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A Theory of Nonenergic N-Ports

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Abstract

A circuit element is nonenergic if the instantaneous power flow into it is always zero. Well-known examples include the ideal diode, transformer, gyrator, and circulator. Most of the interesting nonenergic elements are nonlinear N-ports with $N \geq 2$, and many of their properties are quite counter-intuitive. For example, there exists a surprisingly large class of nonenergic multiport capacitors and inductors, all of which, it turns out, are nonlinear and reciprocal. Nonenergic linear N-ports, on the other hand, are necessarily resistive and antireciprocal.

In this paper we present a rigorous fundamental theory of nonenergic N-ports which results in a general canonical representation. Special canonical forms are developed for nonenergic resistors, capacitors, and inductors, and numerous examples are given.

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Outline

I Introduction

An N-port $\mathcal{N}$ is said to be nonenergic [Duinker, 1959 and 1962] iff the instantaneous power flow into it is always zero, i.e.

$$p(t) = \sum_{k=1}^{N} v_k(t) i_k(t) = 0 \qquad (1\text{-}1)$$

for all admissible pairs $(\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot))$ and for all t.

A nonenergic N-port can be viewed as a device which simply transfers power from one port to another without loss or gain. This point of view is particularly evident, for example, in the scattering matrix representation of a circulator.

Many of the results in this paper are quite counterintuitive, e.g. the existence of nontrivial nonenergic capacitors and inductors. The roles played by reciprocity and linearity considerations are likewise unexpected: it turns out that nonenergic N-port capacitors and inductors must be nonlinear and reciprocal while nonenergic linear N-ports are all resistive and anti-reciprocal. In addition, the choice of representation is far more crucial than the authors had anticipated. Hybrid representations are essential, since continuous voltage-or current-controlled representations of nonenergic reciprocal resistors and continuous voltage- or charge-controlled representations of nonenergic capacitors and inductors are impossible, except in trivial cases.

Perhaps the results are counterintuitive because our intuition is highly developed only for linear N-ports and nonlinear 1-ports, while all the interesting nonenergic N-ports are nonlinear with $N \geq 2$. The others are special and particularly simple in one sense or another. For example, linear, time-invariant, nonenergic devices can all be realized simply by interconnecting ideal gyrators. And the most general sort of nonenergic 1-port will obviously look like either an open circuit or a short, depending perhaps on the operating point and the time as in the case of an ideal diode or a time-varying switch. Yet there are nonlinear multiport resistors, capacitors, and inductors which are nonenergic without being pathological, i.e. that are time-invariant, smooth, reciprocal, and defined on all of $\mathbb{R}^n$. See examples 4-4 and 5-6.

We now give a brief synopsis of the content of the following sections.

In section II it is shown that every linear, time-invariant, nonenergic N-port is resistive and antireciprocal, and that any such device can be synthesized from ideal gyrators.

Section III gives two conditions on the state space representation of $\mathcal{N}$, each of which separately is necessary and sufficient for $\mathcal{N}$ to be nonenergic. The first is a condition on the state equations alone; the second involves the stored energy and the dissipation rate. An example is given of a nonlinear nonenergic R-C network, which serves as an introduction to the considerations of sections IV and V.

Section IV gives a general theory of nonenergic resistive N-ports, both linear and nonlinear. Necessary and sufficient conditions for nonenergicness are derived in terms of the potential functions content, co-content, and hybrid content in the reciprocal case.

Section V establishes that a nonenergic multiport capacitor must be reciprocal. Then necessary and sufficient conditions for nonenergicness are developed in terms of the co-energy and hybrid energy potential functions. An interesting parallel between nonenergic voltage-controlled capacitors and classical thermodynamics is noted.

Section VI extends the results in [Duinker, 1959] by displaying a class of nonenergic Lagrangian N-ports which are not traditors.

Section VII is substantially more abstract than the rest. In parts A and B no assumption is made that $\mathcal{N}$ is time-invariant, lumped, causal, or continuous or that its admissible pairs are measurable functions. Theorem 7-1 provides a canonical form for nonenergic N-ports. Theorem 7-2 builds on theorem 7-1 to show that if a nonenergic N-port is linear, it must be resistive. In part C we drop the "global coordinate assumption," i.e. the assumption that $\mathcal{N}$ is globally voltage-controlled, current-controlled, hybrid, etc. Resistive and capacitive N-ports are viewed as N-dimensional differentiable manifolds in $\mathbb{R}^{2N}$, and the properties of nonenergicness and reciprocity are presented purely in terms of local coordinates.

Sections II through VI deal only with the time-invariant case.

II Linear Time-Invariant N-Ports

A. Definitions and Assumptions

Definition 2-1 Suppose the 2N port voltages and currents of an N-port $\mathcal{N}$ are partitioned into two vectors, $\underset{\sim}{u} \varepsilon \mathbb{R}^N$ and $\underset{\sim}{y} \varepsilon \mathbb{R}^N$, in such a way that either $u_k = v_k$ and $y_k = i_k$ or $u_k = i_k$ and $y_k = v_k$ for each $k \in \{1,2,\cdots N\}$. Then $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a hybrid pair.

It follows immediately from the definition that the instantaneous net power flow into $\mathcal{N}$ can be given simply in terms of $\underset{\sim}{u}$ and $\underset{\sim}{y}$ as

$$p(t) = \sum_{k=1}^{N} v_k(t) i_k(t) = \langle \underset{\sim}{u}(t), \underset{\sim}{y}(t) \rangle \tag{2-1}$$

if $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a hybrid pair, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product operation on $\mathbb{R}^N$.

Although the definitions of reciprocity and antireciprocity are commonly stated in the frequency domain [Penfield, Spence and Duinker], we will require the following equivalent time-domain statement.

Definition 2-2 A linear N-port $\mathcal{N}$ is said to be reciprocal iff $\underset{\sim}{v}'(\cdot)*\underset{\sim}{i}''(\cdot) = \underset{\sim}{v}''(\cdot)*\underset{\sim}{i}'(\cdot)$ and antireciprocal iff $\underset{\sim}{v}'(\cdot)*\underset{\sim}{i}''(\cdot) = -\underset{\sim}{v}''(\cdot)*\underset{\sim}{i}'(\cdot)$, whenever $\left(\underset{\sim}{v}'(\cdot),\underset{\sim}{i}'(\cdot)\right)$ and $\left(\underset{\sim}{v}''(\cdot),\underset{\sim}{i}''(\cdot)\right)$ are admissible pairs[1].

We let $U \subset \mathbb{R}^N$ denote the set of admissible values an input waveform $\underset{\sim}{u}(\cdot)$ is allowed to take. We denote by $\mathcal{U}$ the class of all admissible input waveforms. It may be helpful to note that an admissible waveform $\underset{\sim}{u}(\cdot)$ is in $\mathcal{U}$, whereas at any particular time t, the vector $\underset{\sim}{u}(t)$ is in U.

We require throughout the remainder of section II that $\mathcal{U}$ satisfy the technical assumptions listed in part B of section VII. As explained there, this will occur automatically in every case of practical interest. In addition we require that: i) $\mathcal{U}$ be restricted sufficiently so that there exists a unique output waveform $\underset{\sim}{y}(\cdot)$ for each input waveform $\underset{\sim}{u}(\cdot) \varepsilon \mathcal{U}$, and ii) $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair[2].

---

[1] If $\underset{\sim}{x}(\cdot)$ and $\underset{\sim}{y}(\cdot)$ are two $\mathbb{R}^N$-valued time functions, then their convolution is given by

$$[x(\cdot)*y(\cdot)]_{(t)} \triangleq \sum_{k=1}^{N} \int_{-\infty}^{\infty} x_k(\tau) y_k(t-\tau) d\tau$$

[2] The existence of a representation satisfying condition ii) is a very weak assumption for linear N-ports. It is equivalent to the existence of a scattering matrix representation, which is in turn equivalent to the existence of at least one representation of the form $\underset{\sim}{A}(s)\ \underset{\sim}{V}(s) = \underset{\sim}{B}(s)\ \underset{\sim}{I}(s)$ [Anderson, Newcomb and Zuidweg].

B. Every Nonenergic Linear N-Port is Antireciprocal and Resistive

Theorem 2-1 Suppose $\mathcal{N}$ is a linear N-port characterized by the state equations

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du \qquad (2\text{-}2)$$

or the convolution operator $y(t) = [G(\cdot)*u(\cdot)](t)$, where u and y form a hybrid pair. Then the following three conditions are equivalent:

i) $\mathcal{N}$ is nonenergic.

ii) $\mathcal{N}$ is characterized by $y = Hu$,

where $H \in \mathbb{R}^{N\times N}$ is an antisymmetric constant matrix.

iii) $\mathcal{N}$ is an antireciprocal resistor.

Proof

i) ⇒ ii) This is a consequence of theorems 7-1 and 7-2, applied in the time-invariant case.

ii) ⇒ iii)

$$\left(v'(\cdot)*i''(\cdot)\right) + \left(v''(\cdot)*i'(\cdot)\right)$$

$$= \left[\left(v'(\cdot)+v''(\cdot)\right)*\left(i'(\cdot)+i''(\cdot)\right)\right] - \left[v'(\cdot)*i'(\cdot)\right] - \left[v''(\cdot)*i''(\cdot)\right] \qquad (2\text{-}3)$$

Since $\mathcal{N}$ is linear, each of the three terms on the right hand side of equation (2-3) is the convolution of an admissible voltage waveform with its corresponding current waveform. In order to show that each such term is zero, we first renumber the ports if necessary so that either $u = v$ and $y = i$ or $u = \left[i_1,\cdots i_k,\ v_{k+1},\cdots v_N\right]^T = \left[i_I^T, v_{II}^T\right]^T$ and $y = \left[v_1,\cdots v_k, i_{k+1},\cdots i_N\right]^T =$

$\left[\underset{\sim}{v}_I^T, \underset{\sim}{i}_{II}^T\right]^T$, for some integer k, $1 \leq k \leq N$. We show in part A of the Appendix that such renumbering preserves the antisymmetry of $\underset{\sim}{H}$. Then if $\left(\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\right)$ is any admissible pair,

$$\underset{\sim}{v}(\cdot) * \underset{\sim}{i}(\cdot) = \left(\underset{\sim}{v}_I(\cdot) * \underset{\sim}{i}_I(\cdot)\right) + \left(\underset{\sim}{v}_{II}(\cdot) * \underset{\sim}{i}_{II}(\cdot)\right) =$$

$$\left(\underset{\sim}{v}_I(\cdot) * \underset{\sim}{i}_I(\cdot)\right) + \left(\underset{\sim}{i}_{II}(\cdot) * \underset{\sim}{v}_{II}(\cdot)\right) = \left[\underset{\sim}{v}_I(\cdot)^T, \underset{\sim}{i}_{II}(\cdot)^T\right]^T * \left[\underset{\sim}{i}_I^T(\cdot), \underset{\sim}{v}_{II}^T(\cdot)\right]^T =$$

$$\underset{\sim}{y}(\cdot) * \underset{\sim}{u}(\cdot) = \left(\underset{\sim}{H}\underset{\sim}{u}(\cdot)\right) * \underset{\sim}{u}(\cdot) = \sum_{j=1}^{N} \sum_{i=1}^{N} h_{ij}\left[u_i(\cdot) * u_j(\cdot)\right] = 0,$$

by the antisymmetry of $\underset{\sim}{H}$. The calculation is similar if $\underset{\sim}{u} = \underset{\sim}{v}$ and $\underset{\sim}{y} = \underset{\sim}{i}$. Therefore the right hand side of equation (2-3) is zero and the conclusion follows.

iii) ⇒i) If we let $\underset{\sim}{v}'(\cdot) = \underset{\sim}{v}''(\cdot)$ and $\underset{\sim}{i}'(\cdot) = \underset{\sim}{i}''(\cdot)$, then it follows directly from the definition of antireciprocity that $[\underset{\sim}{v}(\cdot) * \underset{\sim}{i}(\cdot)] \equiv 0$ if $\left(\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\right)$ is an admissible pair and $\mathcal{N}$ is antireciprocal. Let $(\hat{\underset{\sim}{v}}, \hat{\underset{\sim}{i}})$ be a dc admissible pair for $\mathcal{N}$. Then $(\underset{\sim}{0}, \underset{\sim}{0})$ is also an admissible pair since $\mathcal{N}$ is linear. Define the admissible pair of waveforms $\left(\tilde{\underset{\sim}{v}}(\cdot), \tilde{\underset{\sim}{i}}(\cdot)\right)$ by:

$$\tilde{\underset{\sim}{v}}(t) = \begin{cases} \hat{\underset{\sim}{v}}, & 0 \leq t \leq 1 \\ \underset{\sim}{0}, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{\underset{\sim}{i}}(t) = \begin{cases} \hat{\underset{\sim}{i}}, & 0 \leq t \leq 1 \\ \underset{\sim}{0}, & \text{otherwise} . \end{cases}$$

Then

$$\langle \hat{\underset{\sim}{v}}, \hat{\underset{\sim}{i}} \rangle = \int_0^1 \langle \hat{\underset{\sim}{v}}, \hat{\underset{\sim}{i}} \rangle \, d\tau = \int_{-\infty}^{\infty} \langle \tilde{\underset{\sim}{v}}(\tau), \tilde{\underset{\sim}{i}}(1-\tau) \rangle \, d\tau = [\tilde{\underset{\sim}{v}}(\cdot) * \tilde{\underset{\sim}{i}}(\cdot)](t=1) = 0$$

where the first equality follows because $\hat{\underset{\sim}{v}}$ and $\hat{\underset{\sim}{i}}$ are constant vectors, and the second equality follows becasue $\tilde{\underset{\sim}{v}}(t)$ and $\tilde{\underset{\sim}{i}}(t)$ are constant vectors for all $0 \leq t \leq 1$.

∎

Example 7-5 will show that the conclusion that $\mathcal{N}$ must be antireciprocal holds only in the time-invariant case.

Example 2-1 Ideal Transformer

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & -n \\ n & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \qquad (2\text{-}4)$$

Example 2-2 Ideal Gyrator

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad (2\text{-}5)$$

Corollary. Let $\mathcal{N}$ be a nonenergic N-port described by the state equations (2-2). Then $\underset{\sim}{D}$ is antisymmetric. Furthermore, if $\mathcal{N}$ is completely controllable (definition 3-2), it is completely unobservable, i.e. $\underset{\sim}{C} = \underset{\sim}{0}$.

C. Nonenergic Linear N-Ports Which are Both Reciprocal and Antireciprocal. Transformer-Only Synthesis.

The ideal transformer is nonenergic, linear, reciprocal, and antireciprocal [Penfield, Spence, and Duinker], as a simple computation will verify. Theorem 2-2 gives a canonical form for such devices.

Definition 2-3 Suppose $\underset{\sim}{u} \in \mathbb{R}^N$ and $\underset{\sim}{y} \in \mathbb{R}^N$ are a hybrid pair, but $\underset{\sim}{u} \neq \underset{\sim}{v}$ and $\underset{\sim}{u} \neq \underset{\sim}{i}$. Then $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a mixed hybrid pair.

Example 2-3 Consider the following four choices of independent and dependent variables for a 3-port resistor:

i) $\underset{\sim}{u} = \begin{bmatrix} v_1 \\ i_2 \\ v_3 \end{bmatrix}, \; \underset{\sim}{y} = \begin{bmatrix} i_1 \\ v_2 \\ i_3 \end{bmatrix}$ ii) $\underset{\sim}{u} = \begin{bmatrix} v_1 \\ i_1 \\ v_2 \end{bmatrix}, \; \underset{\sim}{y} = \begin{bmatrix} i_2 \\ v_3 \\ i_3 \end{bmatrix}$

iii) $\underset{\sim}{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \; \underset{\sim}{y} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$ iv) $\underset{\sim}{u} = \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}, \; \underset{\sim}{y} = \begin{bmatrix} i_1 \\ i_2 \\ v_3 \end{bmatrix}$.

Choices i), iii) and iv) are all hybrid pairs, but only choices i) and iv) are mixed hybrid pairs.

Theorem 2-2 Suppose $\mathcal{N}$ is a linear N-port which is both nonenergic and reciprocal. If $\mathcal{N}$ is characterized by

$$\underset{\sim}{y} = \underset{\sim}{H}\underset{\sim}{u}, \quad \underset{\sim}{H} \in \mathbb{R}^{N\times N} \tag{2-6}$$

where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair, then exactly one of the following three cases must hold.

i) The input $\underset{\sim}{u} = \underset{\sim}{v}$, $\underset{\sim}{y} = \underset{\sim}{i}$, $\underset{\sim}{H} = \underset{\sim}{0}$ and $\mathcal{N}$ is an N-port open circuit.

ii) The input $\underset{\sim}{u} = \underset{\sim}{i}$, $\underset{\sim}{y} = \underset{\sim}{v}$, $\underset{\sim}{H} = \underset{\sim}{0}$ and $\mathcal{N}$ is an N-port short circuit.

iii) The input $\underset{\sim}{u}$ and output $\underset{\sim}{y}$ form a mixed hybrid pair. After renumbering the ports, if necessary, so that $\underset{\sim}{u} = (\underset{\sim}{i}_I^T, \underset{\sim}{v}_{II}^T)^T$ and $\underset{\sim}{y} = (\underset{\sim}{v}_I^T, \underset{\sim}{i}_{II}^T)^T$, then $\underset{\sim}{H}$ has the form given in the following elaboration of equation (2-6):

$$\begin{bmatrix} \underset{\sim}{v}_I \\ \hline \underset{\sim}{i}_{II} \end{bmatrix} = \left[\begin{array}{c|c} \underset{\sim}{0}^{(k)} & \underset{\sim}{B} \\ \hline -\underset{\sim}{B}^T & \underset{\sim}{0}^{(n-k)} \end{array}\right] \begin{bmatrix} \underset{\sim}{i}_I \\ \hline \underset{\sim}{v}_{II} \end{bmatrix}, \tag{2-7}$$

where $\underset{\sim}{B} \in \mathbb{R}^{k\times(N-k)}$ is arbitrary.

Proof It is well known that a linear N-port given by an impedance or admittance matrix $\underset{\sim}{H}$ is reciprocal iff $\underset{\sim}{H}$ is symmetric and antireciprocal iff $\underset{\sim}{H}$ is antisymmetric. If $\underset{\sim}{H}$ is both symmetric and antisymmetric, then $h_{ij} = h_{ji}$ and $h_{ij} = -h_{ji}$, so $\underset{\sim}{H} = \underset{\sim}{0}$. This proves the conclusion in cases i) and ii). In case iii), after suitable renumbering, we have

$$\begin{bmatrix} \underset{\sim}{i}_I \\ \hline \underset{\sim}{i}_{II} \end{bmatrix} = \left[\begin{array}{c|c} \underset{\sim}{A} & \underset{\sim}{B} \\ \hline -\underset{\sim}{B}^T & \underset{\sim}{C} \end{array}\right] \begin{bmatrix} \underset{\sim}{i}_I \\ \hline \underset{\sim}{v}_{II} \end{bmatrix}, \tag{2-8}$$

where the appearance of $-B^T$ follows from the antisymmetry of H required by theorem 2-1. It follows from definition 2-2 that since $\mathcal{N}$ is both reciprocal and antireciprocal,

$$v'(\cdot)*i''(\cdot) \equiv 0, \text{ for all admissible } v'(\cdot) \text{ and } i''(\cdot). \qquad (2\text{-}9)$$

Substituting equation (2-8) into equation (2-9) and dropping terms which cancel yields

$$\left(Ai_I'(\cdot)\right)*i_I''(\cdot) + v_{II}'(\cdot)*\left(Cv_{II}''(\cdot)\right) \equiv 0,$$

which is possible for all admissible $i_I'(\cdot)$, $i_I''(\cdot)$, $v_{II}'(\cdot)$, and $v_{II}''(\cdot)$ iff $A = 0$ and $C = 0$. ■

Using theorem 2-2, it is easy to synthesize every nonenergic linear reciprocal resistive N-port from ideal transformers. The canonical synthesis is given in Fig. 1, where turns-ratios are expressed in terms of the entries of B in equation (2-7).

D. Scattering Representation

Definition 2-4 A square matrix M of real or complex numbers is orthogonal iff M is nonsingular and $M^T = M^{-1}$.

Theorem 2-3 Let $\mathcal{N}$ be a linear N-port given by the scattering matrix $S(s)$ with respect to real positive port normalization numbers, where $S(s)$ is defined for all s in the open right half plane. Then $\mathcal{N}$ is nonenergic $\Leftrightarrow$ $S(s)$ is real, orthogonal, and independent of s.

Proof ($\Leftarrow$) Since $S(s)$ is orthogonal, $\mathcal{N}$ is antireciprocal [Penfield, Spence, and Duinker]. Since $\mathcal{N}$ has a scattering matrix representation, $\mathcal{N}$ has a hybrid matrix representation [Anderson, Newcomb, and Zuidweg]. And since $S(s)$ is real and independent of s, $\mathcal{N}$ is resistive. The conclusion follows from theorem 2-1.

(⇒) Since $\mathcal{N}$ has a scattering matrix representation in the open right half plane, $\mathcal{N}$ has a hybrid matrix representation [Anderson et.al.]. Then it follows from theorem 2-1 that $\mathcal{N}$ is antireciprocal and resistive. Since $\mathcal{N}$ is antireciprocal, $\underset{\sim}{S}(s)$ is orthogonal [Penfield et.al.], and since $\mathcal{N}$ is resistive, $\underset{\sim}{S}(s)$ is real and independent of s.

■

Example 2-4 It is well-known that every real orthogonal 2×2 matrix can be written in one of the following two forms [Hoffman and Kunze]:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{2-10}$$

or

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2-11}$$

It is not hard to verify that equation (2-10) is the scattering matrix with respect to 1 Ω port normalization numbers of an ideal transformer, where $\tan\theta = \frac{2n}{1-n^2}$. Similarly, equation (2-11) is the scattering matrix with respect to 1 Ω port normalization numbers of an ideal gyrator, where $\tan\theta = \frac{2g}{1-g^2}$.

