ON THE APPROXIMATION OF SOLUTIONS TO MULTIPLE CRITERIA DECISION MAKING PROBLEMS

by

Elijah Polak

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Elijah Polak

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

This paper presents an adaptive precision method based on cubic splines and a new scalarization procedure, which constructs approximations to the surface of noninferior points. It is particularly well suited for the two or three criteria optimization problems, but it can also be used to some extent for high dimensional problems. An important aspect of this method is that it is quite efficient, since it computes no more points than necessary to ensure a prescribed level of precision of approximation.

I. INTRODUCTION

In the past few years, a new scalarization method for computing noninferior points for a multiple criteria decision problem has been presented independently by several researchers [9], [19], [13]. Unlike the earlier characterizations which were based on convex combinations of the criteria, see for example [5], [1], this method does not depend on convexity. However, while a major obstacle has thus been removed, the cost of computing a single noninferior point is still quite high, since it requires that an associated constrained optimization problem be solved. Consequently, we cannot entertain the idea of computing very large numbers of noninferior points, but must, somehow, make do with a relatively small number. In this paper, we present an algorithm (derived from the one in [13]) which constructs an economical grid of noninferior points to be used in conjunction with an interpolation scheme in the value space. The algorithm is specifically designed for the bicriteria case, but it can also be used in higher dimensional situations.

II. CHARACTERIZATION OF NONINFERIOR POINTS

The algorithm we shall present in Sec. III is specialized to the two and three criteria case, with equality and inequality type constraints. Nevertheless it is useful to first consider the multicriteria optimization problem in greater generality. Thus, we assume that we are given a closed constraint set $\Omega \subset \mathbb{R}^n$, a continuously differentiable vector criterion function $f : \mathbb{R}^n \to \mathbb{R}^m$, and a partial order in $\mathbb{R}^m$ defined by $y_1 \leqq y_2$ if (componentwise) $y_1^i \leq y_2^i$, $i = 1,2,\ldots,m$. When $y_1 \leqq y_2$ but $y_1 \neq y_2$ we shall denote this fact by $y_1 \lneq y_2$. Now, let

$$V = \{y \in \mathbb{R}^m \,|\, y \in f(x),\ x \in \Omega\}, \tag{1}$$

be the set of possible values, and for any $y \in \mathbb{R}^m$ let

$$N(y) = \{y' \,|\, y' \lneq y\} \tag{2}$$

be the "negative cone at y."

Then, the multicriteria decision problem consists of constructing the following two sets: (i) the set of <u>noninferior</u> (Pareto optimal) <u>values</u>

$$V_N \triangleq \{y \in V \,|\, N(y) \cap V = \{y\}\} \tag{3}$$

and the set of <u>noninferior</u> (Pareto optimal) <u>points</u>

$$\Omega_N = \{x \in \Omega \,|\, f(x) \in V_N\} \tag{4}$$

Now, let $\bar{f} : \mathbb{R}^n \to \mathbb{R}^{m-1}$ be defined by

$$\bar{f}(x) \triangleq (f^1(x), f^2(x), \ldots, f^{m-1}(x))^T \tag{5}$$

where the f^i, $i = 1,2,\ldots,m-1$, are the first m-1 components of the vector criterion function f, and let

$$\bar{Y} = \{\bar{y} \in \mathbb{R}^{m-1} \,|\, \Omega \cap \{x \,|\, \bar{f}(x) \lneq \bar{y}\} \neq \phi\} \tag{6}$$

We now define the <u>sensitivity function</u> $s : \bar{Y} \to \mathbb{R}^1$ by

$$s(\bar{y}) \triangleq \min\{f^m(x) \,|\, x \in \Omega, \bar{f}(x) \lneq \bar{y}\} \tag{7}$$

Note that this sensitivity function is quite similar the one defined by Geoffrion [7]. We shall denote the graph of s by Γ, i.e.

$$\Gamma \triangleq \{y \in \mathbb{R}^m \,|\, y = (\bar{y}, y^m), \bar{y} \in \bar{Y}, y^m = s(\bar{y})\} \tag{8}$$

Our algorithm is based on the following properties of Γ.

