ON THE APPLICATION OF DEGREE THEORY TO THE ANALYSIS OF RESISTIVE NONLINEAR NETWORKS

by

Leon O. Chua and Niatsu N. Wang

Memorandum No. ERL-M555

17 July 1975

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

ON THE APPLICATION OF DEGREE THEORY TO THE ANALYSIS OF RESISTIVE NONLINEAR NETWORKS

Leon O. Chua and Niantsu N. Wang

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

This paper presents an application of the theory of the degree of a map to the study of the existence of solutions and some related problems for resistive nonlinear networks. Many well-known results in this area have been generalized to allow coupling among the nonlinear resistors. The usual hypothesis requiring the nonlinear resistors to be eventually increasing has been weakened considerably by only requiring the resistors to be eventually passive. Instead of investigating special cases by special techniques, we study the network equations from a geometrical point of view. The concept of homotopy of odd fields provides a unified yet simple approach for analyzing a large class of practical nonlinear networks. Many known results belong to this category and are derived as special cases of our generalized theorems. This approach leads to a much better understanding of the geometric structure of the vector fields associated with the network equations. As a result, in so far as the existence of solutions is concerned, the concept of eventual passivity is shown to be far more basic than that of eventual increasingness. The emphasis on the concept of eventual passivity also leads naturally to the inclusion of coupling among the nonlinear resistors.

The homotopy of odd fields also provides some useful techniques for locating the solutions. Along this line, we also study the bounding region of solutions and discuss the operating range of nonlinear resistors.

Research sponsored by National Science Foundation Grant ENG72-03783, and the Naval Electronic Systems Command Contract N00039-76-C-0022.

I. INTRODUCTION

One of the fundamental problems in resistive nonlinear network analysis is the question of existence and uniqueness of solutions. In this paper we investigate these problems for a large class of networks by applying the theory of the degree of a map. A large class of resistive nonlinear networks can be described by an equation of the form $f(z) = g(z) + Hz - s = 0$, where $z = [v,i]$ denotes the unknown network variables and $g(z)$ is a nonlinear continuous function from $\mathbb{R}^n$ into $\mathbb{R}^n$. The function $f: \mathbb{R}^n \to \mathbb{R}^n$ is then a continuous vector field on $\mathbb{R}^n$. Let D be a bounded open subset of $\mathbb{R}^n$. Roughly speaking,[1] the degree of f relative to $0 \in \mathbb{R}^n$ and D, denoted by $d(f;0,D)$, is equal to the algebraic number of solutions of $f(z) = 0$ in D. In particular, if $d(f;0,D) \neq 0$, then there exists at least one solution of $f(z) = 0$ in D. Because of the special structure of many network equations, the function f, when considered as a vector field on $\mathbb{R}^n$, has the same degree as an odd field f' relative to 0 and some open symmetric region D. Since an odd field f' always satisfies $d(f';0,D) = 1$ or -1, f has at least one solution in D. This simple observation not only leads to a deeper insight into the the structure of network equations but also provides a unified approach for solving a large class of problems. Many network problems, including those containing nonlinear coupled resistors, can be solved very efficiently by this unified yet extremely simple approach.

Since we shall apply it in the proof of almost every theorem in this paper, we define in Sec. II the degree of a map in $\mathbb{R}^n$ and derive some important properties which are of particular relevance to nonlinear network analysis. In Sec. III we investigate the hybrid analysis of a

[1]The precise meaning of this statement will be given in Sec. II.

very general class of resistive nonlinear networks via a "black box" n-port formulation. In particular, as special cases of some more general theorems, we will show that the vector fields $\underset{\sim}{f}$ associated with the class of networks considered by Sandberg and Willson [7,8,9,13] and Roska and Klimo [18] are homotopic to odd fields over some sufficiently large regions symmetric about the origin and hence these networks possess at least one solution. In Sec. IV, we derive the hybrid network equations via a topological formulation. The network equations obtained via this formulation are generally different from those derived by the n-port formulation. The advantage of this formulation is that the constitutive relations of the nonlinear resistors and their interconnections are easily identified from the resulting network equations. This in turn leads to more circuit and graph theoretic hypotheses, rather than mathematical conditions. For example, the results of Desoer and Wu [11] are proved easily via this unified approach. Finally, in Sec. V, we present a method for finding a bounding region which contains the solutions. This is important both from the computational and from the network design point of view.

In this paper, a two-terminal resistor is characterized by either $i = \hat{i}(v)$ or $v = \hat{v}(i)$, where v and i denote the branch voltage and current of the resistor respectively, and where $\hat{v}$ and $\hat{i}$ denote continuous functions from $\mathbb{R}^1$ into $\mathbb{R}^1$. If the resistor can be characterized by $i = \hat{i}(v)$, it is said to be voltage-controlled (v.c.) and if it can be characterized by $v = \hat{v}(i)$ it is said to be current-controlled (c.c.). The graph of either $\hat{v}$ or $\hat{i}$ is called the v-i curve of the resistor.

Let R be a v.c. resistor. R is said to be of type U (Unbounded) if $i \to \infty$ as $v \to \infty$ and $i \to -\infty$ as $v \to -\infty$; of type B (Bounded) if $|\hat{i}(v)| \leq M < \infty$ as $|v| \to \infty$; and of type H (Half-bounded) if $|\hat{i}(v)| \leq M < \infty$ as $v \to \infty$ and

$\hat{i}(v) \to -\infty$ as $v \to -\infty$ [11]. Similar definitions are defined dually for c.c. resistors.

A two-terminal resistor is said to be monotone if its v-i curve is monotone. Let R be a v.c. resistor. R is said to be increasing if $\hat{i}(v)$ is an increasing function of v, and strictly increasing if $\hat{i}(v)$ is a strictly increasing function of v. Obviously, R is monotone if it is increasing or strictly increasing. Similar definitions are defined for c.c. resistors.

A two-terminal resistor is said to be passive if its v-i curve lies in the first and the third quadrants only. The v-i curve of a passive resistor always passes through the origin if it is a continuous function. A resistor is said to be eventually (ultimately) passive if its v-i curve eventually lies in the first and the third quadrants only.

Resistors with constant v-i curves will be considered as independent sources. It is easy to see that any monotone v.c. resistor can be replaced by a parallel combination of an independent current source and a v.c. resistor which is passive. The same fact applies dually to c.c. resistors.

Coupled resistors will be considered as resistive m-ports. Let R be an m-port with port voltage $\underset{\sim}{v}_R \in \mathbb{R}^m$ and port current $\underset{\sim}{i}_R \in \mathbb{R}^m$. R is said to be passive if $\underset{\sim}{v}_R^T \underset{\sim}{i}_R \geq 0$ for all admissible pairs $(\underset{\sim}{v}_R, \underset{\sim}{i}_R)$ of R. In general, R can be characterized by an equation of the form $\underset{\sim}{y}_R = \underset{\sim}{h}(\underset{\sim}{z}_R)$ where $\underset{\sim}{z}_R$ denotes the port voltages and/or currents which are "independent port variables" and $\underset{\sim}{y}_R$ denotes the "dependent port variables." Obviously R is passive if, and only if, $\underset{\sim}{z}_R^T \underset{\sim}{h}(\underset{\sim}{z}_R) = \underset{\sim}{z}_R^T \underset{\sim}{y}_R = \underset{\sim}{v}_R^T \underset{\sim}{i}_R \geq 0$ for all $\underset{\sim}{z}_R \in \mathbb{R}^m$. We say that R is eventually passive (resp., eventually passive with respect to $\underset{\sim}{c}$) if there exists an $M > 0$ such that $\|\underset{\sim}{z}_R\| > M$ implies $\underset{\sim}{z}_R^T \underset{\sim}{h}(\underset{\sim}{z}_R) \geq 0$ (resp. $(\underset{\sim}{z}_R - \underset{\sim}{c})^T \underset{\sim}{h}(\underset{\sim}{z}_R) \geq 0$).

In these cases we also say that the function $\underset{\sim}{h}$ is passive, eventually passive, eventually passive with respect to $\underset{\sim}{c}$, respectively.

Finally, a few remarks concerning the notations: (1) All vectors and matrices are typified with a wiggle under the symbol. We use lower-case letters with subscripts for components of vectors. Thus $\underset{\sim}{z} = [z_1, z_2, \ldots, z_n]^T \in \mathbb{R}^n$ where T denotes transposition. All vectors are defined to be column vectors. We use upper-case letters for matrices. (2) In order to avoid confusion, functions are sometimes denoted by "hats," thus $v = \hat{v}(i)$ means that the variable v is computed via the function $\hat{v}(\cdot)$ at i. (3) We use E and I to denote the sets of indices pertaining to the voltage ports and current ports respectivley. Thus i_k, $k \in E$ means the current associated with the "voltage port k." In a similar fashion, we use U, H and B to denote the sets of indices associated with type U, type H and type B resistors respectively. (4) Let $\underset{\sim}{z} \in \mathbb{R}^n$. By $\bar{\underset{\sim}{z}}$ we mean the vector $[|z_1|, |z_2|, \ldots, |z_n|]^T$. Define $\underset{\sim}{D}_{\underset{\sim}{z}} \triangleq \text{diag}(\text{sgn } z_k)$ where $\text{sgn } z_k = 1$ if $z_k \geq 0$ and $\text{sgn } z_k = -1$ if $z_k < 0$, then $\underset{\sim}{z} = \underset{\sim}{D}_{\underset{\sim}{z}} \bar{\underset{\sim}{z}}$. (5) Let $\underset{\sim}{A} \in \mathbb{R}^{n \times m}$, i.e., an $n \times m$ real matrix. We write $\underset{\sim}{A} > 0$ ($\underset{\sim}{A} \geqq 0$) if all its components are positive (non-negative). By $\underset{\sim}{A} \geq 0$ we mean $\underset{\sim}{A} \geqq 0$ but $\underset{\sim}{A} \neq$ the zero matrix. (6) We use both the ℓ_∞-norm and the ℓ_2-norm in $\mathbb{R}^n$ throughout the paper. Thus $\|\underset{\sim}{z}\| \triangleq \max_i\{|z_i|\}$, and $\|\underset{\sim}{z}\|_2 \triangleq (\sum_i |z_i|^2)^{1/2}$. In ℓ_∞-norm, a sphere $S(\underset{\sim}{0}, r) \triangleq \{\underset{\sim}{z} \in \mathbb{R}^n: \|\underset{\sim}{z}\| = r\}$ centered at $\underset{\sim}{0}$ with radius r becomes the surface of a "cube" defined by $S(\underset{\sim}{0}, r) = \{z \in \mathbb{R}^n: |z_i| \leq r, i = 1, 2, \ldots, n;$ where equality holds for at least one $i\}$. (7) We use $\bar{S}$ and ∂S to denote the closure and the boundary of a set S, respectively. (8) Finally, we denote the difference between a set A and a set B by $A \backslash B$; i.e., $A \backslash B = \{\underset{\sim}{x} \in A: \underset{\sim}{x} \notin B\}$.

II. THE DEGREE OF A MAP

In this section we define the degree of a map in $\mathbb{R}^n$ and derive some of its useful properties. In order to make this section self-contained, we supply short proofs for almost all theorems whenever possible. For more details, see [1]-[5].

2.1 The Degree of a Map

Let D be a bounded, open subset of $\mathbb{R}^n$. Let $C(\bar{D})$ be the space of all continuous functions defined on $\bar{D}$ into $\mathbb{R}^n$ with the topology of uniform convergence.[2] Thus, $C(\bar{D})$ is a normed linear space with a norm defined by $|\underset{\sim}{f}| = \sup_{\underset{\sim}{z} \in D}\{\|\underset{\sim}{f}(\underset{\sim}{z})\|\}$ for all $\underset{\sim}{f} \in C(\bar{D})$. By $C^1(\bar{D})$ we mean the subset of $C(\bar{D})$ consisting of all functions in $C(\bar{D})$ which are continuously differentiable.

<u>Definition 1</u>: Let $\underset{\sim}{p} \in \mathbb{R}^n$, and $\underset{\sim}{f} \in C(\bar{D})$. Assume that $\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{p}$ has no solutions in ∂D. The <u>degree</u> of $\underset{\sim}{f}$ relative to $\underset{\sim}{p}$ and D, denoted by $d(\underset{\sim}{f};\underset{\sim}{p},D)$, is defined by the following algorithm:

<u>Step 1</u>: Assume $\underset{\sim}{f} \in C^1(\bar{D})$ and that the Jacobian matrix $J_{\underset{\sim}{f}}(\underset{\sim}{z}) \triangleq \partial \underset{\sim}{f}(\underset{\sim}{z})/\partial \underset{\sim}{z}$ is nonsingular at each $\underset{\sim}{z} \in \mathcal{S} \triangleq \{\underset{\sim}{z} \in D: \underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{p}\}$, the degree of $\underset{\sim}{f}$ relative to $\underset{\sim}{p}$ and D is defined as the <u>algebraic number</u> of solutions of $\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{p}$ in D, that is,

$$d(\underset{\sim}{f};\underset{\sim}{p},D) = \sum_{\underset{\sim}{z} \in \mathcal{S}} \operatorname{sgn} \det J_{\underset{\sim}{f}}(\underset{\sim}{z}).$$

<u>Remarks</u>: 1. Since $J_{\underset{\sim}{f}}(z)$ is nonsingular at any $\underset{\sim}{z} \in \mathcal{S}$ and $\bar{D}$ is compact, $\mathcal{S}$ is a finite set. Hence $d(\underset{\sim}{f};\underset{\sim}{p},D)$ is a finite integer.

2. There is a "volume integral" representation of $d(\underset{\sim}{f};\underset{\sim}{p},D)$; namely,

---

[2] Recall that $\bar{D} \triangleq$ the closure of D and $\partial D \triangleq$ the boundary of D.

$$d(\underset{\sim}{f};\underset{\sim}{p},D) = \int_D \psi_\varepsilon(\|\underset{\sim}{f}(\underset{\sim}{z})-\underset{\sim}{p}\|)\det \underset{\sim}{J}_f(\underset{\sim}{z})d\underset{\sim}{z}$$

where $\varepsilon > 0$ is a small number and $\psi_\varepsilon: \mathbb{R}_+ \to \mathbb{R}$ is a continuous "bump" function satisfying:[3]

(a) $\psi_\varepsilon(s) = 0$ for all $s \geq \varepsilon$, and

(b) $\int_{\mathbb{R}^n} \psi_\varepsilon(\|\underset{\sim}{x}\|)d\underset{\sim}{x} = 1.$

Proof: Let $\mathcal{S} = \{\underset{\sim}{z}_1, \underset{\sim}{z}_2, \ldots, \underset{\sim}{z}_k\}$. Since $\underset{\sim}{J}_f(\underset{\sim}{z}_i)$ is nonsingular for all $i = 1,2,\ldots,k$, there exists an $\varepsilon > 0$ such that $\underset{\sim}{f}$ is a homeomorphism from each ball $B(\underset{\sim}{z}_i,\varepsilon_i) \subset D$ centered at $\underset{\sim}{z}_i$ with radius ε_i onto $B(\underset{\sim}{p},\varepsilon)$. Then we obtain

$$\int_D \psi_\varepsilon(\|\underset{\sim}{f}(\underset{\sim}{z})-\underset{\sim}{p}\|)\det \underset{\sim}{J}_f(\underset{\sim}{z})d\underset{\sim}{z} = \sum_{i=1}^{k} \int_{B(\underset{\sim}{z}_i,\varepsilon_i)} \psi_\varepsilon(\|\underset{\sim}{f}(\underset{\sim}{z})-\underset{\sim}{p}\|)\det \underset{\sim}{J}_f(\underset{\sim}{z})d\underset{\sim}{z}.$$

Let $\underset{\sim}{x}(\underset{\sim}{z}) = \underset{\sim}{f}(\underset{\sim}{z}) - \underset{\sim}{p}$, then $\underset{\sim}{J}_f(\underset{\sim}{z}) = \partial\underset{\sim}{x}/\partial\underset{\sim}{z}$. Rewriting $\det \underset{\sim}{J}_f(\underset{\sim}{z}) = |\det \underset{\sim}{J}_f(\underset{\sim}{z})| \operatorname{sgn} \det\left(\underset{\sim}{J}_f(\underset{\sim}{z})\right)$ and applying the standard change of variable formula for volume integrals, the above integral becomes

$$\sum_{i=1}^{k} \int_{B(\underset{\sim}{z}_i,\varepsilon_i)} \psi_\varepsilon(\|\underset{\sim}{x}(\underset{\sim}{z})\|)\,|\det \underset{\sim}{J}_f(\underset{\sim}{z})|\operatorname{sgn} \det\left(\underset{\sim}{J}_f(\underset{\sim}{z}_i)\right)d\underset{\sim}{z}$$

$$= \sum_{i=1}^{k} \left[\int_{B(0,\varepsilon)} \psi_\varepsilon(\ \underset{\sim}{x}\ )d\underset{\sim}{x}\right] \operatorname{sgn}\left(\det \underset{\sim}{J}_f(\underset{\sim}{z}_i)\right) = \sum_{i=1}^{k} \operatorname{sgn} \det \underset{\sim}{J}_f(\underset{\sim}{z}_i) = d(\underset{\sim}{f};\underset{\sim}{p},D).$$

where the last equation follows from conditions (a) and (b); namely,

$$\int_{B(\underset{\sim}{0},\varepsilon)} \psi_\varepsilon(\|\underset{\sim}{x}\|)d\underset{\sim}{x} = \int_{\mathbb{R}^n} \psi_\varepsilon(\|\underset{\sim}{x}\|)d\underset{\sim}{x} = 1. \qquad \blacksquare$$

---

[3] $\mathbb{R}_+ \triangleq \{x \in \mathbb{R}: x \geq 0\}$.

3. The degree $d(\underset{\sim}{f};\underset{\sim}{p},D)$ is a continuous function of both $\underset{\sim}{f}$ and $\underset{\sim}{p}$. This is an immediate consequence of the integral representation stated above.

Step 2: Assume $\underset{\sim}{f} \in C^1(\bar{D})$ and that $J_{\underset{\sim}{f}}(\underset{\sim}{z}_i)$ is singular at some $\underset{\sim}{z}_i \in \mathcal{S}$. It follows from Sard's theorem[4] that there exists a sequence $\{\underset{\sim}{p}_m\} \subset \mathbb{R}^n$, $\underset{\sim}{p}_m \to \underset{\sim}{p}$ as $m \to \infty$ such that each solution set $\mathcal{S}_m = \{\underset{\sim}{z} \in D: \underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{p}_m\}$ contains only $\underset{\sim}{z}$'s at which $J_{\underset{\sim}{f}}(\underset{\sim}{z})$ is nonsingular. Define

$$d(\underset{\sim}{f};\underset{\sim}{p},D) = \lim_{m\to\infty} d(\underset{\sim}{f};\underset{\sim}{p}_m,D).$$

Remark: Since $d(\underset{\sim}{f};\underset{\sim}{p}_m,D)$ is continuous in $\underset{\sim}{p}_m$ and integer-valued, the above limit is reached after finitely many steps and is independent of any particular choice of the sequence $\{\underset{\sim}{p}_m\}$.

Step 3: Assume that $\underset{\sim}{f} \in C(\bar{D})$ but $\underset{\sim}{f} \notin C^1(\bar{D})$. In this case, since $C^1(\bar{D})$ is dense in $C(\bar{D})$, there exists a sequence $\{\underset{\sim}{f}_m\} \subset C^1(\bar{D})$, $\underset{\sim}{f}_m \to \underset{\sim}{f}$ as $m \to \infty$ uniformly on D. Define

$$d(\underset{\sim}{f};\underset{\sim}{p},D) = \lim_{m\to\infty} d(\underset{\sim}{f}_m,\underset{\sim}{p},D).$$

Remark: Since $d(\underset{\sim}{f}_m,\underset{\sim}{p},D)$ is continuous in $\underset{\sim}{f}_m$ and integer-valued, the above limit is reached after finitely many steps and is independent of any particular choice of the sequence $\{\underset{\sim}{f}_m\}$.

We now derive a few important properties of the degree.

Property 1: Continuity property. The degree $d(\underset{\sim}{f};\underset{\sim}{p},D)$ of $\underset{\sim}{f}$ relative to $\underset{\sim}{p}$ and D is a continuous function of both $\underset{\sim}{p}$ and $\underset{\sim}{f}$.

Proof. We have already established this property. ∎

---

[4] Sard's theorem: Let $\underset{\sim}{f}: \bar{D} \to \mathbb{R}^n$, $\underset{\sim}{f} \in C^1(\bar{D})$, and let $B \triangleq \{\underset{\sim}{z} \in D: \det J_{\underset{\sim}{f}}(\underset{\sim}{z}) = 0\}$, then $\underset{\sim}{f}(B)$ is of measure zero.

Property 2: Homotopy invariance. A homotopy $\underset{\sim}{h}(\underset{\sim}{z},\lambda)$ over $\bar{D}$ is any continuous function from $\bar{D} \times [0,1] \to \mathbb{R}^n$. Let $\underset{\sim}{h}(\underset{\sim}{z},\lambda)$ be a homotopy over $\bar{D}$. If $\underset{\sim}{h}(\underset{\sim}{z},\lambda) = \underset{\sim}{p}$ has no solution in ∂D for any $\lambda \in [0,1]$, then $d\left(\underset{\sim}{h}(\underset{\sim}{z},\lambda);\ \underset{\sim}{p},D\right)$ is a constant independent of λ.

Proof: Since the function $\hat{d}(\lambda) \triangleq d\left(\underset{\sim}{h}(\underset{\sim}{z},\lambda);\ \underset{\sim}{p},D\right)$ is well defined on $[0,1]$ by assumption and is continuous in λ, it follows from Property 1 that $\hat{d}(\lambda)$ is integer-valued and hence must be a constant independent of λ. ∎

Remark: In this case, $\underset{\sim}{h}(\underset{\sim}{z},0)$ and $\underset{\sim}{h}(\underset{\sim}{z},1)$ are said to be homotopic to each other, and we say that $\underset{\sim}{h}(\underset{\sim}{z},\lambda)$ connects $\underset{\sim}{h}(\underset{\sim}{z},0)$ and $\underset{\sim}{h}(\underset{\sim}{z},1)$ homotopically.

Property 3: Boundary Value Dependence. The degree $d(\underset{\sim}{f};\underset{\sim}{p},D)$ is uniquely determined by the action of $\underset{\sim}{f}$ on the boundary ∂D.

Proof: Let $\underset{\sim}{f}$ and $\tilde{\underset{\sim}{f}}$ be two functions from $\bar{D} \to \mathbb{R}^n$ such that $\underset{\sim}{f}(\underset{\sim}{z}) = \tilde{\underset{\sim}{f}}(\underset{\sim}{z})$ for all $\underset{\sim}{z} \in \partial D$. Define $\underset{\sim}{h}(\underset{\sim}{z},\lambda) = \lambda \underset{\sim}{f}(\underset{\sim}{z}) + (1-\lambda)\tilde{\underset{\sim}{f}}(\underset{\sim}{z})$. Since $\underset{\sim}{h}(\underset{\sim}{z},\lambda) = \underset{\sim}{f}(\underset{\sim}{z}) = \tilde{\underset{\sim}{f}}(\underset{\sim}{z})$ for all $\underset{\sim}{z} \in \partial D$, $\underset{\sim}{h}(\underset{\sim}{z},\lambda) = \underset{\sim}{p}$ has no solution in ∂D. It follows from Property 2 that $d(\underset{\sim}{f};\underset{\sim}{p},D) = d(\tilde{\underset{\sim}{f}};\underset{\sim}{p},D)$. ∎

Since $\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{p}$ if, and only if, $\underset{\sim}{f}(\underset{\sim}{z}) - \underset{\sim}{p} = \underset{\sim}{0}$, there is no loss of generality to consider only $d(\underset{\sim}{f};\underset{\sim}{0},D)$. We then consider $\underset{\sim}{f}$ as a vector field defined on $\bar{D}$ and any $\underset{\sim}{z} \in D$ such that $\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{0}$ is called a singular point of $\underset{\sim}{f}$. Property 2 can then be restated as follows:

Property 2': Suppose the homotopy $\underset{\sim}{h}(\underset{\sim}{z},\lambda) \neq \underset{\sim}{0}$ for all $\underset{\sim}{z} \in \partial D$ and for all $\lambda \in [0,1]$, then $d\left(\underset{\sim}{h}(\underset{\sim}{z},\lambda);\underset{\sim}{0},D\right)$ is a constant independent of λ.

As an application of Property 2', we have:

Property 4: Let $\underset{\sim}{f}$ and $\tilde{\underset{\sim}{f}}$: $\bar{D} \to \mathbb{R}^n$ be continuous functions such that $\underset{\sim}{f}$ and $\tilde{\underset{\sim}{f}}$ never vanish on ∂D. If $\underset{\sim}{f}$ and $\tilde{\underset{\sim}{f}}$ are never opposite to each other on ∂D, i.e.,

$$\frac{\underset{\sim}{f}(\underset{\sim}{z})}{\|\underset{\sim}{f}(\underset{\sim}{z})\|} \neq \frac{-\tilde{\underset{\sim}{f}}(\underset{\sim}{z})}{\|\tilde{\underset{\sim}{f}}(\underset{\sim}{z})\|} \quad \text{for all } \underset{\sim}{z} \in \partial D.$$

then $d(f;0,D) = d(\tilde{f};0,D)$.

Proof: Define the homotopy $h(z,\lambda) = \lambda f(z) + (1-\lambda)\tilde{f}(z)$. By assumption, $h(z,\lambda) \neq 0$ on ∂D for all $\lambda \in [0,1]$. It follows from Property 2' that $d(f;0,D) = d(\tilde{f};0,D)$. ■

Property 5: Let $f \in C(\bar{D})$, where $f = f' + f''$. The component f' is called a principal part of f if $\|f'(z)\| > \|f''(z)\|$ for all $z \in \partial D$. Let f' be a principal part of f, then

$$d(f;0,D) = d(f';0,D).$$

Proof: Suppose $f(z) = -\alpha f'(z)$ for some $\alpha > 0$ and $z \in \partial D$, then $(1+\alpha)f'(z) = -f''(z)$ and we obtain $\|f'(z)\| < \|f''(z)\|$, a contradiction. Hence, f and f' are never opposite to each other on ∂D. It follows from Property 4 that $d(f;0,D) = d(f';0,D)$. ■

Property 6: Let $f \in C(\bar{D})$. Let $c \in \mathbb{R}^n$ be a constant unit vector. We say that the field f omits the direction c if

$$\frac{f(z)}{\|f(z)\|} \neq c \quad \text{for all } z \in \partial D.$$

If f omits the direction c, $d(f;0,D) = 0$.