E. Gyrator-Only Synthesis of General Nonenergic Linear N-Ports

It is clear that any N-port constructed only from ideal gyrators will be linear and nonenergic. The remarkable fact is that the converse is also true. An algorithm is given in section 5.9 of [Carlin and Giordano] for the synthesis of any nonenergic linear N-port with a scattering representation from ideal gyrators and transformers. Since an ideal transformer can be constructed by connecting two ideal gyrators in cascade, this amounts to a gyrator-only synthesis.

III. State Variable Representation

A. Definitions and Assumptions

Definition 3-1 Suppose the N-port $\mathcal{N}$ is characterized by the state equations

$$\dot{x} = f(x,u)$$
$$y = g(x,u). \tag{3-1}$$

If $u \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$ form a hybrid pair (definition 2-1), then equations (3-1) are called the hybrid dynamical system representation for $\mathcal{N}$.

Throughout section III we will require that U be an open set in $\mathbb{R}^N$ and that $\mathcal{U}$ be the class of all piecewise continuous waveforms: $\mathbb{R}_+ \to U$, where $\mathcal{U}$ and U are as defined in part A of section II. We denote the state space of $\mathcal{N}$ by Σ, and Σ will be an open set in $\mathbb{R}^m$.

Definition 3-2 If for each choice of states x_0, $x_1 \in \Sigma$, there exists a time $t_1 \geq 0$ and an admissible input $u(\cdot) \in \mathcal{U}$ that steers $\mathcal{N}$ from state x_0 at $t = 0$ to x_1 at $t = t_1$, then $\mathcal{N}$ is said to be completely controllable.

B. The Output Map $y = g(x,u)$

Theorem 3-1 Suppose that $\mathcal{N}$ is described by the hybrid dynamical system representation (3-1), that $g(\cdot,\cdot)$ is continuous and that $\mathcal{N}$ is completely controllable. Then $\mathcal{N}$ is nonenergic $\Leftrightarrow$ $g(x,u)$ can be written in the form

$$g(x,u) = H(x,u)u, \text{ for all } u \in U \text{ and all } x \in \Sigma, \tag{3-2}$$

where $H(x,u)$ is a function assigning to each point $(x,u) \in \Sigma \times U$ an antisymmetric real N×N matrix, i.e. $H(x,u) = -H^T(x,u)$.

Proof ($\Leftarrow$)

$$p(t) = \langle u(t),y(t) \rangle = u^T(t)\, H\big(x(t),u(t)\big)\, u(t) \equiv 0,$$

since $H(x,u)$ is antisymmetric.

($\Rightarrow$) If $g(x,u)$ and u are orthogonal for all $(x,u) \in \sum \times U$, then the conclusion follows from theorem 7-1 whenever $u \neq 0$ and from the continuity of g when $u = 0$. It remains to show that $g(x,u)$ and u must always be orthogonal. Suppose the contrary, i.e. that there exist $u^* \in U$ and $x^* \in \sum$ such that $\langle u^*, g(x^*,u^*)\rangle = c \neq 0$. Then for any initial state x_0 at $t = 0$, there exist a $t_1 \geq 0$ and a control $\hat{u}(\cdot) \in \mathcal{U}$ that transfers $\mathcal{N}$ from x_0 at $t = 0$ to x^* at $t = t_1$. If $\hat{u}(t_1) = u^*$, we are done. Otherwise, let the control $\tilde{u}(t)$ be given by

$\tilde{u}(t) = \hat{u}(t)$ when $0 \leq t \leq t_1$ and $\tilde{u}(t) = u^*$ when $t > t_1$. Now $x(\cdot)$ is continuous since it satisfies equation (3-1), and $g(\cdot,\cdot)$ is also continuous. Therefore

$$\lim_{t\to t_1^+} \big(p(t)\big) = \lim_{t\to t_1^+} \Big\langle \tilde{u}(t), g\big(x(t),\tilde{u}(t)\big)\Big\rangle =$$

$$\lim_{t\to t_1^+} \Big\langle u^*, g\big(x(t),u^*\big)\Big\rangle = \Big\langle u^*, g(x^*,u^*)\Big\rangle = c \neq 0, \tag{3-3}$$

contrary to the assumption that $\mathcal{N}$ is nonenergic. ∎

See the note which follows the proof of theorem 7-1.

It is not difficult to show that the above theorem remains true if we require that $u(t)$ be continuous, provided U is a connected set in $\mathbb{R}^N$.

Corollary. Every non-energic n-port satisfying the hypotheses of Theorem 3-1 is a memristive n-port [Chua and Kang].

C. Stored Energy and Dissipation Rate

Nonenergicness implies a relation between the state equations, the stored energy, and the dissipation rate of $\mathcal{N}$. Intuitively, the requirement is that power dissipated in the resistors of $\mathcal{N}$ must come from active elements and storage elements inside $\mathcal{N}$, since none of it can come in through the ports.

Suppose $\mathcal{N}$ consists of: i) (multiport) reciprocal capacitors and inductors, ii) nonenergic (multiport) resistors, e.g. ideal diodes, transformers and gyrators, iii) dissipative resistors, e.g. linear 2-terminal resistors, transistors, zener diodes, independent sources, and iv) dependent sources. Then all the (positive or negative) power dissipation occurs in the elements listed under iii) and iv), and all the energy storage takes place in elements listed under i). Let $\mathcal{N}$ be characterized by the dynamical system representation (3-1) in which the components of the state vector $\underset{\sim}{x}$ are capacitor voltages and/or charges and inductor fluxes and/or currents. Let the components of $\underset{\sim}{u}$ and $\underset{\sim}{y}$ be port voltages and/or currents; we don't require that $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair. We must require that the capacitors and inductors be reciprocal so that there will be a stored energy function $E(\underset{\sim}{x})$ defined to within an additive constant [Willems]. If $\mathcal{N}$ is a well-posed network[1], then $\underset{\sim}{x}$ and $\underset{\sim}{u}$ uniquely determine all the voltages and currents at all the ports of all the dissipative resistors and dependent sources, so we can write a function $d(\underset{\sim}{x},\underset{\sim}{u})$ which gives the total power dissipated inside $\mathcal{N}$ as a function of the input and state.

<u>Theorem 3-2</u> Suppose $\mathcal{N}$ is a well-posed network, the above conditions hold, $E(\underset{\sim}{x})$ is C^1, $\underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u})$ and $d(\underset{\sim}{x},\underset{\sim}{u})$ are continuous, and $\mathcal{N}$ is completely controllable. Then $\mathcal{N}$ is nonenergic $\Leftrightarrow$

$$\left\langle \underset{\sim}{\nabla E}\;(\underset{\sim}{x}),\; \underset{\sim}{f}\;(\underset{\sim}{x},\underset{\sim}{u}) \right\rangle + d(\underset{\sim}{x},\underset{\sim}{u}) = 0, \text{ for all } \underset{\sim}{x} \in \Sigma \text{ and all } \underset{\sim}{u} \in U \qquad (3\text{-}4)$$

<u>Proof</u>

($\Leftarrow$) By Tellegen's theorem (or by conservation of energy) we have that $p(t) = \frac{d}{dt} E\big(\underset{\sim}{x}(t)\big) + d\big(\underset{\sim}{x}(t),\underset{\sim}{u}(t)\big)$, and by the chain rule we have that $\frac{d}{dt} E\big(x(t)\big) = \left\langle \underset{\sim}{\nabla E}\big(\underset{\sim}{x}(t)\big),\; \underset{\sim}{f}\big(\underset{\sim}{x}(t),\underset{\sim}{u}(t)\big)\right\rangle$.

($\Rightarrow$) Since $\mathcal{N}$ is completely controllable, the conclusion follows by a continuity argument similar to that leading up to equation (3-3).

∎

[1]It is possible, though quite lengthy, to derive sufficient conditions for well-posedness of general networks in terms of topological matrices and element constitutive relations. It is generally easier to verify well-posedness by ad hoc techniques in particular cases of interest.

D. Examples

Example 3-1 The network in Fig. 2 is a simple example which illustrates the ideas in theorems 3-1 and 3-2.

Let the state be q, the input be i, the output be v, and $U = \mathbb{R}$. The normal form hybrid state equations and the functions E and d become:

$$\begin{aligned} \dot{\underset{\sim}{x}} = f(\underset{\sim}{x},\underset{\sim}{u}) &\rightarrow \dot{q} = i - g\circ f(q) \\ \underset{\sim}{y} = g(x,u) &\rightarrow \quad v = f(q) \\ E(x) &\rightarrow \quad E(q) = \int_0^q f(q')dq' \\ d(\underset{\sim}{x},\underset{\sim}{u}) &\rightarrow \quad d(q,i) = f(q)\cdot[g\circ f(q)], \end{aligned}$$

where "$\circ$" denotes the "composition" operation.

Theorems 3-1 and 3-2 allow us to conclude immediately that if such a network is nonenergic and completely controllable, it is trivial. The requirements of theorem 3-1 can only be met if $f(q) = 0$ for all $q \in \Sigma$, so that $\underset{\sim}{H}(\underset{\sim}{x},\underset{\sim}{u})$ is "the only antisymmetric 1×1 matrix"; namely, zero. Likewise, according to theorem 3-2, $\mathcal{N}$ is nonenergic iff $f(q)\cdot[i - g\circ f(q)] + f(q)[g\circ f(q)] = 0$, or $i\cdot f(q) = 0$ for all $(i, q) \in \mathbb{R}\times\Sigma$, or $f(q) = 0$, for all $q \in \Sigma$. Hence, in either case, we conclude that the one-port must be a short-circuit.

Example 3-2 Let the network in Fig. 3 be represented by a hybrid dynamical system representation with input $\underset{\sim}{u} = (i_1,i_2)^T$, output $\underset{\sim}{y} = (v_1,v_2)^T$ and state $\underset{\sim}{x} = (q_1,q_2)^T$. We do not need to include q_3 in the state, since q_3 is uniquely determined by q_1 and q_2: $q_3 = f_3(v_3) = f_3(v_1-v_2) = f_3\big(f_1(q_1) - f_2(q_2)\big)$.

If we define

$$\underset{\sim}{M}(\underset{\sim}{q}) = \begin{bmatrix} 1 + f_3'\big(f_1(q_1)-f_2(q_2)\big)f_1'(q_1) & - f_3'\big(f_1(q_1)-f_2(q_2)\big)f_2'(q_2) \\ - f_3'\big(f_1(q_1)-f_2(q_2)\big)f_1'(q_1) & 1 + f_3'\big(f_1(q_1)-f_2(q_2)\big)f_2'(q_2) \end{bmatrix},$$

then the hybrid dynamical system representation for this circuit is given by:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \underset{\sim}{M}^{-1}(\underset{\sim}{q}) \begin{bmatrix} i_1 - g_1 \circ f_1(q_1) \\ i_2 - g_2 \circ f_2(q_2) \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} f_1(q_1) \\ f_2(q_2) \end{bmatrix}$$

if $\underset{\sim}{M}(\underset{\sim}{q})$ is invertible, as a straightforward calculation will verify. If the network is completely controllable and if $\underset{\sim}{M}(q)$ is invertible at each point $\underset{\sim}{q} \in \Sigma$, then it follows easily from theorem 3-1 that $\mathcal{N}$ is nonenergic iff $f_1(q_1) = f_2(q_2) = 0$, for all $(q_1, q_2)^T \in \Sigma$, i.e. the network must be trivial.

The surprising fact, however, is that there are nontrivial choices of element constitutive relations for the network of figure 3 which cause it to be nonenergic. Of course, $\underset{\sim}{M}(\underset{\sim}{q})$ is singular in these cases. One such choice is given in Fig. 4, where the network has been separated into resistive and capacitive two-ports, $\mathcal{R}$ and $\mathcal{C}$, connected in parallel. The following calculation shows that $\mathcal{R}$ and $\mathcal{C}$ are individually nonenergic.

1. $\mathcal{R}$ is non-energic:

$$j_1(v_1, v_2) = 1/v_1$$

$$j_2(v_1, v_2) = -1/v_2$$

$$p(t) = v_1 j_1 + v_2 j_2 = 0 \qquad (3\text{-}5)$$

2. $\mathcal{C}$ is non-energic:

$$Q_1(v_1, v_2) = -\ell n\, v_1 + \ell n(v_1 - v_2)$$

$$Q_2(v_1, v_2) = \ell n\, v_2 - \ell n(v_1 - v_2)$$

$$\dot{Q}_1 = \left(\frac{-1}{v_1}\right)\dot{v}_1 + \left(\frac{1}{v_1 - v_2}\right)(\dot{v}_1 - \dot{v}_2)$$

$$\dot{Q}_2 = \left(\frac{1}{v_2}\right)\dot{v}_2 - \left(\frac{1}{v_1 - v_2}\right)(\dot{v}_1 - \dot{v}_2)$$

$$p(t) = v_1 \dot{Q}_1 + v_2 \dot{Q}_2 = 0 \qquad (3\text{-}6)$$

Both $\mathcal{R}$ and $\mathcal{C}$ are somewhat singular networks. $\mathcal{R}$ is well defined only if v_1 and v_2 are nonzero; $\mathcal{C}$ only if $v_1 > v_2 > 0$. This may be due in part to the naive synthesis procedure adopted here or to some fundamental limitation on all syntheses using two-terminal elements and no dependent sources.

However, the authors have discovered certain restrictions which are <u>independent</u> of the synthesis procedure. For example, every nontrivial, nonenergic voltage-controlled capacitor must be discontinuous at the origin, $\underset{\sim}{v} = \underset{\sim}{0}$. Similarly, every nontrivial, nonenergic, reciprocal, voltage- or current- controlled resistor must blow up at the origin, $\underset{\sim}{v} = \underset{\sim}{0}$ or $\underset{\sim}{i} = \underset{\sim}{0}$. These conclusions will emerge as consequences of the theory developed in sections IV and V.

IV. Multiport Resistors

We require throughout section IV that U, the set of admissible input values, be an open set in $\mathbb{R}^N$.

A. General Theory

<u>Theorem 4-1</u> If $\mathcal{N}$ is a resistive N-port characterized by

$$\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{u}), \text{ for all } \underset{\sim}{u} \in U, \tag{4-1}$$

where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a hybrid pair (definition 2-1) and $\underset{\sim}{f}$ is continuous, then $\mathcal{N}$ is nonenergic $\Leftrightarrow$ $\underset{\sim}{f}(\underset{\sim}{u})$ can be written in the form $\underset{\sim}{f}(\underset{\sim}{u}) = \underset{\sim}{H}(\underset{\sim}{u})\underset{\sim}{u}$, where $\underset{\sim}{H}(\cdot)$ assigns to each point of U an antisymmetric matrix in $\mathbb{R}^{N\times N}$.

<u>Proof</u> ($\Leftarrow$) $\langle \underset{\sim}{v},\underset{\sim}{i} \rangle = \langle \underset{\sim}{y},\underset{\sim}{u} \rangle = \underset{\sim}{u}^T\underset{\sim}{H}^T(\underset{\sim}{u})\underset{\sim}{u} = 0$, since $\underset{\sim}{H}(\underset{\sim}{u})$ is antisymmetric.

($\Rightarrow$) Since $\underset{\sim}{u}$ and $\underset{\sim}{f}(\underset{\sim}{u})$ must be orthogonal for all $\underset{\sim}{u} \in \mathcal{U}$, the conclusion follows from theorem 7-1 for nonzero $\underset{\sim}{u}$. Extension to the case where $\underset{\sim}{u} = 0$ follows from the continuity of $\underset{\sim}{f}$. ∎

See the note which follows the proof of theorem 7-1.

<u>Example 4-1</u> The ideal transformer and gyrator, equations (2-4) and (2-5).

Example 4-2 The type II conjunctor [Duinker, 1962], characterized by

$$\begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 & -Bv_3 & 0 \\ Bv_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_3 \end{bmatrix}$$

Example 4-3 The 2-port $\mathcal{R}$ of Fig. 4 and equation (3-5) can be characterized by

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/(v_1 v_2) \\ -1/(v_1 v_2) & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \begin{matrix} v_1 \neq 0 \\ v_2 \neq 0 \end{matrix} . \tag{4-2}$$

Example 4-4

$$\begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -i_2 v_3 \\ 0 & 0 & -i_1 v_3 \\ i_2 v_3 & i_1 v_3 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_3 \end{bmatrix}, \quad \text{for all } (i_1, i_2, v_3) \in \mathbb{R}^3 \tag{4-3}$$

It is easy to verify that this one is reciprocal (definition 4-2).

In the statement of theorem 4-1, the phrase "can be written as" is used advisedly, since such a representation of $\underset{\sim}{f}(\underset{\sim}{u})$ is by no means unique. Equation (4-2) above could also be written as

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1/v_1^2 & 0 \\ 0 & -1/v_2^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \begin{matrix} v_1 \neq 0 \\ v_2 \neq 0 \end{matrix} \tag{4-4}$$

and equation (4-3) as

$$\begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 & -v_3^2 & 0 \\ -v_3^2 & 0 & 0 \\ 0 & 2i_1 v_3 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_3 \end{bmatrix} \tag{4-5}$$

Definition 4-1 Suppose $\mathcal{N}$ satisfies the conditions of theorem 4-1 and $\underset{\sim}{f}$ is C^1. Then $\mathcal{N}$ is said to be locally strictly passive or incrementally strictly passive at a point $\underset{\sim}{u} \in U$ iff the Jacobian of $\underset{\sim}{f}$ is positive definite at that point.

Theorem 4-2 Assume that $\mathcal{N}$ satisfies the conditions of theorem 4-1 and $\underset{\sim}{f}$ is C^1. If $\mathcal{N}$ is nonenergic, then $\mathcal{N}$ is not locally strictly passive at any point of U.

Proof First choose $\underset{\sim}{u} \neq \underset{\sim}{0}$. Then for all $\delta\underset{\sim}{u}$ such that $\underset{\sim}{u} + \delta\underset{\sim}{u} \in U$, $0 = \langle \underset{\sim}{f}(\underset{\sim}{u}+\delta\underset{\sim}{u}), \underset{\sim}{u}+\delta\underset{\sim}{u} \rangle - \langle \underset{\sim}{f}(\underset{\sim}{u}), \underset{\sim}{u} \rangle = \langle \left[J\left(\underset{\sim}{f}(\underset{\sim}{u})\right)\right]\delta\underset{\sim}{u}, u \rangle + \langle \underset{\sim}{f}(\underset{\sim}{u}), \delta\underset{\sim}{u} \rangle + O(\|\delta\underset{\sim}{u}\|^2)$, so $\left\{\underset{\sim}{u}^T\left[J\left(\underset{\sim}{f}(\underset{\sim}{u})\right)\right] + \underset{\sim}{f}^T(\underset{\sim}{u})\right\}\delta\underset{\sim}{u} = O(\|\delta\underset{\sim}{u}\|^2)$. Therefore $\underset{\sim}{u}^T\left[J\left(\underset{\sim}{f}(\underset{\sim}{u})\right)\right] + \underset{\sim}{f}^T(\underset{\sim}{u}) = \underset{\sim}{0}$, and it follows that $\underset{\sim}{u}^T\left[J\left(\underset{\sim}{f}(\underset{\sim}{u})\right)\right]\underset{\sim}{u} + \underset{\sim}{f}^T(\underset{\sim}{u})\underset{\sim}{u} = 0$. Since $\mathcal{N}$ is nonenergic, we know that $\underset{\sim}{f}^T(\underset{\sim}{u})\underset{\sim}{u} = 0$. Combining this with the previous equation proves that the Jacobian of $\underset{\sim}{f}$ is not positive definite at $\underset{\sim}{u}$. The conclusion can be extended to the point $\underset{\sim}{u} = \underset{\sim}{0}$ as well, since $\underset{\sim}{f}$ is C^1. ∎

B. Reciprocity, Antireciprocity, Content, and Co-Content

Definition 4-2 for reciprocity and antireciprocity of nonlinear N-ports is modelled after definition 2-2 for the linear case. The meaning of the term "operating point" is deliberately left somewhat vague here; in the case of resistors, capacitors and inductors there will be no ambiguity.

Definition 4-2 $\mathcal{N}$ is said to be reciprocal iff

$$\delta\underset{\sim}{v}'(\cdot) * \delta\underset{\sim}{i}''(\cdot) = \delta\underset{\sim}{v}''(\cdot) * \delta\underset{\sim}{i}'(\cdot) \tag{4-6}$$

and antireciprocal iff

$$\delta\underset{\sim}{v}'(\cdot) * \delta\underset{\sim}{i}''(\cdot) = -\delta\underset{\sim}{v}''(\cdot) * \delta\underset{\sim}{i}'(\cdot) \tag{4-7}$$

whenever $(\delta\underset{\sim}{v}'(\cdot), \delta\underset{\sim}{i}'(\cdot))$ and $(\delta\underset{\sim}{v}''(\cdot), \delta\underset{\sim}{i}''(\cdot))$ are small-signal variations admissible to the linear approximation to $\mathcal{N}$ about the same operating point.

In the case that $\mathcal{N}$ is resistive, i.e. memoryless, definition 4-2 takes on the following form.

Theorem 4-3 A C^1 resistive N-port $\mathcal{N}$ is reciprocal (definition 4-2) $\Leftrightarrow$

$$\langle \delta\underset{\sim}{v}', \delta\underset{\sim}{i}'' \rangle = \langle \delta\underset{\sim}{v}'', \delta\underset{\sim}{i}' \rangle \tag{4-8}$$

and antireciprocal iff

$$\langle \delta\underset{\sim}{v}', \delta\underset{\sim}{i}'' \rangle = - \langle \delta\underset{\sim}{v}'', \delta\underset{\sim}{i}' \rangle \tag{4-9}$$

whenever $(\delta\underset{\sim}{v}', \delta\underset{\sim}{i}')$ and $(\delta\underset{\sim}{v}'', \delta\underset{\sim}{i}'')$ are small signal[1] variations admissible to the linear approximation to $\mathcal{N}$ about the same operating point $(\underset{\sim}{v}, \underset{\sim}{i})$.