Proposition 1: The set of noninferior values, V_N, is contained in Γ, the graph of $s(\cdot)$.

Proof: First, note that an alternative characterization of $s(\cdot)$ is

$$s(\bar{y}) = \min\{y'^m | y' \in V, \bar{y}' \leqq \bar{y}\} \tag{9}$$

Now suppose that $y = (\bar{y}, y^m) \in V_N$, but $y \notin \Gamma$ (i.e. $y^m \neq s(\bar{y})$). Then from (9), there exists a $y' = (\bar{y}', y'^m) \in V$ such that $\bar{y}' \leqq \bar{y}$ and $y'^m = s(\bar{y}) < y^m$. But this implies that $y' \leq y$, which contradicts our assumption that $y \in V_N$. Hence, $V_N \subset \Gamma$. ¤

Theorem 1: A point $y = (\bar{y}, y^m) \in \Gamma$ is a noninferior value if and only if $y \in V$ and y is a strong global minimizer for (9), i.e.

$$V_N = \Gamma_0 \triangleq \{y = (\bar{y}, y^m) \in \Gamma \cap V | y^m = s(\bar{y}) < y'^m \; \forall y' \in V \cap N(y)\} \tag{10}$$

Proof: $\Rightarrow$ First, recall that by Proposition 1, $V_N \subset \Gamma \cap V$. Now suppose that $y \in V_N$, but $y \notin \Gamma_0$. Then there exists a $y' \in V \cap N(y)$ such that $y'^m < y^m$. Since this contradicts our assumption that $y \in V_N$, we conclude that V_N is contained in Γ_0.

$\Leftarrow$ Now suppose that $y \in \Gamma_0$, but $y \notin V_N$. Then there must exist a $y' \in V$ such that $y' \leq y$. But by definition of Γ_0, $y^m < y'^m$ for all $y' \in V \cap N(y)$ and hence we have a contradiction. Consequently, Γ_0 is contained in V_N. ¤

Corollary: A point $x \in \Omega$ is noninferior if and only if it is a global minimizer of (7), for $y = \bar{f}(x)$, satisfying $f^m(x) < f^m(x')$ for all $x' \in \Omega$ such that $\bar{f}(x') \leq \bar{f}(x)$. ¤

The following result is obvious.

Proposition 2: The sensitivity function $s(\cdot)$ is monotonically decreasing, i.e. $\bar{y}' \geq \bar{y}$ implies that $s(\bar{y}') \leq s(\bar{y})$. ¤

We can now extract a more specialized characterization of points in V_N, as follows.

Theorem 2: Suppose that $s(\cdot)$ is piecewise continuously differentiable. If $y \in \Gamma$ is such that $y \in V$ and $\nabla s(\bar{y}) < 0$, then $y \in V_N$. ¤

In words, theorem 2 states that V_N contains all the points of Γ where the slope is strictly negative. In fact, it is not difficult to see that the difference between V_N and the subset of Γ of nonzero slope points is a set of zero measure, when $s(\cdot)$ is piecewise continuously differentiable.

The following result, stated without proof, follows directly from the properties of differentiable manifolds and from the standard results on sensitivity to parameters (see Luenberger [10, p. 236]).

<u>Theorem 3</u>: Suppose that the criterion function $f(\cdot)$ is twice continuously differentiable, that

$$\Omega = \{x \in \mathbb{R}^n \mid g(x) = 0,\ h(x) \leqq 0\} \tag{11}$$

with $g : \mathbb{R}^n \to \mathbb{R}^\ell$, $h : \mathbb{R}^n \to \mathbb{R}^p$ twice continuously differentiable, and that for all $\bar{y} \in \bar{Y}$, the set

$$\Omega_{\bar{y}} \triangleq \{x \in \mathbb{R}^n \mid g(x) = 0,\ h(x) \leqq 0,\ \bar{f}(x) \leqq \bar{y}\} \tag{12}$$