Proof: Define $h(z,\lambda) = \lambda f(z) + (1-\lambda)\tilde{f}(z)$, $\lambda \in [0,1]$ where $\tilde{f}(z) = c$ for all $z \in \bar{D}$. Since f omits the direction c, $h(z,\lambda) \neq 0$ for all $z \in \partial D$ and $\lambda \in [0,1]$. Since the degree of a constant map is always zero, it follows from Property 2' that $d(f;0,D) = d(\tilde{f};0,D) = 0$. ■

Property 7: Let $f \in C(\bar{D})$ where D is symmetric about the origin. If $f(z) = -f(-z)$ for all z in D, f is called an odd field. If f is an odd field on D, then $d(f;0,D) =$ an odd integer.

Proof: For simplicity, assume the Jacobian matrix $J_f(z)$ is nonsingular at each $\underset{\sim}{z} \in \mathcal{S} \triangleq \{\underset{\sim}{z} \in D: \underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{0}\}$.[5] The solutions of $\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{0}$ can be grouped in pairs $(\underset{\sim}{z}_i, -\underset{\sim}{z}_i)$; $i = 1,2,\ldots,k$ together with $\underset{\sim}{z}_0 = \underset{\sim}{0}$. Since $\underset{\sim}{f}$ is an odd field, $\det \underset{\sim}{J}_f(\underset{\sim}{z}_i) = \det \underset{\sim}{J}_f(-\underset{\sim}{z}_i)$. Hence

$$d(\underset{\sim}{f};\underset{\sim}{0},D) = \sum_{i=1}^{k} [\text{sgn} \det \underset{\sim}{J}_f(\underset{\sim}{z}_i) + \text{sgn} \det \underset{\sim}{J}_f(-\underset{\sim}{z}_i)] + \text{sgn} \det \underset{\sim}{J}_f(\underset{\sim}{z}_0)$$

$$= \text{an even integer} \pm 1 = \text{an odd integer.} \quad ■$$

As an important application of <u>Property 7</u>, we have:

<u>Property 8</u>: Let $\underset{\sim}{f} \in C(\bar{D})$ where D is symmetric about the origin. If

$$\frac{\underset{\sim}{f}(\underset{\sim}{z})}{\|\underset{\sim}{f}(\underset{\sim}{z})\|} \neq \frac{\underset{\sim}{f}(-\underset{\sim}{z})}{\|\underset{\sim}{f}(-\underset{\sim}{z})\|} \quad \text{for all } \underset{\sim}{z} \in \partial D,$$

i.e., $\underset{\sim}{f}(\underset{\sim}{z})$ and $\underset{\sim}{f}(-\underset{\sim}{z})$ are both nonzero and do not point in the same direction, then $d(\underset{\sim}{f};\underset{\sim}{0},D)$ = an odd integer.

<u>Proof</u>: Define $\underset{\sim}{h}(\underset{\sim}{z},\lambda) = \lambda[\underset{\sim}{f}(\underset{\sim}{z}) - \underset{\sim}{f}(-\underset{\sim}{z})] + (1-\lambda)\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{f}(\underset{\sim}{z}) - \lambda\underset{\sim}{f}(-\underset{\sim}{z}); \lambda \in [0,1]$. By assumption, $\underset{\sim}{h}(\underset{\sim}{z},\lambda) \neq \underset{\sim}{0}$ for all $\underset{\sim}{z} \in \partial D$ and $\lambda \in [0,1]$. By homotopy invariance, $d(\underset{\sim}{f};\underset{\sim}{0},D) = d\big(\underset{\sim}{h}(\cdot,1);\underset{\sim}{0},D\big)$. But $\underset{\sim}{h}(\underset{\sim}{z},1) = \underset{\sim}{f}(\underset{\sim}{z}) - \underset{\sim}{f}(-\underset{\sim}{z})$ implies $\underset{\sim}{h}(\cdot,1)$ is an odd field, hence according to <u>Property 7</u>, $d(\underset{\sim}{f},\underset{\sim}{0},D)$ = an odd integer. ■

<u>Remark</u>: If $d(\underset{\sim}{f};\underset{\sim}{0},D)$ = an odd integer, then $\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{0}$ has at least one solution in D. This fact will be applied extensively to network problems in later sections.

<u>Property 9</u>: Let $\underset{\sim}{f} \in C(\bar{D})$. If $d(\underset{\sim}{f};\underset{\sim}{0},D) \neq 1$, then

(i) there exists at least one $\underset{\sim}{z} \in \partial D$ such that $\underset{\sim}{f}(\underset{\sim}{z})$ and $\underset{\sim}{z}$ are in the same direction, and

(ii) there exists at least one $\underset{\sim}{z}' \in \partial D$ such that $\underset{\sim}{f}(\underset{\sim}{z}')$ and $\underset{\sim}{z}'$ are in the opposite directions.

---

[5] The general case can be proved by invoking (2) and (3) of Def. 1.

Proof: Observe that if either (i) or (ii) is violated, $\underset{\sim}{f}$ will be homotopic to the identity map on D and hence $d(\underset{\sim}{f};\underset{\sim}{0},D) = 1$, a contradiction. ∎

Property 10: (A special case of Hopf theorem.) Let $\underset{\sim}{f}$ and $\tilde{\underset{\sim}{f}}$ be two vector fields on $B(\underset{\sim}{0},r) \triangleq \{\underset{\sim}{z} \in \mathbb{R}^n: \|\underset{\sim}{z}\| < r\}$ such that $d(\underset{\sim}{f};\underset{\sim}{0},D) = d(\tilde{\underset{\sim}{f}};\underset{\sim}{0},D)$. Then $\underset{\sim}{f}$ and $\tilde{\underset{\sim}{f}}$ are homotopic over D.

Proof: We will not prove the theorem here. For detailed proof, see [4].

2.2 Index of a Singular Point and Structurally Stable Solutions

Definition 2. Let $\underset{\sim}{f} \in C(\bar{D})$. Let $\underset{\sim}{z}$ be a singular point of $\underset{\sim}{f}$ in D, i.e., $\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{0}$. The singular point $\underset{\sim}{z}$ is said to be isolated if there exists a neighborhood $B \subset D$ of $\underset{\sim}{z}$ such that $\underset{\sim}{z}$ is the only singular point of $\underset{\sim}{f}$ in B. The quantity $d(\underset{\sim}{f};\underset{\sim}{0},B)$ is called the index of $\underset{\sim}{z}$.

A singular point $\underset{\sim}{z}$ of $\underset{\sim}{f}$ in D is said to be structually stable if for any $\varepsilon > 0$, there is a $\delta > 0$ such that there exists at least one singular point in the ball $B(\underset{\sim}{z},\varepsilon)$ for any $\underset{\sim}{f}_\delta \in C(\bar{D})$ such that $\|\underset{\sim}{f}_\delta - \underset{\sim}{f}\| < \delta$.

Property 11: Let $\underset{\sim}{f}$ be a vector field on $\mathbb{R}^n$ with finitely many zeros, and suppose that the sum of the indices of its zeros is 0. Let $B(\underset{\sim}{0},r)$ be a ball containing all $\underset{\sim}{f}^{-1}(\underset{\sim}{0})$. Then there exists a continuous vector field $\hat{\underset{\sim}{f}}$ that has no zeros, yet equal to $\underset{\sim}{f}$ on $\mathbb{R}^n \backslash B(\underset{\sim}{0},r)$.

Proof: We define $\hat{\underset{\sim}{f}}$ in $B(\underset{\sim}{0},r)$ by extending $\underset{\sim}{f}$ on $\partial B(0,r)$ continuously into $B(\underset{\sim}{0},r)$ in such a way that $\hat{\underset{\sim}{f}}(\underset{\sim}{z}) \neq \underset{\sim}{0}$ for all $\underset{\sim}{z} \in B(\underset{\sim}{0},r)$. The complete proof is, however, rather involved. For a detailed outline of the proof, see [5], pp. 146-147. ∎

Remark: Using a similar method, it is easily shown that if $\underset{\sim}{0}$ is a singular point of $\underset{\sim}{f}$ with index equal to 0, then we can always find an $\hat{\underset{\sim}{f}}$ such that $\hat{\underset{\sim}{f}}(\underset{\sim}{z}) = \underset{\sim}{f}(\underset{\sim}{z})$ for all $\underset{\sim}{z} \in \mathbb{R}^n \backslash B(\underset{\sim}{0},r)$ and $\hat{\underset{\sim}{f}}(\underset{\sim}{z}) \neq \underset{\sim}{0}$ for all $\underset{\sim}{z} \in B(\underset{\sim}{0},r)$ where $B(\underset{\sim}{0},r)$ is a ball in which $\underset{\sim}{0}$ is the only singular point of $\underset{\sim}{f}$.

Property 12: An isolated singular point is structurally stable if, and only if, its index is different from zero.

Proof: (If) This is a direct consequence of the continuity property of $d(\underset{\sim}{f},\underset{\sim}{0},D)$ in $\underset{\sim}{f}$.

(Only if) Let $\underset{\sim}{0}$ be an isolated singular point of $\underset{\sim}{f}$ such that $d\left(\underset{\sim}{f},\underset{\sim}{0},B(\underset{\sim}{0},r)\right) = 0$ where $B(\underset{\sim}{0},r)$ is an open ball in which $\underset{\sim}{0}$ is the only singular point. Since $d\left(\underset{\sim}{f},\underset{\sim}{0},B(\underset{\sim}{0},r)\right) = 0$, we can always find an $\hat{\underset{\sim}{f}}$ such that $\hat{\underset{\sim}{f}}(\underset{\sim}{z}) = \underset{\sim}{f}(\underset{\sim}{z})$ for all $\underset{\sim}{z} \in \mathbb{R}^n \backslash B(\underset{\sim}{0},r)$ and $\hat{\underset{\sim}{f}}(\underset{\sim}{z}) \neq 0$ for all $\underset{\sim}{z} \in B(\underset{\sim}{0},r)$. Since $\underset{\sim}{f}$ is continuous and $r > 0$ can be chosen arbitrarily small, for any $\delta > 0$, there is an $\underset{\sim}{f}_\delta$, such that $\|\underset{\sim}{f}(\underset{\sim}{z})-\underset{\sim}{f}_\delta(\underset{\sim}{z})\| < \delta$ for all $\underset{\sim}{z} \in \mathbb{R}^n$ but $\underset{\sim}{f}_\delta(\underset{\sim}{z}) \neq \underset{\sim}{0}$ for all $\underset{\sim}{z} \in B(\underset{\sim}{0},r(\delta))$ where $r(\delta) > 0$ depends on δ. Hence $\underset{\sim}{0}$ is not structurally stable. ■

A structurally stable singular point varies continuously with $\underset{\sim}{f}$. In view of Property 12, we define:

Definition 3: Let $\underset{\sim}{f}$ be a continuous vector field on $\mathbb{R}^n$. A point $\underset{\sim}{z} \in \mathbb{R}^n$ is said to be a structurally stable solution of $\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{0}$ if, and only if, the index of $\underset{\sim}{z}$ is nonzero.

III. HYBRID ANALYSIS VIA N-PORT FORMULATION

3.1 The Network Equations

In this section we investigate properties of network equations by hybrid analysis via an n-port formulation. Let $\mathcal{N}$ be a network containing finitely many nonlinear resistors (coupled or uncoupled to each other), independent sources and linear dependent sources. Extracting all nonlinear resistors and replacing them by ports, the remaining n-port (which contains only linear resistors, linear controlled sources and independent sources) is then described by a hybrid n-port representation. This analysis is particularly suited for networks which contain relatively few nonlinear elements.

Theorem 1. [12] A linear n-port N containing only positive linear resistors and independent sources has a hybrid representation

$$\begin{bmatrix} \mathbf{i}_E \\ \mathbf{v}_I \end{bmatrix} + \begin{bmatrix} \mathbf{H}_{EE} & \mathbf{H}_{EI} \\ \mathbf{H}_{IE} & \mathbf{H}_{II} \end{bmatrix} \begin{bmatrix} \mathbf{v}_E \\ \mathbf{i}_I \end{bmatrix} = \mathbf{s} \, .$$

(where E and I pertain to the voltage ports and current ports, respectively, and $\mathbf{s}$ is a vector accounting for the independent sources) if, and only if, the voltage ports, together with the internal voltage sources do not form any loops and the current ports, together with the internal current sources do not form any cut sets.

Assuming the nonlinear resistor voltages and currents across the ports are related by associated reference directions, the following properties of the hybrid matrix $\mathbf{H} \triangleq \begin{bmatrix} \mathbf{H}_{EE} & \mathbf{H}_{EI} \\ \mathbf{H}_{IE} & \mathbf{H}_{II} \end{bmatrix}$ can be easily verified [12].

[P1]. $\mathbf{H}_{EE}$ and $\mathbf{H}_{II}$ are symmetric, positive semidefinite or positive definite. Nullity of $\mathbf{H}_{EE}$ (resp. $\mathbf{H}_{II}$) is equal to the number of independent cut sets (resp. loops) consisting of voltage and/or current ports only.

[P2]. $\mathbf{H}_{EI} = -\mathbf{H}_{IE}^T$.

[P3]. Elements in $\mathbf{H}_{EI}$ and $\mathbf{H}_{IE}$ are bounded by 1 in magnitude.

[P4]. It follows from [P1] that $\mathbf{H}$ is at least positive semidefinite. It is positive definite, if, and only if, both $\mathbf{H}_{EE}$ and $\mathbf{H}_{II}$ are positive definite. In particular, $\mathbf{H}$ is positive definite and hence nonsingular if the ports do not form any loops or cut sets. Otherwise, it may be singular.

Let the constitutive relations of the nonlinear resistors across the ports be represented by

$$\mathbf{i}_E = \hat{\mathbf{i}}_E(\mathbf{v}_E, \mathbf{i}_I) \text{ and } \mathbf{v}_I = \hat{\mathbf{v}}_I(\mathbf{v}_E, \mathbf{i}_I)$$

Combining these equations with the above hybrid representation, we obtain the network equation:

$$\underset{\sim}{f}(\underset{\sim}{v}_E,\underset{\sim}{i}_I) \triangleq \begin{bmatrix} \hat{\underset{\sim}{i}}_E(\underset{\sim}{v}_E,\underset{\sim}{i}_I) \\ \hat{\underset{\sim}{v}}_I(\underset{\sim}{v}_E,\underset{\sim}{i}_I) \end{bmatrix} + \underset{\sim}{H}\begin{bmatrix} \underset{\sim}{v}_E \\ \underset{\sim}{i}_I \end{bmatrix} - \underset{\sim}{s} = \underset{\sim}{0} \ . \tag{1}$$

In the general case where the n-port N contains also linear controlled sources, the hybrid n-port representation defined by Eq. (1) can be generated efficiently in most cases by topological methods [12-15]. The hybrid matrix $\underset{\sim}{H}$, however, may or may not satisfy [P1]-[P4].

Example 1: Consider the simple transistor circuit shown in Fig. 1(a). Replace the transistor by its Ebers-Moll model as shown in Fig. 1(b). Extracting the diodes as voltage ports and imbedding the linear controlled sources within the 2-port N as shown in Fig. 1(c), we obtain a hybrid representation:

$$\underset{\sim}{f}(v_1,v_2) = \begin{bmatrix} \hat{i}_1(v_1) \\ \hat{i}_2(v_2) \end{bmatrix} + \begin{bmatrix} .1010\times10^{-2} & -.9901\times10^{-3} \\ -.1980\times10^{-6} & .1980\times10^{-4} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} .1198\times10^{-1} \\ -.1369\times10^{-3} \end{bmatrix} = \underset{\sim}{0}.$$

In this case, the nonlinear functions $i_k = \hat{i}_k(v_k) = I_0(e^{v_k/V_T}-1)$ where I_0 and V_T are constants for k = 1.2 are nonlinear diagonal maps, i.e., i_k is a function of v_k only.

Example 2: Consider the same circuit in Example 1. Extract each "diode-controlled source combination" as a voltage port as shown in Fig. 1(d). The hybrid representation is then given by

$$\underset{\sim}{f}(v_1,v_2) = \begin{bmatrix} \hat{i}_1(v_1,v_2) \\ \hat{i}_2(v_1,v_2) \end{bmatrix} + \begin{bmatrix} 1.01\times10^{-3} & 1.0\times10^{-3} \\ -1.0\times10^{-3} & 1.0\times10^{-3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} 1.15\times10^{-3} \\ -1.20\times10^{-3} \end{bmatrix} = \underset{\sim}{0},$$

where $\hat{i}_1(v_1,v_2) = \hat{i}(v_1) - \alpha_F\hat{i}(v_2)$ and $\hat{i}_2(v_1,v_2) = \hat{i}(v_2) - \alpha_R\hat{i}(v_1)$, $\hat{i}(v) \triangleq I_0(e^{v/V_T}-1)$. In this case, the ports are "coupled resistors" and the nonlinear functions $\hat{i}_k(\cdot)$ are no longer diagonal. However, the hybrid matrix $\underset{\sim}{H}$ satisfies all [P1]-[P4].

It is well-known that the nonlinear function $[\hat{i}_1(\cdot),\hat{i}_2(\cdot)]^T$ in this example of transistor models is passive. That is, if $i_k = \hat{i}_k(v_1,v_2)$, then $[v_1,v_2][i_1,i_2]^T \geq 0$ for all $[v_1,v_2] \in \mathbb{R}^2$.

The preceiding examples show that the extraction of elements as ports is not unique. For example, consider the subnetwork $\mathcal{N}'$ shown in Fig. 2(a). We can extract the nonlinear resistor R as a current port. The characteristic of R as shown in Fig. 2(b) is a c.c. type B two-terminal resistor. On the other hand, we can consider $\mathcal{N}'$ as a 3-terminal element and pull out terminals 1-3 and 2-3 as two voltage ports. We then replace $\mathcal{N}'$ with a pair of ports which are "coupled" as shown in Fig. 2(c). In this particular example, we shall see in Theorem 5 that the constitutive relations of the ports satisfy an interesting property.

To simplify notations, we frequently do not differentiate the voltage port variables and the current port variables and simply write $\underset{\sim}{z} \triangleq [\underset{\sim}{v}_E,\underset{\sim}{i}_I]^T \in \mathbb{R}^n$. Equation (1) then becomes

$$\boxed{\underset{\sim}{f}(\underset{\sim}{z}) \triangleq \underset{\sim}{g}(\underset{\sim}{z}) + \underset{\sim}{H}\underset{\sim}{z} - \underset{\sim}{s} = \underset{\sim}{0}} \tag{2}$$

where $\underset{\sim}{g}(\cdot) = [\hat{\underset{\sim}{i}}_E(\cdot),\hat{\underset{\sim}{v}}_I(\cdot)]^T$.

As is implied by Theorem 1, there exist networks which do not possess the hybrid representation defined by Eq. (2). A more general representation is given by [8]:

$$\boxed{\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{H}_1\underset{\sim}{g}(\underset{\sim}{z}) + \underset{\sim}{H}_2\underset{\sim}{z} - \underset{\sim}{s} = \underset{\sim}{0},} \tag{3}$$

where $\underset{\sim}{H}_1$ and $\underset{\sim}{H}_2$ are constant n×n matrices and $\underset{\sim}{z}$ and $\underset{\sim}{g}(\underset{\sim}{z})$ are defined as in Eq. (2).

3.2 The Existence of Solutions

We now present some useful theorems on the existence of solutions of Eqs. (1), (2) and (3). As we have remarked in Sec. I, the theory of the degree of a map will play an important role in the proofs. We will show that the vector field $\tilde{f}$ is homotopic to an odd field over a symmetric region D and hence $d(\tilde{f};\tilde{0},D)$ is an odd integer. It then follows from Property 7 in Sec. II that $\tilde{f}(\tilde{z}) = \tilde{0}$ possesses at least one solution in D. Furthermore, if the Jacobian matrix $J_{\tilde{f}}$ is nonsingular at each solution then there is an odd number of solutions. All of these solutions are structurally stable because their indices are either 1 or -1. Thus, suppose we have already found two solutions $\tilde{z}_1$ and $\tilde{z}_2$ and $J_{\tilde{f}}(\tilde{z}_1)$ and $J_{\tilde{f}}(\tilde{z}_2)$ are both nonsingular, then there is at least one more solution which is also structurally stable.

Solutions which are not structurally stable are not continuously dependent of $\tilde{f}$. A slight perturbation of $\tilde{f}$ may preclude its existence immediately. For example, consider the tunnel diode circuit shown in Fig. 3. The network equation is given by

$$f(v) = \hat{i}(v) + \frac{v}{R} - \frac{E}{R} = 0,$$

where $\hat{i}(v)$ is the characteristic of the tunnel diode shown in Fig. 3(b). Number of solutions of this circuit depends on the values of E and R. For example, for $(E,R) = (E_1,R_1)$ as shown in Fig. 3(c), there are three solutions v_1, v_2 and v_3 with indices 1, -1 and 1 respectively. The degree $d(f,0,[-E_1,E_1]) = 1$ and all solutions are structurally stable. On the other hand, for $(E,R) = (E_2,R_2)$ as shown in Fig. 3(d), there are only two solutions v_1' and v_2'. Solution v_2' is not structurally stable because the index of v_2' is zero. The degree $d(f,0,[-E_2,E_2]) = 1$, but in this case there is only one solution v_1' which continuously varies with E and R.

The preceding example and Property 11 of Sec. II show that a solution which is not structurally stable is pathological. Consequently, our emphasis throughout this paper will be focused on predicting the existence of one or more structurally stable solutions.

Theorem 2. Let $\mathcal{N}$ be a network described by Eq. (2). Assume that

(i) $\underset{\sim}{g}(\cdot)$ is eventually passive, and

(ii) the hybrid matrix $\underset{\sim}{H}$ is positive definite.

Then $\mathcal{N}$ has at least one solution. Furthermore, if the Jacobian matrix is nonsingular at each solution, then there is an odd number of solutions all of which are structurally stable.

Proof: Define $\underset{\sim}{h}(\underset{\sim}{z},\lambda) \overset{\Delta}{=} \lambda\underset{\sim}{f}(\underset{\sim}{z}) + (1-\lambda)\underset{\sim}{z}$, $\lambda \in [0,1]$.

Premultiplying $\underset{\sim}{h}(\underset{\sim}{z},\lambda)$ by $\underset{\sim}{z}^T$, we obtain

$$\underset{\sim}{z}^T\underset{\sim}{h}(\underset{\sim}{z},\lambda) = \lambda\underset{\sim}{z}^T\underset{\sim}{f}(\underset{\sim}{z}) + (1-\lambda)\,\|\underset{\sim}{z}\|_2^2 = \lambda\underset{\sim}{z}^T\underset{\sim}{g}(\underset{\sim}{z}) + \lambda\underset{\sim}{z}^T\underset{\sim}{H}\underset{\sim}{z} - \lambda\underset{\sim}{z}^T\underset{\sim}{s} + (1-\lambda)\|\underset{\sim}{z}\|_2^2 .$$

Since $\underset{\sim}{g}(\cdot)$ is eventually passive, there exists an $r_1 > 0$ such that $\|\underset{\sim}{z}\| > r_1$ implies $\underset{\sim}{z}^T\underset{\sim}{g}(\underset{\sim}{z}) \geq 0$. Since $\underset{\sim}{H}$ is positive definite, $\underset{\sim}{z}^T\underset{\sim}{H}\underset{\sim}{z} \geq \gamma\,\|\underset{\sim}{z}\|^2$ for some $\gamma > 0$. Therefore, there is an $r_2 > 0$ such that $\underset{\sim}{z}^T\underset{\sim}{H}\underset{\sim}{z} - \underset{\sim}{z}^T\underset{\sim}{s} > 0$ for all $\underset{\sim}{z}$, $\|\underset{\sim}{z}\| > r_2$. Let $r = \max\{r_1,r_2\}$, then we have

$$\underset{\sim}{z}^T\underset{\sim}{h}(\underset{\sim}{z},\lambda) > 0 \quad \text{for all } \underset{\sim}{z} \in S(\underset{\sim}{0},r) \text{ and } \lambda \in [0,1].$$

Hence $\underset{\sim}{f}$ is homotopic to the identity map over $B(\underset{\sim}{0},r) \overset{\Delta}{=} \{\underset{\sim}{z} \in \mathbb{R}^n ; \|\underset{\sim}{z}\| < r\}$. This implies that $d\big(\underset{\sim}{f};\underset{\sim}{0},B(\underset{\sim}{0},r)\big) = 1$ and the conclusion follows. ∎

Corollary 1. Let $\mathcal{N}$ be a network containing only two-terminal resistors. If all nonlinear resistors are eventually passive and never form any loops or cut sets, then the conclusion of Theorem 2 is true.

Proof: Extract the nonlinear resistors as ports and derive the hybrid equation Eq. (2). Since the nonlinear resistors do not form any loops or cut sets, $\underset{\sim}{H}$ exists and is positive definite. ∎

Corollary 2. Let $\mathcal{N}$ be a network containing diodes, transistors, positive linear resistors and independent voltage and/or current sources. Replace each transistor by its "passive" Ebers-Moll model [16]. Extract all diodes and each "diode-controlled source combination" as ports as in Example 2 of Sec. 3.1. If the ports do not form any loops or cut sets, then the conclusion of Theorem 2 is true.

Proof: By assumption, the hybrid matrix $\underset{\sim}{H}$ exists and is positive definite. On the other hand, since the extracted diodes and transistors are passive, so is the function $\underset{\sim}{g}(\cdot)$. ∎

Example. Consider the flip-flop circuit shown in Fig. (4), where the transistors are represented by a passive Ebers-Moll circuit model. Upon open circuiting the capacitors, we obtain a resistive circuit satisfying the hypotheses of the Corollary. It follows that except for some exact combination of element values, this circuit must have an odd number of solutions. Indeed, it is well known that depending on the values of the resistances, this circuit can have either one or three solutions.

Theorem 3. Let $\mathcal{N}$ be a network described by Eq. (2). Assume that

(i) $\underset{\sim}{g}(\cdot)$ is bounded in $\mathbb{R}^n$, and

(ii) the hybrid matrix $\underset{\sim}{H}$ is nonsingular.

Then $\mathcal{N}$ has at least one solution. Furthermore, if the Jacobian matrix J_f is nonsingular at each solution, then there is an odd number of solutions all of which are structurally stable.