The proof is given in part B of the Appendix.

Definition 4-3 If a current-controlled resistive N-port $\mathcal{N}$ is characterized by the gradient of a scalar function, i.e. $\underset{\sim}{v} = \underset{\sim}{f}(\underset{\sim}{i}) = \underset{\sim}{\nabla} G(\underset{\sim}{i})$, for all $\underset{\sim}{i} \in U$, then G is called the content function of $\mathcal{N}$.

The co-content $\bar{G}(\underset{\sim}{v})$ is defined similarly in the voltage-controlled case.

Theorem 4-4 Let $\mathcal{N}$ be a current-controlled resistive N-port characterized by $\underset{\sim}{v} = \underset{\sim}{f}(\underset{\sim}{i})$, for all $\underset{\sim}{i} \in U$, where $\underset{\sim}{f}$ is C^1. Then i) $\Rightarrow$ ii) and ii) $\Leftrightarrow$ iii), where statements i), ii) and iii) are:

i) $\mathcal{N}$ has a C^2 content function $G(\underset{\sim}{i})$ defined on U.

ii) The Jacobian of $\underset{\sim}{f}$ is symmetric at each point of U.

iii) $\mathcal{N}$ is reciprocal.

Moreover, if U is simply connected, then i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii).

The proof is given in part B of the Appendix. Equivalent conditions hold for the co-content in the voltage-controlled case.

---

[1] See [Chua and Lam]. Their footnote #7 assigns a mathematically exact meaning from differential geometry to the expression "small signal variations··· same operating point, $(\underset{\sim}{v}, \underset{\sim}{i})$".

C. Homogeneous Functions

Definition 4-4 A set $S \subset \mathbb{R}^n$ is called a cone iff $x \in \mathbb{R}^n \Rightarrow \lambda \underset{\sim}{x} \in \mathbb{R}^n$, for all $\lambda > 0$.

Notation: Throughout part IV, U_c denotes an open cone in $\mathbb{R}^N$. Of course $\mathbb{R}^N$ itself is an open cone. The domains in examples 4-5 through 4-8, 5-1, 5-2 and 6-1 are open cones which are proper subsets of $\mathbb{R}^2$.

Definition 4-5 Φ: $U_c \subset \mathbb{R}^n \to \mathbb{R}$ is k-order homogeneous iff

$$\Phi(\lambda \underset{\sim}{x}) = \lambda^k \Phi(\underset{\sim}{x}), \text{ for all } \lambda > 0,\ \underset{\sim}{x} \neq \underset{\sim}{0} \tag{4-10}$$

Example 4-5 $\Phi(x,y) = \frac{x^2 y^2}{x+y}$

is third order homogeneous on $U_c = \{(x,y) | x + y \neq 0\}$.

Lemma 4-1 Suppose Φ: $U_c \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at all nonzero $\underset{\sim}{x} \in U_c$. Then Φ is k-order homogeneous $\Leftrightarrow$

$$\langle \underset{\sim}{\nabla} \Phi(\underset{\sim}{x}), \underset{\sim}{x} \rangle = k\Phi(\underset{\sim}{x}), \text{ for all } \underset{\sim}{x} \in U_c,\ \underset{\sim}{x} \neq 0. \tag{4-11}$$

Lemma 4-1 is a standard theorem of analysis [Courant and Hilbert].

Lemma 4-2 If Φ: $U_c \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at all nonzero $\underset{\sim}{x} \in U_c$ and k-order homogeneous, then $\frac{\partial \Phi}{\partial x_j}(\underset{\sim}{x})$ is k-1 order homogeneous, $j = 1, 2 \cdots N$.

Proof Differentiating equation (4-10) w.r.t. x_j yields

$$\lambda (D_j \Phi)(\lambda \underset{\sim}{x}) = \lambda^k (D_j \Phi)(\underset{\sim}{x}).$$

■

D. The Reciprocal Case when $\mathcal{N}$ is Current-controlled or Voltage-Controlled

Theorem 4-5 Let $\mathcal{N}$ be characterized by $\underset{\sim}{v}(\underset{\sim}{i}) = \nabla G(\underset{\sim}{i})$, for all $\underset{\sim}{i} \in U_c \subset \mathbb{R}^N$. Then $\mathcal{N}$ is nonenergic iff G is 0-order homogeneous.

Proof $\quad \langle \underset{\sim}{i}, \underset{\sim}{v}(\underset{\sim}{i}) \rangle = \langle \underset{\sim}{i}, \underset{\sim}{\nabla} G(\underset{\sim}{i}) \rangle$

The conclusion follows from lemma 4-1. ∎

Corollary Under the assumptions of theorem 4-5, if $\mathcal{N}$ is nonenergic, then each component of $\underset{\sim}{v}(\underset{\sim}{i})$ is -1 order homogeneous.

Proof. Follows from lemma 4-2. ∎

It follows from the corollary above that if $\mathcal{N}$ is a reciprocal current-controlled nonenergic resistor, then $\mathcal{N}$ either blows up at the origin or else is an N-port short-circuit. For this reason nontrivial, nonenergic, reciprocal, current-controlled resistors will probably not be encountered in practice.

Everything said here applies equally to $\bar{G}(\underset{\sim}{v})$ and $\underset{\sim}{i}(\underset{\sim}{v})$ in the voltage-controlled case.

Example 4-6

$$G(\underset{\sim}{i}) = \frac{i_1}{\sqrt{i_1^2 + i_2^2}}$$

$$v_1 = \frac{\partial G}{\partial i_1} = \frac{i_2^2}{(i_1^2 + i_2^2)^{3/2}}$$

$$v_2 = \frac{\partial G}{\partial i_2} = \frac{-i_1 i_2}{(i_1^2 + i_2^2)^{3/2}}, \quad (i_1, i_2) \neq (0,0).$$

The interest of the next example lies in the fact that it can be synthesized using only 2-terminal nonlinear elements without dependent sources.

Example 4-7 The 2-port resistor $\mathcal{R}$ of equation (3-5) and Fig. 4 is characterized by the content function $G(\underset{\sim}{i}) = \ln\left(\left|\frac{i_1}{i_2}\right|\right)$, $i_1 \neq 0$, $i_2 \neq 0$.

One might suspect that if $\mathcal{N}$ is reciprocal and the components of $\underset{\sim}{v}(\underset{\sim}{i})$ are -1 order homogeneous, then $\mathcal{N}$ is nonenergic. This would be the converse to the corollary to theorem 4-5. The next example shows, however, that this is not the case.

Example 4-8
$$v_1 = \frac{i_1}{i_1^2 + i_2^2}, \quad V_2 = \frac{i_2}{i_1^2 + i_2^2}$$

$$\langle \underset{\sim}{v}, \underset{\sim}{i} \rangle = 1, \; (i_1, i_2) \neq (0,0)$$

$$G(\underset{\sim}{i}) = \frac{\ln(i_1^2 + i_2^2)}{2}$$

E. Reciprocity and Hybrid Content

Suppose we have a resistive N-port characterized by equation (4-1). If $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a mixed hybrid pair (definition 2-3), then, after renumbering the ports if necessary, we can write

$$\underset{\sim}{u} = \left[i_1, \cdots i_k, v_{k+1}, \cdots v_N\right]^T = \left[\underset{\sim}{i}_I^T, \underset{\sim}{v}_{II}^T\right]^T$$

$$\underset{\sim}{y} = \left[v_1, \cdots v_k, i_{k+1}, \cdots i_N\right]^T = \left[\underset{\sim}{v}_I^T, \underset{\sim}{i}_{II}^T\right]^T, \; 1 \leq k \leq N-1 \qquad (4\text{-}12)$$

We adopt the following notation: $\underset{\sim}{i}_I^* \triangleq -\underset{\sim}{i}_I$; the simple function carrying $\underset{\sim}{i}_I$ into $\underset{\sim}{i}_I^*$ is written $\underset{\sim}{i}_I^*(\underset{\sim}{i}_I)$; $\underset{\sim}{u}^* \triangleq \left[\underset{\sim}{i}^{*T}, \underset{\sim}{v}_{II}^T\right]^T$; and the function carrying $\underset{\sim}{u}^*$ into $\underset{\sim}{u}$ is written $\underset{\sim}{u}(\underset{\sim}{u}^*)$.

Definition 4-6 Suppose $\mathcal{N}$ is characterized by equation (4-1). A differentiable scalar function $\Phi(\underset{\sim}{u}^*)$ is called a hybrid content function for $\mathcal{N}$ iff $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{u}) = \underset{\sim}{f}\left(\underset{\sim}{u}(\underset{\sim}{u}^*)\right) = \underset{\sim}{\nabla}\Phi(\underset{\sim}{u}^*)$, for all $\underset{\sim}{u}^*$ such that $\underset{\sim}{u}(\underset{\sim}{u}^*) \in U$.

Theorem 4-6 Let $\mathcal{N}$ be a resistive N-port characterized by equation (4-1). Let $\underset{\sim}{f}$ be C^1, let $\underset{\sim}{u}$ and $\underset{\sim}{y}$ be a mixed hybrid pair, and let the ports be renumbered if necessary so that $\underset{\sim}{u}$ and $\underset{\sim}{y}$ appear as in equation (4-12). Then i) $\Rightarrow$ ii) and ii) $\Leftrightarrow$ iii), where statements i), ii) and iii) are:

i) $\mathcal{N}$ has a C^2 hybrid content function, $\Phi(\underset{\sim}{u}^*)$.

ii) At each point of U the Jacobian of $\underset{\sim}{f}$ is of the form

$$\left[\begin{array}{c|c} \underset{\sim}{A} & \underset{\sim}{B} \\ \hline -\underset{\sim}{B}^T & \underset{\sim}{C} \end{array}\right],$$

where $\underset{\sim}{A} \in \mathbb{R}^{k\times k}$ and $\underset{\sim}{C} \in \mathbb{R}^{(N-k)\times(N-k)}$ are symmetric.

iii) $\mathcal{N}$ is reciprocal.

Moreover, if U is simply connected then i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii).

The proof is in part B of the Appendix.

F. Reciprocal, Nonenergic, Hybrid Resistors

Example 4-9 A direct computation will verify that the resistive 3-port of equations (4-3) and (4-5) has the hybrid content function $\Phi(\underset{\sim}{u}^*) = \Phi(\underset{\sim}{i}^*_I, \underset{\sim}{v}_{II}) = \Phi(i_1^*, i_2^*, v_3) = i_1^* i_2^* v_3^2$.

Notice that in the above example, Φ has the property that $\Phi(\mu^{-1}\underset{\sim}{i}^*_I, \mu\underset{\sim}{v}_{II}) = \Phi(\underset{\sim}{i}^*_I, \underset{\sim}{v}_{II})$, for all $\mu > 0$. Far from being accidental this condition on Φ turns out to be necessary and sufficient for $\mathcal{N}$ to be nonenergic.

Theorem 4-7 Let $\mathcal{N}$ be a resistive N-port characterized by a C^1 hybrid content function $\Phi(\underset{\sim}{i}^*_I, \underset{\sim}{v}_{II})$ defined on all of $\mathbb{R}^N$. Then $\mathcal{N}$ is nonenergic $\Leftrightarrow$ $\Phi(\mu^{-1}\underset{\sim}{i}^*_I, \mu\underset{\sim}{v}_{II}) = \Phi(\underset{\sim}{i}^*_I, \underset{\sim}{v}_{II})$ for all $\mu > 0$ and for all $\left[\underset{\sim}{i}^{*T}_I, \underset{\sim}{v}^T_{II}\right]^T \in \mathbb{R}^N$.

Proof $\mathcal{N}$ is nonenergic iff

$$\langle \underset{\sim}{v}, \underset{\sim}{i} \rangle = \sum_{j=1}^{k} i_j \frac{\partial\Phi}{\partial i_j^*}(\underset{\sim}{i}^*_I, \underset{\sim}{v}_{II}) + \sum_{j=k+1}^{N} v_j \frac{\partial\Phi}{\partial v_j}(\underset{\sim}{i}^*_I, \underset{\sim}{v}_{II}) =$$

$$-\sum_{j=1}^{k} i_j^* \frac{\partial\Phi}{\partial i_j^*} + \sum_{j=k+1}^{N} v_j \frac{\partial\Phi}{\partial v_j} = 0. \qquad (4\text{-}13)$$

$$(\Leftarrow) \quad \left.\frac{d}{d\mu}\right|_{\mu=1} \Phi(\mu^{-1}\underset{\sim}{i}_I^*, \mu \underset{\sim}{v}_{II}) = -\sum_{j=1}^{k} i_j^* \frac{\partial\Phi}{\partial i_j^*} + \sum_{j=k+1}^{N} v_j \frac{\partial\Phi}{\partial v_j} = 0$$

($\Rightarrow$) Define a vector field on $\mathbb{R}^N$ as follows: to each point $\left[\underset{\sim}{i}_I^{*T}, \underset{\sim}{v}_{II}^T\right]^T \in \mathbb{R}^N$ assign the vector $\underset{\sim}{w}\left(\left[\underset{\sim}{i}_I^{*T}, \underset{\sim}{v}_{II}^T\right]^T\right) = \left[-\underset{\sim}{i}_I^{*T}, \underset{\sim}{v}_{II}^T\right]^T = \left[\underset{\sim}{i}_I^T, \underset{\sim}{v}_{II}^T\right]^T$. If $\mathcal{N}$ is nonenergic, then the vector field $\underset{\sim}{\nabla}\Phi$ is everywhere orthogonal to the vector field $\underset{\sim}{w}$, and therefore Φ is constant along the integral curves of $\underset{\sim}{w}$. A straightforward solution of N first order, linear, uncoupled differential equations shows that the integral curve of w passing through a point $\left[\underset{\sim}{i}_I^{*T}, \underset{\sim}{v}_{II}^T\right]^T$ is given by $\left[e^{-\lambda}\underset{\sim}{i}_I^{*T}, e^{\lambda}v_{II}^T\right]^T$. Substituting μ for e^λ yields the theorem. ∎

Corollary Let $\mathcal{N}$ be a reciprocal linear N-port resistor characterized by $\underset{\sim}{y} = \underset{\sim}{H}\underset{\sim}{u}$, for all $\underset{\sim}{u} \in \mathbb{R}^N$, where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a mixed hybrid pair and $\underset{\sim}{H} \in \mathbb{R}^{N\times N}$. Then $\mathcal{N}$ is nonenergic $\Leftrightarrow$ $\mathcal{N}$ has a hybrid content of the form $\Phi(\underset{\sim}{i}_I^*, \underset{\sim}{v}_{II}) = \underset{\sim}{i}_I^{*T} \underset{\sim}{B}\, \underset{\sim}{v}_{II} + \Phi_0$, $\underset{\sim}{B} \in \mathbb{R}^{k\times(N-k)}$.

The proof is a direct computation using theorem 2-2 and equation 2-7.

G. The Antireciprocal Case

Theorem 4-8 Let $\mathcal{N}$ be an antireciprocal resistive N-port characterized by equation (4-1), where $U = \mathbb{R}^n$, $\underset{\sim}{f}$ is C^1, and $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair. Then $\mathcal{N}$ is nonenergic $\Leftrightarrow$ $\mathcal{N}$ is linear.

Proof ($\Leftarrow$) Follows from theorem 2-1.

($\Rightarrow$) Every antireciprocal C^1 resistive N-port is affine [Chua and Lam], i.e. $\underset{\sim}{y} = \underset{\sim}{H}\underset{\sim}{u} + \underset{\sim}{c}$, $\underset{\sim}{H} \in \mathbb{R}^{N\times N}$. Since $\mathcal{N}$ is nonenergic, $\langle \underset{\sim}{y}, \underset{\sim}{u} \rangle = \langle \underset{\sim}{H}\underset{\sim}{u} + \underset{\sim}{c}, \underset{\sim}{u} \rangle = 0$, for all $\underset{\sim}{u} \in \mathbb{R}^N$, which is possible only if $\underset{\sim}{c} = 0$. ∎

V. Multiport Capacitors and Inductors

For simplicity of language, all results in this section are stated in terms of capacitive N-ports and the variables $\underset{\sim}{v}$ and $\underset{\sim}{q}$, but they apply to inductive N-ports as well if $\underset{\sim}{i}$ is substituted for $\underset{\sim}{v}$ and $\underset{\sim}{\phi}$ for $\underset{\sim}{q}$. We will require throughout section V that U, the set of admissible input values, be an open set in $\mathbb{R}^N$.

A. Reciprocity and Coenergy

Theorem 5-1 A C^1 capacitive N-port $\mathcal{N}$ is reciprocal (definition 4-2) $\Leftrightarrow$

$$\langle \delta\underset{\sim}{v}', \delta\underset{\sim}{q}'' \rangle = \langle \delta\underset{\sim}{v}'', \delta\underset{\sim}{q}' \rangle \qquad (5\text{-}1)$$

whenever $(\delta\underset{\sim}{v}', \delta\underset{\sim}{q}')$ and $(\delta\underset{\sim}{v}'', \delta\underset{\sim}{q}'')$ are small signal variations admissible to the linear approximation to $\mathcal{N}$ about the same operating point, $(\underset{\sim}{v}, \underset{\sim}{q})$.

The proof is in part C of the Appendix.

Definition 5-1 If a voltage-controlled capacitive N-port, $\mathcal{N}$, is characterized by the gradient of a differentiable scalar function $\hat{W}(\underset{\sim}{v})$, i.e. if $\underset{\sim}{q}(\underset{\sim}{v}) = \underset{\sim}{\nabla}\hat{W}(\underset{\sim}{v})$ for all $\underset{\sim}{v} \in U$, then $\hat{W}$ is called the electric coenergy function for $\mathcal{N}$.

Theorem 5-2 Let $\mathcal{N}$ be a voltage-controlled capacitive N-port characterized by

$$\underset{\sim}{q} = \underset{\sim}{q}(\underset{\sim}{v}) \text{ for all } \underset{\sim}{v} \in U, \qquad (5\text{-}2)$$

where $q(\cdot)$ is C^1. Then i) $\Rightarrow$ ii) and ii) $\Leftrightarrow$ iii), where statements i), ii) and iii) are:

i) $\mathcal{N}$ has a C^2 electric coenergy function $\hat{W}(\underset{\sim}{v})$ defined on U.
ii) $\mathcal{N}$ is reciprocal.
iii) The Jacobian of $\underset{\sim}{q}(\underset{\sim}{v})$ is symmetric at each point of U.

Moreover, if U is simply connected, then i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii).

The proof is outlined in part C of the Appendix.

B. Charge-Controlled and Voltage-Controlled Capacitors

Theorem 5-3 Suppose $\mathcal{N}$ is a charge-controlled capacitor characterized by $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{q})$ for all $\underset{\sim}{q} \in U$. Then $\mathcal{N}$ is nonenergic $\Leftrightarrow$ $\underset{\sim}{v}(\underset{\sim}{q}) = \underset{\sim}{0}$ for all $\underset{\sim}{q} \in U$.

Proof ($\Leftarrow$) Obvious.

($\Rightarrow$) Suppose there exists a point $\hat{\underset{\sim}{q}} = \left[\hat{q}_1, \cdots, \hat{q}_N\right]^T \in U$ where $\underset{\sim}{v}(\hat{\underset{\sim}{q}}) \neq \underset{\sim}{0}$. Then let $\underset{\sim}{q}(t=0) = \hat{\underset{\sim}{q}}$ and $\underset{\sim}{i}(t) = \hat{\underset{\sim}{i}} \cos 2\pi t$, $t > 0$. Then $\underset{\sim}{q}(t=1) = \hat{\underset{\sim}{q}}$, $\underset{\sim}{i}(t=1) = \hat{\underset{\sim}{i}}$ and $p(t=1) = \langle \underset{\sim}{v}(\hat{\underset{\sim}{q}}), \hat{\underset{\sim}{i}} \rangle$. We can choose $\underset{\sim}{i}$ so that $p(t=1) \neq 0$, contradicting the assumption that $\mathcal{N}$ is nonenergic. ∎

Theorem 5-4 Suppose $\mathcal{N}$ is a voltage-controlled capacitor characterized by

$$\underset{\sim}{q} = \underset{\sim}{q}(\underset{\sim}{v}), \text{ for all } \underset{\sim}{v} \in U_c, \tag{5-3}$$

where U_c is an open cone in $\mathbb{R}^N$ (definition 4-4) and $\underset{\sim}{q}(\underset{\sim}{v})$ is C^1. Then $\mathcal{N}$ is nonenergic $\Leftrightarrow$ $\mathcal{N}$ is reciprocal and $q_j(\underset{\sim}{v})$ is 0-order homogeneous, $j = 1, \cdots N$.

Proof ($\Leftarrow$)

$$p = \langle \underset{\sim}{v}, \dot{\underset{\sim}{q}} \rangle = \underset{\sim}{v}^T\left[\underset{\sim}{J}\left(\underset{\sim}{q}(\underset{\sim}{v})\right)\right]\dot{\underset{\sim}{v}} =$$

$$\dot{\underset{\sim}{v}}^T\left[\underset{\sim}{J}\left(\underset{\sim}{q}(\underset{\sim}{v})\right)\right]^T\underset{\sim}{v} = \dot{\underset{\sim}{v}}^T\left[\underset{\sim}{J}\left(\underset{\sim}{q}(\underset{\sim}{v})\right)\right]\underset{\sim}{v}. \tag{5-4}$$

The j-th component of $\left[\underset{\sim}{J}\left(\underset{\sim}{q}(\underset{\sim}{v})\right)\right]\underset{\sim}{v}$ is

$$\sum_{k=1}^{N} v_k \frac{\partial q_j}{\partial v_k}(\underset{\sim}{v}). \tag{5-5}$$

But by lemma 4-1, expression (5-5) is zero for each value of $j \in \{1, \cdots N\}$, since each component of $\underset{\sim}{q}(\underset{\sim}{v})$ is 0-order homogeneous.

