u) is uniquely determined for all u, whereas $p_j(u)$ is unique for $j \in J_i^Q \cup S_i^Q$ and $u \in [u^*_{i-1}, u^*_i]$.

Proof: (i) By Corollary 2.2 p, r form an equilibrium price and rent system if and only if they sustain the optimal allocation constructed in Theorem 3.1. But for this allocation, the conditions (3.35)-(3.39), corresponds exactly with those given in Theorem 2.1 and so (i) is proved.

(ii) We prove the uniqueness statements by backwards induction on i. First of all we claim that

(3.40) $$p_j(u), \ j \in J_m^Q \cup S_m^Q \text{ and } r(u)$$

are unique for $u \in [u^*_{m-1}, u^*_m]$

To see this note that $p(u^*_m)=0$, hence $r(u^*_m)=0$. Fix $u \in [u^*_{m-1}, u^*_m]$. We use Lemma 3.4 with the following identification: $F=S^Q_m$, $G=J^Q_m$, $H=J-J^Q_m$, and $p=p(u)$, $\bar{r}=r(u)$. Since $S^Q_m \subset J^Q_m$ therefore $J = F \cup G \cup H$ and $F \cap G$ is non-empty, unless S^Q_m and J^Q_m are empty in which case $u^*_{m-1} = u^*_m$ and (3.40) is trivially satisfied. We now verify the conditions (3.30), (3.31), (3.32). If $f \in F=S^Q_m$, then by (3.36) $p_f(u)$ is uniquely determined, in fact $p_f(u)= p_f(u^*\ ) + t_f(u^*_m-u)$ $= t_f(u^*_m-u) = \bar{p}_f$ say. If $g \in G = J^Q_m$, then by (3.35) $p(u)D^g = \alpha_g r(u)$. Finally, if $g \in G = J^Q_m$ and $h \in H = J-J^Q_m$ then $x^m_h =0$, hence $D^g_h =0$. Hence (3.40) follows from Lemma 3.4. As induction hypothesis we now suppose that

(3.41) $\qquad p_j(u),\ j \in J^Q_{i+1} \cup S^Q_{i+1}$ and $r(u)$

are unique for $u \in [u^*_i, u^*_{i+1}]$.

and we shall prove that this also holds for i. We first apply Lemma 3.4 with the following identification: $F = S^Q_{i+1}$, $G = J^Q_i$, $H = [J_i - J^Q_i] \cup [(J - J^Q_i) - S^Q_{i+1}]$, $p = p(u^*_i)$ and $\bar{r} = r(u^*_i)$. By (3.41) $r(u^*_i)$ is already determined and so are $p_f(u^*_i)$, $f \in F$, hence (3.30) holds; (3.31) follows from (3.35); finally if $h \in [J_i - J^Q_i]$ then this good is not produced in ring i and if $h \in [(J - J^Q_i) - S^Q_{i+1}]$ then this good is not imported from rings i+1,..,m into ring i, so that $D^g_h =0$ for $g \in G$, $h \in H$ and hence (3.32) is verified. By Lemma 3.4 therefore $p_j(u^*_i)$ is uniquely determined for $j \in F \cup G = S^Q_{i+1} \cup J^Q_i \supset S^Q_i \cup J^Q_i$. Next fix $u \in [u^*_{i-1}, u^*_i]$. We apply Lemma 3.4 once again but this time with the identification $F = S^Q_i$, $G = J^Q_i$, $H = [J_i - J^Q_i] \cup [(J-J^Q_i) - S^Q_i]$, $p = p(u)$ and $\bar{r} = r(u)$. If $f \in F = S^Q_i$, then $p_f(u)$ is uniquely determined, in fact $p_f(u) = p_f(u^*_i) + t_f(u^*_i - u)$ by (3.36); if $g \in G = J^Q_i$ then $p(u)D^g = \alpha_g r(u)$ by (3.35). Finally, if $h \in H$, then just as above $D^g_h = 0$. By Lemma 3.4 therefore $p_j(u)$, $j \in S^Q_i \cup J^Q_i$ is uniquely determined by $\{p_j(u^*_i) | j \in S^Q_i\}$ and $r(u)$. Moreover if $S^Q_i \cap J^Q_i$ is non-empty then $r(u)$ is unique also and (3.41) hold for i. But $S^Q_i \cap J^Q_i$ is empty if and only if J^Q_i is empty i.e. $u^*_{i-1} = u^*_i$ in which case (3.41) trivially holds for i. Thus (ii) is proved also.