satisfies the Kuhn-Tucker constraint qualification [3]. Furthermore, let $x(\bar{y}) \in \Omega_{\bar{y}}$ be such that $s(\bar{y}) = f^m(x(\bar{y}))$. Then $s(\cdot)$ is differentiable at every point $\bar{y}$ where $x(\cdot)$ is differentiable and $\nabla s(\bar{y}) = -\lambda(\bar{y})$, where $\lambda(\bar{y}) \geqq 0$ is a Kuhn-Tucker multiplier, i.e., it satisfies

$$-\nabla f^m(x(\bar{y})) + \frac{\partial g(x(\bar{y}))}{\partial x}^T \psi(\bar{y}) + \frac{\partial h(x(\bar{y}))}{\partial x}^T \mu(\bar{y}) + \frac{\partial \bar{f}(x(\bar{y}))}{\partial x} \lambda(\bar{y}) = 0 \tag{13}$$

$$\langle \lambda(\bar{y}), \bar{f}(\bar{y}) \rangle = 0 \tag{14}$$

for some $\psi(\bar{y}) \in \mathbb{R}^\ell$, $\mu(\bar{y}) \geqq 0$ in $\mathbb{R}^p$, with $\langle \mu(\bar{y}), h(x(\bar{y})) \rangle$ 0. ¤

We can expect $x(\bar{y})$ to be differentiable at points $\bar{y}$ where the minimizer $x(\bar{y})$ is unique. Thus, at such a point, when we compute $s(\bar{y})$, not only do we compute a point on Γ, but we also get the slope of Γ, provided we use a minimization algorithm which produces the multiplier $\lambda(\bar{y})$ as well as $x(\bar{y})$ and $s(\bar{y})$. Thus, we need to use an algorithm of the penalty function type [6], or a dual method of feasible directions, such as [15], which automatically compute the required multiplier.

An indication of when $x(\bar{y})$ is a unique minimizer can be deduced from Theorem 1 as follows.

<u>Corollary 1</u>: Suppose that $y = (\bar{y}, y^m) \in V_N$ and for any two $x', x'' \in \{x \in \Omega \mid \bar{f}(x) = \bar{y}\}$ $f^m(x') \neq f^m(x'')$, then the minimizer $x(\bar{y})$ of (7) is unique.

<u>Proof</u>: By assumption, for all $x' \in \{x \in \Omega \mid \bar{f}(x) = \bar{y}\}$, $f^m(x') \neq f^m(x(\bar{y}))$, hence $f^m(x')$ $> f^m(x(y))$. Next, by Theorem 1, for any $x' \in \Omega$ such that $\bar{f}(x') \leq \bar{y}$, we must have

$s(\bar{y}) = f^m(x(\bar{y})) < f^m(x')$. Hence $x(\bar{y})$ is unique. □

In fact, it can be shown that $s(\cdot)$ is piecewise continuously differentiable under fairly weak assumptions, though it is not simple to do it since it is a highly non-trivial exercise in differential geometry. The interested reader is referred to [20], [21], for a presentation of available results. In this paper, we shall simply assume that $s(\cdot)$ is several times piecewise continuously differentiable.

Now, when computing $s(\bar{y})$ by solving (7), we may, in fact, find a local minimum and not a global one. We must therefore establish the relation of local minimizers of (7) to the Pareto optimal set. For this purpose we define a point $y \in V$ to be a <u>locally noninferior value</u> if there exists a neighborhood B of y such that $N(y) \cap B \cap V = \{y\}$. Next, we define a point $x \in \Omega$ to be a <u>locally noninferior point</u> if $f(x)$ is a locally noninferior value.

The following result can be proved in an analogous manner to theorem 1 and its corollary.

<u>Theorem 4</u>: A point $y \in V$ is a locally noninferior value if and only if it is a strong local minimizer for (9). Furthermore, a point $x \in \Omega$ is locally noninferior if and only if it is a local minimizer for (7), with $\bar{y} = \bar{f}(x)$, and $f^m(x) < f^m(x')$ for all $x' \in B \cap \Omega$ such that $\bar{f}(x') \leq \bar{f}(x)$, where B is some neighborhood of x. □

Thus, when we solve (7) for any $\bar{y} \in \bar{Y}$ and get an x^* which is a local or global minimizer, we are sure that x^* is at least locally noninferior if $f(x^*) = \bar{y}$ and second order sufficiency conditions of optimality are satisfied at x^*.