($\Rightarrow$) We first establish by contradiction that $\mathcal{N}$ is reciprocal. Suppose, on the contrary, that there is a point $\hat{v} \in U_c$ where the Jacobian of $q(v)$ is not symmetric. Then there is a C^1 closed curve C in a neighborhood of $\hat{v}$ such that $\oint_C q(v) \cdot dv \neq 0$. But since C is closed, $\oint_C q \cdot dv + \oint_C v \cdot dq = \Delta \langle q(v), v \rangle = 0$, so $\oint_C v \cdot dq$, which represents the total energy absorbed by $\mathcal{N}$ over the closed path, is nonzero. This contradicts the assumption that $\mathcal{N}$ is nonenergic. Therefore $\mathcal{N}$ is reciprocal and equation (5-4) still holds. Since at a given instant we can specify v and $\dot{v}$ independently, as in the second part of the proof of theorem 5-3, it follows that

$$\left[J\left(q(v)\right)\right] v = 0 \tag{5-6}$$

for all $v \in U_c$. Expanding equation (5-6) as in expression (5-5) and using lemma 4-1 proves that each component of $q(v)$ is 0-order homogeneous. ∎

It is easy to see that the only continuous zero-order homogeneous functions defined on all of $\mathbb{R}^N$ are the constant functions. (Think about continuity at the origin.) But if we allow $q(v)$ to be defined only on $\mathbb{R}^N - \{0\}$, we can exhibit nontrivial nonenergic capacitors as in the following example.

Example 5-1

$$q_1 = \frac{v_1}{\sqrt{v_1^2 + v_2^2}}$$

$$q_2 = \frac{v_2}{\sqrt{v_1^2 + v_2^2}}, \quad (v_1, v_2) \neq (0,0) \tag{5-7}$$

The next example, while it is only defined on a portion of $\mathbb{R}^N - \{0\}$, is of interest because it can be synthesized from 2-terminal nonlinear capacitors without using dependent sources.

Example 5-2 The capacitive 2-port $\mathcal{C}$ of Fig. 4 and equations (3-6) is characterized by

$$Q_1 = \ln\left(\frac{v_1 - v_2}{v_1}\right)$$

$$Q_2 = \ln\left(\frac{v_2}{v_1 - v_2}\right), \quad v_1 > v_2 > 0 \tag{5-8}$$

If the constitutive relations of the individual capacitors in $\mathcal{C}$ are extended as follows:

$$C_1: \quad q_1 = -\ln|v_1|$$

$$C_2: \quad q_2 = \ln|v_2|$$

$$C_3: \quad q_3 = \ln|v_3| \tag{5-9}$$

Then equations (5-8) can be extended to become

$$Q_1 = \ln\left|\frac{v_1 - v_2}{v_1}\right|$$

$$Q_2 = \ln\left|\frac{v_2}{v_1 - v_2}\right|, \quad v_1 \neq 0,\ v_2 \neq 0,\ v_1 \neq v_2. \tag{5-10}$$

<u>Example 5-3</u> The type II traditor [Duinker, 1959] can be written as a capacitive 3-port as follows:

$$q_1 = \frac{-v_2}{Av_3}$$

$$q_2 = \frac{-v_1}{Av_3}$$

$$q_3 = \frac{v_1 v_2}{Av_3^2}, \quad v_3 \neq 0. \tag{5-11}$$

<u>Corollary</u> Suppose $\mathcal{N}$ satisfies the conditions of theorem 5-4.

Then $\mathcal{N}$ is nonenergic $\Leftrightarrow \hat{W}(\underset{\sim}{v})$ exists and is, to within an additive constant, a C^2 first order homogeneous function.

Proof (⇐) Then $\underset{\sim}{q}(\underset{\sim}{v})$ is C^1 and reciprocal and it follows from lemma 4-2 that each component is 0-order homogeneous. The conclusion then follows from theorem 5-4.

(⇒) From theorem 5-4, $\underset{\sim}{q}(\underset{\sim}{v})$ is reciprocal. The coenergy is given by

$$\hat{W}(\underset{\sim}{v}) = \langle \underset{\sim}{v}, \underset{\sim}{q}(\underset{\sim}{v}) \rangle , \tag{5-12}$$

since $\underset{\sim}{\nabla} \langle \underset{\sim}{v}, \underset{\sim}{q}(\underset{\sim}{v}) \rangle = \underset{\sim}{q}(\underset{\sim}{v}) + \left[\underset{\sim}{J}\left(\underset{\sim}{q}(\underset{\sim}{v})\right)\right]^T \underset{\sim}{v} = \underset{\sim}{q}(\underset{\sim}{v}) + \left[\underset{\sim}{J}\left(\underset{\sim}{q}(\underset{\sim}{v})\right)\right] \underset{\sim}{v} = \underset{\sim}{q}(\underset{\sim}{v})$, where the last equality follows from lemma 4-1 and the fact that each component of $\underset{\sim}{q}(\underset{\sim}{v})$ is 0-order homogeneous. Since each component of $\underset{\sim}{q}(\underset{\sim}{v})$ is 0-order homogeneous, $W(\underset{\sim}{v})$ as given by equation (5-12) is first order homogeneous. ∎

Examples 5-4 For examples 5-1, 5-2 (equation (5-8)), and 5-3, $\hat{W}(\underset{\sim}{v})$ is given respectively by:

5-1) $\hat{W}(v) = \sqrt{v_1^2 + v_2^2}$, $(v_1, v_2) \neq (0,0)$

5-2) $\hat{W}(\underset{\sim}{v}) = v_2 \ell n v_2 - v_1 \ell n v_1 + (v_1 - v_2) \ell n (v_1 - v_2)$, $v_1 > v_2 > 0$

5-3) $\hat{W}(\underset{\sim}{v}) = -\left(\frac{v_1 v_2}{A v_3}\right)$, $v_3 \neq 0$.

C. Parallel to Classical Thermodynamics

There is an interesting relationship between our voltage-controlled nonenergic capacitors and classical thermodynamics. The Euler relation [Callen], $W = TS - PV + \mu N$, can be thought of as a relation between the stored energy W, the "port voltages" T, P, and μ, and the "port charges" S, V and N of a reciprocal, 3-port, charge-controlled capacitor [Oster and Perelson]. It is a fundamental axiom of classical thermodynamics that W(S,V,N) is first-order homogeneous, at least for homogeneous fluid systems. It follows from the dual of the preceding corollary that the energy $W(\underset{\sim}{q})$ of a reciprocal charge-controlled capacitor is first order homogeneous to within an additive constant ⇔ the capacitor is "non-coenergic," i.e. the rate of change of coenergy, $\langle \underset{\sim}{q}, \dot{\underset{\sim}{v}} \rangle$, is always zero. In this sense we can say that classical thermodynamics is the study of "non-coenergic" 3-port capacitors.

D. Reciprocity and the Hybrid Energy Function

Definition 5-2 If an N-port capacitor is characterized by

$$\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{u}) \quad \text{for all } \underset{\sim}{u} \in U, \tag{5-13}$$

where $\underset{\sim}{u} \neq \underset{\sim}{q}$ and $\underset{\sim}{u} \neq \underset{\sim}{v}$ but for each $j \in \{1,2\cdots N\}$ either $u_j = q_j$ and $y_j = v_j$ or $u_j = v_j$ and $y_j = q_j$, then by analogy with definition 2-3 we say that $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a mixed $\underset{\sim}{q}$-$\underset{\sim}{v}$ hybrid pair.

Suppose $\mathcal{N}$ is an N-port capacitor characterized by equation (5-13), where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a mixed $\underset{\sim}{q}$-$\underset{\sim}{v}$ hybrid pair. Then, after renumbering the ports if necessary, we can write

$$\underset{\sim}{u} = \left[q_1, \cdots q_k, v_{k+1}, \cdots v_N\right]^T = \left[\underset{\sim}{q}_I^T, \underset{\sim}{v}_{II}^T\right]^T$$

$$\underset{\sim}{y} = \left[v_1, \cdots v_k, q_{k+1}, \cdots q_N\right]^T = \left[\underset{\sim}{v}_I^T, \underset{\sim}{q}_{II}^T\right]^T, \quad 1 \leq k \leq N-1. \tag{5-14}$$

We define $\underset{\sim}{q}_I^*$ by $\underset{\sim}{q}_I^* \triangleq -\underset{\sim}{q}_I$, the simple function carrying $\underset{\sim}{q}_I$ into $\underset{\sim}{q}_I^*$ as $\underset{\sim}{q}_I^*(\underset{\sim}{q}_I)$, $\underset{\sim}{u}^*$ by $\underset{\sim}{u}^* = \left[\underset{\sim}{q}_I^{*T}, \underset{\sim}{v}_{II}^T\right]^T$, and the function carrying $\underset{\sim}{u}^*$ into $\underset{\sim}{u}$ by $u(u^*)$.

Definition 5-3 Suppose $\mathcal{N}$ is characterized by equation (5-13) and $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a mixed $\underset{\sim}{q}$-$\underset{\sim}{v}$ hybrid pair. If there exists a differentiable scalar function $\Phi(\underset{\sim}{u}^*)$ such that $\underset{\sim}{y} = \underset{\sim}{f}\left(\underset{\sim}{u}(\underset{\sim}{u}^*)\right) = \underset{\sim}{\nabla}\Phi(\underset{\sim}{u}^*)$ for all $\underset{\sim}{u}^*$ such that $\underset{\sim}{u}(\underset{\sim}{u}^*) \in U$, then Φ is called a hybrid energy function for $\mathcal{N}$.

Theorem 5-5 Suppose $\mathcal{N}$ is characterized by equation (5-13), $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a mixed $\underset{\sim}{q}$-$\underset{\sim}{v}$ hybrid pair, and f is C^2. Then i) $\Rightarrow$ ii) and ii) $\Leftrightarrow$ iii), where statements i), ii) and iii) are:

i) $\mathcal{N}$ has a C^2 hybrid energy function Φ.

ii) $\mathcal{N}$ is reciprocal.

iii) At every point $\underset{\sim}{u} \in U$, the Jacobian of $\underset{\sim}{f}(\underset{\sim}{u})$ has the form

$$\left[\begin{array}{c|c} \underset{\sim}{A} & \underset{\sim}{B} \\ \hline -\underset{\sim}{B}^T & \underset{\sim}{C} \end{array}\right],$$

where $\underset{\sim}{A} \in \mathbb{R}^{k\times k}$ and $\underset{\sim}{C} \in \mathbb{R}^{(N-k)\times(N-k)}$ are symmetric. Moreover, if U is simply connected, then i) ⇔ ii) ⇔ iii).

The proof is outlined in part C of the Appendix.

<u>Theorem 5-6</u> Let $\mathcal{N}$, $\underset{\sim}{u}$, $\underset{\sim}{y}$ and $\underset{\sim}{f}$ satisfy the conditions of theorem 5-5 and U_c be an open cone in $\mathbb{R}^N$. Then $\mathcal{N}$ is nonenergic ⇔ f is of the form

$$\underset{\sim}{v}_I = -\left[\frac{\partial g}{\partial q_I}(q_I, \underset{\sim}{v}_{II})\right]^T \underset{\sim}{v}_{II}$$

$$\underset{\sim}{q}_{II} = \underset{\sim}{g}(\underset{\sim}{q}_I, \underset{\sim}{v}_{II})$$

where $g\colon U_c \to \mathbb{R}^{N-k}$ is any C^1 function such that

i) $\left[\dfrac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{v}_{II}}(q_I, \underset{\sim}{v}_{II})\right]$ is symmetric at each point of U, and

ii) each component of $\underset{\sim}{g}(\underset{\sim}{q}_I, \underset{\sim}{v}_{II})$ is 0-order homogeneous in $\underset{\sim}{v}_{II}$ for every value of $\underset{\sim}{q}_I$.

<u>Proof</u> Suppose $\mathcal{N}$ is characterized by

$$\underset{\sim}{v}_I = \underset{\sim}{h}(q_I, \underset{\sim}{v}_{II})$$

$$\underset{\sim}{q}_{II} = \underset{\sim}{g}(\underset{\sim}{q}_I, \underset{\sim}{v}_{II}),$$

where for the moment we make no assumptions about $\underset{\sim}{h}$ and $\underset{\sim}{g}$ except that they are C^1. Then $p = \langle \underset{\sim}{v}_I, \dot{q}_I \rangle + \langle \underset{\sim}{v}_{II}, \underset{\sim}{q}_{II} \rangle = \left\{\underset{\sim}{h}^T(\underset{\sim}{q}_I, \underset{\sim}{v}_{II}) + \underset{\sim}{v}_{II}^T \left[\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{q}_I}(\underset{\sim}{q}_I, \underset{\sim}{v}_{II})\right]\right\} \dot{\underset{\sim}{q}}_I + \left\{ \underset{\sim}{v}_{II}^T \left[\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{v}_{II}}(q_I, \underset{\sim}{v}_{II})\right]\right\} \dot{\underset{\sim}{v}}_{II}$. Since we can specify $\dot{\underset{\sim}{q}}_I$ and $\dot{\underset{\sim}{v}}_{II}$ independently from each other, from $\underset{\sim}{v}_I$, and from $\underset{\sim}{q}_{II}$, as in the second part of the proof of theorem 5-3, $\mathcal{N}$ is nonenergic iff the two expressions enclosed by $\{\cdot\}$ are always zero. The first expression yields $\underset{\sim}{h}$ in terms of $\underset{\sim}{g}$. The product of the second expression and $\dot{\underset{\sim}{v}}_{II}$ is the net power flow into the N-k port voltage-controlled capacitor obtained by open circuiting the first k ports of $\mathcal{N}$. It follows from theorem 5-4 that conditions i) and ii) of theorem 5-6 are necessary and sufficient for this N-k port to be nonenergic.

■

Example 5-5 Since the zero function is the only linear zero-order homogeneous function, all the linear examples can be generated by requiring that $\underset{\sim}{g}(\underset{\sim}{q}_I,\underset{\sim}{v}_{II}) = -\underset{\sim}{B}^T\underset{\sim}{q}_I$, $\underset{\sim}{B} \in \mathbb{R}^{k\times(N-k)}$. Then

$$\begin{aligned} \underset{\sim}{v}_I &= \underset{\sim}{B}\underset{\sim}{v}_{II} \\ \underset{\sim}{q}_{II} &= -\underset{\sim}{B}^T\underset{\sim}{q}_I. \end{aligned} \tag{5-15}$$

We recognize equation (5-15) as a restatement of equation (2-7) in terms of charge rather than current. Equation (5-15) represents a multiport ideal transformer, as predicted by Theorems 2-1 and 2-2. A realization is given in Fig. 1.

Example 5-6 A simple class of examples arises if we require $\underset{\sim}{g}(\underset{\sim}{q}_I,\underset{\sim}{v}_{II})$ to be independent of $\underset{\sim}{v}_{II}$, but otherwise arbitrary. Then we have

$$\begin{aligned} \underset{\sim}{v}_I &= -\left[\underset{\sim}{J}(\underset{\sim}{g}(\underset{\sim}{q}_I))\right]^T\underset{\sim}{v}_{II} \\ \underset{\sim}{q}_{II} &= \underset{\sim}{g}(\underset{\sim}{q}_I), \end{aligned} \tag{5-16}$$

where $\underset{\sim}{g}$ is any C^1 function: $\mathbb{R}^k \to \mathbb{R}^{N-k}$. The type II traditor of equation (5-11), for example, can be written as a hybrid capacitor with $\underset{\sim}{g}(\underset{\sim}{q}_I) = Aq_1q_2$, or

$$\begin{aligned} v_1 &= -Aq_2v_3 \\ v_2 &= -Aq_1v_3 \\ q_3 &= Aq_1q_2 \end{aligned} \tag{5-17}$$

Example 5-7

$$v_1 = -\sqrt{v_2^2 + v_3^2}$$
$$q_2 = q_1 v_2 / \sqrt{v_2^2 + v_3^2}$$
$$q_3 = q_1 v_3 / \sqrt{v_3^2 + v_3^2}, \quad (v_2, v_3) \neq (0,0).$$

Corollary 1 Under the conditions of theorem 5-6, $\mathcal{N}$ is nonenergic $\Leftrightarrow$ $\mathcal{N}$ has a hybrid energy function of the form

$$\Phi(\mathbf{q}_I^*, \mathbf{v}_{II}) = \left\langle \mathbf{g}(-\mathbf{q}_I^*, \mathbf{v}_{II}), \mathbf{v}_{II} \right\rangle + c, \tag{5-18}$$

where $g\colon U \to \mathbb{R}^{N-k}$ is any C^1 function which satisfies conditions i) and ii) of theorem 5-6, and c is an arbitrary constant.

Proof ($\Leftarrow$)

$$\mathbf{v}_I(\mathbf{q}_I^*, \mathbf{v}_{II}) = \nabla_{\mathbf{q}_I^*}\Phi = \left\{ \left[\frac{\partial g}{\partial(-\mathbf{q}_I^*)} (-\mathbf{q}_I^*, \mathbf{v}_{II}) \right] \cdot \left[-\mathbf{1}^{(k)} \right] \right\}^T \mathbf{v}_{II}, \tag{5-19}$$

where $\mathbf{1}^{(k)}$ denotes a kxk identity matrix, and

$$\mathbf{q}_{II}(\mathbf{q}_I^*, \mathbf{v}_{II}) = \nabla_{\mathbf{v}_{II}}\Phi = \mathbf{g}(-\mathbf{q}_I^*, \mathbf{v}_{II}) + \left[\frac{\partial g}{\partial \mathbf{v}_{II}} (-\mathbf{q}_I^*, \mathbf{v}_{II}) \right]^T \mathbf{v}_{II}. \tag{5-20}$$

The last term on the right hand side of equation (5-20) is zero because of assumptions i) and ii) on $\mathbf{g}$ and lemma 4-1. Equations (5-19) and (5-20) become

$$\mathbf{v}_I\left(\mathbf{q}_I^*(\mathbf{q}_I), \mathbf{v}_{II}\right) = -\left[\frac{\partial g}{\partial(-\mathbf{q}_I^*)} \left(-\mathbf{q}_I^*(\mathbf{q}_I), \mathbf{v}_{II}\right) \right]^T \mathbf{v}_{II} = -\left[\frac{\partial \mathbf{g}}{\partial \mathbf{q}_I} (\mathbf{q}_I, \mathbf{v}_{II}) \right]^T \mathbf{v}_{II} \tag{5-21}$$

$$\mathbf{q}_{II}\left(\mathbf{q}_I^*(\mathbf{q}_I), \mathbf{v}_{II}\right) = \mathbf{g}\left(-\mathbf{q}_I^*(\mathbf{q}_I), \mathbf{v}_{II}\right) = \mathbf{g}(\mathbf{q}_I, \mathbf{v}_{II}), \tag{5-22}$$

which guarantees nonenergicness by theorem 5-6.

(⇒) If $\mathcal{N}$ is nonenergic, then by theorem 5-6 $\mathcal{N}$ is characterized by equations (5-21) and (5-22). The hybrid energy function is produced by use of the inner product as in equation (5-18).

∎

Examples 5-8 The hybrid energy functions for examples 5-5; 5-6, equation (5-16); 5-6, equation (5-17); and 5-7 are, respectively:

(5-5): $\Phi(\underset{\sim}{q}_I^*, \underset{\sim}{v}_{II}) = \left\langle \underset{\sim}{B}^T \underset{\sim}{q}_I^*, \underset{\sim}{v}_{II} \right\rangle = \underset{\sim}{q}_I^{*T} \underset{\sim}{B} \underset{\sim}{v}_{II}$

(5-6), equation (5-16): $\Phi(\underset{\sim}{q}_I^*, \underset{\sim}{v}_{II}) = \left\langle \underset{\sim}{g}(-\underset{\sim}{q}_I^*), \underset{\sim}{v}_{II} \right\rangle$

(5-6), equation (5-17): $\Phi(\underset{\sim}{q}_I^*, \underset{\sim}{v}_{II}) = \left\langle A(-q_1^*)(-q_2^*), v_3 \right\rangle = A q_1^* q_2^* v_3$

(5-7): $\Phi(\underset{\sim}{q}_I^*, \underset{\sim}{v}_{II}) = \left\langle \frac{-q_1^*}{\sqrt{v_2^2 + v_3^2}} (v_2, v_3),\ (v_2, v_3) \right\rangle = -\ q_1^* \sqrt{v_2^2 + v_3^2}$

Corollary 2 Under the conditions of theorem 5-6, if $\mathcal{N}$ is nonenergic then $\mathcal{N}$ is reciprocal.

Proof It follows from corollary 1 and theorem 5-5.

∎

VI. Lagrangian N-Ports

We call $\mathcal{N}$ a Lagrangian N-port [Duinker, 1959] if $\mathcal{N}$ is characterized by Lagrange's equations,

$$y_k = \frac{d}{dt} \frac{\partial L(\underset{\sim}{x}, \dot{\underset{\sim}{x}})}{\partial \dot{x}_k} - \frac{\partial L(\underset{\sim}{x}, \dot{\underset{\sim}{x}})}{\partial x_k}, \quad k = 1, 2, \cdots N, \tag{6-1}$$

where $\dot{\underset{\sim}{x}}$ and $\underset{\sim}{y}$ form a hybrid pair (definition (2-1)). $\mathcal{N}$ is called a traditor [Duinker, 1959] if the Lagrangian is of the form $L = a\dot{x}_j f(\underset{\sim}{x})$, $j \in \{1, \cdots N\}$, where f is arbitrary.

It is easy to show that every traditor is nonenergic, but Duinker's theorem that every nonenergic Lagrangian N-port is a traditor depends on the particular assumption that the Lagrangian is of the form $L = \dot{x}_1^{\alpha_1} \dot{x}_2^{\alpha_2} \cdots \dot{x}_N^{\alpha_N} f(\underset{\sim}{x})$, where the α's are nonnegative integers. The following theorem shows that if this assumption is relaxed, we can discover new sorts of nonenergic Lagrangian N-ports.