Proof: Consider the vector field on $\mathbb{R}^n$:

$$\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{g}(\underset{\sim}{z}) + \underset{\sim}{H}\underset{\sim}{z} - \underset{\sim}{s}.$$

Premultiply $\underset{\sim}{f}(\underset{\sim}{z})$ by $(\underset{\sim}{H}\underset{\sim}{z})^T$, we obtain

$$\underset{\sim}{z}^T\underset{\sim}{H}^T\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{z}^T\underset{\sim}{H}^T\underset{\sim}{g}(\underset{\sim}{z}) + \underset{\sim}{z}^T\underset{\sim}{H}^T\underset{\sim}{H}\underset{\sim}{z} - \underset{\sim}{z}^T\underset{\sim}{H}^T\underset{\sim}{s}$$

Since $\underset{\sim}{H}$ is nonsingular, there exists a $\gamma > 0$ such that $\underset{\sim}{z}^T\underset{\sim}{H}^T\underset{\sim}{H}\underset{\sim}{z} \geq \gamma\|\underset{\sim}{z}\|^2$. Since $\underset{\sim}{g}(\cdot)$ is bounded in $\mathbb{R}^n$ there is an $r > 0$ such that

$$\underset{\sim}{z}^T\underset{\sim}{H}^T\underset{\sim}{f}(\underset{\sim}{z}) > 0 \quad \text{for all } \underset{\sim}{z} \in S(\underset{\sim}{0},r).$$

Hence, on the sphere $S(\underset{\sim}{0},r)$, $\underset{\sim}{f}(\underset{\sim}{z})$ and $\underset{\sim}{f}(-\underset{\sim}{z})$ are both nonzero and do not point in the same direction. Hence the conclusion follows from Property 8 of Sec. II. ∎

Remarks. 1. Condition (i) is true if all nonlinear resistors are two-terminal type B resistors, i.e., the v-i curves saturate and become bounded both from above and below.

2. Even if the nonlinear resistors are not of type B, we can often invoke the no-gain property for the nonlinear element [13] and replace each type U or each type H resistor by a type B resistor such that their v-i curves coincide within a bounded region which contains the solutions. This region is obtained by considering the no-gain properties and the magnitudes of the independent sources. Then as long as $\underset{\sim}{H}$ is nonsingular, $\mathcal{N}$ has at least one solution. (See Sec. V for an application to transistor circuits.)

Theorem 4. Let $\mathcal{N}$ be a network described by Eq. (1). Assume that the nonlinear resistors across the ports are not coupled and that the dc conductances of all v.c. resistors and the dc resistances of all c.c. resistors tend to infinity, i.e.,

$$\lim_{|v_k|\to\infty} \hat{i}_k(v_k)/v_k = \infty \qquad k \in E \text{ , and}$$

$$\lim_{|i_\ell|\to\infty} \hat{v}_\ell(i_\ell)/i_\ell = \infty \qquad \ell \in I.$$

Then $\mathcal{N}$ has at least one solution. Furthermore, if the Jacobian matrix is nonsingular at each solution, then there is an odd number of solutions all of which are structurally stable.

Proof: First, there exists a $\gamma > 0$ such that $\left\|\underset{\sim}{H}\begin{bmatrix}\underset{\sim}{v}_E\\ \underset{\sim}{i}_I\end{bmatrix}\right\| < \gamma\left\|\begin{bmatrix}\underset{\sim}{v}_E\\ \underset{\sim}{i}_I\end{bmatrix}\right\|$. By assumption, for any $\gamma' > \gamma$ there is an $r_1 > 0$ such that $\left\|\begin{bmatrix}\underset{\sim}{v}_E\\ \underset{\sim}{i}_I\end{bmatrix}\right\| > r_1$ implies $\left\|\begin{bmatrix}\hat{\underset{\sim}{i}}_E(\underset{\sim}{v}_E)\\ \hat{\underset{\sim}{v}}_I(\underset{\sim}{i}_I)\end{bmatrix}\right\| > \gamma'\left\|\begin{bmatrix}\underset{\sim}{v}_E\\ \underset{\sim}{i}_I\end{bmatrix}\right\|$ Furthermore, there exists an $r_2 > r_1$ such that $\begin{bmatrix}\hat{\underset{\sim}{i}}_E(\underset{\sim}{v}_E)\\ \hat{\underset{\sim}{v}}_I(\underset{\sim}{i}_I)\end{bmatrix}$ and $\begin{bmatrix}\hat{\underset{\sim}{i}}_E(-\underset{\sim}{v}_E)\\ \hat{\underset{\sim}{v}}_I(-\underset{\sim}{i}_I)\end{bmatrix}$ are both nonzero and do not point in the same direction for all $\begin{bmatrix}v_E\\ i_I\end{bmatrix}$ satisfying $\left\|\begin{bmatrix}\underset{\sim}{v}_E\\ \underset{\sim}{i}_I\end{bmatrix}\right\| \geq r_2$. Moreover, since γ' is arbitrary, the first term in Eq. (1) eventually dominates the remaining terms. Hence $\begin{bmatrix}\hat{\underset{\sim}{i}}_E(\cdot)\\ \hat{\underset{\sim}{v}}_I(\cdot)\end{bmatrix}$ is the principal part of $\underset{\sim}{f}$ on $S(\underset{\sim}{0},r_2)$. Therefore $\underset{\sim}{f}$ is homotopic to an odd field and the conclusion follows from Property 7 of Sec. II. ■

Remarks: 1. This theorem represents a slight generalization of an existence theorem first proved by Roska and Klimo [18].

2. This theorem obviously remains valid when the limit $+\infty$ in the hypotheses is changed to $-\infty$.

Theorem 5. Let $\mathcal{N}$ be a network described by Eq. (2). Assume that

(i) each component $g_j(z_1,z_2,\ldots,z_n)$ of $\underset{\sim}{g}$ satisfies

$$\lim_{z_j\to\pm\infty} g_j(z_1,z_2,\ldots,z_n) = \pm\infty \text{ , respectively,}$$

(ii) there exists an $M > 0$ and a $B > 0$ such that

$$|z_j| > M \Rightarrow z_j g_j(z_1,\ldots,z_n) > 0 \qquad j = 1,2,\ldots,n,$$

and

$$|z_j| \le M \Rightarrow |g_j(z_1,\ldots,z_n)| < B$$

(iii) for each diagonal matrix $\underset{\sim}{D} = \text{diag}\,(d_j)$ in which $d_j = 1$ or -1 for each j, there exists a $\underset{\sim}{p} \in \mathbb{R}^n$, $\underset{\sim}{p} > 0$ such that $\underset{\sim}{p}^T \underset{\sim}{D}\underset{\sim}{H}\underset{\sim}{D} \geqq 0$. Then $\mathcal{N}$ has at least one solution. Furthermore, if the Jacobian matrix $\underset{\sim}{J}_f$ is non-singular at each solution, then there is an odd number of solutions all of which are structurally stable.

Proof: There are 2^n distinct diagonal matrices $\underset{\sim}{D}_k$ in (iii) and we have to consider at most 2^{n-1} distinct corresponding $\underset{\sim}{p}_k$'s. Pick these $\underset{\sim}{p}_k$'s and fix them. Let $\underset{\sim}{z} \in \mathbb{R}^n$, $\bar{\underset{\sim}{z}} \triangleq \underset{\sim}{D}_z \bar{\underset{\sim}{z}}$ where $\underset{\sim}{D}_z = \text{diag}\,(\text{sgn}\, z_j)$ and $\bar{\underset{\sim}{z}} = [|z_1|,|z_2|,\ldots,|z_n|]^T$. Let $\underset{\sim}{p}_z$ be the corresponding vector such that $\underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{H} \underset{\sim}{D}_z \geqq 0$. Consider the quantity

$$\phi(\underset{\sim}{z}) \triangleq \underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{g}(\underset{\sim}{z}) - |\underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{s}| \quad ,$$

since $(\underset{\sim}{p}_z^T \underset{\sim}{D}_z)_j z_j > 0$ for all $z_j \neq 0$, $j = 1,2,\ldots,n$, by assumptions (i) and (ii), $\underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{g}(\underset{\sim}{z})$ is bounded from below for all $\underset{\sim}{z} \in \mathbb{R}^n$. Since there are only finitely many distinct $\underset{\sim}{p}_z$ and $\underset{\sim}{D}_z$, it follows from assumption (i) that there exists an $r > 0$ such that

$$\phi(\underset{\sim}{z}) > 0 \quad \text{for all } \underset{\sim}{z} \in S(\underset{\sim}{0},r).$$

Now consider the vector field

$$\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{g}(\underset{\sim}{z}) + \underset{\sim}{H}\underset{\sim}{z} - \underset{\sim}{s}\,.$$

Premultiply $\underset{\sim}{f}(\underset{\sim}{z})$ by $\underset{\sim}{p}_z^T \underset{\sim}{D}_z$, we obtain

$$\underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{g}(\underset{\sim}{z}) + \underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{H} \underset{\sim}{D}_z \bar{\underset{\sim}{z}} - \underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{s} \geq \underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{g}(\underset{\sim}{z}) - \underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{s} \quad \text{by (iii)}$$

$$\geq \phi(\underset{\sim}{z}) > 0 \quad \text{for all } \underset{\sim}{z} \in S(\underset{\sim}{0},r)$$

Similarly, $-\underset{\sim}{p}_z^T \underset{\sim}{D}_z \underset{\sim}{f}(-\underset{\sim}{z}) \geq \phi(-\underset{\sim}{z}) > 0$, for all $\underset{\sim}{z} \in S(\underset{\sim}{0},r)$. Therefore, $\underset{\sim}{f}(\underset{\sim}{z})$ and $\underset{\sim}{f}(-\underset{\sim}{z})$ are both nonzero and do not point in the same direction for all $\underset{\sim}{z} \in S(\underset{\sim}{0},r)$. Hence, the conclusion follows from Property 8 in Sec. II. ∎

Remarks: 1. Conditions (i) and (ii) are satisfied by many practical circuit. For example, consider the two-port shown in Fig. 2(c). Since by KVL

$$i_j = \frac{v_j - \tanh(i_j + i_j')}{R_j} \qquad j = 1,2$$

where $i_j' = i_2$ if $j = 1$ and $i_j' = i_1$ if $j = 2$, it is easy to see that

$$\lim_{v_j \to \pm\infty} \hat{i}_j(v_1,v_2) = \pm\infty \quad \text{respectively, and}$$

$$|v_j| > 1 \Rightarrow v_j \hat{i}_j(v_1,v_2) > 0, \text{ and}$$

$$|v_j| \leq 1 \Rightarrow |\hat{i}_j(v_1,v_2)| < \frac{2}{R_j} \qquad j = 1,2.$$

2. Both positive semi-definite matrices and diagonally dominant matrices satisfy condition (iii). [6,9].

Example. Let $\mathcal{N}$ be a network containing only two-terminal nonlinear and positive linear resistors and independent sources. Let all nonlinear resistors be grounded and let $\mathcal{N}$ be described by a conductance representation $\underset{\sim}{f}(\underset{\sim}{v}) = \underset{\sim}{g}(\underset{\sim}{v}) + \underset{\sim}{G}\underset{\sim}{v} - \underset{\sim}{s} = \underset{\sim}{0}$, where $\underset{\sim}{g}(\underset{\sim}{v})$ is the constitutive relation of all nonlinear resistors considered as voltage ports and $\underset{\sim}{G}$ is the conductance matrix. If all nonlinear resistors are of type U, then the conclusion of Theorem 5 is true. Notice that in this case $\underset{\sim}{G}$ is diagonally dominant.

Before we present the next theorem, let us first introduce a special class of matrices.

Definition 4. An n×n matrix A is said to belong to "the class P_0" if A satisfies one of the following equivalent conditions:

(i) All principal minors of A are nonnegative.

(ii) To each $x \in \mathbb{R}^n$, $x \neq 0$; there exists an index k such that $x_k \neq 0$ and $x_k(Ax)_k \geq 0$. (In particular, $a_{ii} \geq 0$, i = 1,2,...,n.)

(iii) To each $x \in \mathbb{R}^n$, $x \neq 0$; there exists a diagonal matrix $D_x \geq 0$ such that $\langle x, D_x x \rangle > 0$ and $\langle Ax, D_x x \rangle \geq 0$, where $\langle x,y \rangle \triangleq \sum_{j=1}^{n} x_j y_j$.

(iv) Every real eigenvalue of A and of each principal submatrix of A is nonnegative.

(v) For each diagonal matrix $D > 0$, $\det(A+D) \neq 0$.

Properties of P_0 matrices have been discussed in detail in [6] and [7].

Theorem 6. Let $\mathcal{N}$ be a network described by Eq. (2). Assume that (i) for each k > 0, there exists a constant (which may depend on K) M > 0 such that $|z_j| > M \Rightarrow z_j g_j(z) > 0$ and $|g_j(z)| > K$ for all j = 1,2,...,n.

(ii) the hybrid matrix $H \in P_0$.

Then $\mathcal{N}$ has at least one solution. Furthermore, if the Jacobian matrix J_f is nonsingular at each solution, then there is an odd number of solutions all of which are structurally stable.

Proof: As in the previous theorem, we prove that on a sphere $S(0,r)$ with r sufficiently large, $f(z) \neq 0$ and $f(z)$ and $f(-z)$ do not point in the same direction. To simplify the notation, we will prove the theorem for the case n=2 and 3. The same procedure applies, mutatis mutandis, for the general case. Consider first n=2 and consider the vector field associated with Eq. (2):

$$f(z_1,z_2) = \begin{bmatrix} g_1(z_1,z_2) \\ g_2(z_1,z_2) \end{bmatrix} + \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

Define the constants $M_j^{(k)}$, $j = 1,2$; $k = 1,2$ as follows:

(1) According to assumption (i), for each $j = 1,2$, there exists a constant $M_j^{(1)} > 0$ such that

$$z_j g_j(\underset{\sim}{z}) > 0 \text{ and } |g_j(\underset{\sim}{z})| > |s_j| \quad \text{for all } |z_j| \geq M_j^{(1)} .$$

(2) Since $\underset{\sim}{H} \in P_0$, $h_{jj} \geq 0$. For each $j = 1,2$, there exists a constant $M_j^{(2)} \geq M_j^{(1)}$ such that

$$\left| |g_j(\underset{\sim}{z}) + h_{jj}z_j| - |h_{j\ell}|\cdot M_\ell^{(1)} \right| > |s_j|, \text{where } \ell \neq j$$

for all $|z_j| \geq M_j^{(2)}$.

Let $r = \max\{M_j^{(2)}, j = 1,2\}$. Consider $\underset{\sim}{f}(\underset{\sim}{z})$ on the sphere $S(\underset{\sim}{0},r)$. Since $\underset{\sim}{H} \in P_0$, by (ii) of Def. 4, there exists at least one component of $\underset{\sim}{z} \in S(\underset{\sim}{0},r)$, say z_2, $z_2 \neq 0$ and $z_2(\underset{\sim}{H}\underset{\sim}{z})_2 \geq 0$. There are two possibilities:

(a) $|z_2| \geq M_2^{(1)}$. In this case, by (1)

$$|g_2(\underset{\sim}{z}) + (\underset{\sim}{H}\underset{\sim}{z})_2| \geq |g_2(\underset{\sim}{z})| > |s_2|.$$

Hence, $\underset{\sim}{f}(\underset{\sim}{z})$ and $\underset{\sim}{f}(-\underset{\sim}{z})$ are both nonzero and do not point in the same direction since their second components are nonzero and assume opposite signs.

(b) $|z_2| < M_2^{(1)}$. In this case $|z_1| = r > M_1^{(2)}$. By (2)

$$|g_1(\underset{\sim}{z}) + h_{11}z_1 + h_{12}z_2| \geq \left| |g_1(\underset{\sim}{z}) + h_{11}z_1| - |h_{12}|\cdot M_2^{(1)} \right| > |s_1|.$$

Hence, $\underset{\sim}{f}(\underset{\sim}{z})$ and $\underset{\sim}{f}(-\underset{\sim}{z})$ are both nonzero and do not point in the same direction since now their first components are nonzero and assume opposite signs. In any case, $\underset{\sim}{f}(\underset{\sim}{z}) \neq \underset{\sim}{0}$, $\underset{\sim}{f}(\underset{\sim}{z})$ and $\underset{\sim}{f}(-\underset{\sim}{z})$ never point in the same direction on $S(\underset{\sim}{0},r)$.

We now present the proof for the case n=3 in greater detail because the same procedure can be readily extended to the general case. Define the constant $M_j^{(k)}$, j = 1,2,3 and k = 1,2,3 successively as follows:

(1) By assumption (i), for each j = 1,2,3, there exists a constant $M_j^{(1)} > 0$ such that

$$z_j g_j(\underset{\sim}{z}) > 0 \text{ and } |g_j(\underset{\sim}{z})| > |s_j|, \text{ for all } |z_j| \geq M_j^{(1)} .$$

(2) By assumptions (i) and (ii), for each j = 1,2,3, there exists a constant $M_j^{(2)} \geq M_j^{(1)}$ such that

$$|g_j(\underset{\sim}{z})| > [|s_j| + \max_{\ell \neq j} |h_{j\ell}| \cdot M_\ell^{(1)}], \text{ for all } |z_j| \geq M_j^{(2)} .$$

(3) Similarly, for each j = 1,2,3, there exists a constant $M_j^{(3)} \geq M_j^{(2)}$ such that

$$|g_j(\underset{\sim}{z})| > [|s_j| + \sum_{\ell \neq j} |h_{j\ell}| \cdot M_\ell^{(2)}], \text{ for all } |z_j| \geq M_j^{(3)} .$$

Now consider $\underset{\sim}{f}(\underset{\sim}{z})$ on the sphere $S(\underset{\sim}{0},r)$ where $r = \max\{M_j^{(3)}, j = 1,2,3\}$. Since $\underset{\sim}{H} \in P_0$, there is at least one component of $\underset{\sim}{z}$, say $z_3 \neq 0$ and $z_3(\underset{\sim}{H}\underset{\sim}{z})_3 \geq 0$. There are two possibilities:

(a) $|z_3| \geq M_3^{(1)}$. Then, by (1) we obtain

$$|g_3(\underset{\sim}{z}) + (\underset{\sim}{H}\underset{\sim}{z})_3| \geq |g_3(\underset{\sim}{z})| > |s_3|.$$

Hence $f_3(\underset{\sim}{z})f_3(-\underset{\sim}{z}) < 0$.

(b) $|z_3| < M_3^{(1)}$. Then consider the subsystem:

$$\underset{\sim}{f}'(\underset{\sim}{z}) = \begin{bmatrix} g_1(\underset{\sim}{z}) \\ g_2(\underset{\sim}{z}) \end{bmatrix} + \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \left(\begin{bmatrix} h_{13} \\ h_{23} \end{bmatrix} z_3 - \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \right)$$

Let $\underset{\sim}{z}' = [z_1, z_2]^T$ and $\underset{\sim}{H}' = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$.

Again, since the submatrix $\underset{\sim}{H}'$ of $\underset{\sim}{H}$ is also in P_0, there is at least one component of $\underset{\sim}{z}'$, say $z_2 \neq 0$ and $z_2(\underset{\sim}{H}'\underset{\sim}{z}')_2 \geq 0$. Again there are two cases:

(a') $|z_2| \geq M_2^{(2)}$. By (2),

$$|g_2(\underset{\sim}{z}') + (\underset{\sim}{H}'\underset{\sim}{z}')_2| \geq |g_2(\underset{\sim}{z}')| > [|s_2| + |h_{23}|\cdot M_3^{(1)}] \geq [|s_2| + |h_{23}|\cdot|z_3|]$$

Hence $f_2(\underset{\sim}{z})f_2(-\underset{\sim}{z}) < 0$.

(b') $|z_2| < M_2^{(2)}$. Then $|z_1| = r \geq M_1^{(3)}$. Consider the subsystem

$$f_1(\underset{\sim}{z}) = g_1(\underset{\sim}{z}) + h_{11}z_1 + (h_{12}z_2+h_{13}z_3-s_1)$$

By (3), we obtain

$$|g_1(\underset{\sim}{z}) + h_{11}z_1| \geq |g_1(\underset{\sim}{z})| > [|s_1| + \sum_{\ell\neq 1} |h_{1\ell}|\cdot M_\ell^{(2)}]$$

$$\geq [|s_1| + \sum_{\ell\neq 1} |h_{1\ell}|\cdot|z_\ell|].$$

Hence $f_1(\underset{\sim}{z})f_1(-\underset{\sim}{z}) < 0$. Therefore, the conclusion is true for n=3.

Now, for the general case, define $M_j^{(k)} > 0$, $M_j^{(k+1)} \geq M_j^{(k)}$; $j = 1,2,\ldots,n$, $k = 1,2,\ldots,n$ as follows:

$$|z_j| > M_j^{(k)} \Rightarrow |g_j(\underset{\sim}{z})| > [|s_j| + \max_{\substack{\ell_m\in\{1,2,..,n\}\\ \ell_m\neq j}} \sum_{m=1}^{k-1} |h_{j\ell_m}|\cdot M_{\ell_m}^{(k-1)}]$$

where it is understood that there is no summation sign for k=1. By the same procedure we can prove that $\underset{\sim}{f}(\underset{\sim}{z}) \neq \underset{\sim}{0}$ and $\underset{\sim}{f}(\underset{\sim}{z})$ and $\underset{\sim}{f}(-\underset{\sim}{z})$ do not point in the same direction for all $\underset{\sim}{z} \in S(\underset{\sim}{0},r)$, where $r = \max_{1\leq j\leq n}\{M_j^{(n)}\}$. ∎

As a special case of Theorem 6, we have

Corollary 3. Let $\mathcal{N}$ be a network described by Eq. (2). Assume that (i) $g(\cdot)$ is diagonal and each component $g_j(z_j)$ of $\underset{\sim}{g}$ is of type U, and

(ii) $\underset{\sim}{H} \in P_0$.

<u>Then the conclusion of Theorem 6 is true</u>.

To illustrate the application of the preceding results, we now present two well-known results in the form of examples and give a new and simple proof for each.

<u>Example 1</u>. Let $\mathcal{N}$ be a network containing strictly increasing, type U, uncoupled resistors and independent sources. Pick a tree $\mathcal{J}$ and the corresponding co-tree $\mathcal{L}$. Then the fundamental loop matrix[6] $\underset{\sim}{B} \triangleq [\underset{\sim}{1}_{\mathcal{L}\mathcal{L}}, \underset{\sim}{B}_{\mathcal{L}\mathcal{J}}]$ and the network equation is given by

$$\underset{\sim}{f}(\underset{\sim}{v}_{\mathcal{J}}, \underset{\sim}{i}_{\mathcal{L}}) = \begin{bmatrix} \hat{\underset{\sim}{i}}_{\mathcal{J}}(\underset{\sim}{v}_{\mathcal{J}}) \\ \hat{\underset{\sim}{v}}_{\mathcal{L}}(\underset{\sim}{i}_{\mathcal{L}}) \end{bmatrix} + \begin{bmatrix} 0 & -\underset{\sim}{B}^T_{\mathcal{L}\mathcal{J}} \\ \underset{\sim}{B}_{\mathcal{L}\mathcal{J}} & 0 \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}} \\ \underset{\sim}{i}_{\mathcal{L}} \end{bmatrix} - \begin{bmatrix} \underset{\sim}{j} \\ \underset{\sim}{e} \end{bmatrix} = \underset{\sim}{0},$$

where $[\underset{\sim}{j}, \underset{\sim}{e}]$ is the source vector and the nonlinear functions $\hat{\underset{\sim}{i}}_{\mathcal{J}}(\cdot)$ and $\hat{\underset{\sim}{v}}_{\mathcal{L}}(\cdot)$ are type U diagonal maps. Hence condition (i) of the corollary is satisfied. On the other hand, since a skew-symmetric matrix is always positive semidefinite and hence belongs to P_0, condition (ii) of the preceding corollary is also satisfied. It follows from the uniqueness property of monotone networks (Theorem 10) that there exists exactly one solution and this solution is structurally stable.

<u>Example 2</u>. Suppose now that the resistors are strictly increasing but are not necessarily of type U. Let each nonlinear resistor R_k be defined within a "box" $|v_k| \leq V_k$ and $|i_k| \leq I_k$, where V_k and I_k are any pre-specified maximum voltage and current bounds. If the resistors are passive and if each independent current (resp. voltage) source is connected in $\mathcal{N}$ via soldering-iron (resp. pliers) entry, then as long as the magnitudes of the sources are small enough, $\mathcal{N}$ has at least one structurally stable

---

[6] We number the branches in $\mathcal{L}$ first. $\underset{\sim}{1}_{\mathcal{L}\mathcal{L}}$ denotes an identity matrix.

solution.

Proof: Since all current sources are connected via soldering-iron entries, we can apply the i-shift and the nonlinear Norton's theorems to transform the current sources into equivalent voltage sources in series with passive nonlinear resistors [19] as long as their magnitudes are small enough. For example, a current source I in parallel with a resistor R_I characterized by $v_I = \hat{v}_I(i_I)$, as shown in Fig. 5(a) is equivalent to a voltage source $E = \hat{v}_I(I)$ in series with a passive nonlinear resistor R_E characterized by $i_E = \hat{i}_E(v_E) = \{\hat{v}_I^{-1}\left(v_E+\hat{v}_I(I)\right) - I\}$ as long as $|I|$ is small enough so that there exists a nontrivial range for v_E in which $\hat{v}_I^{-1}\left(v_E+\hat{v}_I(I)\right)$ is well-defined. It is easy to see that $\hat{i}_E(\cdot)$ is passive and strictly increasing. Hence, there is no loss of generality in our assuming that all sources are independent voltage sources. Since the resistors are passive, the no-gain property holds. Hence there exists a constant C_1 such that as long as the sum of the magnitudes of the voltage sources is less than C_1, the solution (if it exists) of each resistor R_k will fall within its range $|v_k| \leq V_k$. Similarly, there exists a constant C_2 such that as long as the sum of the magnitudes of the current sources is less than C_2, the solution (if it exists) of each resistor R_k will fall within its range $|i_k| \leq I_k$. Assume the sources satisfy these conditions. Now, if we extend the characteristic of each resistor beyond the box $|v_k| \leq V_k$, $|i_k| \leq I_k$ by a strictly increasing type U function, we have not perturbed any solution (if it exists). But by example 1 the new circuit has at least one structurally stable solution. Therefore, so does $\mathcal{N}$. ∎

For an alternative proof of example 2, see [13]. The bounds we obtained, however, are more flexible.

Remark: Condition (iii) in Theorem 5 defines a subclass of P_0 matrices by the following lemma.

Lemma 1. Let $\underset{\sim}{H} \in \mathbb{R}^{n\times n}$. If for each $\underset{\sim}{D} = \text{diag}(d_j)$, $d_j = \pm 1$, there exists a $\underset{\sim}{p} \in \mathbb{R}^n$, $\underset{\sim}{p} > 0$ such that $\underset{\sim}{p}^T\underset{\sim}{D}\underset{\sim}{H}\underset{\sim}{D} \geqq 0$, then $\underset{\sim}{H} \in P_0$.