III. INTERPOLATION OF Γ

Let us now restrict ourselves to the case with Ω as in (11) and all functions at least five times differentiable. Also we assume that the assumptions in [21] are satisfied, where $m = 2$, i.e. to the bicriteria case. Then $y = (y^1, y^2)$, so that $\bar{y} = y^1$ and Γ is a piecewise continuously differentiable curve. As we have already explained, when we compute two points on Γ, $(y_1^1, s(y_1^1))$, $(y_2^1, s(y_2^1))$, we also obtain the derivatives $s(y_1^1)$ and $s(y_2^1)$ in the process. Therefore, it is natural to interpolate such points by means of Hermite cubic polynomials [8]. Thus, given y_j^1, y_k^1 in $\bar{Y}$, with $s_j \triangleq s(y_j^1)$, $s_k \triangleq s(y_k^1)$, $s_j' \triangleq s'(y_j^1)$, $s_k' = s_k'(y^1)$, the Hermite interpolating cubic is defined by

$$H^{j,k}(y^1) = s_j \frac{(y^1-y_k^1)^2}{(y_j^1-y_k^1)^2}\left[1 - \frac{2(y^1-y_j^1)}{(y_j^1 - y_k^1)}\right] + s_k \frac{(y^1-y_j^1)^2}{(y_j^1-y_k^1)^2}\left[1 - \frac{2(y^1-y_k^1)}{(y_k^1 - y_j^1)}\right]$$

$$+ s_j' \frac{(y^1-y_j^1)(y^1-y_k^1)^2}{(y_j^1-y_k^1)^2} + s_k' \frac{(y^1 y_k^1)(y^1-y_j^1)^2}{(y_j^1-y_k^1)^2} \tag{15}$$

Assuming that $s(\cdot)$ is four times continuously differentiable on $[y_j^1, y_k^1]$, the interpolation error is bounded [see [8]),

$$|s(y^1) - H_{j,k}(y^1)| \leq \left|\frac{d^4 s(\eta)}{d(y')^4}\right| (y_j^1-y^1)^2 (y_k^1-y^1)^2 \leq \frac{1}{16}\left|\frac{d^4 s(\eta)}{d(y')^4}\right| (y_j^1-y_k^1)^4 \tag{16}$$

for some $\eta \in [y_j^1, y_k^1]$.

To make use of formula (16) so as to determine whether $H_{j,k}(\cdot)$ is a sufficiently good approximation to $s(\cdot)$ over the interval $[y_j^1, y_k^1]$, we need to know the number $\frac{d^4 s(\eta)}{d(y^1)^4}$, or at least to have a reasonable estimate of it. To compute such an estimate, we propose the following simple scheme which should be adequate for our purpose. Given three points $y_n^1 < y_{n+1}^1 < y_{n+2}^1$, we shall assume that for all $y^1 \in [y_n^1, y_{n+2}^1]$,

$$\left|\frac{d^4 s(y^1)}{d(y^1)^4}\right| \leq 4|s(y_{n+1}^1) - H_{n,n+2}(y_{n+1}^1)| / (y_{n+1}^1-y_n^1)^2 (y_{n+2}^1-y_{n+1}^1)^2 \tag{17}$$

If we now interpolate between the points y_n^1, y_{n+2}^1, we can expect that (see (16)) for all $y^1 \in [y_n^1, y_{n+2}^1]$,

$$|s(y^1) - H_{n,n+2}(y^1)| \leq e(y_n^1, y_{n+1}^1, y_{n+2}^1) \triangleq \frac{4|s(y_{n+1}^1) - H_{n,n+2}(y_{n+1}^1)|}{(y_{n+1}^1-y_n^1)^2 (y_{n+2}^1-y_{n+1}^1)^2} \; \frac{1}{16}(y_{n+2}^1-y_n^1)^4$$

$$= \frac{|s(y_{n+1}^1) - H_{n,n+2}(y_{n+1}^1)| (y_{n+2}^1-y_n^1)^4}{4(y_{n+1}^1-y_n^1)^2 (y_{n+2}^1-y_{n+1}^1)^2} \tag{18}$$