Theorem 6-1 Suppose $\mathcal{N}$ is a Lagrangian N-port. If L is independent of $\underset{\sim}{x}$, $L(\dot{\underset{\sim}{x}})$ is defined on an open cone (definition 4-4), and $L(\dot{\underset{\sim}{x}})$ is C^2 and first-order homogeneous, then $\mathcal{N}$ is nonenergic.

Proof Expanding equation (6-1) yields in this case

$$\underset{\sim}{y} = \left[\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} (\dot{\underset{\sim}{x}})\right] \ddot{\underset{\sim}{x}},$$

and therefore

$$p = \underset{\sim}{y}^T \dot{\underset{\sim}{x}} = \ddot{\underset{\sim}{x}}^T \left[\frac{\partial^2 L}{\partial \dot{x}_j \partial \dot{x}_i} (\dot{\underset{\sim}{x}})\right] \dot{\underset{\sim}{x}} .$$

The i-th entry of $\left[\frac{\partial^2 L}{\partial \dot{x}_j \partial \dot{x}_i} (\dot{\underset{\sim}{x}})\right] \dot{\underset{\sim}{x}}$ is

$$\sum_{j=1}^{N} \dot{x}_j \frac{\partial}{\partial x_j} \left(\frac{\partial L(\dot{\underset{\sim}{x}})}{\partial \dot{x}_i}\right) . \qquad (6\text{-}2)$$

Since $L(\dot{\underset{\sim}{x}})$ is first order homogeneous, $\frac{\partial L}{\partial \dot{x}_i}$ is 0-order homogeneous by lemma 4-2. Therefore expression (6-2) is identically zero by lemma 4-1.

∎

The following example is a 2-port which satisfies the assumptions of the above theorem. One can easily verify by direct calculation that it is nonenergic.

Example 6-1 $L(x,\dot{x}) = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\left(\dot{x}_1^2 + \dot{x}_2^2\right)^{3/2}} \begin{bmatrix} \dot{x}_2^2 & -\dot{x}_1\dot{x}_2 \\ -\dot{x}_1\dot{x}_2 & \dot{x}_1^2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}, \quad (\dot{x}_1,\dot{x}_2) \neq (0,0) \qquad (6\text{-}3)$$

We can choose to let (x_1,x_2) represent the port charges (q_1,q_2). Then in the above example y_1 and y_2 are port voltages, which depend in equation (6-3) on the port currents and their derivatives.

The following theorem shows how to generate more complex nonenergic Lagrangian N-ports from those introduced in theorem 6-1 and from traditors.

Theorem 6-2 If $L_1(\underset{\sim}{x},\dot{\underset{\sim}{x}}), \cdots, L_k(\underset{\sim}{x},\dot{\underset{\sim}{x}})$ are Lagrangians of nonenergic N-ports, then any linear combination of L_1 through L_k is the Lagrangian of a nonenergic N-port.

The proof is immediate.

VII. Fundamental Theory

Notation In part VII we will use superscripts to indicate vector components, i.e. $\underset{\sim}{v} = \left[v^1,v^2,\cdots v^N\right]^T$, etc.

The restriction to the time-invariant case is dropped throughout part VII.

A. Operator Theory of Nonenergic N-Ports. A Canonical Form.

Definition 7-1 Suppose $\underset{\sim}{u} \in \mathbb{R}^N$ and $\underset{\sim}{y} \in \mathbb{R}^N$ form a hybrid pair (definition 2-1). Let $\mathcal{U}$ be the class of all admissible input waveforms, $\underset{\sim}{u}(\cdot)$, for $\mathcal{N}$. If

1) there exists an operator $\underset{\sim}{\mathcal{H}}$ assigning to each input waveform $\underset{\sim}{u}(\cdot) \in \mathcal{U}$ a unique output waveform $\underset{\sim}{y}(\cdot)$, i.e. $\underset{\sim}{y}(\cdot) = \underset{\sim}{\mathcal{H}}\,\underset{\sim}{u}(\cdot)$, $\underset{\sim}{y}(t) = \left(\underset{\sim}{\mathcal{H}}\underset{\sim}{u}(\cdot)\right)(t)$, and

2) every admissible pair $\left(\underset{\sim}{v}(\cdot),\ \underset{\sim}{i}(\cdot)\right)$ of $\mathcal{N}$ can be represented, after perhaps a suitable reordering of components, as $\left(\underset{\sim}{\mathcal{H}}\underset{\sim}{u}(\cdot),\underset{\sim}{u}(\cdot)\right)$ for some $\underset{\sim}{u}(\cdot) \in \mathcal{U}$, then we call $\underset{\sim}{\mathcal{H}}$ a hybrid operator for $\mathcal{N}$.

If $\mathcal{N}$ has a hybrid operator $\underset{\sim}{\mathcal{H}}$, then $\mathcal{N}$ is nonenergic iff

$$p(t) = \left\langle \underset{\sim}{u}(t),\underset{\sim}{y}(t) \right\rangle = \left\langle \underset{\sim}{u}(t),\left(\underset{\sim}{\mathcal{H}}\underset{\sim}{u}(\cdot)\right)(t) \right\rangle = 0, \text{ for all } \underset{\sim}{u}(\cdot) \in \mathcal{U} \text{ and for all } t \qquad (7\text{-}1)$$

Notice that we cannot discuss the ideal diode in this framework. Its v-i relation is such that there is no representation in which either v or i can be chosen globally as the independent variable.

Definition 7-2 Let $\mathcal{V}_n$ be any class of functions: $\mathbb{R} \to \mathbb{R}^n$. Let $\mathbb{R}_a^{n\times n}$ be the set of all antisymmetric n×n real matrices. Let A_n be the class of all functions: $\mathbb{R} \to \mathbb{R}_a^{n\times n}$. We will denote by $\mathcal{A}$ an operator: $\mathcal{V}_n \to A_n$.

If $u(\cdot) \in \mathcal{V}_n$, then $\mathcal{A}u(\cdot)$ is an antisymmetric-matrix-valued time function $\Lambda(\cdot)$, i.e. for each value of t, $\Lambda(t) \in \mathbb{R}_a^{n\times n}$. Thus we write $\mathcal{A}u(\cdot) = \Lambda(\cdot) \in A_n$ to indicate that the time function $\Lambda(\cdot)$ is the image under $\mathcal{A}$ of the time function $u(\cdot)$. And we write $\left(\mathcal{A}u(\cdot)\right)_{(t)} = \Lambda(t) \in \mathbb{R}_a^{n\times n}$ to indicate that the image under $\mathcal{A}$ of the waveform $u(\cdot)$, when evaluated at the time t, is a certain n×n antisymmetric matrix $\Lambda(t)$.

Example 7-1 Let $\mathcal{V}_2$ be the class of all C^1 functions: $\mathbb{R} \to \mathbb{R}^2$. Then one operator $\mathcal{A}$ on $\mathcal{V}_2$ is given by

$$\left(\mathcal{A}x(\cdot)\right)_{(t)} = \begin{bmatrix} 0 & \dot{x}^1(t)x^2(t+1) \\ -\dot{x}^1(t)x^2(t+1) & 0 \end{bmatrix},$$

for all $x(\cdot) = \left(x^1(\cdot),\ x^2(\cdot)\right)^T \in \mathcal{V}_2$ and for all t.

In the special case n = 1, the intended interpretation is that the scalar $0 \in \mathbb{R}$ is the only antisymmetric 1×1 matrix. And an operator carrying $\mathcal{V}_1$ into A_1 just assigns the zero function, $0(t) = 0$ for all t, to each waveform in $\mathcal{V}_1$.

Lemma 7-1 Let x and y be any two vectors in $\mathbb{R}^n$ such that $\langle x,y \rangle = 0$ and $x \neq 0$. Then there exists a matrix $\Lambda \in \mathbb{R}_a^{n\times n}$ such that $y = \Lambda x$.

The proof is in part E of the Appendix.

Suppose that $u(\cdot) \in \mathcal{V}_n$ and $\mathcal{A}u(\cdot) = \Lambda(\cdot)$. Then we let $\left(\mathcal{A}u(\cdot)\right)_{(t)}u(t) = \Lambda(t)u(t)$ represent the vector in $\mathbb{R}^n$ obtained by operating on $u(t)$, a vector in $\mathbb{R}^n$, with the n×n antisymmetric matrix $\left(\mathcal{A}u(\cdot)\right)_{(t)} = \Lambda(t)$. See example 7-2.

Theorem 7-1 Let $\mathcal{N}$ be an N-port characterized by a hybrid operator $\mathcal{H}$ defined on a class $\mathcal{U}$ of admissible input waveforms: $\mathbb{R} \to \mathbb{R}^n$. Then $\mathcal{N}$ is nonenergic ⇔ there exists an operator $\mathcal{A}$ defined on $\mathcal{U}$ such that the following equation holds for every choice of $\left(u(\cdot),t\right) \in \mathcal{U} \times \mathbb{R}$ such that $u(t) \neq 0$:

$$\mathbf{y}(t) = \left(\boldsymbol{\mathcal{H}}\mathbf{u}(\cdot)\right)_{(t)} = \left(\boldsymbol{\mathcal{A}}\,\mathbf{u}(\cdot)\right)_{(t)}\mathbf{u}(t) \tag{7-2}$$

<u>Proof</u> (⇐) Choose any waveform $\mathbf{u}(\cdot) \in \mathcal{U}$ and any time t. If it turns out that $\mathbf{u}(t) = \mathbf{0}$, then equation (7-2) need not hold, but $\langle \mathbf{u}(t), \mathbf{y}(t) \rangle = 0$ nonetheless. If it turns out that $\mathbf{u}(t) \neq \mathbf{0}$, then equation (7-2) does hold and $\left(\boldsymbol{\mathcal{A}}\mathbf{u}(\cdot)\right)_{(t)}$ is some n×n antisymmetric matrix $\boldsymbol{\Lambda}$. Therefore

$$\left\langle \mathbf{u}(t), \left(\boldsymbol{\mathcal{H}}\mathbf{u}(\cdot)\right)_{(t)} \right\rangle = \left\langle \mathbf{u}(t), \left(\boldsymbol{\mathcal{A}}\,\mathbf{u}(\cdot)\right)_{(t)}\mathbf{u}(t) \right\rangle = \left\langle \mathbf{u}(t), \boldsymbol{\Lambda}\mathbf{u}(t) \right\rangle = 0,$$

and $\left\langle \mathbf{u}(t), \mathbf{y}(t) \right\rangle = 0$ in this case as well.

(⇒) Since $\mathcal{N}$ is nonenergic, equation (7-1) must hold. It follows from lemma 7-1 that for each choice of $\left(\mathbf{u}(\cdot), t\right) \in \mathcal{U} \times \mathbb{R}$ such that $\mathbf{u}(t) \neq \mathbf{0}$, there exists a matrix $\boldsymbol{\Lambda} \in \mathbb{R}_a^{N\times N}$ such that

$$\mathbf{y}(t) = \left(\boldsymbol{\mathcal{H}}\mathbf{u}(\cdot)\right)_{(t)} = \boldsymbol{\Lambda}\mathbf{u}(t). \tag{7-3}$$

There may be more than one such matrix, but for each choice of $\left(\mathbf{u}(\cdot), t\right) \in \mathcal{U} \times \mathbb{R}$ such that $\mathbf{u}(t) \neq \mathbf{0}$ we choose one and denote it $\boldsymbol{\Lambda}\left(\mathbf{u}(\cdot), t\right)$. This choice of $\boldsymbol{\Lambda}$ then satisfies equation (7-3). And for each choice of $\left(\mathbf{u}(\cdot), t\right) \in \mathcal{U} \times \mathbb{R}$ such that $\mathbf{u}(t) = \mathbf{0}$ we (arbitrarily) choose some matrix $\boldsymbol{\Lambda} \in \mathbb{R}_a^{\tilde{N}\times N}$ and denote it $\boldsymbol{\Lambda}\left(\mathbf{u}(\cdot), t\right)$ as well, even though in this case our choice of $\boldsymbol{\Lambda}$ will not satisfy equation (7-3). This construction has yielded a map $\boldsymbol{\Lambda}\left(\mathbf{u}(\cdot), t\right): \mathcal{U} \times \mathbb{R} \to \mathbb{R}_a^{N\times N}$ which satisfies equation (7-3) at each point $\left(\mathbf{u}(\cdot), t\right) \in \mathcal{U} \times \mathbb{R}$ such that $\mathbf{u}(t) \neq \mathbf{0}$. And this map is our desired operator $\boldsymbol{\mathcal{A}}$.

∎

Note that the construction used in the second part of the above proof makes implicit use of the axiom of choice. Notice also that there is no claim that $\boldsymbol{\mathcal{A}}$ is unique.

<u>Note</u> The exact statement of theorem 7-1, that there exists an operator $\boldsymbol{\mathcal{A}}$ such that $\mathbf{y}(t) = \left(\boldsymbol{\mathcal{A}}\,\mathbf{u}(\cdot)\right)_{(t)}\mathbf{u}(t)$ whenever u(t) is nonzero, is paraphrased in theorems 3-1 and 4-1 with words to the effect that the maps in question in those cases "can be written in the form" $\left(\boldsymbol{\mathcal{A}}\mathbf{u}(\cdot)\right)_{(t)}\mathbf{u}(t)$. The statements of theorems 3-1 and 4-1 take on a particularly simple form, however, because the maps in question are memoryless and therefore $\boldsymbol{\mathcal{A}}$ is required to be memoryless as well.

Example 7-2 For the voltage-controlled nonenergic capacitor of example 5-1, let $\mathcal{U}$ be the class of all C^1 voltage waveforms $\left(v^1(\cdot),v^2(\cdot)\right)^T$ which never pass through the origin of $\mathbb{R}^2$. Then the 2-port of equation (5-7) can be represented in terms of an operator $\mathcal{A}$ as follows:

$$\begin{bmatrix} i^1(t) \\ i^2(t) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\dot{v}^1(t)v^2(t) - v^1(t)\dot{v}^2(t)}{\left[\left(v^1(t)\right)^2 + \left(v^2(t)\right)^2\right]^{3/2}} \\ \dfrac{v^1(t)\dot{v}^2(t) - \dot{v}^1(t)v^2(t)}{\left[\left(v^1(t)\right)^2 + \left(v^2(t)\right)^2\right]^{3/2}} & 0 \end{bmatrix} \begin{bmatrix} v^1(t) \\ v^2(t) \end{bmatrix} \tag{7-4}$$

Theorem 7-1 can be interpreted as providing the following canonical representation for nonenergic N-ports characterized by a hybrid operator $\mathcal{H}$:

$$y(t) = \left(\mathcal{H}u(\cdot)\right)(t) = \begin{cases} \left(\mathcal{A}u(\cdot)\right)(t)\,u(t), & \text{for all } t \text{ such that } u(t) \neq 0 \\ \left(\mathcal{B}u(\cdot)\right)(t), & \text{for all } t \text{ such that } u(t) = 0 \end{cases} \tag{7-5}$$

where $\mathcal{B}$ is an arbitrary operator: $\mathcal{U} \to$ {the class of all functions: $\mathbb{R} \to \mathbb{R}^N$}. Examples 2-1, 2-2, 4-2, 4-3, 4-4, 7-2, 7-3, 7-4, and 7-5 are all given in this canonical form, but $\mathcal{B}$ is the zero operator in all but examples 7-3 and 7-4. Since any expression of the form $\left(\mathcal{A}u(\cdot)\right)(t)u(t)$ is zero whenever $u(t)$ is zero, no explicit mention of $\mathcal{B}$ was needed except in examples 7-3 and 7-4. This canonical representation is of course not the only possible representation, as comparison of equations (4-2) and (4-4), equations (4-3) and (4-5), or equations (5-7) and (7-4) will demonstrate.

Example 7-3 This one is nonlinear, time-varying, anticipative, and discontinuous.

$$
\begin{bmatrix} v^1(t) \\ v^2(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & t\left[e^{-(\cdot)^2} * i^1(\cdot)\right]_{(t)} \\ -t\left[e^{-(\cdot)^2} * i^1(\cdot)\right]_{(t)} & 0 \end{bmatrix} \begin{bmatrix} i^1(t) \\ i^2(t) \end{bmatrix}, & \text{for all } t \text{ such that } \underset{\sim}{i}(t) \neq \underset{\sim}{0} \\ \begin{bmatrix} i^1(t-1) \\ i^2(t+1) \end{bmatrix}, & \text{for all } t \text{ such that } \underset{\sim}{i}(t) = \underset{\sim}{0} \end{cases}
$$

The assumptions behind theorem 7-1 were very weak. In particular we have not required that $\mathcal{N}$ be linear, time-invariant, lumped, causal or continuous or that its inputs be continuous or even measurable functions of time.

B. Proof That Every Nonenergic Linear N-Port is Resistive

We assume throughout part B that $\mathcal{U}$, the class of admissible input waveforms: $\mathbb{R} \to \mathbb{R}^N$, is closed under addition and scalar multiplication, so that linearity will make sense. In addition $\mathcal{U}$ must satisfy any one of the following three (mutually exclusive) conditions.

Standing Assumptions Either

i) N is arbitrary, but $\mathcal{U}$ contains only the zero waveform $\underset{\sim}{0}(t)$, or

ii) $N = 1$, i.e. $\mathcal{N}$ is a 1-port, and if $u_1(\cdot)$ is any waveform in $\mathcal{U}$ and t' is is any time such that $u_1(t') = 0$, then there exists some other waveform $u_2(\cdot) \in \mathcal{U}$ such that $u_2(t') \neq 0$, or

iii) $N \geq 2$, i.e. $\mathcal{N}$ is a multiport, and in addition $\mathcal{U}$ satisfies a), b), and c) below:

a) Let $\underset{\sim}{u}_1(\cdot)$ and $\underset{\sim}{u}_2(\cdot)$ be any two waveforms in $\mathcal{U}$ and let t' be any time such that the vectors $\underset{\sim}{u}_1(t')$ and $\underset{\sim}{u}_2(t') \in \mathbb{R}^N$ are linearly dependent but neither is zero. Then there is some other waveform $\underset{\sim}{u}_3(\cdot) \in \mathcal{U}$ such that the vectors $\underset{\sim}{u}_1(t')$ and $\underset{\sim}{u}_3(t')$ are linearly independent.

b) If $\underset{\sim}{u}_1(\cdot)$ is any waveform in $\mathcal{U}$ and t' is any time such that $\underset{\sim}{u}_1(t') = 0$, then there exists some other waveform $\underset{\sim}{u}_2(\cdot) \in \mathcal{U}$ such that $\underset{\sim}{u}_2(t') \neq \underset{\sim}{0}$.

c) If $\underset{\sim}{u}(\cdot) = \left(u^1(\cdot), u^2(\cdot), \cdots u^N(\cdot)\right) \in \mathcal{U}$, then the waveforms $\left(u^1(\cdot), 0, \cdots 0\right)$, $\left(0, u^2(\cdot), 0, \cdots 0\right)$ and $\left(0, 0, \cdots u^N(\cdot)\right)$ are also in $\mathcal{U}$.

Note that the converse of condition iii-c) is guaranteed by our previous assumption that $\mathcal{U}$ is closed under addition.

The standing assumption is given in such detailed and explicit form in order to make clear exactly what is necessary to prove theorem 7-2. It is extremely weak, and every case of interest will satisfy it. For any fixed value of N, for example, the following classes of functions: $\mathbb{R} \to \mathbb{R}^N$ will all satisfy it: 1) all functions, 2) all measurable functions, 3) all piecewise continuous functions, 4) all C^k functions, $k = 0, 1, 2, \cdots$ or $k = \infty$, 5) all L^p functions, $p = 1, 2, \cdots$ or $p = \infty$, 6) the intersection or the span of any two of the above. Condition i) is included for the sole purpose of allowing the (multiport) nullator, which is linear and nonenergic, to fit our scheme.

<u>Theorem 7-2</u> Suppose that $\mathcal{N}$ is a nonenergic N-port characterized by a hybrid operator $\underset{\sim}{\mathcal{H}}$ and that $\mathcal{U}$ satisfies the standing assumption. If $\mathcal{N}$ is linear, that is, if

$$\underset{\sim}{\mathcal{H}}\left(a\underset{\sim}{u}_1(\cdot) + b\underset{\sim}{u}_2(\cdot)\right) = a\underset{\sim}{\mathcal{H}}\,\underset{\sim}{u}_1(\cdot) + b\underset{\sim}{\mathcal{H}}\,\underset{\sim}{u}_2(\cdot), \text{ for all } u_1(\cdot), u_2(\cdot) \in \mathcal{U} \text{ and all } a, b \in \mathbb{R}. \tag{7-6}$$

then $\mathcal{N}$ is resistive, i.e. memoryless.

We are now in a position to show the necessity of the provision of condition ii) of the standing assumption which also appears as part b) of condition iii). The following example shows that when this condition is violated, theorem 7-2 no longer holds.

<u>Example 7-4</u> Let $\mathcal{N}$ be a 1-port with input $i(\cdot)$, output $v(\cdot)$, and admissible input class $\mathcal{U}$ consisting of all waveforms $i(\cdot)$ that pass through the origin at $t = 1$. If $\mathcal{N}$ is characterized by

$$\underset{\sim}{\mathcal{H}}: \; i(\cdot) \mapsto v(\cdot);\; v(t) = \left(\underset{\sim}{\mathcal{H}}\, i(\cdot)\right)_{(t)} = \begin{cases} 0 & , t \neq 1 \\ i(t-1), & t = 1, \end{cases}$$

then $\mathcal{N}$ is linear and nonenergic, but not memoryless.

Proof of Theorem 7-2 $\mathcal{U}$ satisfies either condition i), condition ii), or condition iii) of the standing assumption.

Case i) $\mathcal{U}$ contains only the zero waveform, $\underset{\sim}{0}(\cdot)$.