4. <u>The Supply Function</u>. We study some elementary properties of the supply function of the spatial economy described above.

<u>Definition4.1</u>: For $Q \geq 0$ let $p = P(Q)$ be the price vector given by $p=p^*(u_0)$ where p^* is defined in (3.21). The function $P(Q)$ is called the (Mashallian) <u>supply</u> <u>function</u> at the CBD.

<u>Lemma 4.1.</u> Let $p(u)$ be any price system which sustains an optimal allocation with final demand Q. If $Q_j > 0$ then $p_j(u_0) = P_j(Q)$.

<u>Proof</u>: If $Q_j > 0$ then $j \in S^Q_1$, hence by Theorem 3.3 $p_j(u_0)$ is uniquely determined. In particular $p_j(u_0) = P_j(Q)$.

Thus the supply function $P(Q)$ is uniquely determined over the domain $\{Q >> 0\}$ and, for $Q_j = 0$, $P_j(Q)$ is defined by continuation. Now when the technology is linear the supply prices are usually not unique so that the uniqueness result might be surprising. It can be traced to the fact that because of transport costs our spatial economy, regarded as an aggregate production unit, exhibits strictly decreasing returns-to-scale. We now give a formal proof of this fact.

For $Q \geq 0$ let $x(Q) = (x^1(Q), .., x^m(Q))$ be the unique solution of (3.14), (3.15) i.e.

$$[Dx^i(Q)]_j = \begin{cases} 0 & \text{, if } j \in J_{i-1} \\ Q_j + [A(x^1(Q)+\dots+x^{i-1}(Q))]_j, & \text{if } j \in J_i - J_{i-1} \end{cases},$$

$$x^i_j(Q) = 0 \qquad \text{, if } j \notin J_i$$

From these equations it follows that x^i is a linear function of Q, $x^i(Q) = C^i Q$ where the matrix $C^i = (C^i_{jk})$ satisfies

(4.1) $\qquad C^i \geq 0,\ C^i_{jk} = 0$ if $j \notin J_i$ and $C^i_{jk} > 0$ if $j,k \in J_i - J_{i-1}$

Let $\gamma^i = \alpha' C^i$. Then since $\alpha \gg 0$ by assumption we may conclude from (4.1) that

(4.2) $\qquad \gamma^1 + \dots + \gamma^m = \alpha'(C^1 + \dots + C^m) \gg 0$

Let $L(Q) = (L_1(Q),\dots,L_m(Q))$ be given by

(4.3) $\qquad L_i(Q) = \alpha' x^i(Q),$

and $u(Q) = (u_1(Q),\dots,u_m(Q))$ be given by

(4.4) $\qquad \int_{u_{i-1}(Q)}^{u_i(Q)} \theta(u)\,du = L_i(Q)$

Thus $S_i(Q)$ is the amount of land in the ith ring and $u_i(Q)$ is the farthest distance in the ith ring. ($u_0(Q) = u_0$, the radius of the CBD). With this notation the function $P(Q)$ is given by (cf. (3.21))

(4.5) $\qquad P(Q) = \sum_{i=1}^{m} (u_i(Q) - u_{i-1}(Q))t^i = \sum_{i=1}^{m} u_i(Q)(t^i - t^{i+1})$

where $t^{m+1} = 0$. Differentiating (4.4) and using (4.3) yields

$$\frac{\partial u_i}{\partial Q_j}(Q)\ \theta(u_i(Q)) - \frac{\partial u_{i-1}}{\partial Q_j}(Q)\ \theta(u_{i-1}(Q)) = \frac{\partial L_i}{\partial Q_j}(Q) =$$
$$= [\alpha' C^i]_j = \gamma^i_j ,$$

and since $\frac{\partial u_0}{\partial Q} = 0$, therefore

(4.6) $\qquad \frac{\partial u_i}{\partial Q_j}(Q)\ \theta(u_i(Q)) = \sum_{k=1}^{i} [\alpha' C^i]_j \geq 0,$

Moreover, by (4.1), the inequality is strict if $j \in J_i$. We summarize this result as a lemma. A similar result for a residential economy is given in Hartwick, Schweizer and Varaiya [1975].

<u>Lemma 4.2</u>. If $\tilde{Q} \geq Q$ and $\tilde{Q}_j > Q_j$ where $j \in J_i$, then $u_k(\tilde{Q}) \geq u_k(Q)$ for $k=1,..,i-1$ and $u_k(\tilde{Q}) > u_k(Q)$ for $k = i,..m$.