Proof: We prove that for any diagonal matrix $\underset{\sim}{E} > 0$, $\det(\underset{\sim}{H}+\underset{\sim}{E}) \neq 0$. Then by (v) of Def. 3, $\underset{\sim}{H} \in P_0$. Suppose $\underset{\sim}{H} \notin P_0$. Let $\underset{\sim}{E}' > 0$ be a diagonal matrix such that $\det(\underset{\sim}{H}+\underset{\sim}{E}') = 0$. Hence there exists a $\underset{\sim}{z} \in \mathbb{R}^n$, $\underset{\sim}{z} \neq \underset{\sim}{0}$ such that $(\underset{\sim}{H}+\underset{\sim}{E}')\underset{\sim}{z} = \underset{\sim}{0}$. Since $\underset{\sim}{z} \triangleq \underset{\sim}{D}_z\bar{\underset{\sim}{z}}$, we have $\underset{\sim}{H}\underset{\sim}{D}_z\bar{\underset{\sim}{z}} + \underset{\sim}{E}'\underset{\sim}{D}_z\bar{\underset{\sim}{z}} = \underset{\sim}{0}$. By assumption, there is a $\underset{\sim}{p} > 0$ such that $\underset{\sim}{p}^T\underset{\sim}{D}_z\underset{\sim}{H}\underset{\sim}{D}_z \geqq 0$. Hence we obtain $\underset{\sim}{p}^T\underset{\sim}{D}_z\underset{\sim}{H}\underset{\sim}{D}_z\bar{\underset{\sim}{z}} + \underset{\sim}{p}^T\underset{\sim}{D}_z\underset{\sim}{E}'\underset{\sim}{D}_z\bar{\underset{\sim}{z}} = 0$. But $\underset{\sim}{p}^T\underset{\sim}{D}_z\underset{\sim}{H}\underset{\sim}{D}_z\bar{\underset{\sim}{z}} \geq 0$ and $\underset{\sim}{p}^T\underset{\sim}{D}_z\underset{\sim}{E}'\underset{\sim}{D}_z\bar{\underset{\sim}{z}} = \underset{\sim}{p}^T\underset{\sim}{E}'\bar{\underset{\sim}{z}} > 0$. Hence, we have a contradiction and $\det(\underset{\sim}{H}+\underset{\sim}{E}) \neq 0$. ∎

Remark: The class of matrices $\underset{\sim}{A}$ such that there exists an $\underset{\sim}{x} \geqq 0$, $\underset{\sim}{x} \neq \underset{\sim}{0}$ for which $\underset{\sim}{A}\underset{\sim}{x} \geqq 0$ is called S_0 [6]. If $\underset{\sim}{A} \in S_0$ and no submatrix of $\underset{\sim}{A}$ obtained by omitting at least one column belongs to S_0, then $\underset{\sim}{A}$ is said to be in class M. It can be shown that any matrix in the set $P_0 \cap M$ satisfies condition (iii) of Theorem 5. For example, any matrix -- also known as an M-matrix -- having non-positive off-diagonal elements and whose inverse matrix has only non-negative elements belongs to the class $P_0 \cap M$ [2].

In case the network contains also type H and type B nonlinar resistors, the following theorem is a generalization of Theorem 3 and a special case of Theorem 5.

Definition 5: Let $\underset{\sim}{g}: \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear diagonal map. The set $\mathcal{B}(\underset{\sim}{g})$ is defined as $\mathcal{B}(\underset{\sim}{g}) \triangleq \{\underset{\sim}{z} \in \mathbb{R}^n: \|\underset{\sim}{g}(\eta\underset{\sim}{z})\| \leq M < \infty \text{ as } \eta \to \infty\}$. It is evident that $\mathcal{B}(\underset{\sim}{g}) = \prod_{k=1}^{n} I_k$ is a product of intervals where $I_k = \{0\}$ for all $k \in U$, $I_k = [0,\infty)$ for all $k \in H$ and $I_k = (-\infty,\infty)$ for all $k \in B$.

Theorem 7. Let $\mathcal{N}$ be a network described by Eq. (2). Assume that

(i) $g(\cdot)$ is diagonal and all components $g_j(z_j)$ of g are eventually passive,

(ii) for each $D = \text{diag}(d_j)$ where $d_j = +1$ or -1, there exists a $p \in \mathbb{R}^n$, $p > 0$ such that $p^T DHD \geq 0$, and

(iii) $\mathcal{R}(g) \cap \mathcal{N}(H) = 0$, where $\mathcal{N}(H)$ is the null space of H; i.e., the zero vector is the only point in common.

Then $\mathcal{N}$ has at least one solution. Furthermore, if the Jacobian matrix J_f is nonsingular at each solution, then there is an odd number of solutions all of which are structurally stable.

Proof: Consider the vector field

$$f(z) = g(z) + Hz - s = g(z) + HD_z\bar{z} - s.$$

For each $z \in \mathbb{R}^n$, let $p_z > e \triangleq [2,2,\ldots,2]^T$ be a vector in $\mathbb{R}^n$ such that $p_z^T D_z H D_z \geq 0$. Since p_z is determined only by D_z, there are finitely many p_z's we have to consider. Choose these p_z's and fix them. Premultiply $f(z)$ by $p_z^T D_z$, we obtain

$$p_z^T D_z f(z) = p_z^T D_z g(z) + p_z^T D_z H D_z \bar{z} - p_z^T D_z s.$$

Since all $g_j(\cdot)$'s are eventually passive, by an argument similar to that used in the proof of Theorem 5, there exists an $r_1 > 0$ such that

$$p_z^T D_z f(z) > 0 \quad \text{for all} \quad z \in \mathcal{U} \tag{4}$$

where $\mathcal{U} \triangleq \{z \in \mathbb{R}^n$: there exists at least one component z_j, $|z_j| > r_1$ for some $j \in U$ or $z_j < -r_1$ for some $j \in H\}$.[7] Evidently $\mathcal{U} \cap \mathcal{R}(g) = \phi$.

[7] Here H means type H (half-bounded) resistors, not the hybrid matrix H. See Fig. 6 for a geometrical interpretation of the set $\mathcal{U}$.

It follows from $\mathcal{B}(g) \cap \mathcal{N}(H) = 0$ and the fact that $\mathcal{B}(g) = \prod_{k=1}^{n} I_k$ and $\mathcal{N}(H)$ is a subspace of $\mathbb{R}^n$ that there is an $r_2 > 0$ such that $\|z\| \geq r_2$ and $z \in \mathcal{N}(H) \Rightarrow z \in \mathcal{U}$. (See Appendix (1) for a detailed proof.) Let $r_3 > \max\{r_1, r_2\}$, then $\mathcal{N}(H) \cap S(0,r_3) \subset \mathcal{U}$. Let $\bar{\mathcal{U}} \triangleq \{z \in \mathbb{R}^n : z \notin \mathcal{U}\}$ be the complement of $\mathcal{U}$. Then there is a $\mu > 0$ such that $z^T H^T H z \geq \mu \|z\|^2$ for all $z \in \bar{\mathcal{U}} \cap S(0,r_3)$. The quantity μ depends on r_3. Since $\mathcal{N}(H)$ is a subspace of $\mathbb{R}^n$, $\mu = \mu(r_3)$ is an increasing function of r_3 with $\lim_{r_3 \to \infty} \mu = \mu_0 \triangleq \inf\{z^T H^T H z : z \in \mathcal{B}(g) \cap S(0,1)\}$. (See Appendix (2).) The sets $\mathcal{U}$, $\bar{\mathcal{U}}$, $\mathcal{B}(g)$ and $\mathcal{N}(H)$ are shown in Fig. 6 for the two dimensional case where g_1 is of type U and g_2 is of type H.

Now let $M \triangleq \inf_{z \in \mathcal{U}}\{z^T H^T g(z)\}$, clearly $M > -\infty$. Consider the vector field $f(z)$ on $\bar{\mathcal{U}}$, premultiplying $f(z)$ by $z^T H^T$, we obtain

$$z^T H^T f(z) = z^T H^T g(z) + z^T H^T H z - z^T H^T s.$$

Since $z^T H^T g(z) \geq M > -\infty$, there exists an $r_4 \geq r_3$ such that

$$z^T H^T f(z) > 0 \qquad \text{for all } z \in \bar{\mathcal{U}} \cap S(0,r_4). \tag{5}$$

Claim: There is an $r > 0$ such that $f(z) \neq 0$ and $f(z)$ and $f(-z)$ do not point in the same direction for all $z \in S(0,r)$.

Proof: Let $r \geq r_4$, consider $f(z)$ on $S(0,r)$. There are three cases:

(i) $z \in \mathcal{U}$ and $-z \in \mathcal{U}$: By Eq. (4), both $p_z^T D_z f(z) > 0$ and $-p_z^T D_z f(-z) > 0$ and hence the claim is true.

(ii) $z \in \bar{\mathcal{U}}$ and $-z \in \bar{\mathcal{U}}$: By Eq. (5), both $z^T H^T f(z) > 0$ and $-z^T H^T f(-z) > 0$ and hence the claim is true.

(iii) $z \in \bar{\mathcal{U}}$ and $-z \in \mathcal{U}$: In this case, consider

$$\left[\frac{1}{\|z^T H^T\|} z^T H^T + p_z^T D_z\right] f(z) = \frac{1}{\|z^T H^T\|} z^T H^T g(z) + p_z^T D_z g(z) + \frac{1}{\|z^T H^T\|} z^T H^T H z$$

$$+ p_z^T D_z H D_z \bar{z} - \frac{1}{\|z^T H^T\|} z^T H^T s - p_z^T D_z s \,. \qquad (6)$$

and

$$-\left[\frac{1}{\|z^T H^T\|} z^T H^T + p_z^T D_z\right] f(-z) = \frac{1}{\|z^T H^T\|}\left[-z^T H^T g(-z)\right] + \left[-p_z^T D_z g(-z)\right]$$

$$+ \frac{1}{\|z^T H^T\|} z^T H^T H z + p_z^T D_z H D_z \bar{z} + \frac{1}{\|z^T H^T\|} z^T H^T s + p_z^T D_z s. \qquad (7)$$

In Eq. (6), since every term on the right-hand side is bounded except $(1/\|z^T H^T\|) z^T H^T H z \geq \mu \|z\|$ and $p_z^T D_z H D_z \bar{z} \geq 0$, there exists an $r_5 \geq r_4$ such that $\|z\| \geq r_5$ implies $[(1/\|z^T H^T\|) z^T H^T + p_z^T D_z] f(z) > 0$. In Eq. (7), since $p_z > e \triangleq [2,2,\ldots,2]^T$ and the components of g are eventually passive, there is an $r_6 \geq r_5$ such that $\|z\| \geq r_6$ implies

$$-p_z^T D_z g(-z) > \frac{1}{\|z^T H^T\|}\left\{\left|-z^T H^T g(-z)\right| + \left|z^T H^T s\right|\right\} + \left|p_z^T D_z s\right| .$$

This inequality is true because the last two terms on the right-hand side are bounded by a constant, and the coefficient of each component of $g(-z)$ in the first term is less than one in magnitude. Hence, we have

$$-\left[\frac{1}{\|z^T H^T\|} z^T H^T + p_z^T D_z\right] f(-z) > 0.$$

Hence, let $r = r_6$ and the claim is true. Therefore, the conclusion of Theorem 7 follows from Property 8 in Sec. II. ∎

Theorem 8: Let $\mathcal{N}$ be a network described by Eq. (2). Assume that (i) the function $g(z) = [g_U(z_U), g_H(z_H), g_B(z_B)]^T$ satisfies

(a) there exists an $M > 0$ and a $B > 0$ such that

$$|z_j| > M \Rightarrow z_j g_j(\underset{\sim}{z}_U) > 0,$$

$$|z_j| \leq M \Rightarrow |g_j(\underset{\sim}{z}_U)| < B,$$

and

$$\lim_{z_j \to \pm\infty} g_j(\underset{\sim}{z}_U) = \pm\infty \text{ respectively, for all } j \in U.$$

(b) $\underset{\sim}{g}_H(\underset{\sim}{z}_H)$ is diagonal and eventually passive and each component $\underset{\sim}{g}_j(\underset{\sim}{z}_j)$ is of type H.

(c) $\underset{\sim}{g}_B(\underset{\sim}{z}_B)$ is bounded in $\mathbb{R}^n$ and eventually passive,

(ii) conditions (ii) and (iii) of Theorem 7 are satisfied.

Then $\mathcal{N}$ has at least one solution. Furthermore, if the Jacobian matrix $J_{\underset{\sim}{f}}$ is nonsingular at each solution, then there is an odd number of solutions all of which are structurally stable.

Proof: Similar to that given for Theorem 7.

Theorem 8 can be further generalized to the case described by Eq. (3); namely, networks described by equations of the form $\underset{\sim}{H}_1\underset{\sim}{g}(\underset{\sim}{z}) + \underset{\sim}{H}_2\underset{\sim}{z} - \underset{\sim}{s} = \underset{\sim}{0}$.

Theorem 9: Let $\mathcal{N}$ be a network described by Eq. (3). Assume that

(i) condition (i) of Theorem 8 is satisfied.

(ii) for each $\underset{\sim}{D} = \mathrm{diag}(d_j)$ where $d_j = \pm 1$, there exists a $\underset{\sim}{p} \in \mathbb{R}^n$, $\underset{\sim}{p} \geq 0$, such that $\underset{\sim}{p}^T\underset{\sim}{D}\underset{\sim}{H}_1\underset{\sim}{D} \geqq 0$, $\underset{\sim}{p}^T\underset{\sim}{D}\underset{\sim}{H}_2\underset{\sim}{D} \geqq 0$ and $\underset{\sim}{p}^T\underset{\sim}{D}(\underset{\sim}{H}_1+\underset{\sim}{H}_2)\underset{\sim}{D} > 0$.

(iii) $\mathcal{B}(\underset{\sim}{g}) \cap \mathcal{N}(\underset{\sim}{H}_2) = \underset{\sim}{0}$.

Then the conclusion for Theorem 8 holds.

Proof: This proof requires only a minor modification of that given for Theorem 7 and is therefore omitted.

Remarks: 1. As a special case of practical interest, a pair of matrices

$(\underset{\sim}{H}_1,\underset{\sim}{H}_2)$ is called a passive pair if $\underset{\sim}{H}_1\underset{\sim}{x} = \underset{\sim}{H}_2\underset{\sim}{y} \Rightarrow \underset{\sim}{x}^T\underset{\sim}{y} \geq 0$. A passive pair $(\underset{\sim}{H}_1,\underset{\sim}{H}_2)$ satisfies condition (ii) of Theorem 9 [7]. Passive pairs occur in many practical situations. For example, consider the network equation $\underset{\sim}{f}(\underset{\sim}{v}) = \underset{\sim}{H}_1\hat{\underset{\sim}{i}}(\underset{\sim}{v}) + \underset{\sim}{H}_2\underset{\sim}{v} - \underset{\sim}{s} = \underset{\sim}{0}$. Setting $\underset{\sim}{s} = \underset{\sim}{0}$ by nullifying all independent sources, we obtain $\underset{\sim}{H}_1\hat{\underset{\sim}{i}}(\underset{\sim}{v}) = \underset{\sim}{H}_2(-\underset{\sim}{v})$. The pair $(\underset{\sim}{H}_1,\underset{\sim}{H}_2)$ being a passive pair then implies $\hat{\underset{\sim}{i}}^T(\underset{\sim}{v})\underset{\sim}{v} \leq 0$. Since the reference current direction is assumed leaving the ports, the "sourceless" n-port cannot deliver positive power to the external nonlinear resistors connected across the ports.

2. Let us partition $\underset{\sim}{z} = [\underset{\sim}{z}_U,\underset{\sim}{z}_H,\underset{\sim}{z}_B]$ and $\underset{\sim}{g} = [\underset{\sim}{g}_U,\underset{\sim}{g}_H,\underset{\sim}{g}_B]$ as in Theorem 8. Partition the columns of any compatible matrix A accordingly, i.e., $\underset{\sim}{A} = [\underset{\sim}{A}^U,\underset{\sim}{A}^H,\underset{\sim}{A}^B]$. The following lemma is an obvious observation:

<u>Lemma 2</u>: Consider a network $\mathcal{N}$ described by Eq. (3). Assume $\underset{\sim}{g}$ satisfies condition (i) of Theorem 8. Then $\mathcal{N}$ has at least one solution for <u>any</u> $\underset{\sim}{s} \in \mathbb{R}^n$ <u>only if</u> the column vectors of the augmented rectangular matrix $(\underset{\sim}{H}_1^U,\underset{\sim}{H}_1^H,\underset{\sim}{H}_2)$ spans the space $\mathbb{R}^n$; i.e.,

$$\mathrm{SPAN}(\underset{\sim}{H}_1^U,\underset{\sim}{H}_1^H,\underset{\sim}{H}_2) = \mathbb{R}^n.$$

<u>Proof</u>: Suppose $\mathrm{SPAN}(\underset{\sim}{H}_1^H,\underset{\sim}{H}_1^H,\underset{\sim}{H}_2) \neq \mathbb{R}^n$. Let $\bar{\underset{\sim}{s}} \in \mathbb{R}^n$, $\bar{\underset{\sim}{s}} \neq 0$ such that $\bar{\underset{\sim}{s}}^T[\underset{\sim}{H}_1^U,\underset{\sim}{H}_1^H,\underset{\sim}{H}_2] = 0$. Then for any $\underset{\sim}{s} = \eta\bar{\underset{\sim}{s}}$, $\eta \in \mathbb{R}$, we obtain $\underset{\sim}{s}^T[\underset{\sim}{H}_1\underset{\sim}{g}(\underset{\sim}{z}) + \underset{\sim}{H}_2\underset{\sim}{z} - \underset{\sim}{s}] = \eta\bar{\underset{\sim}{s}}^T\underset{\sim}{H}_1^B\underset{\sim}{g}_B(\underset{\sim}{z}_B) - \eta^2\bar{\underset{\sim}{s}}^T\bar{\underset{\sim}{s}}$. Since g_B is bounded, there is an $M < \infty$ such that

$$|\bar{\underset{\sim}{s}}^T\underset{\sim}{H}_1^B\underset{\sim}{g}_B(\underset{\sim}{z}_B)| < M \quad \text{for all } \underset{\sim}{z}_B.$$

Therefore there is an η_0 such that

$$\underset{\sim}{s}^T[\underset{\sim}{H}_1\underset{\sim}{g}(\underset{\sim}{z}) + \underset{\sim}{H}_2\underset{\sim}{z} - \underset{\sim}{s}] < 0$$

for all $\underset{\sim}{s} = n\bar{\underset{\sim}{s}}$, $|n| \geq n_0$. That is, Eq. (3) does not have a solution for any $\underset{\sim}{s} = n\bar{\underset{\sim}{s}}$, $|n| \geq n_0$. ∎

This observation provides a simple check for the necessary conditions for the existence of solutions.

(iii) In view of Theorems 7,8,9 and Lemma 2, we conclude that if $(\underset{\sim}{H}_1,\underset{\sim}{H}_2)$ is a passive pair and if $\mathcal{B}(\underset{\sim}{g}) \cap \mathcal{N}(\underset{\sim}{H}_2) = \underset{\sim}{0}$, then $\text{SPAN}(\underset{\sim}{H}_1^U,\underset{\sim}{H}_1^H,\underset{\sim}{H}_2) = \mathbb{R}^n$. Let us prove this fact in the following lemma.[8]

Lemma 3: Consider Eq. (3). If $(\underset{\sim}{H}_1,\underset{\sim}{H}_2)$ is a passive pair and if $\mathcal{B}(\underset{\sim}{g}) \cap \mathcal{N}(\underset{\sim}{H}_2) = \underset{\sim}{0}$, then $\text{SPAN}\ (\underset{\sim}{H}_1^U,\underset{\sim}{H}_1^H,\underset{\sim}{H}_2) = \mathbb{R}^n$.

Proof: First, since $[\underset{\sim}{0},\underset{\sim}{0},\underset{\sim}{z}_B]^T \in \mathcal{B}(\underset{\sim}{g})$ for any $\underset{\sim}{z}_B$ and since $\mathcal{B}(\underset{\sim}{g}) \cap \mathcal{N}(\underset{\sim}{H}_2) = \underset{\sim}{0}$, it follows that the columns of $\underset{\sim}{H}_2^B$ are linearly independent. Let $\underset{\sim}{x} = [\underset{\sim}{x}_U,\underset{\sim}{x}_H,\underset{\sim}{0}]^T$, $\underset{\sim}{y} = [\underset{\sim}{x}_U,\underset{\sim}{x}_H,\underset{\sim}{y}_B]^T$ where $[\underset{\sim}{x}_U,\underset{\sim}{x}_H] \neq \underset{\sim}{0}$, then $\underset{\sim}{x}^T\underset{\sim}{y} > 0$. Since $(\underset{\sim}{H}_1,\underset{\sim}{H}_2)$ is a passive pair, $\underset{\sim}{H}_1\underset{\sim}{x} + \underset{\sim}{H}_2\underset{\sim}{y} \neq \underset{\sim}{0}$. Thus,

$$(\underset{\sim}{H}_1^U+\underset{\sim}{H}_2^U)\underset{\sim}{x}_U + (\underset{\sim}{H}_1^H+\underset{\sim}{H}_2^H)\underset{\sim}{x}_H + \underset{\sim}{H}_2^B\underset{\sim}{y}_B \neq \underset{\sim}{0},$$

for any $[\underset{\sim}{x}_U,\underset{\sim}{x}_H] \neq \underset{\sim}{0}$. If $[\underset{\sim}{x}_U,\underset{\sim}{x}_H] = \underset{\sim}{0}$, then since $\underset{\sim}{H}_2^B$ is of full rank, $\underset{\sim}{H}_2^B\underset{\sim}{y}_B \neq \underset{\sim}{0}$ for any $\underset{\sim}{y}_B \neq \underset{\sim}{0}$ and the above inequality is still true. This proves that $\{\underset{\sim}{H}_1^U + \underset{\sim}{H}_2^U,\ \underset{\sim}{H}_1^H + \underset{\sim}{H}_2^H,\ \underset{\sim}{H}_2^B\}$ is a set of linearly independent vectors. Hence $\text{SPAN}(\underset{\sim}{H}_1^U+\underset{\sim}{H}_2^U,\underset{\sim}{H}_1^H+\underset{\sim}{H}_2^H,\underset{\sim}{H}_2^B) = \mathbb{R}^n \Rightarrow \text{SPAN}(\underset{\sim}{H}_1^U,\underset{\sim}{H}_1^H,\underset{\sim}{H}_2) = \mathbb{R}^n$. ∎

Remark: The following observations follow directly from Lemma 3 and can be used to check whether the condition $\mathcal{B}(\underset{\sim}{g}) \cap \mathcal{N}(\underset{\sim}{H}_2)$ can be satisfied.

1. If $\underset{\sim}{H}_2^B$ is not of full rank, then $\mathcal{B}(\underset{\sim}{g}) \cap \mathcal{N}(\underset{\sim}{H}_2) \neq \underset{\sim}{0}$.
2. If $\text{SPAN}(\underset{\sim}{H}_1^U,\underset{\sim}{H}_1^H,\underset{\sim}{H}_2) \neq \mathbb{R}^n$ but $(\underset{\sim}{H}_1,\underset{\sim}{H}_2)$ is a passive pair then $\mathcal{B}(\underset{\sim}{g}) \cap \mathcal{N}(\underset{\sim}{H}_2) \neq \underset{\sim}{0}$.

Example: Consider the simple circuit shown in Fig. 7. Assume R_1 and R_3

---

[8] We prove this fact since it helps us to understand the structure of $\mathcal{B}(\underset{\sim}{g}) \cap \mathcal{N}(\underset{\sim}{H}_2)$.

are c.c. but R_2 is v.c. The network equation is given by

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{v}_1(i_1) \\ \hat{i}_2(v_2) \\ \hat{v}_3(i_3) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \\ i_3 \end{bmatrix} - \begin{bmatrix} E \\ -I \\ 0 \end{bmatrix} = \underset{\sim}{0}.$$

If the circuit has at least one solution for any input, then it follows from Lemma 2 and Remark 1 that R_2 cannot be a type B resistor, and R_1 and R_3 cannot both be type B. It follows also from Remark 2 that the two resistors belonging to either $\{R_1,R_2\}$ or $\{R_1,R_3\}$ cannot both be type B.

3.3 The Uniqueness of Solutions

A resistive network is said to be monotone if all its resistors are monotone (necessarily 2-terminal, by definition) resistors. Many monotone networks have unique solutions. As we remarked in Sec. I, any monotone resistor can be replaced by a combination of an independent source and a passive resistor.

Theorem 10: [8][9] Consider a network described by Eq. (2). If (i) $\underset{\sim}{g}$ is diagonal and each component $g_j(z_j)$ is strictly increasing, and (ii) $\underset{\sim}{H} \in P_0$,

then $\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{0}$ can have at most one solution

Proof: Suppose there are two solutions $\underset{\sim}{z}'$ and $\underset{\sim}{z}''$. Then we have

$$\underset{\sim}{g}(\underset{\sim}{z}') - \underset{\sim}{g}(\underset{\sim}{z}'') = \underset{\sim}{H}(\underset{\sim}{z}''-\underset{\sim}{z}') .$$

Since $\underset{\sim}{H} \in P_0$, there exists an index j such that $(\underset{\sim}{z}''-\underset{\sim}{z}')_j \neq 0$ and that $(\underset{\sim}{H}(\underset{\sim}{z}''-\underset{\sim}{z}'))_j$ and $(\underset{\sim}{z}''-\underset{\sim}{z}')_j$ are of the same sign. But then $g_j(z_j'') - g_j(z_j')$

[9] Theorems 10 and 11 are proved by Sandberg and Willson. We give here a simpler proof using the ideas developed in the preceding sections.

and $z_j'' - z_j'$ would be of opposite sign, contradicting the fact that g_j is strictly increasing. ∎

Theorem 11: [8] Consider a network described by Eq. (3). Assume that

(i) g is diagonal and each component $g_j(z_j)$ is strictly increasing.

(ii) For each $\underset{\sim}{D} = \text{diag}(d_j)$, $d_j = \pm 1$, there exists a $\underset{\sim}{p} \in \mathbb{R}^n$, $\underset{\sim}{p} \geq 0$ such that $\underset{\sim}{p}^T \underset{\sim}{D}\underset{\sim}{H}_1\underset{\sim}{D} \geqq 0$, $\underset{\sim}{p}^T\underset{\sim}{D}\underset{\sim}{H}_2\underset{\sim}{D} \geqq 0$ and $\underset{\sim}{p}^T\underset{\sim}{D}(\underset{\sim}{H}_1+\underset{\sim}{H}_2)\underset{\sim}{D} > 0$. Then $\mathcal{N}$ can have at most one solution.