By equation (7-5), $\underset{\sim}{\mathcal{H}}\underset{\sim}{0}(\cdot) = \underset{\sim}{\mathcal{H}}\left(\underset{\sim}{0}(\cdot) + \underset{\sim}{0}(\cdot)\right) = \underset{\sim}{\mathcal{H}}\underset{\sim}{0}(\cdot) + \underset{\sim}{\mathcal{H}}\underset{\sim}{0}(\cdot)$, so $\underset{\sim}{\mathcal{H}}\underset{\sim}{0}(\cdot) = \underset{\sim}{0}(\cdot)$ and $\underset{\sim}{\mathcal{H}}$ is the zero operator, which represents a memoryless N-port.

Case ii) $\mathcal{N}$ is a 1-port and $\mathcal{U}$ satisfies condition ii) of the standing assumption.

It follows from theorem 7-1 that $y(t) = \left(\underset{\sim}{\mathcal{H}}u(\cdot)\right)_{(t)} = \left(\underset{\sim}{\mathcal{Q}}\,u(\cdot)\right)_{(t)}u(t) = 0$ for all $(u(\cdot),t) \in \mathcal{U} \times \mathbb{R}$ such that $u(t) \neq 0$, since $\left(\underset{\sim}{\mathcal{Q}}\,u(\cdot)\right)_{(t)}$ is the only 1×1 antisymmetric matrix, namely zero. It turns out that y(t) is also zero for all values of t such u(t) = 0, and the proof is exactly parallel that to be given for the vector-valued case in equations (7-12) and (7-13), so we won't repeat it here. Therefore $\underset{\sim}{\mathcal{H}}$ is again the zero operator, which is memoryless.

Case iii) $N \geq 2$ and $\mathcal{U}$ satisfies condition iii) of the standing assumption.

We deal first with the case N = 2. In this case we can write the operator $\underset{\sim}{\mathcal{Q}}$ in terms of its component operators $\mathcal{Q}_{ij}$: $\mathcal{U} \to$ {the class of all functions: $\mathbb{R} \to \mathbb{R}$} as follows:

$$\left(\underset{\sim}{\mathcal{Q}}\,\underset{\sim}{u}(\cdot)\right)_{(t)} = \begin{bmatrix} 0 & \left(\mathcal{Q}_{12}\underset{\sim}{u}(\cdot)\right)_{(t)} \\ -\left(\mathcal{Q}_{12}\underset{\sim}{u}(\cdot)\right)_{(t)} & 0 \end{bmatrix}. \tag{7-7}$$

See example 7-1. Let $\underset{\sim}{u}_1(\cdot) = \left[u_1^1(\cdot), u_1^2(\cdot)\right]^T$ and $\underset{\sim}{u}_2(\cdot) = \left[u_2^1(\cdot), u_2^2(\cdot)\right]^T$ be any two waveforms in $\mathcal{U}$, and let t be any time such that $\underset{\sim}{u}_1(t)$ and $\underset{\sim}{u}_2(t)$ are linearly independent vectors in $\mathbb{R}^2$. Then from equations (7-5) and (7-6) we have

$$\left(\underset{\sim}{\mathcal{Q}}\,\underset{\sim}{u}_1(\cdot)\right)_{(t)}\underset{\sim}{u}_1(t) + \left(\underset{\sim}{\mathcal{Q}}\,\underset{\sim}{u}_2(\cdot)\right)_{(t)}\underset{\sim}{u}_2(t) = \left[\underset{\sim}{\mathcal{Q}}\left(\underset{\sim}{u}_1(\cdot)+\underset{\sim}{u}_2(\cdot)\right)\right]_{(t)}\left(\underset{\sim}{u}_1(t)+\underset{\sim}{u}_2(t)\right) \tag{7-8}$$

Let the matrix S be given by

$$\underset{\sim}{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Then using equation (7-7) we can rewrite equation (7-8) as

$$\underset{\sim}{S}\left(\mathcal{A}_{12}\underset{\sim}{u}_1(\cdot)\right)_{(t)}\underset{\sim}{u}_1(t) + \underset{\sim}{S}\left(\mathcal{A}_{12}\underset{\sim}{u}_2(\cdot)\right)_{(t)}\underset{\sim}{u}_2(t)$$
$$= \underset{\sim}{S}\left\{\mathcal{A}_{12}\left(\underset{\sim}{u}_1(\cdot)+\underset{\sim}{u}_2(\cdot)\right)\right\}_{(t)}\left(\underset{\sim}{u}_1(t)+\underset{\sim}{u}_2(t)\right). \qquad (7\text{-}9)$$

Since $\underset{\sim}{S}$ is nonsingular, equation (7-9) implies that

$$\left(\mathcal{A}_{12}\underset{\sim}{u}_1(\cdot)\right)_{(t)}\underset{\sim}{u}_1(t) + \left(\mathcal{A}_{12}\underset{\sim}{u}_2(\cdot)\right)_{(t)}\underset{\sim}{u}_2(t)$$
$$= \left\{\mathcal{A}_{12}\left(\underset{\sim}{u}_1(\cdot)+\underset{\sim}{u}_2(\cdot)\right)\right\}_{(t)}\left(\underset{\sim}{u}_1(t)+\underset{\sim}{u}_2(t)\right).$$

Since we have assumed that $\underset{\sim}{u}_1(t)$ and $\underset{\sim}{u}_2(t)$ are linearly independent, it follows that $\left(\mathcal{A}_{12}\underset{\sim}{u}_1(\cdot)\right)_{(t)} = \left\{\mathcal{A}_{12}\left(\underset{\sim}{u}_1(\cdot)+\underset{\sim}{u}_2(\cdot)\right)\right\}_{(t)}$ and $\left(\mathcal{A}_{12}\underset{\sim}{u}_2(\cdot)\right)_{(t)} = \left\{\mathcal{A}_{12}\left(\underset{\sim}{u}_1(\cdot)+\underset{\sim}{u}_2(\cdot)\right)\right\}_{(t)}$. Therefore

$$\left(\mathcal{A}_{12}\underset{\sim}{u}_1(\cdot)\right)_{(t)} = \left(\mathcal{A}_{12}\underset{\sim}{u}_2(\cdot)\right)_{(t)}. \qquad (7\text{-}10)$$

Now let t' be any time such that $\underset{\sim}{u}_1(t')$ and $\underset{\sim}{u}_2(t')$ are linearly dependent, but neither is zero. Then by condition iii-a) of the standing assumption there exists a waveform $\underset{\sim}{u}_3(\cdot) \in \mathcal{U}$ such that $\underset{\sim}{u}_1(t')$ and $\underset{\sim}{u}_3(t')$ are linearly independent. It follows that $\underset{\sim}{u}_2(t')$ and $\underset{\sim}{u}_3(t')$ are also linearly independent. Therefore the pairs $\left(\underset{\sim}{u}_1(\cdot),\underset{\sim}{u}_3(\cdot)\right)$ and $\left(\underset{\sim}{u}_2(\cdot),\underset{\sim}{u}_3(\cdot)\right)$ are linearly independent at $t = t'$. Substituting these pairs into equation (7-8) and repeating the reasoning leading up to equation (7-10) shows that $\left(\mathcal{A}_{12}\underset{\sim}{u}_1(\cdot)\right)_{(t')} = \left(\mathcal{A}_{12}\underset{\sim}{u}_3(\cdot)\right)_{(t')}$ and $\left(\mathcal{A}_{12}\underset{\sim}{u}_2(\cdot)\right)_{(t')} = \left(\mathcal{A}_{12}\underset{\sim}{u}_3(\cdot)\right)_{(t')}$. Therefore $\left(\mathcal{A}_{12}\underset{\sim}{u}_1(\cdot)\right)_{(t')} = \left(\mathcal{A}_{12}\underset{\sim}{u}_2(\cdot)\right)_{(t')}$ and equation (7-10) holds at $t = t'$ as well. The previous two results together show that equation (7-10) holds for every value of t such that $\underset{\sim}{u}_1(t)$ and $\underset{\sim}{u}_2(t)$ are both nonzero.

We can, if we like, think of the component operator $\mathcal{A}_{12}: \mathcal{U} \to \{$the class of all functions: $\mathbb{R} \to \mathbb{R}\}$ instead as a map: $\{\mathcal{U} \times \mathbb{R}\} \to \mathbb{R}$, i.e. $\mathcal{A}_{12}: \left(\underset{\sim}{u}(\cdot),t\right) \mapsto \mathcal{A}_{12}\left(\underset{\sim}{u}(\cdot),t\right) \in \mathbb{R}$. Then since equation (7-10) holds for arbitrary $\underset{\sim}{u}_1(\cdot)$ and $\underset{\sim}{u}_2(\cdot) \in \mathcal{U}$ and for each value of t such that $\underset{\sim}{u}_1(t)$ and $\underset{\sim}{u}_2(t)$ are both nonzero, it follows that if we fix t, then $\mathcal{A}_{12}\left(\underset{\sim}{u}(\cdot),t\right)$ is independent of its first argument so long as its first argument is not a waveform which passes through the origin at t. Referring back to theorem 7-1 and the canonical representation it prescribes, we see that we can write the operator $\underset{\sim}{\mathcal{H}}$ in the following form:

$$\left(\underset{\sim}{\mathcal{H}}\,\underset{\sim}{u}(\cdot)\right)_{(t)} = \begin{cases} \begin{bmatrix} 0 & \mathcal{A}_{12}(t) \\ -\mathcal{A}_{12}(t) & 0 \end{bmatrix} \begin{bmatrix} u^1(t) \\ u^2(t) \end{bmatrix}, & \text{for all t such that } \underset{\sim}{u}(t) \neq \underset{\sim}{0} \\ \left(\underset{\sim}{\mathcal{B}}\,\underset{\sim}{u}(\cdot)\right)_{(t)}, & \text{for all t such that } u(t) = 0, \end{cases} \tag{7-11}$$

for all $\underset{\sim}{u}(\cdot) \in \mathcal{U}$. We have no information about $\underset{\sim}{\mathcal{B}}$ as yet. The component operators of $\underset{\sim}{\mathcal{A}}$ are written as $\pm\mathcal{A}_{12}(t)$ to show that they are independent of $\underset{\sim}{u}(\cdot)$, but could conceivably vary with t.

We need to show finally that $\left(\underset{\sim}{\mathcal{B}}\,\underset{\sim}{u}(\cdot)\right)_{(t)}$ is zero whenever $\underset{\sim}{u}(t)$ is zero. We will show that this follows from equation (7-5). Let $\underset{\sim}{u}_1(\cdot)$ be any waveform in $\mathcal{U}$ that passes through the origin and let $\hat{t}$ be any time such that $\underset{\sim}{u}_1(\hat{t}) = \underset{\sim}{0}$. Then by condition iii-b) of the standing assumption there exists another waveform $\underset{\sim}{u}_2(\cdot) \in \mathcal{U}$ such that $\underset{\sim}{u}_2(\hat{t}) \neq \underset{\sim}{0}$. Then

$$\left[\underset{\sim}{\mathcal{H}}\left(\underset{\sim}{u}_1(\cdot)+\underset{\sim}{u}_2(\cdot)\right)\right]_{(\hat{t})} = \left[\underset{\sim}{\mathcal{A}}\left(\underset{\sim}{u}_1(\cdot)+\underset{\sim}{u}_2(\cdot)\right)\right]_{(\hat{t})}\left(\underset{\sim}{u}_1(\hat{t})+\underset{\sim}{u}_2(\hat{t})\right) =$$
$$\mathcal{A}_{12}(\hat{t})\begin{bmatrix} u_1^2(\hat{t})+u_2^2(\hat{t}) \\ -u_1^1(\hat{t})-u_2^1(\hat{t}) \end{bmatrix} = \mathcal{A}_{12}(\hat{t})\begin{bmatrix} u_2^2(\hat{t}) \\ -u_2^1(\hat{t}) \end{bmatrix} \tag{7-12}$$

and

$$\left[\underset{\sim}{\mathcal{H}}\left(\underset{\sim}{u}_1(\cdot)+\underset{\sim}{u}_2(\cdot)\right)\right]_{(\hat{t})} = \left[\underset{\sim}{\mathcal{H}}\,\underset{\sim}{u}_1(\cdot)\right]_{(\hat{t})} + \left[\underset{\sim}{\mathcal{H}}\,\underset{\sim}{u}_2(\cdot)\right]_{(\hat{t})} =$$
$$\left(\underset{\sim}{\mathcal{B}}\,\underset{\sim}{u}_1(\cdot)\right)_{(\hat{t})} + \mathcal{A}_{12}(\hat{t})\begin{bmatrix} u_2^2(\hat{t}) \\ -u_2^1(\hat{t}) \end{bmatrix}. \tag{7-13}$$

Comparing the last term in equation (7-12) with the last term in equation (7-13) shows that $\left(\mathcal{B}\,\underset{\sim}{u}_1(\cdot)\right)_{(\hat{t})} = \underset{\sim}{0}$. Thus we have shown that $\left(\underset{\sim}{\mathcal{H}}\,\underset{\sim}{u}(\cdot)\right)_{(\hat{t})}$ is zero for any choice of $(\underset{\sim}{u}(\cdot),t) \in \mathcal{U} \times \mathbb{R}$ such that $\underset{\sim}{u}(t) = \underset{\sim}{0}$. Since any expression of the form $\left(\underset{\sim}{\mathcal{A}}\,\underset{\sim}{u}(\cdot)\right)_{(t)}\underset{\sim}{u}(t)$ is zero whenever $\underset{\sim}{u}(t)$ is zero, equation (7-11) simplifies to

$$\underset{\sim}{y}(t) = \left(\underset{\sim}{\mathcal{H}}\,\underset{\sim}{u}(\cdot)\right)_{(t)} = \begin{bmatrix} 0 & \mathcal{A}_{12}(t) \\ -\mathcal{A}_{12}(t) & 0 \end{bmatrix} \begin{bmatrix} u^1(t) \\ u^2(t) \end{bmatrix}, \text{ for all } \underset{\sim}{u}(\cdot) \in \mathcal{U} \text{ and for all } t. \qquad (7\text{-}14)$$

But equation (7-14) represents a memoryless system. This proves the theorem when N=2.

We can reduce the case $N \geq 3$ to the case N=2. Linearity of $\mathcal{N}$ and condition iii-c) of the standing assumption allow us to break up $\underset{\sim}{\mathcal{H}}$ into its component operators $\mathcal{H}_{jk}$, representing signal transfer from port k to port j. And we can then represent the response $\underset{\sim}{y}(\cdot)$ to any input $\underset{\sim}{u}(\cdot) \in \mathcal{U}$ in terms of the responses of the operators $\mathcal{H}_{jk}$ to the component waveforms $u^k(\cdot)$ of $\underset{\sim}{u}(\cdot)$.

Let $j \in \{1,2,\cdots N\}$ be arbitrary and let $\underset{\sim}{u}(\cdot)$ be any waveform in $\mathcal{U}$. The decomposition goes as follows:

$$\begin{aligned} y^j(\cdot) &= \left\{\underset{\sim}{\mathcal{H}}\,\underset{\sim}{u}(\cdot)\right\}^j = \left\{\underset{\sim}{\mathcal{H}}\left[u^1(\cdot),u^2(\cdot),\cdots,u^N(\cdot)\right]\right\} = \\ &\left\{\underset{\sim}{\mathcal{H}}\left[(u^1(\cdot),0,\cdots,0) + (0,u^2(\cdot),\cdots,0) + \cdots + (0,0,\cdots,u^N(\cdot))\right]\right\}^j = \\ &\left\{\underset{\sim}{\mathcal{H}}\,(u^1(\cdot),0,\cdots,0)\right\}^j + \left\{\underset{\sim}{\mathcal{H}}\,(0,u^2(\cdot),\cdots,0)\right\}^j + \cdots + \left\{\underset{\sim}{\mathcal{H}}\,(0,0,\cdots,u^N(\cdot))\right\}^j = \\ &\mathcal{H}_{j1}u^1(\cdot) + \mathcal{H}_{j2}u^2(\cdot) + \cdots + \mathcal{H}_{jN}u^N(\cdot). \end{aligned} \qquad (7\text{-}15)$$

Equation (7-15) can be viewed as a rigorous definition of the component operators $\mathcal{H}_{jk}$. We next show that the theorem holds for each component operator separately. Let $\mathcal{H}_{jk}$ be any component operator. If j=k then we investigate $\mathcal{H}_{jj}$ by driving port j alone, i.e. by considering only inputs of the form $\underset{\sim}{u}(\cdot) = \left(0,\cdots,u^j(\cdot),\cdots,0\right)$. Then we have effectively a 1-port, and we have already shown that the hybrid operator of a nonenergic linear 1-port must be the zero operator. If $j \neq k$ then we investigate $\mathcal{H}_{jk}$ by driving only ports j and k, i.e. by considering only inputs of the form $\underset{\sim}{u}(\cdot) = (0,\cdots,u^j(\cdot),\cdots,u^k(\cdot),\cdots 0)$. The proof given for the case N=2 shows that

$$\left\{ \begin{bmatrix} \mathcal{H}_{jj} & \mathcal{H}_{jk} \\ \mathcal{H}_{kj} & \mathcal{H}_{kk} \end{bmatrix} \circ \begin{bmatrix} u^j(\cdot) \\ u^k(\cdot) \end{bmatrix} \right\}_{(t)} = \begin{bmatrix} 0 & \mathcal{A}_{jk}(t) \\ -\mathcal{A}_{jk}(t) & 0 \end{bmatrix} \begin{bmatrix} u^j(t) \\ u^k(t) \end{bmatrix}$$

for all $\underset{\sim}{u}(\cdot) = (0,\cdots,u^j(\cdot),\cdots,u^k(\cdot),\cdots 0) \in \mathcal{U}$ and for all t. Therefore the operator $\mathcal{H}_{jk}$ does nothing more than multiply $u^k(t)$ by $\mathcal{A}_{jk}(t)$, a memoryless operation. Since j, k and $N \geq 3$ were arbitrary, and since the input-output behavior of $\mathcal{N}$ is completely characterized by the set of operators $\{\mathcal{H}_{jk}\}$ as shown in equation (7-15), this completes the proof. ■

Perhaps the proof of theorem 7-2 is so long because the assumptions were so weak. $\mathcal{N}$ was not required to be time-invariant, lumped or causal and its inputs were not required to be continuous or even measurable functions of time.

Statement iii) of theorem 2-1 claims that a nonenergic linear N-port is not only resistive, but antireciprocal as well. The following example shows that the second part of this claim is true only in the time-invariant case.

Example 7-5 Let $\mathcal{N}$ be the linear, time-varying nonenergic 2-port for which equation (7-14) takes on the following form:

$$\begin{bmatrix} v^1(t) \\ v^2(t) \end{bmatrix} = \begin{bmatrix} 0 & \sin t \\ -\sin t & 0 \end{bmatrix} \begin{bmatrix} i^1(t) \\ i^2(t) \end{bmatrix} .$$

Let $\underset{\sim}{i}'(\cdot)$ and $\underset{\sim}{i}''(\cdot)$ be two input waveforms given by

$$\underset{\sim}{i}'(t) = \begin{bmatrix} e^{-t}u(t) \\ 0 \end{bmatrix}, \quad \underset{\sim}{i}''(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{for all } t,$$

where u(t) equals 0 for $t \leq 0$ and 1 otherwise. Then the corresponding output waveforms $\underset{\sim}{v}'(\cdot)$ and $\underset{\sim}{v}''(\cdot)$ are given by

$$\underset{\sim}{v}'(t) = \begin{bmatrix} 0 \\ -e^{-t} \sin t \; u(t) \end{bmatrix}, \quad \underset{\sim}{v}''(t) = \begin{bmatrix} \sin t \\ 0 \end{bmatrix}, \quad \text{for all } t.$$

Then

$$[\underset{\sim}{v}'(\cdot)*\underset{\sim}{i}''(\cdot)]_{(t)} = \int_{-\infty}^{\infty} \left\langle \underset{\sim}{v}'(\tau), \underset{\sim}{i}''(t-\tau) \right\rangle d\tau = -\int_{-\infty}^{\infty} e^{-\tau} \sin\tau \, u(\tau) d\tau =$$

$$-\int_0^{\infty} e^{-\tau} \sin\tau d\tau = \frac{1}{2} e^{-\tau} (\sin\tau + \cos\tau)\Big|_0^{\infty} = -\frac{1}{2}, \text{ for all } t,$$

and

$$[\underset{\sim}{v}''(\cdot)*\underset{\sim}{i}'(\cdot)]_{(t)} = \int_{-\infty}^{\infty} \left\langle \underset{\sim}{v}''(\tau), \underset{\sim}{i}'(t-\tau) \right\rangle d\tau = \int_{-\infty}^{\infty} \sin\tau \, e^{-(t-\tau)} u(t-\tau) d\tau =$$

$$e^{-t} \int_{-\infty}^{t} \sin\tau \, e^{\tau} \, d\tau = \frac{1}{2} e^{-t} e^{\tau} (\sin\tau - \cos\tau)\Big|_{\tau=-\infty}^{\tau=t} = \frac{1}{2} (\sin t - \cos t), \text{ for all } t.$$

Comparison with definition 2-2 shows that $\mathcal{N}$ is not antireciprocal.

C. Properties of Resistive, Capacitive and Inductive N-Ports as N-Dimensional Manifolds in $\mathbb{R}^{2N}$.

Definition 7-3 (Adapted from [Spivak], p. 111.) A subset M of $\mathbb{R}^n$ is an ℓ-dimensional C^K manifold in $\mathbb{R}^n$ iff for each point $\underset{\sim}{p} \in M$ there exist an open set U in $\mathbb{R}^n$ with $\underset{\sim}{p} \in U$, an open set $V \subset \mathbb{R}^{\ell}$, and a 1-1 C^K function $\underset{\sim}{f} : V \to \mathbb{R}^n$ such that :

1) $\underset{\sim}{f}(V) = M \cap U$,

2) $(\underset{\sim}{D}\underset{\sim}{f})_{(\underset{\sim}{x})}$ has rank ℓ for each $\underset{\sim}{x} \in V$,

3) $\underset{\sim}{f}^{-1} : \underset{\sim}{f}(V) \to V$ is continuous.

Such a function $\underset{\sim}{f}$ is called a coordinate system around $\underset{\sim}{p}$.