Since $t^m >> 0$ and since we have shown that $\frac{\partial u_m}{\partial Q_j} > 0$ for all j, therefore from (4.5) we conclude that

$$(4.7) \qquad \frac{\partial P_i}{\partial Q_j} > 0 \qquad \text{for all } i,j$$

We shall assume from now on that the land available for production increases with distance from the CBD i.e.

$$(4.7) \qquad \frac{d\theta}{du} > 0$$

Differentiation of (4.6) leads to

$$(4.8) \qquad \frac{\partial^2 u_i}{\partial Q_j \partial Q_k} = -\frac{\gamma_j^1+..+\gamma_j^i}{\theta^2(u_i(Q))}\frac{d\theta}{du}\frac{\partial u_i}{\partial Q_k} = -\frac{d\theta}{du}\frac{1}{\theta^3(u_i(Q))}(\gamma_j^1+..+\gamma_j^i)(\gamma_k^1+..+\gamma_k^i)$$

If we regard γ^i as a row vector then using (4.8) we may express the Hessian of u_i as

$$(4.9) \qquad \frac{\partial^2 u_i}{\partial Q_j \partial Q_k} = -\frac{d\theta}{du}\frac{1}{\theta^3(u_i(Q))}(\gamma^1+..+\gamma^i)'(\gamma^1+..+\gamma^i)$$

Since $\frac{d\theta}{du} > 0$ therefore this matrix is negative semidefinite so that $u_i(Q)$ is a concave function of Q. In fact from (4.2) and (4.9) we can conclude that $u_m(Q)$ is strictly concave. The next result follows from this fact and (4.5)

<u>Theorem 4.1</u>. The supply function $P_j(Q)$ has a strictly positive gradient. Moreover if $\frac{d\theta}{du} > 0$, then it is a strictly concave function of Q.

In the special case where the spatial region is pie-shaped we can compute the elasticity.

Theorem 4.2. Suppose $\theta(u) = \theta u$ where $\theta>0$ is a constant. Fix $Q = \bar{Q} > 0$. Then

$$\frac{\lambda}{P(\lambda\bar{Q})} \frac{\partial P}{\partial \lambda}(\lambda \bar{Q}) = 1/2$$

Proof: From (4.3), (4.4) we easily get

$$[u_i(\lambda \bar{Q}) - u_{i-1}(\lambda \bar{Q})]^2 = \lambda [u_i(\bar{Q}) - u_{i-1}(\bar{Q})]^2$$

which, together with (4.5) yields $P(\lambda \bar{Q}) = \lambda^{1/2} P(\bar{Q})$. The result follows upon differentiation.

Of course this last result is obvious. Since the land area available for production increases with the square of the city size, therefore an increase in production levels at the margin by δ% increases the city size by 1/2 δ% and since the transportation cost is linear in distance therefore the marginal cost of production also increases by 1/2 δ%.

5. A remark about imports. Above we have studied the optimal pattern of production which yields a prespecified bundle of goods Q at the CBD. We have shown in particular that this pattern can be sustained as a competitive equilibrium with price P(Q) at the CBD. Suppose now that a fixed price vector at the CBD is exogenously specified. Suppose further that producers are free to export to or import from the CBD at these prices. What will the equilibrium pattern of production be? Evidently if $p \in \mathcal{P} = \{P(Q) | Q \geq 0\}$ then the equilibrium pattern will once again be an optimal allocation with final demand Q where Q is such that $p = P(Q)$. But if $p \notin \mathcal{P}$ then it means that some producers close to the CBD find it more profitable to import at least one input, say j, from

the CBD rather than from producers of j located further away from the CBD. It follows that the price of good j will increase with distance from the CBD so long as local production costs are larger than importing from the CBD. Secondly, the order in which producers will be located can no longer be determined a priori since their profitability, and hence the order, will depend in part on the CBD prices. Thus both qualitative properties described in the previous sections are destroyed. Of course once we go so far from the CBD that it is no longer profitable to import from the CBD, then these properties again continue to hold. Goldstein and Moses [1975] have exhaustively investigated the behavior of the equilibrium for the case of two goods (with a Leontief technology). When we consider three goods it appears that the number of special cases that need to be studied becomes very large and a classification of these seems quite uninformative. It may be that when there are three or more goods nothing more interesting can be said than is possible in a general equilibrium setting. (See e.g. Schweizer, Varaiya and Hartwick [1974].)