Proof: Let $\underset{\sim}{z}'$ and $\underset{\sim}{z}''$ be solutions. Then we have

$$\underset{\sim}{H}_1\left(\underset{\sim}{g}(\underset{\sim}{z}')-\underset{\sim}{g}(\underset{\sim}{z}'')\right) + \underset{\sim}{H}_2(\underset{\sim}{z}'-\underset{\sim}{z}'') = \underset{\sim}{0} .$$

Without loss of generality, assume all nonlinear resistors are passive, we obtain $\underset{\sim}{H}_1\underset{\sim}{D}\overline{\left(\underset{\sim}{g}(\underset{\sim}{z}')-\underset{\sim}{g}(\underset{\sim}{z}'')\right)} + \underset{\sim}{H}_2\underset{\sim}{D}\overline{(\underset{\sim}{z}'-\underset{\sim}{z}'')} = \underset{\sim}{0}$, where $\underset{\sim}{D} = \text{diag}\left(\text{sgn}(z_j'-z_j'')\right)$ and the bar denotes taking the absolute value of each component of the vector indicated. By (ii), there exists a $\underset{\sim}{p} > 0$ such that $\underset{\sim}{p}^T\underset{\sim}{D}\underset{\sim}{H}_1\underset{\sim}{D} \geqq 0$, $\underset{\sim}{p}^T\underset{\sim}{D}\underset{\sim}{H}_2\underset{\sim}{D} \geqq 0$ and $\underset{\sim}{p}^T\underset{\sim}{D}(\underset{\sim}{H}_1+\underset{\sim}{H}_2)\underset{\sim}{D} > 0$. But this contradicts the identity

$$\underset{\sim}{p}^T\underset{\sim}{D}\underset{\sim}{H}_1\underset{\sim}{D}\overline{\left(\underset{\sim}{g}(\underset{\sim}{z}')-\underset{\sim}{g}(\underset{\sim}{z}'')\right)} + \underset{\sim}{p}^T\underset{\sim}{D}\underset{\sim}{H}_2\underset{\sim}{D}\overline{(\underset{\sim}{z}'-\underset{\sim}{z}'')} = 0. \quad ∎$$

The following theorems are due to Sandberg and Willson [8,9].

Theorem 12: Let $\mathcal{N}$ be a network described by Eq. (2). Assume that $\underset{\sim}{g}$ is diagonal and each component of $\underset{\sim}{g}$ is strictly increasing and of type $\underline{U}$. Then $\mathcal{N}$ has a unique solution for any $\underset{\sim}{g}$ and $\underset{\sim}{s}$ if, and only if, $\underset{\sim}{H} \in P_0$.

Proof: (If) Implied by Corollary 3, Theorem 6 and Theorem 10.

(Only If) By constructing counterexamples, see [9]. ∎

Theorem 13: Let $\mathcal{N}$ be a network described by Eq. (3). Assume that

(i) g is diagonal and each component of $\underset{\sim}{g}$ is strictly increasing.

(ii) For each $\underset{\sim}{D} = \text{diag}(d_j)$, $d_j = \pm 1$, there exists a $\underset{\sim}{p} \in \mathbb{R}^n$, $\underset{\sim}{p} \geq 0$ such that $\underset{\sim}{p}^T\underset{\sim}{D}\underset{\sim}{H}_1\underset{\sim}{D} \geqq 0$, $\underset{\sim}{p}^t\underset{\sim}{D}\underset{\sim}{H}_2\underset{\sim}{D} \geqq 0$ and $\underset{\sim}{p}^T\underset{\sim}{D}(\underset{\sim}{H}_1+\underset{\sim}{H}_2)\underset{\sim}{D} > 0$.

Then $\mathcal{N}$ has a unique solution for any $\underset{\sim}{g}$ and $\underset{\sim}{s}$ if, and only if, $\mathcal{R}(g) \cap \mathcal{N}(\underset{\sim}{H}_2) = \underset{\sim}{0}$.

Proof: (If) Implied by Theorem 9 and Theorem 11.

(Only If) By constructing counterexamples, see [8]. ∎

3.4 Boundedness and Continuous Dependence of Solutions

All solutions discussed in Sec. 3.2 and 3.3 are bounded because in the proofs we actually constructed the "bounding spheres" $S(\underset{\sim}{0},r)$. As to the continuous dependence of the solution on the parameters of the network, we have:

Theorem 14: The solutions discussed in Theorems 12 and 13 depend continuously on the function $\underset{\sim}{f}$ (and hence the parameters of the network).

Proof: Since the solutions are unique, their indices are equal to 1. By Property 11 of Sec. II, they are continuous functions of $\underset{\sim}{f}$. ∎

IV. THE HYBRID ANALYSIS BY TOPOLOGICAL FORMULATION

4.1 The Network Equations

In this section we derive the network equations by topological formulation. For simplicity, let us first assume that there are no controlled sources. Each branch in $\mathcal{N}$ is considered as a composite branch as shown in Fig. 8. A composite branch is said to be v.c. (or c.c.) if the resistor is v.c. (or c.c.). Let us choose a special tree $\mathcal{T}$ by picking first a subtree $\mathcal{T}_1 \subset \mathcal{T}$ containing as many v.c. resistors as possible. The remaining set $\mathcal{L}_1$ of v.c. resistors then necessarily form loops with branches in $\mathcal{T}_1$ and must therefore form a part of the cotree $\mathcal{L}$. The c.c. resistors are partitioned into $\mathcal{T}_2$ and $\mathcal{L}_2$ such that $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}$ and $\mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{L}$. Any tree constructed by this algorithm will henceforth be referred to as a hybrid tree. The fundamental loop matrix $\underset{\sim}{B}$ with respect to this hybrid tree can then be partitioned as follows:

$$\underset{\sim}{B} = \begin{bmatrix} \underset{\sim}{1}_{\mathcal{L}_1\mathcal{L}_1} & \underset{\sim}{0}_{\mathcal{L}_1\mathcal{L}_2} & \underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1} & \underset{\sim}{0}_{\mathcal{L}_1\mathcal{J}_2} \\ \underset{\sim}{0}_{\mathcal{L}_2\mathcal{L}_1} & \underset{\sim}{1}_{\mathcal{L}_2\mathcal{L}_2} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2} \end{bmatrix} \begin{matrix} \}\mathcal{L}_1 \\ \}\mathcal{L}_2 \end{matrix}$$

$$\qquad \underbrace{\quad}_{\mathcal{L}_1} \quad \underbrace{\quad}_{\mathcal{L}_2} \quad \underbrace{\quad}_{\mathcal{J}_1} \quad \underbrace{\quad}_{\mathcal{J}_2}$$

where $\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_2} = \underset{\sim}{0}_{\mathcal{L}_1\mathcal{J}_2}$ because each branch in $\mathcal{L}_1$ forms a fundamental loop with branches in $\mathcal{J}_1$ only. If follows from this property that each branch in $\mathcal{J}_2$ forms a fundamental cut set with branches in $\mathcal{L}_2$ only. Let the resistors be characterized by

$$\underset{\sim}{i}_{\mathcal{L}_1} = \hat{\underset{\sim}{i}}_{\mathcal{L}_1}(\underset{\sim}{v}_{\mathcal{L}_1}) \quad \text{and} \quad \underset{\sim}{v}_{\mathcal{L}_2} = \hat{\underset{\sim}{v}}_{\mathcal{L}_2}(\underset{\sim}{i}_{\mathcal{L}_2})$$

$$\underset{\sim}{i}_{\mathcal{J}_1} = \hat{\underset{\sim}{i}}_{\mathcal{J}_1}(\underset{\sim}{v}_{\mathcal{J}_1}) \quad \text{and} \quad \underset{\sim}{v}_{\mathcal{J}_2} = \hat{\underset{\sim}{v}}_{\mathcal{J}_2}(\underset{\sim}{i}_{\mathcal{J}_2}),$$

where the $\hat{\underset{\sim}{i}}$ and $\hat{\underset{\sim}{v}}$ are diagonal maps. Substituting the branch characteristics into the KCL and KVL constraints, we obtain

$$\underset{\sim}{v}_{\mathcal{L}_1} = -\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1} + \underset{\sim}{e}^1 \quad \text{and} \quad \underset{\sim}{i}_{\mathcal{J}_2} = \underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{i}_{\mathcal{L}_2} + \underset{\sim}{j}^2.$$

and the network equation:

$$\underset{\sim}{f}(\underset{\sim}{v}_{\mathcal{J}_1},\underset{\sim}{i}_{\mathcal{L}_2}) \triangleq \begin{bmatrix} \hat{\underset{\sim}{i}}_{\mathcal{J}_1}(\underset{\sim}{v}_{\mathcal{J}_1}) \\ \hat{\underset{\sim}{v}}_{\mathcal{L}_2}(\underset{\sim}{i}_{\mathcal{L}_2}) \end{bmatrix} + \begin{bmatrix} -\underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{J}_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{i}}_{\mathcal{L}_1}(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}+\underset{\sim}{e}^1) \\ \hat{\underset{\sim}{v}}_{\mathcal{J}_2}(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{i}_{\mathcal{L}_2}+\underset{\sim}{j}^2) \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{0} & -\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1} \\ \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} & \underset{\sim}{0} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \\ \underset{\sim}{i}_{\mathcal{L}_2} \end{bmatrix} + \begin{bmatrix} \underset{\sim}{j}^1 \\ \underset{\sim}{e}^2 \end{bmatrix} = \underset{\sim}{0}.$$

where j^k and e^k, $k = 1,2$ are related to sources and are defined by

$$j^1 = -j_{\mathcal{T}_1} + B^T_{\mathcal{L}_1\mathcal{T}_1} j_{\mathcal{L}_1} + B^T_{\mathcal{L}_2\mathcal{T}_1} j_{\mathcal{L}_2} \quad \text{and} \quad j^2 = j_{\mathcal{T}_2} - B^T_{\mathcal{L}_2\mathcal{T}_2} j_{\mathcal{L}_2}$$

$$e^1 = e_{\mathcal{L}_1} + B_{\mathcal{L}_1\mathcal{T}_1} e_{\mathcal{T}_1} \quad \text{and} \quad e^2 = -e_{\mathcal{L}_2} - B_{\mathcal{L}_2\mathcal{T}_1} e_{\mathcal{T}_1} - B_{\mathcal{L}_2\mathcal{T}_2} e_{\mathcal{T}_2}.$$

The above equation can be generalized to allow couplings between the nonlinear resistors belonging to $\mathcal{T}_1$ and $\mathcal{L}_2$, and between the nonlinear resistors belonging to $\mathcal{L}_1$ and $\mathcal{T}_2$; namely,

$$i_{\mathcal{T}_1} = \hat{i}_{\mathcal{T}_1}(v_{\mathcal{T}_1}, i_{\mathcal{L}_2}) \quad \text{and} \quad v_{\mathcal{L}_2} = \hat{v}_{\mathcal{L}_2}(v_{\mathcal{T}_1}, i_{\mathcal{L}_2})$$

$$v_{\mathcal{T}_2} = \hat{v}_{\mathcal{T}_2}(v_{\mathcal{L}_1}, i_{\mathcal{T}_2}) \quad \text{and} \quad i_{\mathcal{L}_1} = \hat{i}_{\mathcal{L}_1}(v_{\mathcal{L}_1}, i_{\mathcal{T}_2}).$$

The corresponding network equation now takes the form

$$f(v_{\mathcal{T}_1}, i_{\mathcal{L}_2})$$

$$\triangleq \begin{bmatrix} \hat{i}_{\mathcal{T}_1}(v_{\mathcal{T}_1}, i_{\mathcal{L}_2}) \\ \hat{v}_{\mathcal{L}_2}(v_{\mathcal{T}_1}, i_{\mathcal{L}_2}) \end{bmatrix} + \begin{bmatrix} -B^T_{\mathcal{L}_1\mathcal{T}_1} & 0 \\ 0 & B_{\mathcal{L}_2\mathcal{T}_2} \end{bmatrix} \begin{bmatrix} \hat{i}_{\mathcal{L}_1}(-B_{\mathcal{L}_1\mathcal{T}_1} v_{\mathcal{T}_1} + e^1, B^T_{\mathcal{L}_2\mathcal{T}_2} i_{\mathcal{L}_2} + j^2) \\ \hat{v}_{\mathcal{T}_2}(-B_{\mathcal{L}_1\mathcal{T}_1} v_{\mathcal{T}_1} + e^1, B^T_{\mathcal{L}_2\mathcal{T}_2} i_{\mathcal{L}_2} + j^2) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -B^T_{\mathcal{L}_2\mathcal{T}_1} \\ B_{\mathcal{L}_2\mathcal{T}_1} & 0 \end{bmatrix} \begin{bmatrix} v_{\mathcal{T}_1} \\ i_{\mathcal{L}_2} \end{bmatrix} + \begin{bmatrix} j^1 \\ e^2 \end{bmatrix} = 0 . \qquad (8)$$

Remarks: 1. Note that since $v_{\mathcal{L}_1} = -B_{\mathcal{L}_1\mathcal{T}_1} v_{\mathcal{T}_1} + e^1$, if we multiply $-B^T_{\mathcal{L}_1\mathcal{T}_1} \hat{i}_{\mathcal{L}_1}(v_{\mathcal{L}_1}, i_{\mathcal{T}_2})$ by $v^T_{\mathcal{T}_1}$, we obtain

$$-v^T_{\mathcal{T}_1} B^T_{\mathcal{L}_1\mathcal{T}_1} \hat{i}_{\mathcal{L}_1}(v_{\mathcal{L}_1}, i_{\mathcal{T}_2}) = [v_{\mathcal{L}_1} - e^1]^T \hat{i}_{\mathcal{L}_1}(v_{\mathcal{L}_1}, i_{\mathcal{T}_2}).$$

Similarly, $\underset{\sim}{i}^T_{\mathcal{L}_2} B_{\mathcal{L}_2\mathcal{J}_2} \hat{\underset{\sim}{v}}_{\mathcal{J}_2}(\underset{\sim}{v}_{\mathcal{L}_1}, \underset{\sim}{i}_{\mathcal{J}_2}) = [\underset{\sim}{i}_{\mathcal{J}_2} - \underset{\sim}{j}^2]^T \hat{\underset{\sim}{v}}_{\mathcal{J}_2}(\underset{\sim}{v}_{\mathcal{L}_1}, \underset{\sim}{i}_{\mathcal{J}_2})$.

2. By definition, $[\hat{\underset{\sim}{i}}_{\mathcal{L}_1}(\cdot), \hat{\underset{\sim}{v}}_{\mathcal{J}_2}(\cdot)]$ is eventually passive with respect to $[\underset{\sim}{e}^1, \underset{\sim}{j}^2]$ if

$$[(\underset{\sim}{v}_{\mathcal{L}_1} - \underset{\sim}{e}^1)^T,\ (\underset{\sim}{i}_{\mathcal{J}_2} - \underset{\sim}{j}^2)^T] \begin{bmatrix} \hat{\underset{\sim}{i}}_{\mathcal{L}_1}(\underset{\sim}{v}_{\mathcal{L}_1}, \underset{\sim}{i}_{\mathcal{J}_2}) \\ \hat{\underset{\sim}{v}}_{\mathcal{J}_2}(\underset{\sim}{v}_{\mathcal{L}_1}, \underset{\sim}{i}_{\mathcal{J}_2}) \end{bmatrix} \geq 0$$

whenever $\|[\underset{\sim}{v}_{\mathcal{L}_1}, \underset{\sim}{i}_{\mathcal{J}_2}]\| \geq K$ for some $K < \infty$. It is easy to see that if $\hat{\underset{\sim}{i}}_{\mathcal{L}_1}$ and $\hat{\underset{\sim}{v}}_{\mathcal{J}_2}$ are diagonal maps, then eventual passivity implies eventual passivity with respect to any $[\underset{\sim}{e}^1, \underset{\sim}{j}^2]$.

4.2 <u>The Existence of Solutions</u>

<u>Theorem 15</u>: <u>Let $\mathcal{N}$ be a network described by Eq. (8). Assume that</u>

(i)

$$\lim_{\rho\to\infty} [\underset{\sim}{v}^T_{\mathcal{J}_1}, \underset{\sim}{i}^T_{\mathcal{L}_2}] \begin{bmatrix} \hat{\underset{\sim}{i}}_{\mathcal{J}_1}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}) \\ \hat{\underset{\sim}{v}}_{\mathcal{L}_2}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}) \end{bmatrix} = \infty, \text{ where } \rho = \|[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}]\|.$$

(ii) $[\hat{\underset{\sim}{i}}_{\mathcal{L}_1}, \hat{\underset{\sim}{v}}_{\mathcal{J}_2}]$ <u>is eventually passive with respect to</u> $[\underset{\sim}{e}^1, \underset{\sim}{j}^2]$.

<u>Then $\mathcal{N}$ has at least one solution. Furthermore, if the Jacobian matrix $\underset{\sim}{J}_f$ is nonsingular at each solution, then there is an odd number of solutions all of which are structurally stable</u>.

<u>Proof</u>: Define the homotopy $\underset{\sim}{h}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}, \lambda)$ by

$$\underset{\sim}{h}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}, \lambda) \triangleq \lambda \underset{\sim}{f}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}) + (1-\lambda) \begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \\ \underset{\sim}{i}_{\mathcal{L}_2} \end{bmatrix}, \quad \lambda \in [0,1]$$

Premultiply $\underset{\sim}{h}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}, \lambda)$ by $[\underset{\sim}{v}^T_{\mathcal{J}_1}, \underset{\sim}{i}^T_{\mathcal{L}_2}]$, we obtain

$$[\underset{\sim}{v}^T_{\mathcal{J}_1}, \underset{\sim}{i}^T_{\mathcal{L}_2}] \underset{\sim}{h}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}, \lambda) = \lambda(s_1 + s_2 + s_3) + (1-\lambda) \left\| \begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \\ \underset{\sim}{i}_{\mathcal{L}_2} \end{bmatrix} \right\|_2^2$$

where $s_1 \triangleq [\underset{\sim}{v}^T_{\mathcal{J}_1}, \underset{\sim}{i}^T_{\mathcal{L}_2}] \begin{bmatrix} \hat{\underset{\sim}{i}}_{\mathcal{J}_1} \\ \hat{\underset{\sim}{v}}_{\mathcal{L}_2} \end{bmatrix}$

$$s_2 \triangleq [(\underset{\sim}{v}_{\mathcal{L}_1} - \underset{\sim}{e}^1)^T, (\underset{\sim}{i}_{\mathcal{J}_2} - \underset{\sim}{j}^2)^T] \begin{bmatrix} \hat{\underset{\sim}{i}}_{\mathcal{L}_1}(\underset{\sim}{v}_{\mathcal{L}_1}, \underset{\sim}{i}_{\mathcal{J}_2}) \\ \hat{\underset{\sim}{v}}_{\mathcal{J}_2}(\underset{\sim}{v}_{\mathcal{L}_1}, \underset{\sim}{i}_{\mathcal{J}_2}) \end{bmatrix}$$

and $s_3 \triangleq [\underset{\sim}{v}^T_{\mathcal{J}_1}, \underset{\sim}{i}^T_{\mathcal{L}_2}] \begin{bmatrix} \underset{\sim}{j}^1 \\ \underset{\sim}{e}^2 \end{bmatrix}$.

Now since $[\hat{\underset{\sim}{i}}_{\mathcal{L}_1}, \hat{\underset{\sim}{v}}_{\mathcal{J}_2}]$ is eventually passive with respect to $[\underset{\sim}{e}^1, \underset{\sim}{j}^2]$, there exists an $M > -\infty$ such that $s_2 > M$ for all $[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}]$. By Schwarz inequality, $|s_3| \le c\|[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}]\|$ where $c = \|[\underset{\sim}{j}^1, \underset{\sim}{e}^2]\|$. Let $K' > c$ be a given constant, then there is an $r' > 0$ such that $\|[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}]\| > r' \Rightarrow K'\|[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}]\| > M + c\|[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}]\|$. According to (i), there exists an $r > r'$ such that $\|[\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}]\| > r \Rightarrow s_1 > K'$. Consider $\underset{\sim}{h}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}, \lambda)$ on the sphere $S(\underset{\sim}{0}, r)$, we then have

$$[\underset{\sim}{v}^T_{\mathcal{J}_1}, \underset{\sim}{i}^T_{\mathcal{L}_2}]\underset{\sim}{h}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}, \lambda) = \lambda \sum_{j=1}^{3} s_j + (1-\lambda) \left\| \begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \\ \underset{\sim}{i}_{\mathcal{L}_2} \end{bmatrix} \right\|_2^2$$

$$> \lambda \left(K' \left\| \begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \\ \underset{\sim}{i}_{\mathcal{L}_2} \end{bmatrix} \right\| - M - c \left\| \begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \\ \underset{\sim}{i}_{\mathcal{L}_2} \end{bmatrix} \right\| \right) + (1-\lambda) \left\| \begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \\ \underset{\sim}{i}_{\mathcal{L}_2} \end{bmatrix} \right\|_2^2$$

$$> 0 \quad , \quad \text{for all } \lambda \in [0,1].$$

Therefore $\underset{\sim}{f}(\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}) \neq \underset{\sim}{0}$ and is homotopic to the identity map on $S(\underset{\sim}{0}, r)$. Hence the conclusion follows. ∎

Corollary: Let $\mathcal{N}$ be a network described by Eq. (8). Assume that

(i) All resistors are uncoupled and eventually passive, and

(ii) the resistors in $\mathcal{J}_1$ and $\mathcal{L}_2$ are of type U.

Then $\mathcal{N}$ has at least one solution for any set of independent sources.

Furthermore, if the Jacobian matrix is nonsingular at each solution, then there is an odd number of solutions all of which are structurally stable.

Remark: Condition (ii) of the corollary is satisfied if there exists a tree (resp. cotree) consisting of v.c. (resp. c.c.) type U resistors only. For in this case, we simply choose $\mathcal{J}_2$ (resp. $\mathcal{L}_1$) to be the empty set.

We now consider a more complicated case where both type H and type B resistors are present. For simplicity, let us assume first that all resistors are two-terminal, uncoupled resistors. Let us recast the network equation in the form $\underset{\sim}{f}(\underset{\sim}{v}_{\mathcal{J}_1},\underset{\sim}{i}_{\mathcal{L}_2}) \triangleq \underset{\sim}{g}(\underset{\sim}{v}_{\mathcal{J}_1},\underset{\sim}{i}_{\mathcal{L}_2}) + \underset{\sim}{A}\begin{bmatrix} \underset{\sim}{v}_{\mathcal{J}_1} \\ \underset{\sim}{i}_{\mathcal{L}_2} \end{bmatrix} + \begin{bmatrix} \underset{\sim}{j}^1 \\ \underset{\sim}{e}^2 \end{bmatrix}$ where

$$\underset{\sim}{g} \triangleq \begin{bmatrix} \hat{\underset{\sim}{i}}_{\mathcal{J}_1}(\underset{\sim}{v}_{\mathcal{J}_1}) \\ \hat{\underset{\sim}{v}}_{\mathcal{L}_2}(\underset{\sim}{i}_{\mathcal{L}_2}) \end{bmatrix} + \begin{bmatrix} -\underset{\sim}{B}^T_{\mathcal{L}_1\mathcal{J}_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_2} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{i}}_{\mathcal{L}_1}(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}+\underset{\sim}{e}^1) \\ \hat{\underset{\sim}{v}}_{\mathcal{J}_2}(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{i}_{\mathcal{L}_2}+\underset{\sim}{j}^2) \end{bmatrix}$$

is the nonlinear part of $\underset{\sim}{f}$ and

$$\underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{0} & -\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_1} \\ \underset{\sim}{B}_{\mathcal{L}_2\mathcal{J}_1} & \underset{\sim}{0} \end{bmatrix}$$

is a skew-symmetric matrix. Furthermore, let us define

$$\underset{\sim}{g}'(\underset{\sim}{v}_{\mathcal{J}_1},\underset{\sim}{i}_{\mathcal{L}_2}) \triangleq [\hat{\underset{\sim}{i}}_{\mathcal{J}_1}(\underset{\sim}{v}_{\mathcal{J}_1}),\hat{\underset{\sim}{v}}_{\mathcal{L}_2}(\underset{\sim}{i}_{\mathcal{L}_2}),\hat{\underset{\sim}{i}}_{\mathcal{L}_1}(-\underset{\sim}{B}_{\mathcal{L}_1\mathcal{J}_1}\underset{\sim}{v}_{\mathcal{J}_1}+\underset{\sim}{e}^1),\hat{\underset{\sim}{v}}_{\mathcal{J}_2}(\underset{\sim}{B}^T_{\mathcal{L}_2\mathcal{J}_2}\underset{\sim}{i}_{\mathcal{L}_2}+\underset{\sim}{j}^2)]^T$$

Definition 5: The set $\mathcal{B}(\underset{\sim}{g})$ is defined by $\mathcal{B}(\underset{\sim}{g}) \triangleq \{[\underset{\sim}{v}_{\mathcal{J}_1},\underset{\sim}{i}_{\mathcal{L}_2}] : \|\underset{\sim}{g}(\eta(v_{\mathcal{J}_1},i_{\mathcal{L}_2}))\| < m < \infty \text{ as } \eta \to \infty\}$. The set $\mathcal{B}(\underset{\sim}{g}')$ is defined in a similar fashion.

In general, $\mathcal{B}(\underset{\sim}{g}) \neq \mathcal{B}(\underset{\sim}{g}')$. However, we can state:

Lemma 4: If all resistors in $\mathcal{N}$ are eventually passive, then $\mathcal{B}(\underset{\sim}{g}) = \mathcal{B}(\underset{\sim}{g}')$.