Assumptions and Definitions. In part C we will deal explicitly with resistive and capacitive N-ports only. Properties of inductive N-ports can be obtained from those of capacitive N-ports by substituting $\underset{\sim}{\phi}$ for $\underset{\sim}{q}$ and $\underset{\sim}{i}$ for $\underset{\sim}{v}$. We call the set of d.c. admissible pairs, $(\underset{\sim}{v},\underset{\sim}{i})$ or $(\underset{\sim}{v},\underset{\sim}{q})$, of $\mathcal{N}$ the graph of $\mathcal{N}$. We assume throughout that the graph of $\mathcal{N}$ does form a manifold in $\mathbb{R}^{2N}$. This is not very restrictive. It amounts to assuming that the graph of $\mathcal{N}$ is smooth enough, has no self-intersections (like a figure 8), and never folds back infinitely close to itself (like a figure 6 made by bending an open interval back on itself). We say that an N-port

$\mathcal{N}$ is regular [Chua and Lam] if its graph is an N-dimensional manifold in $\mathbb{R}^{2N}$, i.e. if $\mathcal{N}$ has "N degrees of freedom," and that $\mathcal{N}$ is C^K if its graph is a C^K manifold in $\mathbb{R}^{2N}$.

In order to model the graph of $\mathcal{N}$ as a manifold in $\mathbb{R}^{2N}$ we will have to keep track of the types of the 2N quantities involved (voltages, charges, etc.) and of the port number with which each quantity is associated. For a resistive N-port we will label the coordinates of $\mathbb{R}^{2N}$ as $[v_1,\ldots,v_N,i_1,\ldots i_N]^T$ and for a capacitive N-port the coordinates will be ordered as $[v_1,\ldots,v_N,q_1,\ldots,q_N]^T$. This ordering then provides a natural way of breaking up any coordinate system $\underset{\sim}{f} : U \to \mathbb{R}^{2N}$ into two functions $\underset{\sim}{f}_1$ and $\underset{\sim}{f}_2 : U \to \mathbb{R}^N$. In the resistive case, $\underset{\sim}{f}_1 : \underset{\sim}{x} \mapsto \underset{\sim}{v}$ and $\underset{\sim}{f}_2 : \underset{\sim}{x} \mapsto \underset{\sim}{i}$. In the capacitive case, $\underset{\sim}{f}_1 : \underset{\sim}{x} \mapsto \underset{\sim}{v}$ and $\underset{\sim}{f}_2 : \underset{\sim}{x} \mapsto \underset{\sim}{q}$.

<u>Lemma 7-2</u>. Suppose $\mathcal{N}$ is a regular C^1 resistive N-port. Then $\mathcal{N}$ is nonenergic $\Leftrightarrow$ for every local coordinate system $\underset{\sim}{f} = (\underset{\sim}{f}_1,\underset{\sim}{f}_2)$, $\langle \underset{\sim}{f}_1(\underset{\sim}{x}),\underset{\sim}{f}_2(\underset{\sim}{x})\rangle = 0$ at each point $\underset{\sim}{x}$ in the domain of $\underset{\sim}{f}$. And $\mathcal{N}$ is reciprocal $\Leftrightarrow$ for every local coordinate system $\underset{\sim}{f} = (\underset{\sim}{f}_1,\underset{\sim}{f}_2)$, $\left[\underset{\sim}{J}\left(\underset{\sim}{f}_2(\underset{\sim}{x})\right)\right]^T \left[\underset{\sim}{J}\left(\underset{\sim}{f}_1(\underset{\sim}{x})\right)\right]$ is symmetric at each point $\underset{\sim}{x}$ in the domain of $\underset{\sim}{f}$.

The first proof is immediate; the second is a direct application of equation (4-8). In the current-controlled case (that is, when there exists an $\underset{\sim}{f} = (\underset{\sim}{f}_1,\underset{\sim}{f}_2)$ such that $\underset{\sim}{f}_2$ is the identity and the range of $\underset{\sim}{f}$ is the graph of $\mathcal{N}$), then the reciprocity condition above reduces to the familiar requirement that the incremental resistance matrix $\underset{\sim}{R}(\underset{\sim}{i})$ be symmetric at each point $\underset{\sim}{i}$.

<u>Lemma 7-3</u>. Suppose $\mathcal{N}$ is a regular C^1 capacitive N-port. Then $\mathcal{N}$ is nonenergic $\Leftrightarrow$ for every local coordinate system $\underset{\sim}{f}$, $\left[\underset{\sim}{J}\left(\underset{\sim}{f}_2(\underset{\sim}{x})\right)\right]^T \underset{\sim}{f}_1(\underset{\sim}{x}) = \underset{\sim}{0}$ at each point $\underset{\sim}{x}$ in the domain of $\underset{\sim}{f}$. And $\mathcal{N}$ is reciprocal $\Leftrightarrow$ for every local coordinate system $\underset{\sim}{f}$, the matrix $\left[\underset{\sim}{J}\left(\underset{\sim}{f}_2(\underset{\sim}{x})\right)\right]^T \left[\underset{\sim}{J}\left(\underset{\sim}{f}_1(\underset{\sim}{x})\right)\right]$ is symmetric at each point $\underset{\sim}{x}$ in the domain of $\underset{\sim}{f}$.

The first proof is immediate; the second is a direct application of equation (5-1).

<u>Theorem 7-3</u>. Suppose $\mathcal{N}$ is a regular C^2 capacitive N-port. If $\mathcal{N}$ is nonenergic, then $\mathcal{N}$ is reciprocal.

Proof Let $\underset{\sim}{p}$ be any point in the graph of $\mathcal{N}$ and let $\underset{\sim}{f} = (\underset{\sim}{f}_1, \underset{\sim}{f}_2)$ be any coordinate system around $\underset{\sim}{p}$. Since $\mathcal{N}$ is nonenergic, $\left[\underset{\sim}{J}\left(\underset{\sim}{f}_2(\underset{\sim}{x})\right)\right]^T \underset{\sim}{f}_1(\underset{\sim}{x}) = \underset{\sim}{0}$ at each $\underset{\sim}{x}$ in the domain of $\underset{\sim}{f}$. Taking the Jacobian of both sides of this equation yields

$$\left[\underset{\sim}{J}\left(\underset{\sim}{f}_2(\underset{\sim}{x})\right)\right]^T \left[\underset{\sim}{J}\left(\underset{\sim}{f}_1(\underset{\sim}{x})\right)\right] + \sum_{k=1}^{N} f_1^k(\underset{\sim}{x}) \left[\frac{\partial^2 f_2^k}{\partial x_i \partial x_j}(\underset{\sim}{x})\right] = \underset{\sim}{0} \in \mathbb{R}^{N\times N}, \qquad (7\text{-}16)$$

where f_1^k denotes the k-th component function of $\underset{\sim}{f}_1$ and f_2^k denotes the k-th component function of $\underset{\sim}{f}_2$. Since at each point $\underset{\sim}{x}$ in the domain of $\underset{\sim}{f}$ the second term on the left hand side of equation (7-16) is a symmetric N×N matrix, it follows that the first term on the left hand side must be symmetric as well. Then by Lemma 7-3, $\mathcal{N}$ is reciprocal.

■

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A. Renumbering the Ports of $\mathcal{N}$ Preserves the Antisymmetry of $\mathbf{H}$

Lemma A-1 Let $\mathcal{N}$ be a linear, time-invariant resistive N-port characterized by $\mathbf{y} = \mathbf{H}\,\mathbf{u}$, where $\mathbf{H} \in \mathbb{R}^{N\times N}$ is antisymmetric and $\mathbf{u}$ and $\mathbf{y}$ form a hybrid pair (definition 2-1). Suppose $u_j = i_j$ for some $j \in \{1,2,\cdots N\}$, i.e. $\mathbf{u} \neq \mathbf{v}$. Then there exists a renumbering of the ports of $\mathcal{N}$ such that, in the new numbering system, $\hat{\mathbf{u}} = \{i_1,\cdots i_k, v_{k+1},\cdots v_N\}$, $\mathbf{y} = \{v_1,\cdots v_k, i_{k+1},\cdots i_N\}$, $1 \leq k \leq N$, and $\hat{\mathbf{y}} = \hat{\mathbf{H}}\,\hat{\mathbf{u}}$, where $\hat{\mathbf{H}}$ is antisymmetric.

Proof Order the entries of $\mathbf{u}$ which are currents as follows: $u_{j_1} = i_{j_1}$, $u_{j_2} = i_{j_2}$, $\cdots$, $u_{j_k} = i_{j_k}$; $j_1 < j_2 < \cdots < j_k$, $1 \leq k \leq N$. Order the remaining entries of $\mathbf{u}$, which are voltages, as follows: $u_{j_{k+1}} = v_{j_{k+1}}$, $u_{j_{k+2}} = v_{j_{k+2}}$, $\cdots$ $u_{j_N} = v_{j_N}$; $j_{k+1} < j_{k+2} < \cdots < j_N$. Now renumber the ports of $\mathcal{N}$ so that port # j_m becomes port # m, for all $m \in \{1,2,\cdots N\}$. Since $\hat{u}_m = u_{j_m}$ and $\hat{y}_m = y_{j_m}$, it follows that $\mathbf{Q}$, the matrix for change of coordinates, given by $\hat{u}_m = \sum_{\ell=1}^{N} q_{m\ell} u_\ell$ and $\hat{y}_m = \sum_{\ell=1}^{N} q_{m\ell} y_\ell$ has its m-th row given by $\boldsymbol{\varepsilon}_{j_m}^T$. Since $\langle \boldsymbol{\varepsilon}_{j_\ell}, \boldsymbol{\varepsilon}_{j_m} \rangle^T = \delta_{\ell m}$, it follows that $\mathbf{Q}$ is orthogonal. Therefore $\hat{\mathbf{H}} = \mathbf{QHQ}^{-1} = \mathbf{QHQ}^T$, and $\hat{\mathbf{H}}^T = (\mathbf{QHQ}^T)^T = \mathbf{QH}^T\mathbf{Q}^T = \mathbf{Q}(-\mathbf{H})\mathbf{Q}^T = -\hat{\mathbf{H}}$, so $\hat{\mathbf{H}}$ is antisymmetric.

$\blacksquare$

Example. Let $\mathbf{u} = (v_1, i_2, v_3, i_4)^T$, $\mathbf{y} = (i_1, v_2, i_3, v_4)^T$, and

$$\mathbf{H} = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$

Then $j_1 = 2$, $j_2 = 4$, $j_3 = 1$, $j_4 = 3$,

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\hat{\mathbf{H}} = \mathbf{QHQ}^T = \begin{bmatrix} 0 & e & -a & d \\ -e & 0 & -c & -f \\ a & c & 0 & b \\ -d & f & -b & 0 \end{bmatrix}.$$

B. Reciprocity. Proofs of Theorems 4-3, 4-4, 4-6, 5-1, 5-2, and 5-5

Proof of Theorem 4-3 (⇒) Let $(\underset{\sim}{v},\underset{\sim}{i})$ be any operating point of the resistive N-port $\mathcal{N}$, and let $(\delta\underset{\sim}{v}',\delta\underset{\sim}{i}')$ and $(\delta\underset{\sim}{v}'',\delta\underset{\sim}{i}'')$ be d.c. small signal variations admissible to the linear approximation to $\mathcal{N}$ about $(\underset{\sim}{v},\underset{\sim}{i})$. Let $(\delta\underset{\sim}{v}'(t),\delta\underset{\sim}{i}'(t))$ equal $(\delta\underset{\sim}{v}',\delta\underset{\sim}{i}')$ for $0 \le t \le 1$ and be zero otherwise. Similarly, let $(\delta\underset{\sim}{v}''(t),\delta\underset{\sim}{i}''(t))$ equal $(\delta\underset{\sim}{v}'',\delta\underset{\sim}{i}'')$ for $0 \le t \le 1$ and be zero otherwise. Since $\mathcal{N}$ is reciprocal (by definition 4-2),

$$\langle \delta\underset{\sim}{v}',\delta\underset{\sim}{i}'' \rangle = \int_{-\infty}^{\infty} \Big\langle \delta\underset{\sim}{v}'(\tau),\delta\underset{\sim}{i}''(1-\tau) \Big\rangle d\tau = \Big[\delta\underset{\sim}{v}'(\cdot)*\delta\underset{\sim}{i}''(\cdot)\Big]_{(t=1)} =$$
$$\Big[\delta\underset{\sim}{v}''(\cdot)*\delta\underset{\sim}{i}'(\cdot)\Big]_{(t=1)} = \langle \delta\underset{\sim}{v}'',\delta\underset{\sim}{i}' \rangle . \qquad \text{(B-1)}$$

(⇐) Let $(\underset{\sim}{v},\underset{\sim}{i})$ be any operating point. Let $\Big(\delta\underset{\sim}{v}'(\cdot),\delta\underset{\sim}{i}'(\cdot)\Big)$ and $\Big(\delta\underset{\sim}{v}''(\cdot),\delta\underset{\sim}{i}''(\cdot)\Big)$ be any two almost everywhere continuous small signal variations about $(\underset{\sim}{v},\underset{\sim}{i})$ such that the convolutions in equation (4-6) are finite for all values of t. Then using equation (4-8) and the definition of the Riemann integral we have

$$\Big[\delta\underset{\sim}{v}'(\cdot)*\delta\underset{\sim}{i}''(\cdot)\Big]_{(t)} = \int_{-\infty}^{\infty} \Big\langle \delta\underset{\sim}{v}'(\tau),\delta\underset{\sim}{i}''(t-\tau)\Big\rangle d\tau = \lim_{L\to\infty} \int_{-L}^{L} \Big\langle \delta\underset{\sim}{v}'(\tau),\delta\underset{\sim}{i}''(t-\tau) \Big\rangle d\tau =$$
$$\lim_{L\to\infty}\lim_{n\to\infty} \frac{2L}{n}\sum_{k=1}^{n} \Big\langle \delta\underset{\sim}{v}'\Big(-L + \frac{2L(k-1/2)}{n}\Big),\ \delta\underset{\sim}{i}''\Big(t + L - \frac{2L(k-1/2)}{n}\Big) \Big\rangle =$$
$$\lim_{L\to\infty}\lim_{n\to\infty} \frac{2L}{n}\sum_{k=1}^{n}\Big\langle \delta\underset{\sim}{v}''\Big(t + L - \frac{2L(k-1/2)}{n}\Big),\ \delta\underset{\sim}{i}'\Big(-L + \frac{2L(k-1/2)}{n}\Big)\Big\rangle =$$
$$\int_{-\infty}^{\infty} \Big\langle \delta\underset{\sim}{v}''(t-\tau),\delta\underset{\sim}{i}'(\tau) \Big\rangle d\tau = \Big[\partial\underset{\sim}{v}''(\cdot)*\delta\underset{\sim}{i}'(\cdot)\Big]_{(t)}. \qquad \text{(B-2)}$$

The proof for antireciprocity is similar.

■

Proof of Theorem 4-4

i) ⇒ ii) $$\frac{\partial v_j}{\partial i_k} = \frac{\partial^2 \Phi}{\partial i_k \partial i_j} = \frac{\partial^2 \Phi}{\partial i_j \partial i_k} = \frac{\partial v_k}{\partial i_j}$$

ii) ⇒ iii) At any operating point $\underset{\sim}{i}$ we have $\delta\underset{\sim}{v}' = \Big[\underset{\sim}{J}\Big(\underset{\sim}{f}(\underset{\sim}{i})\Big)\Big]\delta\underset{\sim}{i}'$ and $\delta\underset{\sim}{v}'' = \Big[\underset{\sim}{J}\Big(\underset{\sim}{f}(\underset{\sim}{i})\Big)\Big]\delta\underset{\sim}{i}''$. Then $\langle \delta\underset{\sim}{v}',\delta\underset{\sim}{i}'' \rangle = \delta\underset{\sim}{i}'^{T}\Big[\underset{\sim}{J}\Big(\underset{\sim}{f}(\underset{\sim}{i})\Big)\Big]^{T}\delta\underset{\sim}{i}'' = \delta\underset{\sim}{i}''^{T}\Big[\underset{\sim}{J}\Big(\underset{\sim}{f}(\underset{\sim}{i})\Big)\Big]^{T}\delta\underset{\sim}{i}' = \langle \delta\underset{\sim}{v}'',\delta\underset{\sim}{i}' \rangle$, which proves $\mathcal{N}$ is reciprocal as in theorem 4-3.

iii) ⇒ ii) $\langle \delta v', \delta i'' \rangle - \langle \delta v'', \delta i' \rangle = 0 = \delta i'^T \left\{ \left[J\left(f(i)\right)\right]^T - \left[J\left(f(i)\right)\right]\right\} \delta i''$. Since $\delta i'$ and $\delta i''$ are arbitrary, the term in brackets must be zero.

ii) ⇒ i) if U is simply connected. This is a standard theorem of analysis [Protter and Morrey].

Proof of Theorem 4-6

i) ⇒ ii) Adopting the notation $g(u^*) = f(u(u^*)) = \nabla\Phi(u^*)$, we have by the chain rule

$$J\left(g(u^*)\right) = \left[J\left(f(u(u^*))\right)\right] \cdot \left[J\left(u(u^*)\right)\right], \tag{B-3}$$

or

$$J\left(f(u(u^*))\right) = \left[J\left(g(u^*)\right)\right] \cdot \left[J\left(u(u^*)\right)\right]^{-1}. \tag{B-4}$$

Write the Jacobian of g(i.e. the Hessian of Φ) at any point u^* as

$$J\left(g(u^*)\right) = \left[\begin{array}{c|c} -A & B \\ \hline B^T & C \end{array}\right], \tag{B-5}$$

where $-A \in \mathbb{R}^{k\times k}$ and $C \in \mathbb{R}^{(N-k)\times(N-k)}$ are symmetric since the Jacobian of g is. The Jacobian of $u(u^*)$ (defined in part E of section IV) at any point u^* is given by

$$J\left(u(u^*)\right) = \left[\begin{array}{c|c} -1^{(k)} & 0 \\ \hline 0 & 1^{(N-k)} \end{array}\right]. \tag{B-6}$$

Combining equations (B-4) through (B-6) yields the result.

ii) ⇒ i), if U is simply connected. It follows from equations (B-4) through (B-6) that the Jacobian of g is symmetric. The remainder is a standard theorem of analysis [Protter and Morrey].

iii) ⇒ ii) About any operating point u we have $\delta y = \left[J\left(f(u)\right)\right] \delta u$, or

$$\begin{bmatrix} \delta v_I \\ \hline \delta i_{II} \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \begin{bmatrix} \delta i_I \\ \hline \delta v_{II} \end{bmatrix}$$

where $A \in \mathbb{R}^{k\times k}$, etc.

$$\langle \delta\mathbf{v}', \delta\mathbf{i}'' \rangle = \langle \delta\mathbf{v}_I', \delta\mathbf{i}_I'' \rangle + \langle \delta\mathbf{v}_{II}', \delta\mathbf{i}_{II}'' \rangle =$$
$$(\delta\mathbf{i}_I')^T \mathbf{A}^T (\delta\mathbf{i}_I'') + (\delta\mathbf{v}_{II}')^T \mathbf{B}^T (\delta\mathbf{i}_I'') + (\delta\mathbf{v}_{II}')^T \mathbf{C}(\delta\mathbf{i}_I'') + (\delta\mathbf{v}_{II}')^T \mathbf{D}(\delta\mathbf{v}_{II}'') \tag{B-7}$$

$$\langle \delta\mathbf{v}'', \delta\mathbf{i}' \rangle = \langle \delta\mathbf{v}_I'', \delta\mathbf{i}_I' \rangle + \langle \delta\mathbf{v}_{II}'', \delta\mathbf{i}_{II}' \rangle =$$
$$(\delta\mathbf{i}_I'')^T \mathbf{A}^T (\delta\mathbf{i}_I') + (\delta\mathbf{v}_{II}'')^T \mathbf{B}^T (\delta\mathbf{i}_I') + (\delta\mathbf{v}_{II}'')^T \mathbf{C}(\delta\mathbf{i}_I') + (\delta\mathbf{v}_{II}'')^T \mathbf{D}(\delta\mathbf{v}_{II}') \tag{B-8}$$

Since $\delta\mathbf{i}_I'$, $\delta\mathbf{i}_I''$, $\delta\mathbf{v}_{II}'$, and $\delta\mathbf{v}_{II}''$ are independent, it follows from equation (4-6) and a term by term comparison of equations (B-7) and (B-8) that

$$\mathbf{A} = \mathbf{A}^T, \quad \mathbf{C} = -\mathbf{B}^T, \quad \mathbf{D} = \mathbf{D}^T. \tag{B-9}$$

ii) $\Rightarrow$ iii) This follows immediately upon substituting equation (B-9) into equations (B-7) and (B-8).

$\blacksquare$

<u>Proof of Theorem 5-1</u> ($\Rightarrow$) In definition 4-2, let $(\mathbf{v},\mathbf{q})$ be an operating point and let $(\delta\mathbf{v}'(\cdot),\delta\mathbf{i}'(\cdot))$ and $(\delta\mathbf{v}''(\cdot),\delta\mathbf{i}''(\cdot)$ be small signal variations about the operating point such that $\|\delta\mathbf{v}'(t)\|$, $\|\delta\mathbf{v}''(t)\|$, $\|\delta\mathbf{q}'(t)\| = \|\int_{-\infty}^{t} \delta\mathbf{i}'(\tau)d\tau\|$, and $\|\delta\mathbf{q}''(t)\| = \|\int_{-\infty}^{t} \delta\mathbf{i}''(\tau)d\tau\|$ are small for all t. For any two such small-signal variations, reciprocity implies that $\int_{-\infty}^{\infty} \left\langle \delta\mathbf{v}'(\tau), \delta\mathbf{i}''(t-\tau) \right\rangle d\tau =$ $\int_{-\infty}^{\infty} \left\langle \delta\mathbf{v}''(\tau), \delta\mathbf{i}'(t-\tau) \right\rangle d\tau$. Therefore

$$\int_{-\infty}^{\infty} \left\langle \delta\mathbf{v}'(\tau), \delta\mathbf{q}''(t-\tau) \right\rangle d\tau = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \left\langle \delta\mathbf{v}'(\tau), \delta\mathbf{i}''(t'-\tau) \right\rangle d\tau dt' =$$
$$\int_{-\infty}^{t} \int_{-\infty}^{\infty} \left\langle \delta\mathbf{v}''(\tau), \delta\mathbf{i}'(t'-\tau) \right\rangle d\tau dt' = \int_{-\infty}^{\infty} \left\langle \delta\mathbf{v}''(\tau), \delta\mathbf{q}'(t-\tau) \right\rangle d\tau, \tag{B-10}$$

for any choice of t.