6. Summary: We have studied the spatial structure of production when there are interdependencies among production units. These interdependencies have been characterized by a Leontief technology in which all goods may be intermediate goods.

It has been shown that the optimal production pattern can be sustained as a competitive equilibrium. This result is, however, a trivial consequence of a much more general existence theorem due to Schweizer, Hartwick and Varaiya [1974] where it is shown to depend only upon the fact that the technology set is convex. Less trivial is the result that the optimal pattern and the equilibrium price and rents are essentially unique.

Of much greater interest and surprise is the result that independent of demand the various goods will be produced in rings which can be ranked <u>a priori</u> according to the distance from the market (CBD) at which they are located. Furthermore, in the optimum goods will be shipped only towards the CBD and never towards the periphery. As a consequence, as one moves away from the CBD each ring will be increasingly "self-sufficient" in the sense that it will import fewer inputs. The proof of the result made crucial use of the absence of joint production and lack of substitute activities of production.

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Appendix.

1. <u>Proof of Theorem 2.1</u>. The necessity of the conditions follow from known results (see e.g. Hestenes [1966], p.354) and so we shall only prove the sufficiency. Let $\omega = (u, x(\cdot), f(\cdot), \phi(\cdot))$ by any allocation with final output Q, and let $s(u) \geq 0$, $\sigma(u) \geq 0$ be the solutions of (2.1), (2.2). We will show that

$$(A.1) \qquad T(\omega^*) \leq T(\omega)$$

From (2.7) we conclude

(A.2) $$\dot{p}^*(u) + t \geq 0 \quad , \quad [\dot{p}^*(u) + t]\, s^*(u) = 0$$

(A.3) $$-\dot{p}^*(u) + t \geq 0 \quad , \quad [-\dot{p}^*(u) + t]\sigma^*(u) = 0$$

It follows that

$$0 = [\dot{p}^*(u) + t]\, s^*(u) + [-\dot{p}^*(u) + t]\,\sigma^*(u)$$
$$\leq [\dot{p}^*(u) + t]\, s\,(u) + [-\dot{p}^*(u) + t]\,\sigma(u),$$

and hence integration gives us

$$0 = \int_{u_0}^{\bar{u}^*} p^*(u)\,[s^*(u) - \sigma^*(u)] + T(\omega^*)$$

(A.4) $$\leq \int_{u_0}^{\bar{u}^*} p^*(u)\,[s(u) - \sigma(u)] + T(\omega)$$

If we evaluate the integrals by parts using the boundary conditions of (2.1), (2.2) and (2.6) we get

$$\int_{u_0}^{\bar{u}^*} p^*(u)[s^*(u) - \sigma^*(u)]du = -p^*(u_0)Q + \int_{u_0}^{\bar{u}^*} p^*(u)[\dot{s}^*(u) - \dot{\sigma}^*(u)]du$$

(A.5) $$= -p^*(u_0)Q + \int_{u_0}^{\bar{u}^*} p^*(u)\, Dx^*(u)du,$$

and similarly,

(A.6) $$\int_{u_0}^{\bar{u}^*} \dot{p}^*(u)[s(u) - \sigma(u)]\, du = -p^*(u_0)Q - p^*(\bar{u})\sigma(\bar{u}) + \int_{u_0}^{\bar{u}} p^*(u)Dx(u)du$$

Substitution of (A.5) and (A.6) into (A.4) leads to

(A.7) $$T(\omega^*) \leq T(\omega) - p^*(\bar{u})\sigma(\bar{u}) - \int_{u_0}^{\hat{u}} p^*(u)[Dx^*(u) - Dx(u)]du$$

where $\hat{u} = \max\,(u^*, u)$. Now $p^*(\bar{u})\,\sigma(\bar{u}) \geq 0$ and $p^*(u)[Dx^*(u) - Dx(u)] \geq 0$ by (2.4) and so (A.1) has been proved.

2. <u>Proof of Corollary 2.2</u>. Let $\omega = (\bar{u}, x(\cdot), f(\cdot), \phi(\cdot))$ be another optimal allocation with final output Q. Then of course we have $T(\omega^*) = T(\omega)$ and so we must have equality in (A.7). From this it follows immediately that ω satisfies (2.4). Retracing the equality in (A.7) backwards we can also deduce equality in (A.4) from which we can conclude that ω satisfies (2.5). The assertion is proved.

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