Proof: It is clear that $\mathcal{B}(\tilde{g}') \subset \mathcal{B}(\tilde{g})$. To prove that $\mathcal{B}(\tilde{g}) \subset \mathcal{B}(\tilde{g}')$, suppose the contrary. Then there exists a $\tilde{v}_{\mathcal{J}_1}$ such that[10]

$$\|\hat{\tilde{i}}_{\mathcal{L}_1}(-\tilde{B}_{\mathcal{L}_1\mathcal{J}_1}(\eta\tilde{v}_{\mathcal{J}_1})+\tilde{e}^1)\| \to \infty \text{ but } \|-\tilde{B}^T_{\mathcal{L}_1\mathcal{J}_1}\hat{\tilde{i}}_{\mathcal{L}_1}(-\tilde{B}_{\mathcal{L}_1\mathcal{J}_1}(\eta\tilde{v}_{\mathcal{J}_1})+\tilde{e}^1)\| < M < \infty$$

as $\eta \to \infty$. Recalling Remark 1, premultiply $-\tilde{B}^T_{\mathcal{L}_1\mathcal{J}_1}\hat{\tilde{i}}_{\mathcal{L}_1}(-\tilde{B}_{\mathcal{L}_1\mathcal{J}_1}(\eta\tilde{v}_{\mathcal{J}_1})+\tilde{e}^1)$ by $\tilde{v}^T_{\mathcal{J}_1}$, and note that $-\tilde{B}_{\mathcal{L}_1\mathcal{J}_1}\tilde{v}_{\mathcal{J}_1} = \tilde{v}_{\mathcal{L}_1} - \tilde{e}^1$, we obtain

$$(\tilde{v}_{\mathcal{L}_1}-\tilde{e}^1)^T\hat{\tilde{i}}_{\mathcal{L}_1}\left(\eta(\tilde{v}_{\mathcal{L}_1}-\tilde{e}^1)+\tilde{e}^1\right) = \sum_{\ell_1\in\mathcal{L}_1}\left(\tilde{v}_{\ell_1}-\tilde{e}^1_{\ell_1}\right)^T\hat{\tilde{i}}_{\ell_1}\left(\eta(\tilde{v}_{\ell_1}-\tilde{e}^1_{\ell_1})+\tilde{e}^1_{\ell_1}\right) < \infty$$

as $\eta \to \infty$.

Since all resistors are eventually passive, each term in the summation will be nonnegative when η is sufficiently large. But by assumption, $\|\hat{\tilde{i}}_{\mathcal{L}_1}(\eta(\tilde{v}_{\mathcal{L}_1}-\tilde{e}^1)+\tilde{e}^1)\| \to \infty$ as $\eta \to \infty$, and the above summation cannot be bounded by any constant. Hence we have a contradiction. ∎

Let $[\tilde{v}_{\mathcal{J}_1},\tilde{i}_{\mathcal{L}_2}] \triangleq \tilde{z} = [z_1,z_2,\ldots,z_n]^T \in \mathbb{R}^n$, then the set $\mathcal{B}(\tilde{g}') = B_1 \cap B_2 \cap B_3$. The set $B_1 = \prod_{k=1}^{n} I_k$, where $I_k = \{0\}$ for all $k \in U$, $I_k = [0,\infty)$ for all $k \in H$, and $I_k = (-\infty,\infty)$ for all $k \in B$. The set B_2 represents an intersection of hyperplanes defined by linear equations, for example; $(-\tilde{B}_{\mathcal{L}_1\mathcal{J}_1})_\ell\tilde{v}_{\mathcal{J}_1} = 0$ where $(\hat{\tilde{i}}_{\mathcal{L}_1})_\ell$ is a type U resistor.[11] The set B_3 represents the intersection of half spaces defined by linear inequalities, for example: $(\tilde{B}^T_{\mathcal{L}_2\mathcal{J}_2})_m\tilde{i}_{\mathcal{L}_2} \geq 0$ where $(\hat{\tilde{v}}_{\mathcal{J}_2})_m$ is a type H resistor.

Theorem 16: Let $\mathcal{N}$ be a network described by Eq. (8). Assume that

(i) all resistors are uncoupled and eventually passive, and

(ii) $\mathcal{B}(\tilde{g}) \cap \mathcal{N}(\tilde{A}) = \tilde{0}$, where $\mathcal{N}(\tilde{A})$ is the null space of $\tilde{A}$. Then

---

[10] Or if there exists an $\tilde{i}_{\mathcal{L}_2}$, then we consider $\hat{\tilde{v}}_{\mathcal{J}_2}(\tilde{B}^T_{\mathcal{L}_2\mathcal{J}_2}\tilde{i}_{\mathcal{L}_2}+\tilde{j}^2)$.

[11] $(-\tilde{B}_{\mathcal{L}_1\mathcal{J}_1})_\ell \triangleq$ the ℓ-th row of $-\tilde{B}_{\mathcal{L}_1\mathcal{J}_1}$.

$\mathcal{N}$ has at least one solution for any independent sources. Furthermore, if the Jacobian matrix J_f is nonsingular at each solution, then there is an odd number of solutions all of which are structurally stable.

Proof: Consider the vector field $f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})$. Premultiply f by $[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]$, we obtain

$$[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]f = v_{\mathcal{J}_1}^T \hat{i}_{\mathcal{L}_1}(v_{\mathcal{J}_1}) + i_{\mathcal{L}_2}^T \hat{v}_{\mathcal{L}_2}(i_{\mathcal{L}_2}) + (v_{\mathcal{L}_1} - e^1)^T \hat{i}_{\mathcal{L}_1}(v_{\mathcal{L}_1})$$

$$+ (i_{\mathcal{J}_2} - j^2)^T \hat{v}_{\mathcal{J}_2}(i_{\mathcal{J}_2}) + v_{\mathcal{J}_1}^T j^1 + i_{\mathcal{L}_2}^T e^2$$

$$= [v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T] g(v_{\mathcal{J}_1}, i_{\mathcal{L}_2}) + v_{\mathcal{J}_1}^T j^1 + i_{\mathcal{L}_2}^T e^2 .$$

Since all resistors are uncoupled and eventually passive (i.e., each term above is bounded from below, except $v_{\mathcal{J}_1}^T j^1$ and $i_{\mathcal{L}_2}^T e^2$), for any $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \notin \mathcal{B}(g)$; $\eta[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T] f[\eta(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})] > 0$ and can be arbitrarily large whenever $\eta > 0$ is sufficiently large. Now consider the set $\mathcal{B}(g)$. Let $\varepsilon > 0$ be a fixed constant, define $\mathcal{B}_\varepsilon(g) \triangleq \{z \in \mathbb{R}^n: d(z, \mathcal{B}(g)) \leq \varepsilon\}$ where $d(z, \mathcal{B}(g)) = \inf\{\|z - z'\|: z' \in \mathcal{B}(g)\}$. Because of the structure of $\mathcal{B}(g) = \mathcal{B}(g')$ and the fact that $\mathcal{B}(g) \cap \mathcal{N}(A) = 0$, by a similar proof as that in Theorem 7, it can be shown that there exists an $r_1 > 0$ such that $\mathcal{B}_\varepsilon(g) \cap S(0,r)$ and $\mathcal{N}(A) \cap S(0,r)$ are disjoint for any $r \geq r_1$. Let $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in \mathcal{B}_\varepsilon(g) \cap S(0,r)$, $r \geq r_1$, premultiply f by $\frac{1}{\rho}[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]A^T$, where $\rho \triangleq \|[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}]\|$, we obtain

$$\frac{1}{\rho}[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]A^T f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2}) = \frac{1}{\rho}[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]A^T g(v_{\mathcal{J}_1}, i_{\mathcal{L}_2}) + \frac{1}{\rho}\left\| A \begin{bmatrix} v_{\mathcal{J}_1} \\ i_{\mathcal{L}_2} \end{bmatrix} \right\|_2^2$$

$$+ \frac{1}{\rho}(v_{\mathcal{J}_1}^T j^1 + i_{\mathcal{L}_2}^T e^2).$$

Since $\mathcal{B}_\varepsilon(g) \cap S(0,r)$ is disjoint from $\mathcal{N}(A) \cap S(0,r)$, there exists a $\mu > 0$ such that $\frac{1}{\rho}\left\| A\begin{bmatrix} v_{\mathcal{J}_1} \\ i_{\mathcal{L}_2} \end{bmatrix}\right\|_2^2 \geq \mu \left\| \begin{bmatrix} v_{\mathcal{J}_1} \\ i_{\mathcal{L}_2} \end{bmatrix}\right\|$ for all $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in \mathcal{B}_\varepsilon(g) \cap S(0,r)$. Furthermore, μ is an increasing function of r. (The proof is similar to that given in Appendix 2.) Notice that since $g(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})$ is bounded by a constant for all $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in \mathcal{B}_\varepsilon(g)$, there is an $r_2 \geq r_1$ such that $r \geq r_2$ implies $\frac{1}{\rho}[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]A^T f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2}) > 0$ for all $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in \mathcal{B}_\varepsilon(g) \cap S(0,r)$. On the other hand, since $\eta[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T] f[\eta(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})]$ can be aribtrarily large by increasing η for any $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \notin \mathcal{B}_\varepsilon(g)$, there exists an $r_3 \geq r_2$ such that $[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T] f > 0$ for all $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r) \backslash \mathcal{B}_\varepsilon(g)$ for any $r \geq r_3$.

<u>Claim</u>: There exists an $r_0 \geq r_3$ such that $f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2}) \neq 0$ and $f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})$ and $f\left(-(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})\right)$ do not point in the same direction for all $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r_0)$.

<u>Proof</u>: Let $r \geq r_3$ and consider $f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})$ on $S(0,r)$. There are three cases:

(i) $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r) \backslash \mathcal{B}_\varepsilon(g)$ and $-[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r) \backslash \mathcal{B}_\varepsilon(g)$

In this case, both $[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T] f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2}) > 0$ and $-[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T] f\left(-(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})\right) > 0$. Hence, the claim is true.

(ii) $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r) \cap \mathcal{B}_\varepsilon(g)$ and $-[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r) \cap \mathcal{B}_\varepsilon(g)$.

In this case, both $\frac{1}{\rho}[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]A^T f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2}) > 0$ and $-\frac{1}{\rho}[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]A^T f\left(-(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})\right) > 0$. Hence the claim is true.

(iii) $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r) \backslash \mathcal{B}_\varepsilon(g)$ and $-[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r) \cap \mathcal{B}_\varepsilon(g)$.

Premultiply $f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})$ by $[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T] + \frac{1}{\rho}[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]A^T$, we obtain

$$([v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T] + \frac{1}{\rho}[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]A^T) f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2}) = ([v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T] + \frac{1}{\rho}[v_{\mathcal{J}_1}^T, i_{\mathcal{L}_2}^T]A^T) \times$$

$$g(v_{\mathcal{J}_1}, i_{\mathcal{L}_2}) + \frac{1}{\rho}\left\| A\begin{bmatrix} v_{\mathcal{J}_1} \\ i_{\mathcal{L}_2} \end{bmatrix}\right\|_2^2 + \frac{\rho+1}{\rho}\left(v_{\mathcal{J}_1}^T j^1 + i_{\mathcal{L}_2}^T e^2\right).$$

Notice that $[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}] A \begin{bmatrix} v_{\mathcal{J}_1} \\ i_{\mathcal{L}_2} \end{bmatrix} = 0$ because A is skew symmetric. Since all resistors are eventually passive and since $\frac{1}{\rho}[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}]A^T$ is bounded ($\rho \triangleq \| [v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \|$), there exists an $r_4 \geq r_3$ such that $\{[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}] + \frac{1}{\rho}[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}]A^T\} \, f(v_{\mathcal{J}_1}, i_{\mathcal{L}_2}) > 0$ for all $[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r) \setminus \mathcal{B}_\varepsilon(g)$, $r \geq r_4$. Similarly, premultiply $f\left(-(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})\right)$ by $-\left\{[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}] + \frac{1}{\rho}[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}]A^T\right\}$, we obtain

$$
\begin{aligned}
& - \left\{[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}] + \frac{1}{\rho}[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}]A^T\right\} f\left(-(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})\right) \\
= & - \left\{[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}] + \frac{1}{\rho}[v^T_{\mathcal{J}_1}, i_{\mathcal{L}_2}]A^T\right\} g\left(-(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})\right) + \frac{1}{\rho} \left\| A \begin{bmatrix} v_{\mathcal{J}_1} \\ i_{\mathcal{L}_2} \end{bmatrix} \right\|_2^2 \\
& - \frac{\rho+1}{\rho} \left[v^T_{\mathcal{J}_1} j^1 + i^T_{\mathcal{L}_2} e^2 \right] .
\end{aligned}
$$

Since the resistors are eventually passive, $-[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}] g\left(-(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})\right)$ is bounded from below and since $g\left(-(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})\right)$ is bounded for all $-[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r) \cap \mathcal{B}_\varepsilon(g)$, there exists an $r_5 \geq r_4$ such that $-\left\{[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}] + \frac{1}{\rho}[v^T_{\mathcal{J}_1}, i^T_{\mathcal{L}_2}]A^T\right\} f\left(-(v_{\mathcal{J}_1}, i_{\mathcal{L}_2})\right) > 0$ for all $-[v_{\mathcal{J}_1}, i_{\mathcal{L}_2}] \in S(0,r) \cap \mathcal{B}(g)$, $r \geq r_5$. Hence, the claim is proved upon choosing $r_0 = r_5$. The conclusion follows from Property 8 of Sec. II. ∎

Since there are no controlled sources, condition (ii) of Theorem 16 can be checked by the following topological conditions [10]:

Lemma 5: Let $\mathcal{N}$ be a network satisfying the following conditions:

(i) Every loop of c.c. resistors either contains a type U resistor or else it contains at least two type H resistors which are not similarly directed.[12]

[12] Two type H resistors are similarly directed if their reference directions both agree or disagree with the orientation of the loop.

(ii) Every cut set of v.c. resistors either contains a type U resistor or else it contains at least two type H resistors which are not similarly directed.

Then $\mathcal{B}(\underset{\sim}{g}') \cap \mathcal{N}(\underset{\sim}{A}) = \underset{\sim}{0}$.

Proof: This important observation is proved in [10] by Desoer and Wu.

Remark: Recall that $\mathcal{B}(\underset{\sim}{g}) = \mathcal{B}(\underset{\sim}{g}')$ if all resistors are uncoupled and eventually passive.

Theorem 16 can be generalized to networks containing coupled resistors:

Corollary 4: Let $\mathcal{N}$ be a network described by Eq. (8) and let $\underset{\sim}{y}(\underset{\sim}{z}) = [\underset{\sim}{y}_U(\underset{\sim}{z}_U), \underset{\sim}{y}_H(\underset{\sim}{z}_H), \underset{\sim}{y}_B(\underset{\sim}{z}_B)]$.

(i) Let $\underset{\sim}{z} = [\underset{\sim}{v}_{\mathcal{J}_1}, \underset{\sim}{i}_{\mathcal{L}_2}]$ and $\underset{\sim}{y} = [\underset{\sim}{\hat{i}}_{\mathcal{J}_1}, \underset{\sim}{\hat{v}}_{\mathcal{L}_2}]$. Let the function $\underset{\sim}{y}$ satisfy following equations:

(a) there exists an $M > 0$ and a $B > 0$ such that

$$|z_j| > M \Rightarrow z_j y_j(\underset{\sim}{z}_U) > 0, \quad |z_j| \leq M \Rightarrow |y_j(\underset{\sim}{z}_U)| < B, \text{ and}$$

$$\lim_{z_j \to \pm\infty} y_j(\underset{\sim}{z}_U) = \pm\infty \text{ respectively, for all } j \in U.$$

(b) $\underset{\sim}{y}_H(\underset{\sim}{z}_H)$ is diagonal and eventually passive, each component $y_j(z_j)$ is of type H.

(c) $\underset{\sim}{y}_B(\underset{\sim}{z}_B)$ is bounded and eventually passive.

(ii) Let $\underset{\sim}{z}' \triangleq [\underset{\sim}{v}_{\mathcal{L}_1}, \underset{\sim}{i}_{\mathcal{J}_2}]$ and $\underset{\sim}{y}' = [\underset{\sim}{\hat{i}}_{\mathcal{L}_1}, \underset{\sim}{\hat{v}}_{\mathcal{J}_2}]$. Assume that there exist $M' > 0$ and $B' > 0$ such that each component y_j' of $\underset{\sim}{y}'$ satisfies the condition

$$|z_j'| > M' \Rightarrow z_j' y_j'(z') > 0 \quad \text{and} \quad |z_j'| \leq M' \Rightarrow |y_j'(\underset{\sim}{z}')| < B'.$$

Then the conclusion of Theorem 16 holds.

Proof: Same as the proof for Theorem 16. Notice that condition (ii) insures $\mathcal{B}(\underset{\sim}{g}) = \mathcal{B}(\underset{\sim}{g}')$. ∎

4.3 The Uniqueness of Solutions

It is well known that many monotone networks have unique solutions. The following lemma can be proved by Tellegen's theorem. [11]

Lemma 6: Let $\mathcal{N}$ be a network described by Eq. (8). Assume that all resistors are strictly increasing. Then if $\mathcal{N}$ has a solution, it is unique.

The following theorem is due to Desoer and Wu [11]:

Theorem 17: Let $\mathcal{N}$ be a network described by Eq. (8). Assume that

(i) All resistors are uncoupled and strictly increasing.

(ii) The topological conditions in Lemma 5 are satisfied.

Then $\mathcal{N}$ has a unique solution for any independent sources.

Proof: (If) Implied by Theorem 16, Lemma 5 and Lemma 6.

(Only if) By constructing counterexamples, see [11].

4.4 Boundedness and Continuous Dependence of Solutions

All solutions predicted from the preceding theorems and lemmas are bounded since we actually constructed the bounding spheres $S(\underset{\sim}{0},r)$. As to the continuous dependence of the solution on the parameters of the network, we have

Theorem 18: The solution of Eq. (8) in Theorem 17 depends continuously on the function $\underset{\sim}{f}$ (and hence the parameters of the network).

Proof: See the proof of Theorem 14.

V. THE BOUNDING REGION OF SOLUTIONS

5.1 The Bounds on the Solutions

It is often pointed out that most iterative methods, e.g., the Newton-Raphson method, for finding the solutions of nonlinear algebraic equations converge only when the initial guess is sufficiently close to the solution. The hybrid analysis via n-port formulation suggests a natural and easy way to find a region which contains the solutions. This method is best illustrated by the following example.

Example: Consider the simple transistor circuit and its Ebers-Moll model shown in Fig. 1(c). The equations obtained via an n-port formulation of the circuit are given by

$$\underset{\sim}{f}(i_1,i_2) = \begin{bmatrix} \hat{v}_1(i_1) \\ \hat{v}_2(i_2) \end{bmatrix} + \begin{bmatrix} 1.0\times10^3 & 5.0\times10^4 \\ 1.0\times10 & 5.1\times10^4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} -5 \\ 7 \end{bmatrix} = \underset{\sim}{0}$$

where $\hat{i}_j(v_j) = I_0(e^{v_j/V_T}-1)$ and $\hat{v}_j = \hat{i}_j^{-1}$. It follows from Corollary 2 of Theorem 2 and Theorem 10 that this circuit has a unique solution. Let us now find a region which contains the solution. The Ebers-Moll model represents a no-gain element [12]. Hence $|v_k| \leq 5 + 12 = 17$ v, and $|i_k| \leq (5/10^5) + (12/10^3) = 1.21 \times 10^{-2}$A, k = 1,2. Thus the solution $[i_1,i_2]^*$ must be in the set D_1 as shown in Fig. 9(a).

On the other hand, if we redefine $\hat{v}_k(i_k)$ as shown in Fig. 9(b), the new circuit will have the same solution. Retaining the same notation for $\hat{v}_k$, let us rewrite the network equation in the form

$$\underset{\sim}{f}(i_1,i_2) = \begin{bmatrix} y_1(i_1,i_2) \\ y_2(i_1,i_2) \end{bmatrix} = \begin{bmatrix} \hat{v}_1(i_1) \\ \hat{v}_2(i_1) \end{bmatrix} + \underset{\sim}{L}(i_1,i_2) = \underset{\sim}{0}$$

where

$$\underset{\sim}{L}(i_1,i_2) = \begin{bmatrix} 1.0\times10^3 & 5.0\times10^4 \\ 1.0\times10 & 5.1\times10^4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} -5 \\ 7 \end{bmatrix}$$

is the affine part and the $\hat{v}_k$'s are redefined functions. The image of any $[i_1,i_2]^T$ under $\underset{\sim}{f}$ is the image $\underset{\sim}{L}(i_1,i_2)$ superimposed by the image of $[\hat{v}_1,\hat{v}_2]^T$. Let us consider first the action of the affine map $\underset{\sim}{L}(i_1,i_2)$. Consider the set of points lying on the straight line $y_1 = 17$ in the y_1-y_2 plane. The inverse image of this line under $\underset{\sim}{L}$, denoted by $\underset{\sim}{L}^{-1}(y_1=17)$ is also a straight line in the i_1-i_2 plane. That is, any $[i_1,i_2] \in \underset{\sim}{L}^{-1}(y_1=17)$ will be mapped into the y_1-y_2 plane with $y_1 = 17$. Similarly, we find $\underset{\sim}{L}^{-1}(y_1=-17)$, $\underset{\sim}{L}^{-1}(y_2=17)$ and $\underset{\sim}{L}^{-1}(y_2=-17)$. Thus we have found a parallelogram D_2 bounded by $\underset{\sim}{L}^{-1}(y_1=\pm17)$ and $\underset{\sim}{L}^{-1}(y_2=\pm17)$ in the i_1-i_2 plane which is mapped by $\underset{\sim}{L}$ into a square $Q \triangleq \{\underset{\sim}{y} \in \mathbb{R}^2: \|\underset{\sim}{y}\| = 17\}$ in the y_1-y_2 plane. Now let us consider the whole function $\underset{\sim}{f}$ on the boundary of D_2. To be specific, consider $\underset{\sim}{f}(i_1,i_2)$ on $\underset{\sim}{L}^{-1}(y_1=17)$. The function $\underset{\sim}{f}$ has two parts: the affine part $\underset{\sim}{L}$ maps $\underset{\sim}{L}^{-1}(y_1=17)$ into the line $y_1 = 17$ and the nonlinear part $[\hat{v}_1,\hat{v}_2]$ which maps $\underset{\sim}{L}^{-1}(y_1=17)$ into a curve superimposed on top of the line $y_1 = 17$. However, since $|v_1| \leq 17$, the total image $\underset{\sim}{f}\left(\underset{\sim}{L}^{-1}(y_1=17)\right)$ will be a curve which never crosses the $y_2 = 0$ axis. Similarly, $\underset{\sim}{f}\left(\underset{\sim}{L}^{-1}(y_1=-17)\right)$ and $\underset{\sim}{f}\left(\underset{\sim}{L}^{-1}(y_2=\pm17)\right)$ will be curves which never cross $y_2 = 0$ and $y_1 = 0$ axes, respectively. In other words, $\underset{\sim}{f}$ maps the boundary of D_2 into a closed curve in the y_1-y_2 plane which contains the origin. See Fig. 9(c). That is, the parallelogram D_2 contains the solution $[i_1,i_2]^*$. But D_1 also contains the solution. Consequently, the intersection $D_1 \cap D_2$ will give us a less conservative region which contains the solution. Observe that $D_1 \cap D_2$ as shown in Fig. 9(d) is much smaller than either D_1 or D_2, and any point within this region will provide a fairly good initial guess for the solution of the circuit.

Knowing that the solution is contained in $D_1 \cap D_2$ we can proceed further to modify the characteristics of the nonlinear resistors according to the new constraints imposed by $D_1 \cap D_2$. In this example, however, this step is unnecessary since for $[i_1,i_2] \in D_1 \cap D_2$ the constraints remain the same, i.e., $|v_j(i_j)| \leq 17$, $j = 1,2$.

For the general cases, the following algorithm can be used to find a point close to the solutions:

Let the network equation be given by

$$\underset{\sim}{f}(\underset{\sim}{z}) = \begin{bmatrix} g_1(z_1) \\ g_2(z_2) \\ \vdots \\ g_n(z_n) \end{bmatrix} + \underset{\sim}{L}(\underset{\sim}{z}) = \underset{\sim}{0}$$

where $\underset{\sim}{L}$ denotes an affine map.

<u>Step 1</u>: Find bounds for each variable z_k with the help of the <u>no-gain property</u> [13] or other techniques. Hence $z_{k_\ell}^{(1)} \leq z_k \leq z_{k_u}^{(1)}$.

<u>Step 2</u>: Modify the characteristics $g_k(z_k)$ as in the preceding example in accordance with the bounds on z_k, thus we obtain

$$|g_k(z_k)| \leq b_k^{(1)} \qquad k = 1,2,\ldots,n$$

The set $D_1 \cap D_2$ is then described by

$$-b_k^{(1)} \leq L_k(\underset{\sim}{z}) \leq b_k^{(1)}$$

$$z_{k_\ell}^{(1)} \leq z_k \leq z_{k_u}^{(1)} \qquad k = 1,2,\ldots,n,$$

where L_k is the k-th component of $\underset{\sim}{L}$.

Step 3: Find "improved" bounds $z_{k_\ell}^{(2)}$ and $z_{k_u}^{(2)}$ on the components of all $\underset{\sim}{z}$ in $D_1 \cap D_2$ by solving the following associated linear programming problem [19]:

Find max z_k and min z_k

subject to $-b_{k'}^{(1)} \leq L_{k'}(z) \leq b_{k'}^{(1)}$, $z_{k'_\ell}^{(1)} \leq z_{k'} \leq z_{k'}^{(1)}$; $k' = 1,2,\ldots,n$;

for all $k = 1,2,\ldots,n$.

Step 4: Modify the characteristics $g_k(z_k)$ in accordance with the bounds $z_{k_\ell}^{(2)}$ and $z_{k_u}^{(2)}$ on z_k, thus we obtain

$$|g_k(z_k)| \leq b_k^{(2)} \quad , \qquad k = 1,2,\ldots,n.$$

Set $M = 0$.

Step 5: For each k, if $b_k^{(2)} < b_k^{(1)}$, set $M = 1$. Set $b_k^{(1)} = b_k^{(2)}$, $z_{k_\ell}^{(1)} = z_{k_\ell}^{(2)}$ and $z_{k_u}^{(1)} = z_{k_u}^{(2)}$, $k = 1,2,\ldots,n$.

Step 6: If $M = 1$, go to Step 3, else, go to Step 7.

Step 7: Solve the following associated linear programming problem:

Final a feasible solution $\underset{\sim}{z}^{(0)}$

subject to $-b_k^{(1)} \leq L_k(\underset{\sim}{z}) \leq b_k^{(1)}$, $z_{k_\ell}^{(1)} \leq z_k \leq z_{k_u}^{(1)}$; $k = 1,2,\ldots,n$.

If there is no feasible solution, the network $\mathcal{N}$ does not have a solution.

The feasible solution $\underset{\sim}{z}^{(0)}$ will provide a good initial guess for some iterative method for finding the exact solutions.