Now let $(\delta\mathbf{v}',\delta\mathbf{q}')$ and $(\delta\mathbf{v}'',\delta\mathbf{q}'')$ be any two d.c. small-signal variations about the operating point. Let the waveform $\left(\delta\mathbf{v}'(t),\delta\mathbf{q}'(t)\right)$ equal $(\delta\mathbf{v}',\delta\mathbf{q}')$ for $0 \leq t \leq 1$ and be zero otherwise. Similarly, let $\left(\delta\mathbf{v}''(t),\delta\mathbf{q}''(t)\right)$ equal $(\delta\mathbf{v}'',\delta\mathbf{q}'')$ for $0 \leq t \leq 1$ and be zero otherwise. Since these waveforms are discontinuous and we shall require C^1 waveforms, let $\left\{\left(\delta\mathbf{v}_k'(t),\delta\mathbf{q}_k'(t)\right)\right\}$ be a sequence of C^1 waveforms converging in the L^1 sense to $\left(\delta\mathbf{v}'(t),\delta\mathbf{i}'(t)\right)$ and let $\left\{\left(\delta\mathbf{v}_k''(t),\delta\mathbf{q}_k''(t)\right)\right\}$ be a sequence of C^1 waveforms converging in the L^1 sense to $\left(\delta\mathbf{v}''(t),\delta\mathbf{i}''(t)\right)$.

Then $\langle \delta v', \delta q'' \rangle = \int_{-\infty}^{\infty} \left\langle \delta v'(\tau), \delta q''(1-\tau) \right\rangle d\tau = \lim_{k\to\infty} \int_{-\infty}^{\infty} \left\langle \delta v_k'(\tau), \delta q_k''(1-\tau) \right\rangle d\tau =$ $\lim_{k\to\infty} \int_{-\infty}^{\infty} \left\langle \delta v_k''(\tau), \delta q_k'(1-\tau) \right\rangle d\tau = \int_{-\infty}^{\infty} \left\langle \delta v''(\tau), \delta q'(1-\tau) \right\rangle d\tau = \langle \delta v'', \delta q' \rangle$, where the third equality follows from equation (B-10).

($\Leftarrow$) Equation (5-1) implies that

$$\int_{-\infty}^{\infty} \left\langle \delta v'(\tau), \delta q''(t-\tau) \right\rangle d\tau = \int_{-\infty}^{\infty} \left\langle \delta v''(\tau), \delta q'(t-\tau) \right\rangle d\tau, \tag{B-11}$$

as can be shown by writing each integral as the limit of a sum as in equation (B-2). In order that definition 4-2 make sense, we require that $\delta q'(\cdot)$ and $\delta q''(\cdot)$ be differentiable. The proof then follows by differentiating equation (B-11) with respect to t. ∎

Theorems 5-2 and 5-5 The proof of theorem 4-4, though given for the current-controlled case, remain practically the same if the roles of v and i are interchanged. After this interchange a further substitution of q for i, justified by theorem 5-1, provides a proof of theorem 5-2. Effecting this same substitution of q for i in the proof of theorem 4-6 provides a proof of theorem 5-5.

C. Proof of Lemma 7-1

If n = 1, then y = 0, Λ = 0, and we are done.

If $y = 0$, let Λ be the zero matrix and we are done. If $y \neq 0$, then x and y are nonzero orthogonal vectors and therefore linearly independent. Since any orthonormal set of vectors can be extended to an orthonormal basis, we can form an ordered orthonormal basis for $\mathbb{R}^n$ of the following sort $\{x/\|x\|, y/\|y\|, \alpha_3, \cdots, \alpha_n\}$. With respect to this basis we have that $x = (\|x\|, 0, \cdots 0)^T$ and $y = (0, \|y\|, 0, \cdots 0)^T$. And in this new basis the following antisymmetric matrix carries x into y:

$$\hat{\Lambda} = \left[\begin{array}{cc|c} 0 & -\|y\|/\|x\| & 0 \\ \|y\|/\|x\| & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right].$$

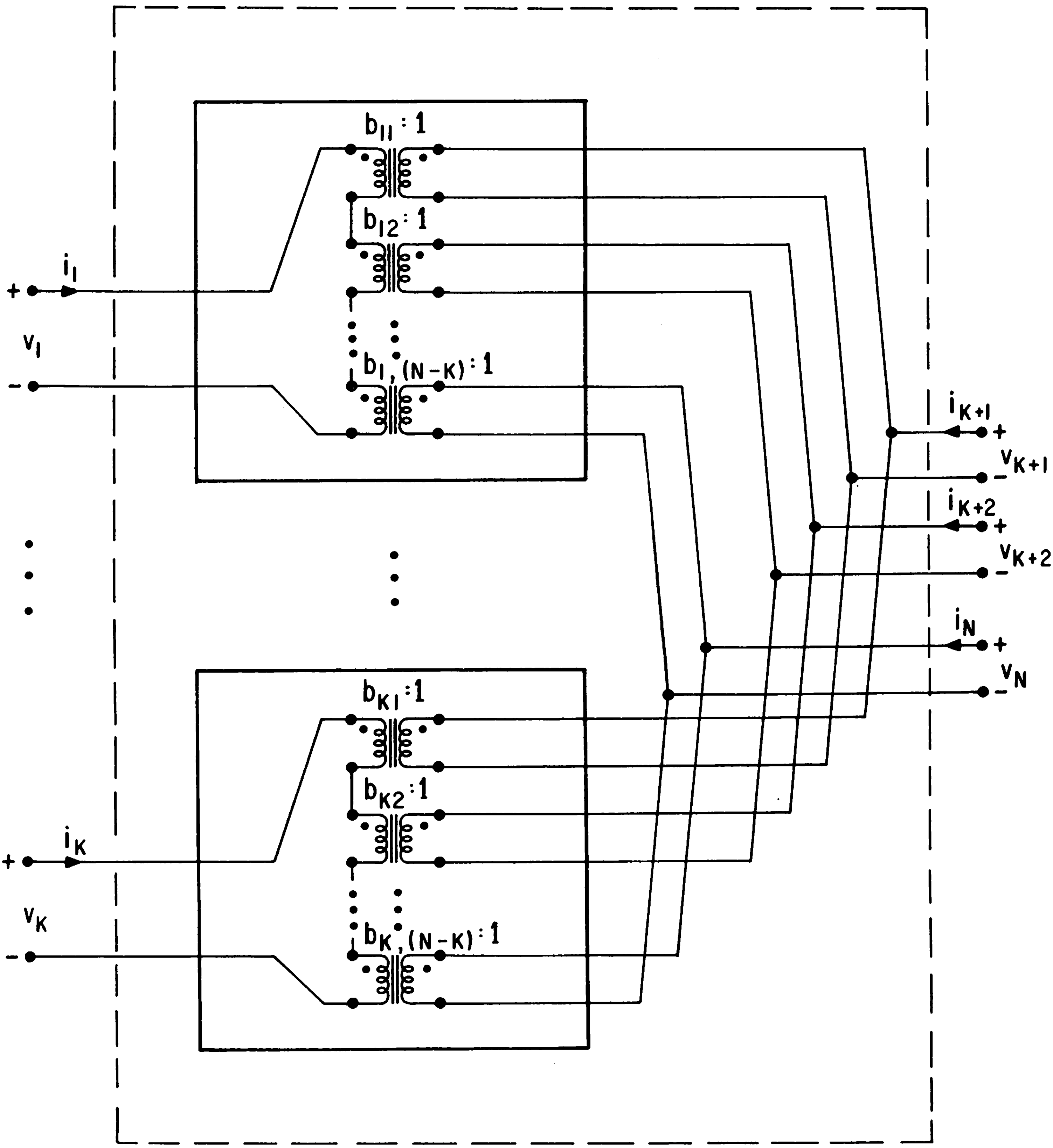
Let $\underset{\sim}{\varepsilon}_j$ represent the j-th vector in the standard ordered basis of $\mathbb{R}^n$, $1 \leq j \leq n$. To find $\underset{\sim}{\Lambda}$, the representation of the matrix $\underset{\sim}{\hat{\Lambda}}$ in the standard basis, we will need the matrix $\underset{\sim}{P}$ whose element P_{ki} is defined implicitly by $\underset{\sim}{\varepsilon}_i = \sum_{k=1}^{n} P_{ki} \underset{\sim}{\alpha}_k$, or

$$\underset{\sim}{1}^{(n)} = \begin{bmatrix} \underset{\sim}{\varepsilon}_1^T \\ \underset{\sim}{\varepsilon}_2^T \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{\varepsilon}_n^T \end{bmatrix} = \underset{\sim}{P}^T \begin{bmatrix} \underset{\sim}{\alpha}_1^T \\ \underset{\sim}{\alpha}_2^T \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{\alpha}_n^T \end{bmatrix} = \underset{\sim}{P}^T \underset{\sim}{A} \tag{C-1}$$

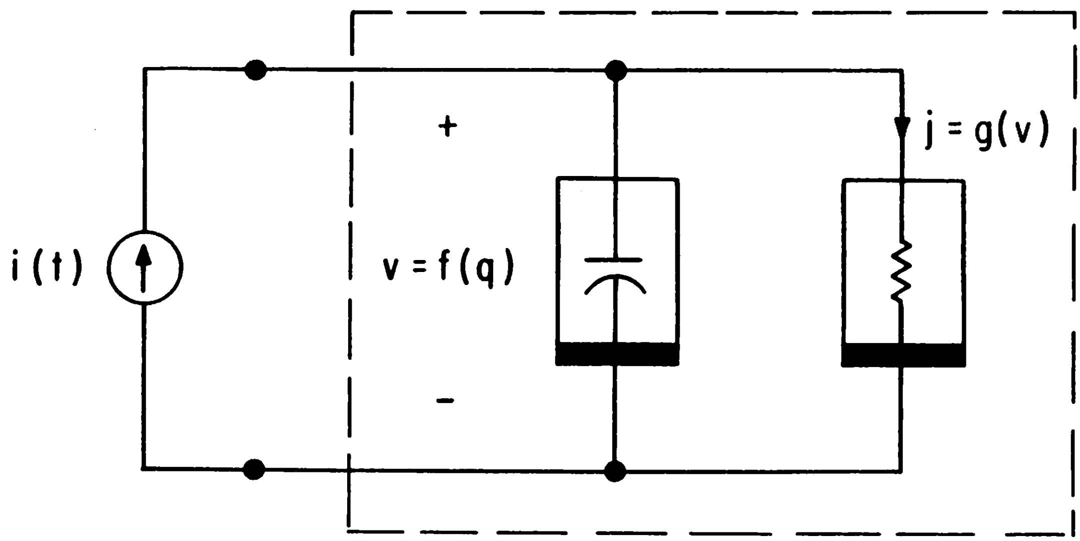
where $\underset{\sim}{\alpha}_1 = \underset{\sim}{x}/\|\underset{\sim}{x}\|$ and $\underset{\sim}{\alpha}_2 = \underset{\sim}{y}/\|\underset{\sim}{y}\|$ and equation (C-1) defines $\underset{\sim}{A}$. Since $\{\underset{\sim}{\alpha}_1, \cdots, \underset{\sim}{\alpha}_n\}$ forms a basis, $\underset{\sim}{A}$ is nonsingular; and since the rows of $\underset{\sim}{A}$ and orthonormal, $\underset{\sim}{A}$ is an orthogonal matrix (definition 2-4). Therefore $\underset{\sim}{P}^T = \underset{\sim}{A}^{-1} = \underset{\sim}{A}^T$, so $\underset{\sim}{P} = \underset{\sim}{A}$. Furthermore $\underset{\sim}{\Lambda} = \underset{\sim}{P}^{-1} \underset{\sim}{\hat{\Lambda}} \underset{\sim}{P}$ is antisymmetric, since $\underset{\sim}{\Lambda}^T = \underset{\sim}{P}^T \underset{\sim}{\hat{\Lambda}}^T \underset{\sim}{P}^{-1^T} = \underset{\sim}{P}^T(-\underset{\sim}{\hat{\Lambda}})\underset{\sim}{P} = -\underset{\sim}{P}^{-1} \underset{\sim}{\hat{\Lambda}} \underset{\sim}{P} = -\underset{\sim}{\Lambda}$.

Figure Captions

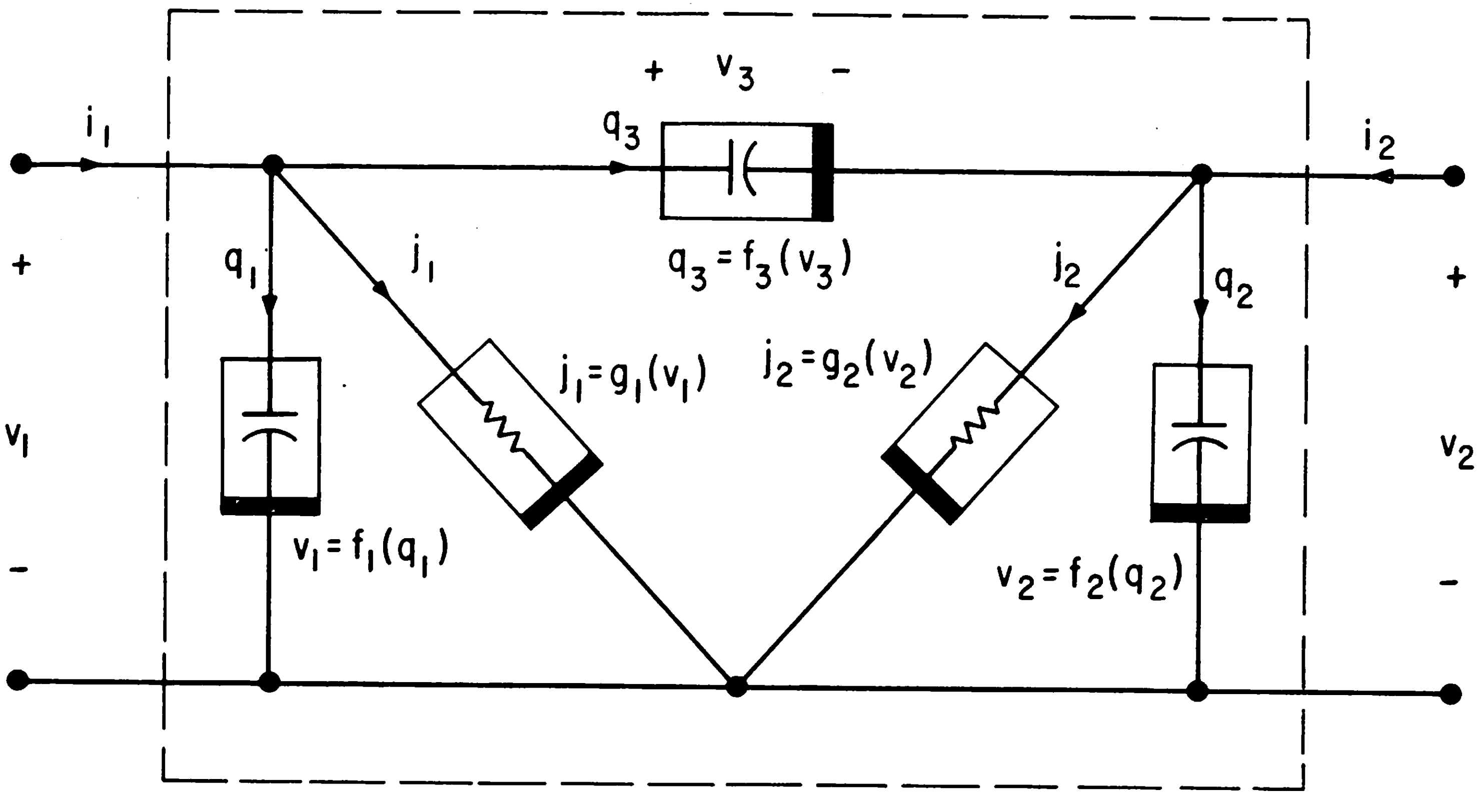
1. Canonical synthesis of linear reciprocal nonenergic N-port from ideal transformers.
2. Figure 2. See example 3-1.
3. Figure 3. See example 3-2.
4. The 2-port network of figure 3 drawn as the parallel connection of a resistive 2-port $\mathcal{R}$ and a capacitive 2-port $\mathcal{C}$. With the element constitutive relations indicated, $\mathcal{R}$ and $\mathcal{C}$ are individually nonenergic (see equations (3-5) and (3-6)). Therefore the combined network is nonenergic as well.



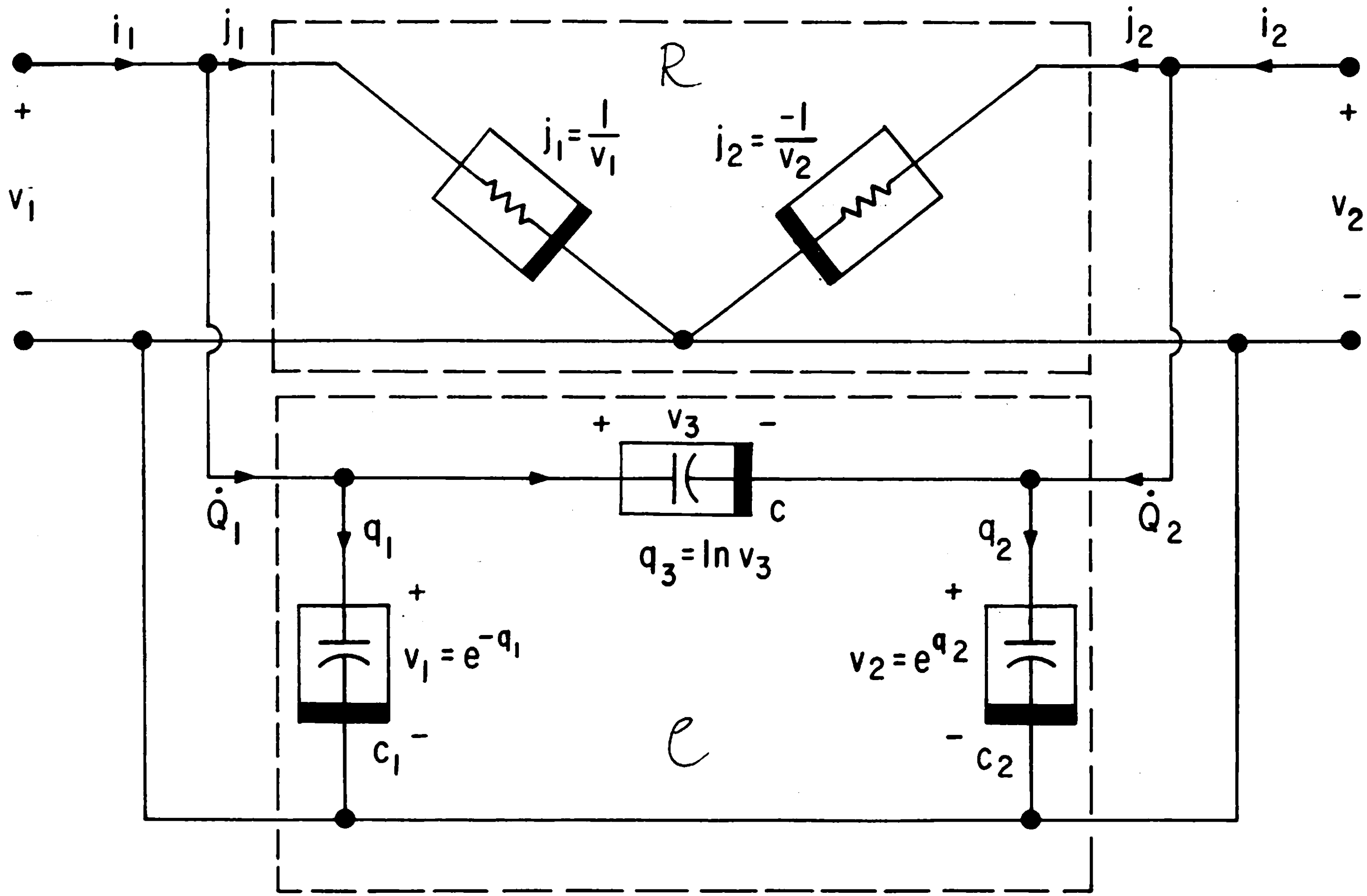
b11 : 1
b12 : 1
b1, (N-K) : 1
i1
+
v1
-
bK1 : 1
bK2 : 1
bK, (N-K) : 1
iK
+
vK
-
iK+1
+
vK+1
-
iK+2
+
vK+2
-
iN
+
vN
-



i (t)
+
v = f (q)
−
j = g(v)



i1
i2
+ v3 -
q3
q3 = f3(v3)
q1
j1
j2
q2
+
+
v1
v2
j1=g1(v1)
j2=g2(v2)
v1 = f1(q1)
v2 = f2(q2)
-
-



i_1
j_1
j_2
i_2
$\mathcal{R}$
$j_1 = \frac{1}{v_1}$
$j_2 = \frac{-1}{v_2}$
v_1
v_2
+
-
v_3
c
$q_3 = \ln v_3$
$\dot{Q}_1$
$\dot{Q}_2$
q_1
q_2
$v_1 = e^{-q_1}$
$v_2 = e^{q_2}$
c_1
c_2
$\mathcal{C}$