5.2 The Operating Range

The maximum power rating of all devices imposes a physical constraint on the maximum permissible range of their operating branch voltages or currents. Let $\mathcal{N}$ be a network containing two-terminal resistors and independent sources. Assume that each resistor R_j in $\mathcal{N}$

is characterized by $v_j = \hat{v}_j(i_j)$ with a maximum operating current $|i_j| < \bar{M}_j < \infty$, or $i_j = \hat{i}_j(v_j)$ with a maximum operating voltage $|v_j| < M_j < \infty$; $j = 1,2,\ldots,b$, where b is the number of branches in $\mathcal{N}$. The Cartesian product of the operating ranges of the resistors, e.g., $\prod_{j=1}^{b} B_j$ will henceforth be referred to as the operating range of $\mathcal{N}$. It is an important design problem to determine a region S in the space of independent sources so that the solutions of $\mathcal{N}$ corresponding to any set of points in S will fall within the operating range of $\mathcal{N}$. In [13] Wu presented a method for finding bounds on the independent sources but the estimate is usually too conservative to be useful. In this final section, we will present a different approach based upon the proof of Theorem 6 in Sec. III. We will use the following two examples to illustrate our method.

Example 1: Consider the linear circuit shown in Fig. 10(a). Assume $R_1 = 4K$, $R_2 = 1K$. Determine a region S in the E-I plane so that $(E,I) \in S$ ensures that the solution falls within the operating range defined by $|i_j| \leq 10^{-3}A$, $j = 1,2$.

Since the circuit is linear, superposition is valid. By considering E and I separately, we obtain

$$|i_1| = |\frac{1}{5} I + \frac{1}{5} E| < 1 \quad \text{and} \quad |i_2| = |\frac{4}{5} I - \frac{1}{5} E| < 1$$

where E is in volts and I in mA. The above inequalities then define a region S shown in Fig. 10(b). In the linear case, the set S is the "maximal set" in the sense that any $(E,I) \notin S$ will force the solution to fall outside of the operating range of $\mathcal{N}$. The generalization of the above technique to more complicated linear circuits is obvious.

For nonlinear circuits, however, the maximal set S is extremely difficult (if not impossible) to determine, and only a subset of the maximal region can usually be found. Since a nonlinear function is generally not odd symmetric $\left(\underset{\sim}{f}(\underset{\sim}{z}) \neq -\underset{\sim}{f}(-\underset{\sim}{z})\right)$, we must fix both the locations and the polarities of the independent sources before investigating the operating range.

Example 2: Consider the nonlinear circuit shown in Fig. 11(a). Find a region S in the E-I plane such that any $(E,I) \in S$ insures that the solution satisfies the following constraint:

$$|v_1| < .6V. \quad \text{and} \quad |i_2| < 1 \times 10^{-3}A.$$

The network equation of the circuit is given by

$$\underset{\sim}{f}(v_1,i_2) = \begin{bmatrix} \hat{i}_1(v_1) \\ Ri_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} -I \\ -E \end{bmatrix} \tag{9}$$

$$\triangleq \begin{bmatrix} \hat{i}_1(v_1) \\ Ri_1 \end{bmatrix} + \underset{\sim}{H}\underset{\sim}{z} + \begin{bmatrix} -I \\ -E \end{bmatrix} = \underset{\sim}{0}$$

where $\underset{\sim}{z} = [v_1 i_2]^T$. Notice that $\underset{\sim}{H}$ is skew-symmetric and hence $\underset{\sim}{H} \in P_0$. The operating ranges of the resistors define a rectangle D in the z-plane. Considering $\underset{\sim}{f}$ as a vector field defined on D, let us find a region S so that $(E,I) \in S$ implies that $\underset{\sim}{f}(\underset{\sim}{z}) \neq \underset{\sim}{0}$ and $\underset{\sim}{f}(\underset{\sim}{z})$ and $\underset{\sim}{f}(-\underset{\sim}{z})$ will not point in the same direction for all $\underset{\sim}{z} \in \partial D$. This then insures that $\underset{\sim}{f}(\underset{\sim}{z}) = \underset{\sim}{0}$ has a solution in D. We shall show that either $f_1(\underset{\sim}{z})f_1(-\underset{\sim}{z}) < 0$ or $f_2(\underset{\sim}{z})f_2(-\underset{\sim}{z}) < 0$ for all $\underset{\sim}{z} \in \partial D$.

(i) Since $H \in P_0$, there exists a z_k, say $z_k = z_1 = v_1$, $z_k \neq 0$ and $(\underset{\sim}{H}\underset{\sim}{z})_1 z_1 \geq 0$. It is easily seen from Eq. (9) that[13] $f_1(\underset{\sim}{z})f_1(-\underset{\sim}{z}) < 0$ if

[13] That is, $\hat{i}_1(v_1) - I \neq 0$ and has the same sign as v_1.

$(v_1)[\hat{i}_1(v_1) - I] > 0$. Thus we obtain the condition

$$v_1 < 0 \quad \text{or} \quad v_1 > \hat{i}_1^{-1}(I) \triangleq M_1^{(1)} \tag{10}$$

(ii) Similarly, if $z_k = z_2 = i_2$ then $f_2(\underset{\sim}{z})f_2(-\underset{\sim}{z}) < 0$ if

$$i_2 < 0 \quad \text{or} \quad i_2 > \frac{E}{R} = M_2^{(2)} . \tag{11}$$

(iii) Now, suppose (i) is true but (10) does not hold. Consider $f_2(\underset{\sim}{z}) = Ri_2 - v_1 - E$, $0 < v_1 < M_1^{(1)}$. In order to have $f_2(\underset{\sim}{z})f_2(-\underset{\sim}{z}) < 0$ we require $i_2f_2 > 0$. This then implies:

$$i_2 < 0 \quad \text{or} \quad i_2 > \frac{E+M_1^{(1)}}{R}$$

Thus, $|i_2| > \frac{E+M_1^{(1)}}{R} \Rightarrow f_2(\underset{\sim}{z})f_2(-\underset{\sim}{z}) < 0$. By assumption, $|i_2| < 1 \times 10^{-3}$A. Therefore we obtain the inequality

$$\frac{E+M_1^{(1)}}{R} \leq 1 \times 10^{-3}$$

or

$$E + i_1^{-1}(I) \leq 1 . \tag{12}$$

(iv) Suppose (ii) is true but Eq. (11) does not hold. Consider $f_1(\underset{\sim}{z}) = \hat{i}_1(v_1) + i_2 - I$, $0 < i_2 < M_2^{(1)}$. In order to have $f_1(\underset{\sim}{z})f_1(-\underset{\sim}{z}) < 0$, we require $v_1f_1 > 0$. This is satisfied if:

$$v_1 < 0 \Rightarrow \begin{cases} \hat{i}_1(v_1) < I - M_2^{(1)} & \text{if } I - M_2^{(1)} \leq 0 \\ \text{no more restriction} & \text{if } I - M_2^{(1)} < 0. \end{cases} \tag{13}$$

or, $v_1 > 0 \Rightarrow v_1 > \hat{i}_1^{-1}(I) = M_1^{(1)}$ (same as (10).)

Since $\hat{i}_1(v_1) > 10^{-12}$A for $v_1 \geq .6$V, we obtain, from (13), that $I - M_2^{(1)} \geq -10^{-12}$ and $M_1^{(1)} < .6$, or

$$I - \frac{E}{R} \geq -10^{-12} \text{ and } \hat{i}_1^{-1}(I) < .6 \tag{14}$$

Inequalities (12) and (14) then define a region S in the E-I plane as shown in Fig. 11(b). The solution of the circuit corresponding to any "input" $(E,I) \in S$ will fall within the operating range of the circuit. Any point not in S may or may not give rise to a (v_1, i_2) in the operating range. For example, $\underset{\sim}{p} = (.7,.15) \notin S$ makes $|v_1| > .6V$.

This technique can be applied to more complicated circuits. The degree of success, however, usually depends upon the individual problems.

VI. CONCLUDING REMARKS

In this paper we have applied the degree theory to the analysis of a large class of resistive nonlinear networks. In particular, we study the structure of the network equations by homotopy of odd fields. The form of network equations (both via the n-port and topological formulations), together with some circuit-theoretic conditions, such as eventual passivity, form a network function which is homotopic to an odd field. This is a natural consequence of the fact that both KCL and KVL are linear constraints and that eventual passivity is implied by odd functions. Thus the degree theory provides a fairly general and unified approach for analyzing a large class of practical networks. All theorems in this paper, although applying to different cases, deal with the homotopy classes of odd fields. Even though most known results can be proved easily by this unified approach we only mentioned a few in the paper because many other results are rather special cases from this point of view.

We would like to point out that insofar as the existence of solutions is concerned, the concept of eventual passivity is much more basic than the so-called eventual increasingness. Many known results, such as those by Sandberg and Willson [7,8,9,13] and Desoer and Wu [11], deal with strictly monotone increasing nonlinear elements first and then extend to cases including eventually strictly increasing nonlinearities. The lengthy proofs of these theorems are basically analytical in nature and therefore do not reveal a clear picture of the geometric structure of the vector fields associated with the network equations. As a result, most of these theorems are confined to eventually increasing networks and cannot be extended easily to networks containing nonlinear coupled elements.

Since most theorems on the existence of solutions are sufficient conditions, it is important to realize the great flexibility in formulating the network equation. For example, consider the Ebers-Moll model of a transistor. We can either extract the diodes in the model to obtain two type H, uncoupled nonlinear resistors, or treat the whole transistor as a passive three-terminal element. In many cases, a proper choice of the ports and the network variables enables us to assert the existence of solutions immediately by inspection.

Some applications to the problem of finding bounding regions of solutions are also discussed in this paper. The method presented in Sec. 5.1 is particularly suited for practical transistor circuits where each transistor is either forward or reverse biased. Standard techniques from linear programming, such as parametric programming and sensitivity analysis [20], can thus be readily adopted to the design of nonlinear networks.

APPENDIX

(1) (Proof for the existence of r_2.) Let us partition $\underset{\sim}{z} \triangleq [\underset{\sim}{z}_U, \underset{\sim}{z}_H, \underset{\sim}{z}_B]$ where each component z_j of z_X corresponds to type X resistors for X = U, H and B. Partition the columns of $\underset{\sim}{H} \triangleq [\underset{\sim}{H}^U, \underset{\sim}{H}^H, \underset{\sim}{H}^B]$ accordingly. Thus $\mathcal{N}(\underset{\sim}{H}) = \{\underset{\sim}{z} \in \mathbb{R}^n : \underset{\sim}{H}^U \underset{\sim}{z}_U + \underset{\sim}{H}^H \underset{\sim}{z}_H + \underset{\sim}{H}^B \underset{\sim}{z}_B = \underset{\sim}{0}\}$. Let $\underset{\sim}{z} = [\underset{\sim}{0}, \underset{\sim}{z}_H, \underset{\sim}{z}_B] \neq \underset{\sim}{0}$ with $\underset{\sim}{z}_H \geq 0$, then $\underset{\sim}{z} \in \mathcal{B}(\underset{\sim}{g})$. Since $\mathcal{B}(\underset{\sim}{g}) \cap \mathcal{N}(\underset{\sim}{H}) = \underset{\sim}{0}$, $\underset{\sim}{H}\underset{\sim}{z} \neq \underset{\sim}{0}$. Hence, we have $\underset{\sim}{H}^H \underset{\sim}{z}_H + \underset{\sim}{H}^B \underset{\sim}{z}_B \neq \underset{\sim}{0}$ for all $[\underset{\sim}{z}_H, \underset{\sim}{z}_B] \neq \underset{\sim}{0}$, $\underset{\sim}{z}_H \geq 0$. Therefore, there exists an $\alpha > 0$ such that $\|\underset{\sim}{H}^H \underset{\sim}{z}_H + \underset{\sim}{H}^B \underset{\sim}{z}_B\| \geq \alpha \|[\underset{\sim}{z}_H, \underset{\sim}{z}_B]\|$ for all $[\underset{\sim}{z}_H, \underset{\sim}{z}_B] \neq \underset{\sim}{0}$, $\underset{\sim}{z}_H \geq 0$. Let $C_1 \triangleq \sup\{\|\underset{\sim}{H}^U \underset{\sim}{z}_U + \underset{\sim}{H}^H \underset{\sim}{z}_H\| : \|[\underset{\sim}{z}_U, \underset{\sim}{z}_H]\| \leq r_1\}$, obviously $C_1 < \infty$. Now, let $\underset{\sim}{z} = [\underset{\sim}{z}_U, \underset{\sim}{z}_H, \underset{\sim}{z}_B] \in \bar{\mathcal{U}}$, write $\underset{\sim}{z}_H = \underset{\sim}{z}_{H'} + \underset{\sim}{z}_{H''}$ where $\underset{\sim}{z}_{H'} < 0$, $\|\underset{\sim}{z}_{H'}\| \leq r_1$ and $\underset{\sim}{z}_{H''} \geq 0$. Consider

$$\|\underset{\sim}{H}\underset{\sim}{z}\| = \|\underset{\sim}{H}^U \underset{\sim}{z}_U + \underset{\sim}{H}^{H'} \underset{\sim}{z}_{H'} + \underset{\sim}{H}^{H''} \underset{\sim}{z}_{H''} + \underset{\sim}{H}^B \underset{\sim}{z}_B\| \geq |\alpha \|[\underset{\sim}{z}_H, \underset{\sim}{z}_B]\| - C_1|.$$

Let $r_2 > \max\{C_1/\alpha, r_1\}$, then for any $\underset{\sim}{z} \in \bar{\mathcal{U}}$, $\|\underset{\sim}{z}\| \geq r_2 \Rightarrow \|[\underset{\sim}{z}_H, \underset{\sim}{z}_B]\| \geq r_2 \Rightarrow \|\underset{\sim}{H}\underset{\sim}{z}\| \neq 0$. That is, $\underset{\sim}{z} \notin \mathcal{N}(\underset{\sim}{H})$. ∎

(2) (Property of μ.) The quantity μ can be defined as $\mu(r_3) = \bar{\mathcal{U}}$ $\inf\{\underset{\sim}{z}^T \underset{\sim}{H}^T \underset{\sim}{H}\underset{\sim}{z} : \underset{\sim}{z} \in \mathcal{P}(r_3)\}$ where $\mathcal{P}(r_3) \triangleq$ the radial projection of $\bar{\mathcal{U}} \cap S(\underset{\sim}{0}, r_3)$ onto the unit sphere $S(\underset{\sim}{0}, 1)$. Since the boundary of $\bar{\mathcal{U}}$ consists of hyperplanes with constant coordinates $\pm r_1$, the set $\mathcal{P}(r_3)$ is a decreasing function of r_3, i.e., $\mathcal{P}(r_3') \subset \mathcal{P}(r_3'')$ if $r_3' > r_3''$. On the other hand, it is easy to see that $\lim_{r_3 \to \infty} \mathcal{P}(r_3) = \mathcal{B}(\underset{\sim}{g}) \cap S(\underset{\sim}{0}, 1)$. Hence $\mu(r_3)$ increases with r_3 and $\lim_{r_3 \to \infty} \mu(r_3) = \mu_0 = \inf\{\underset{\sim}{z}^T \underset{\sim}{H}^T \underset{\sim}{H}\underset{\sim}{z} : \underset{\sim}{z} \in \mathcal{B}(\underset{\sim}{g}) \cap S(\underset{\sim}{0}, 1)\}$. ∎

REFERENCES

1. Berger, M. and M. Berger, Perspectives in Nonlinearity, W. A. Benjamin, Inc., New York, N.Y., 1968.

2. Ortega, J. M. and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1969.

3. Krasnosel'skiy, M. A., Topological Methods in the Theory of Nonlinear Integral Equations, Pergamon Press, New York, N.Y., 1964.

4. Cronin, J., Fixed Points and Topological Degree in Nonlinear Analysis, Amer. Math. Soc., Providence, R.I., 1964.

5. Guillemin, V. and A. Pollack, Differential Topology, Prentice-Hall, Englewood Cliffs, New Jersey, 1974.

6. Fielder, M. and V. Ptak, "Some generalizations of positive definiteness and monotonicity," Numerische Mathematic 9, pp. 163-172 (1966).

7. Willson, A. N., Jr., "New theorems on the equations of nonlinear dc transistor networks," Bell Syst. Tech. J., vol. 49, pp. 1713-1738, Oct. 1970.

8. Sandberg, I. W. and A. N. Willson, Jr., "Existence and uniqueness of solutions for equations of nonlinear dc networks," SIAM J. Appl. Math., vol. 22, pp. 173-186, March 1972.

9. Willson, A. N., Jr., "Some aspects of the theory of nonlinear networks," Proc. of the IEEE, vol. 61, no.8, pp. 1092-1113, Aug. 1973.

10. Duffin, R. J., "Nonlinear networks IIa," Bull. Amer. Math. Soc., vol. 53, pp. 963-971, Oct. 1947.

11. Desoer, C. A. and F. F. Wu, "Nonlinear monotone networks," SIAM J. Appl. Math., vol. 26, pp. 315-333, March 1974.

12. So, H. C., "On the hybrid description of a linear n-port resulting from extraction of arbitrarily specified elements," IEEE Trans. on Circuit Theory, vol. CT-12, pp. 381-387, Sept. 1965.

13. Willson, A. N., Jr., "The no-gain property for networks containing three terminal elements," IEEE Trans. on Circuits and Systems, vo. CAS-22, pp. 678-687, August 1975.

14. Wu, F. F., "Existence of an operating point for a nonlinear circuit using the degree of mapping," IEEE Trans. on Circuits and Systems, vol. CAS-21, no. 5, pp. 671-677, Sept. 1974.

15. Chua, L. O. and P-M Lin, Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.

16. Gopinath, B. and D. Mitra, "When are transistors passive?" Bell Syst. Tech. J. vol. 50, no. 8, pp. 2835-2847, October 1971.

17. Chua, L. O. and D. N. Green, "Graph-theoretic properties of dynamic nonlinear networks," Electronics Research Laboratory, University of California, Memorandum ERL-M507, March 14, 1975.

18. Roska, T. and J. Klimo, "On the solvability of dc equations and the implicit integration formula," International Journal on Circuit Theory and Applications, vol. 1, pp. 273-280, 1973.

19. Chua, L. O., Introduction to Nonlinear Network Theory, McGraw-Hill, New York, N. Y. 1969.

20. Dantzig, G. B., Linear Programming and Extensions, Princeton University Press, 1963, Princeton, New Jersey.

FIGURE CAPTIONS

Fig. 1 An example illustrating the extraction of nonlinear resistors as ports.

Fig. 2 An example illustrating the various techniques for creating an appropriate set of external ports.

Fig. 3 An example showing the structurally stable and unstable solutions.

Fig. 4 A flip-flop circuit.

Fig. 5 Equivalence between voltage and current sources.

Fig. 6 A geometrical interpretation of the sets $\mathcal{U}$ (hatched region), $\bar{\mathcal{U}}$ (unhatched region), and $\mathcal{N}(\underset{\sim}{H})$ in the two-dimensional case. The function g_1 is of type U and the function g_2 is of type H.

Fig. 7 A nonlinear network.

Fig. 8 A composite branch.

Fig. 9 The bounding region of solutions.

(a) The set D_1 obtained by the no-gain property.

(b) Modification of g_j.

(c) The affine map $\underset{\sim}{L}$.

(d) The set $D_1 \cap D_2$.

Fig. 10 An example for illustrating the operating range of a linear network. (a) The linear circuit, (b) The operating range for the circuit in (a) with $|i_j| \leq 1mA$, $j = 1,2$.

Fig. 11 An example for illustrating the operating range of a nonlinear network.

(a) The nonlinear circuit, (b) The operting range for the circuit in (a) with $|v_1| < 0.6V$ and $|i_2| < 1$ mA.

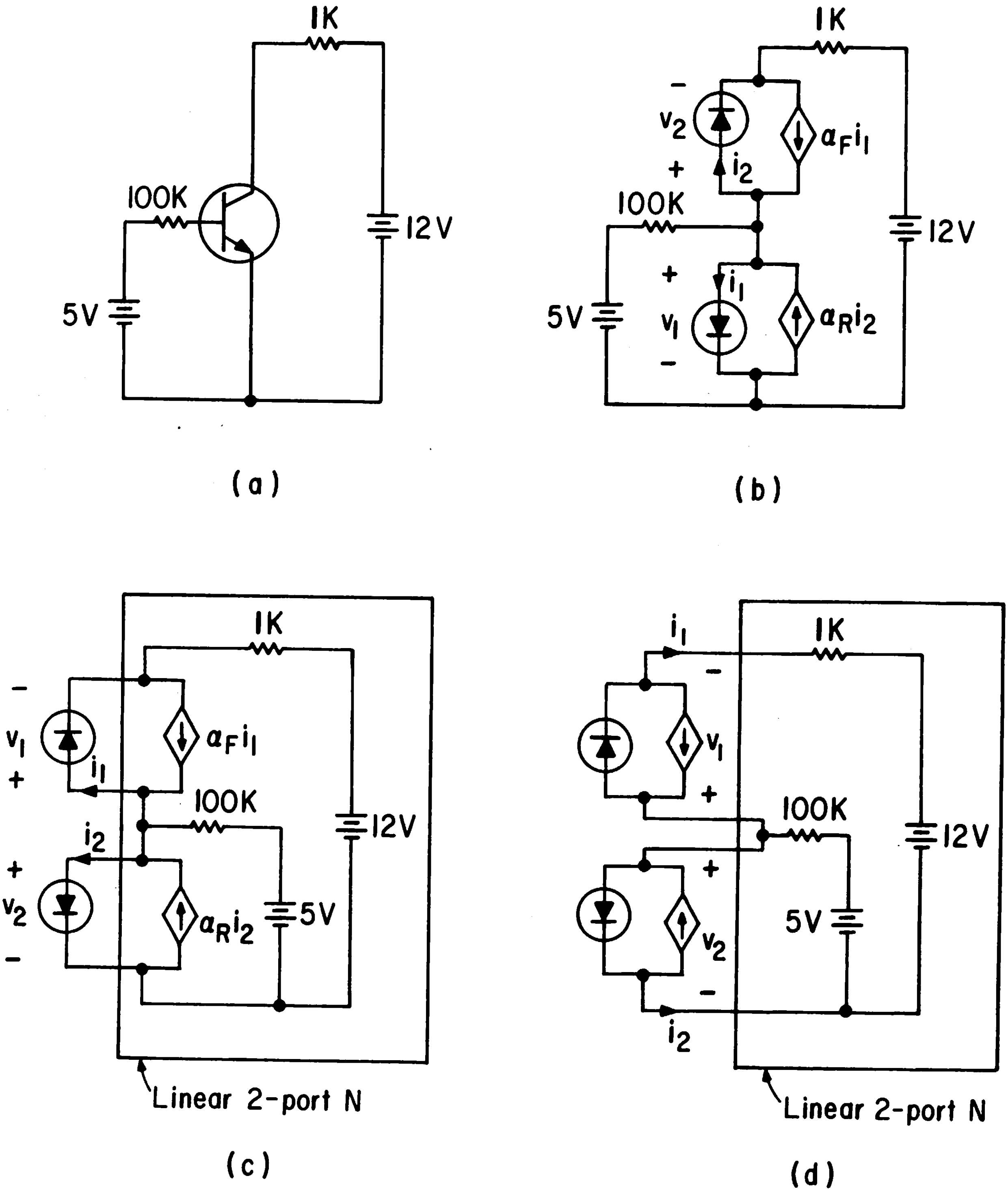
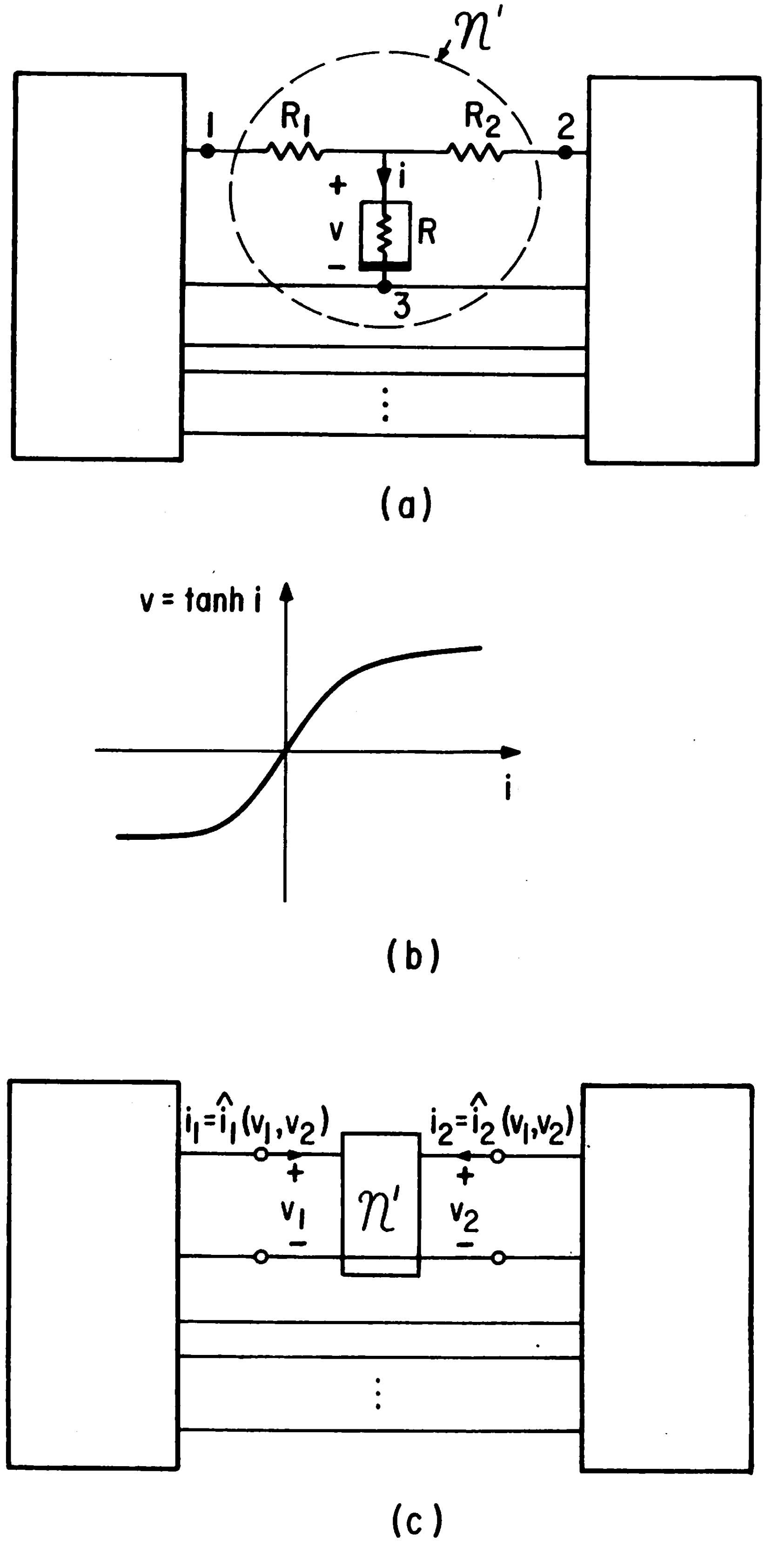

1K
100K
5V
12V
(a)
1K
v2
i2
αF i1
100K
12V
5V
v1
i1
αR i2
(b)
1K
v1
i1
αF i1
100K
12V
i2
v2
αR i2
5V
Linear 2-port N
(c)
i1
1K
v1
100K
12V
5V
v2
i2
Linear 2-port N
(d)

Fig. 1



$\mathcal{N}'$
1
R_1
R_2
2
i
+
v
−
R
3
(a)
v = tanh i
i
(b)
$i_1=\hat{i}_1(v_1,v_2)$
$i_2=\hat{i}_2(v_1,v_2)$
+
v_1
−
$\mathcal{N}'$
+
v_2
−
(c)

Fig. 2

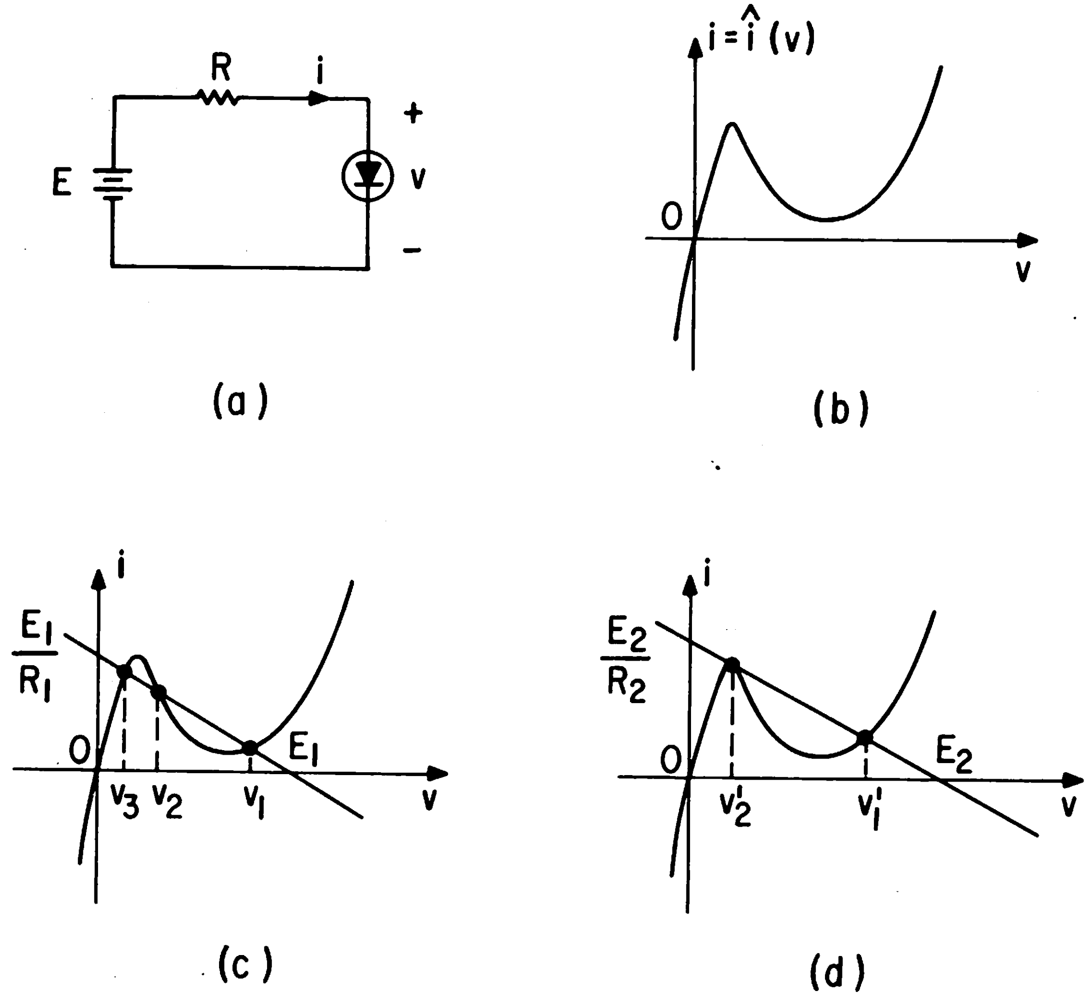


Fig. 3

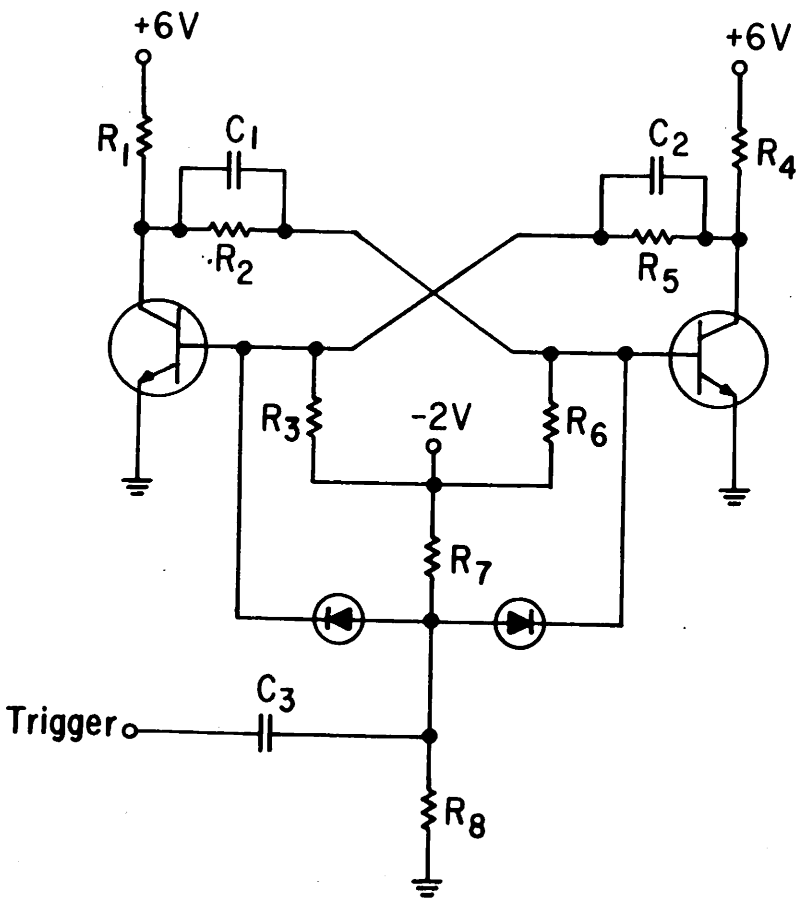


Fig. 4

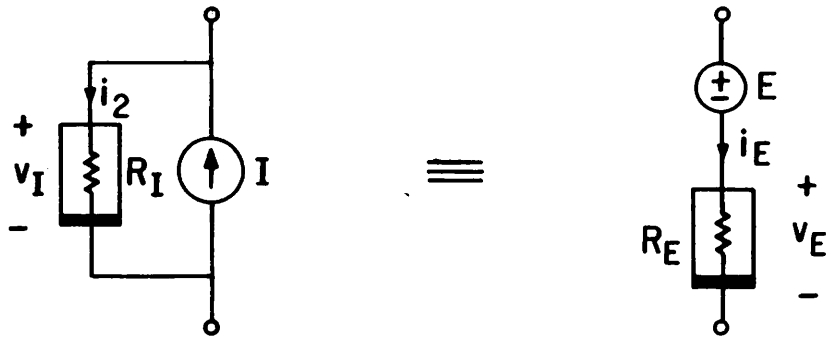

i2
+
vI
RI
I
−
E
iE
+
RE
vE
−

Fig. 5

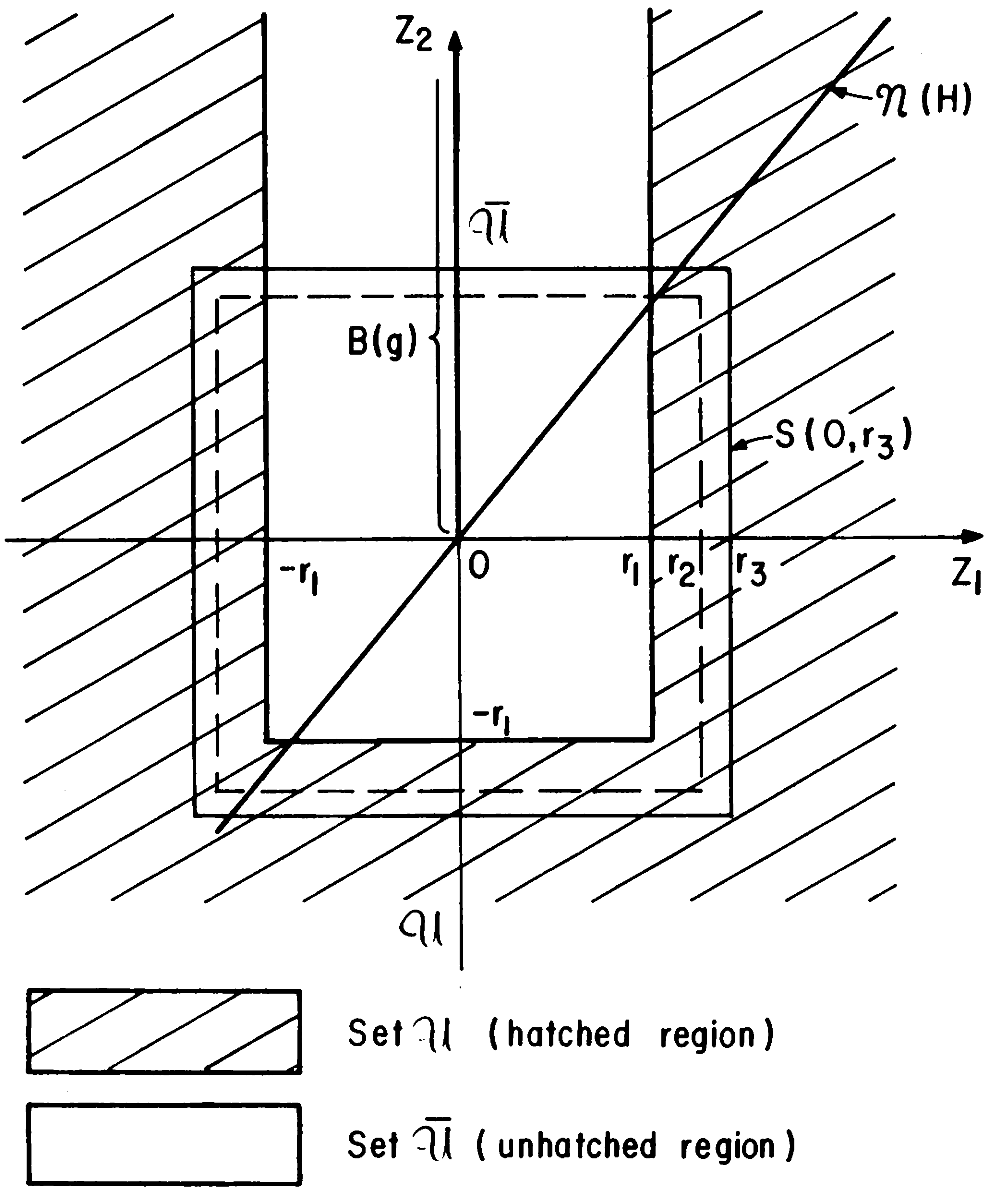
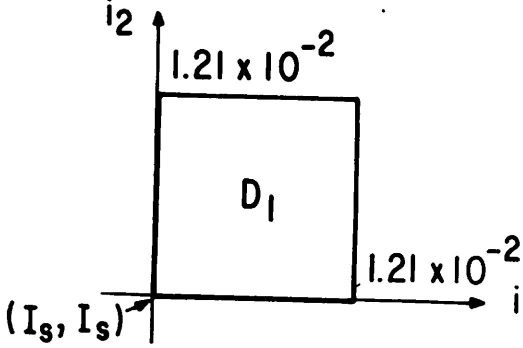
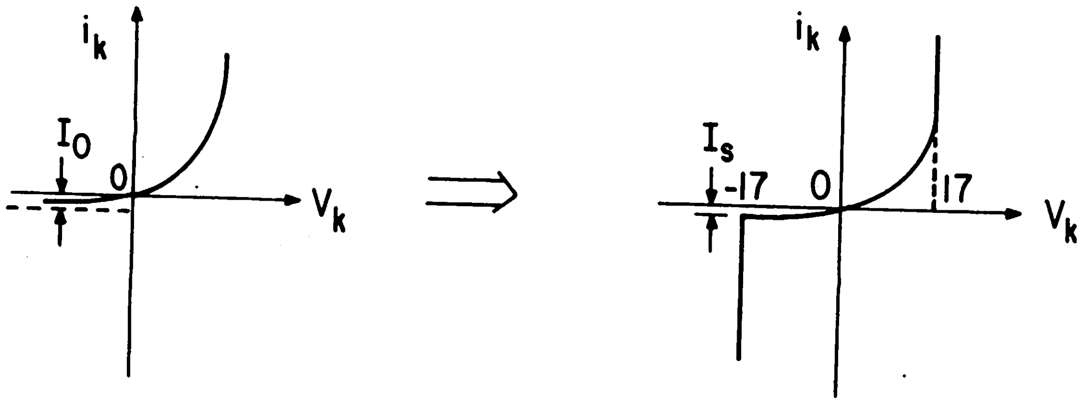

Z2
Ū
η (H)
B(g)
S (0, r3)
−r1
0
r1
r2
r3
Z1
−r1
U
Set U (hatched region)
Set Ū (unhatched region)

Fig. 6



$$I_s = I_0 (e^{-17/V_T} - 1) < 0$$

(a)



(b)

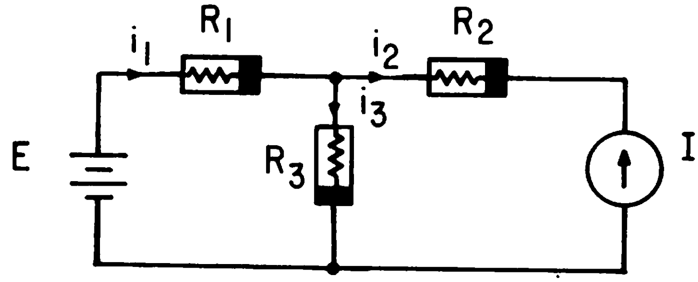


Fig. 7

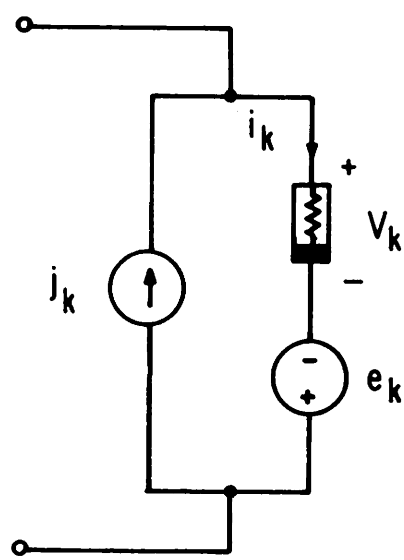
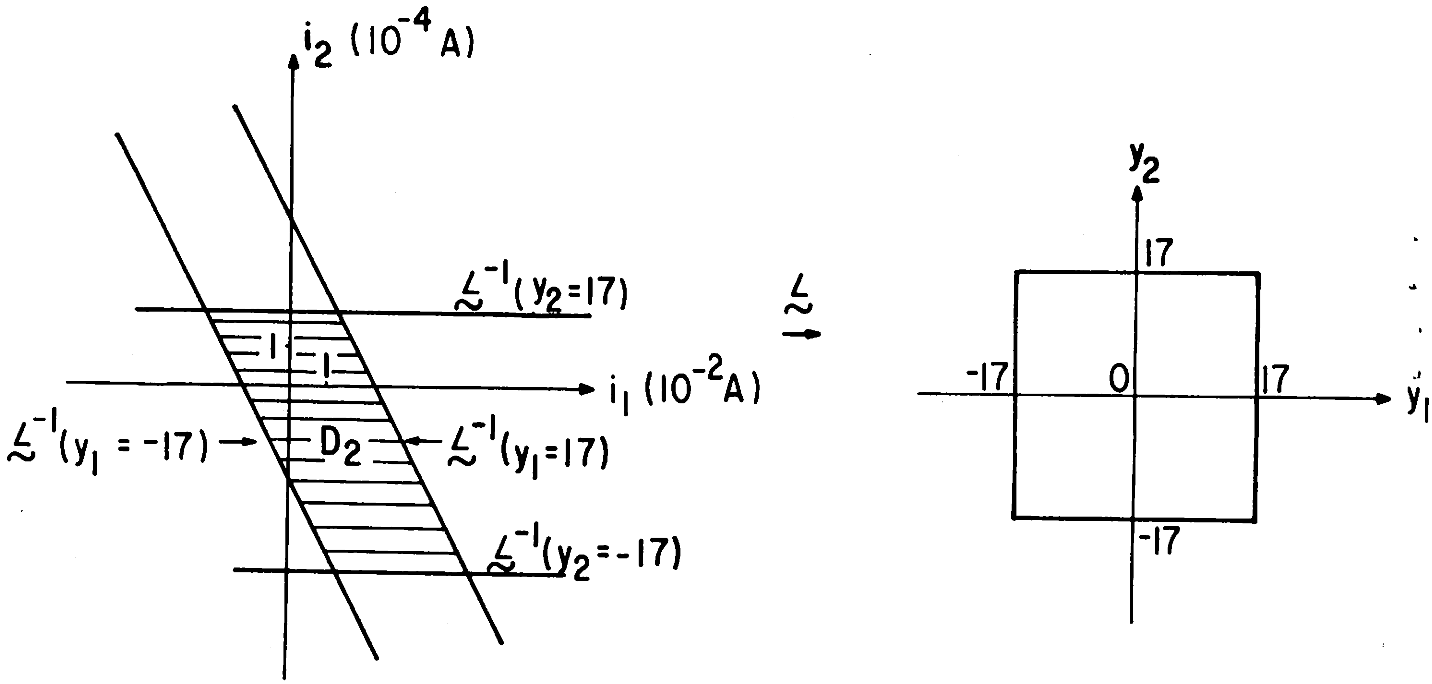
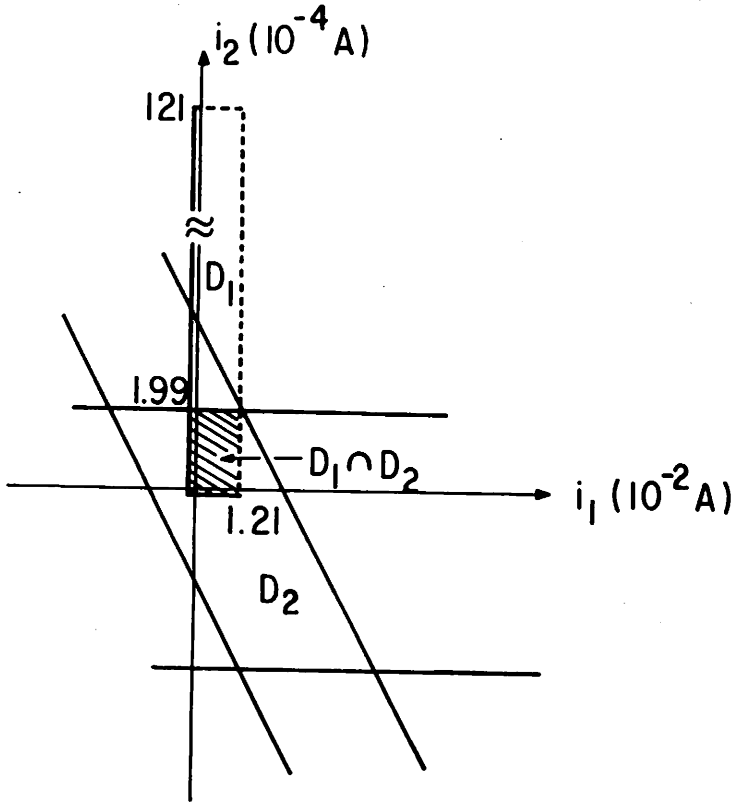


Fig. 8



i_2 (10^{-4} A)
$\underset{\sim}{L}^{-1}(y_2 = 17)$
1
1
i_1 (10^{-2} A)
$\underset{\sim}{L}^{-1}(y_1 = -17)$
D_2
$\underset{\sim}{L}^{-1}(y_1 = 17)$
$\underset{\sim}{L}^{-1}(y_2 = -17)$
$\underset{\sim}{L}$
y_2
17
-17
0
17
y_1
-17

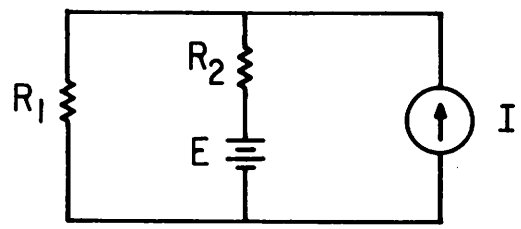
(c)



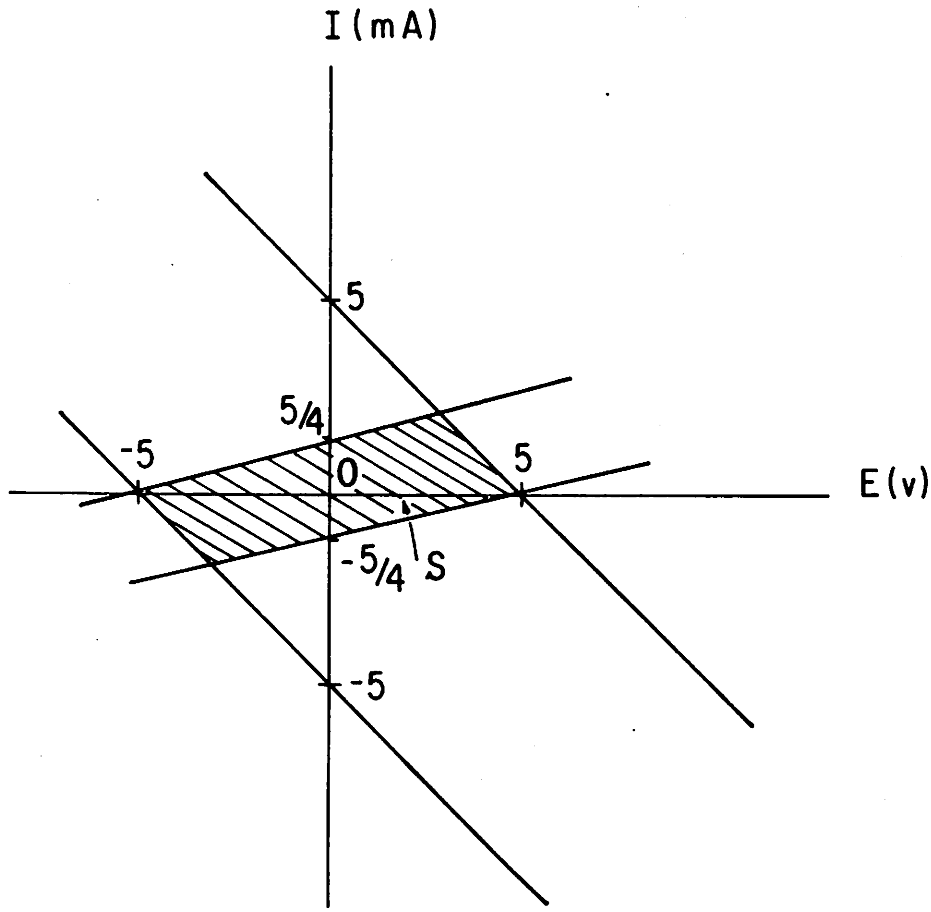
i_2 (10^{-4} A)
121
D_1
1.99
$D_1 \cap D_2$
i_1 (10^{-2} A)
1.21
D_2

(d)

Fig. 9

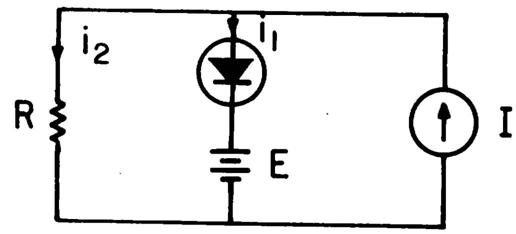


(a)



(b)

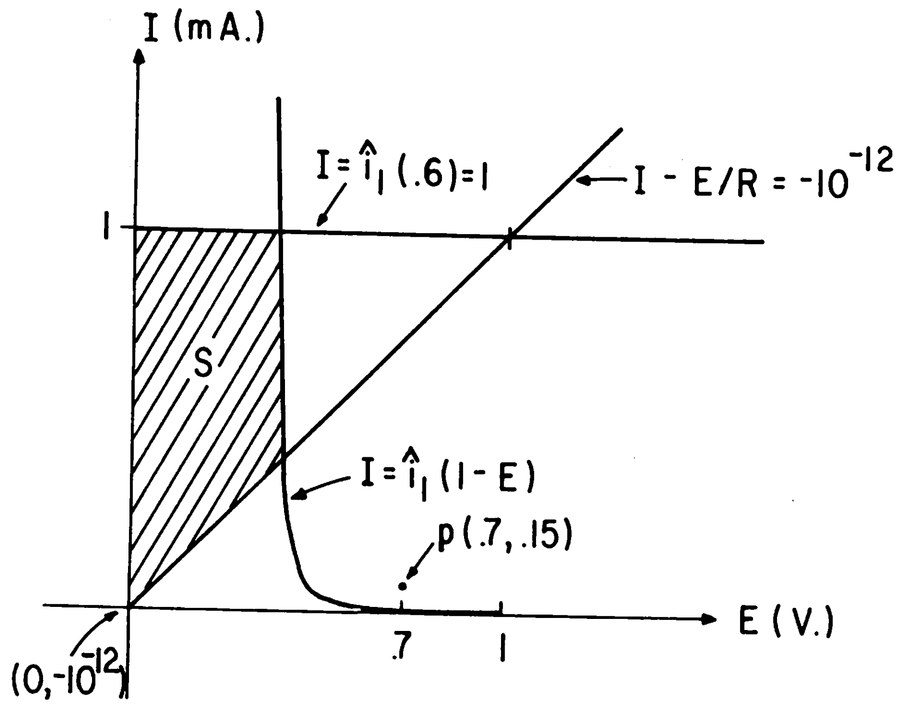
Fig. 10



$R = 1K,\ E, I > 0$

$i_1 = \hat{i}_1(v_1) = 10^{-13}(e^{40v_1} - 1)$

(a)



(b)

Fig. 11