Note that if we now interpolate by means of $H_{n,n+1}(\cdot)$ on $[y_n^1, y_{n+1}^1]$ and $H_{n+1,n+2}(\cdot)$ on $[y_{n+1}^1, y_{n+2}^1]$, the error will be substantially smaller than $e(y_n^1, y_{n+1}^1, y_{n+2}^1)$. In the case where $y_{n+1}^1 = \frac{1}{2}(y_n^1 + y_{n+2}^1)$, the error will now be $\frac{1}{16}\, e(y_n^1, y_{n+1}^1, y_{n+2}^1)$, as can be seen from (16). Thus, if we assume that

$$\max_{y^1 \in [y_{n+1}^1, y_{n+2}^1]} |s(y^1) - H_{n,n+1}(y^1)| \leq e(y_n^1, y_{n+1}^1, y_{n+2}^1) \tag{19a}$$

$$\max_{y^1 \in [y^1_{n+1}, y^1_{n+2}]} |s(y^1) - H_{n+1,n+2}(y^1)| \leq e(y^1_n, y^1_{n+1}, y^1_{n+2}) \tag{19b}$$

we are being conservative.

The interpolation algorithm which we shall present in the next section constructs an economical grid $y^1_1, y^1_2, \ldots, y^1_N$ in $\bar{Y}$ such that $e(y^1_n, y^1_{n+1}, y^1_{n+2}) \leq E$, where E is the desired precision for the interpolation of Γ, by means of Hermite cubics, and such that $H'_{n,n+1}(y^1) \leq 0$ for all $y^1 \in [y^1_n, y^1_{n+1}]$, $n = 1,2,\ldots,N$, since, as we have shown in the preceding section, $s'(y^1) \leq 0$ for almost all $y^1 \in \bar{Y}$.

IV. AN ALGORITHM FOR THE TWO CRITERIA CASE

The algorithm below requires two pieces of data: E, the desired precision for interpolation , and $\theta \in (0,1]$ which serves as a fudge factor for deciding the tentative step length $\Delta\bar{y}^1$, as follows. First, suppose that the points $y^1_1, y^1_2, \ldots y^1_N$ are equispaced, i.e., $y^1_{n+1} - y^1_n = \Delta y^1_0$, $n = 1,2,\ldots,N-1$. Then, by (18) and (16) for some $K > 0$

$$e_0 \triangleq e(y^1_{n+2}, y^1_{n+1}, y^1_n) = 4|s(y^1_{n+1}) - H_{n,n+2}(y^1_{n+1})| = K(\Delta y^1_0)^4 \tag{20}$$

If we now change the spacing between points to Δy^1_1, then the corresponding error bound

$$e_1 = K(\Delta y^1_1)^4, \tag{21}$$

assuming that we are still using the same bound as on the fourth derivative of $s(\cdot)$. Hence, if we want to choose an economical step size Δy^1_1, we should make $e_1 = E$, so that

$$\frac{e_1}{e_0} = \frac{E}{e_0} = \left(\frac{\Delta y^1_1}{\Delta y^1_0}\right)^4 \tag{22}$$

which leads to

$$\Delta\bar{y}^1 = \Delta y^1_1 = \Delta y^1_0 (E/e_0)^{1/4} \tag{23}$$

Since our spacing is not uniform and since the fourth derivative will vary, further tests to ensure the desired level of precision are necessary. These are incorporated in the algorithm below, laid out in terms of five blocks: Initialization, Error check, Monotonicity check, Grid refinement, Grid continuation.

Interpolation Algorithm: (Two criteria f^1, f^2)

Data: E, $0 < \theta \leq 1$, $0 < \Delta y^1_{min} < \Delta y^1_{max}$.

(Initialization)

Step 1: Compute

$$y^2_{min} = \min\{f^2(x) | x \in \Omega\}, \tag{24}$$

$$x^* \in \arg\min\{f^2(x) \; x \in \Omega\} \tag{25}$$

$$y^1_{min} = \min\{f^1(x) | x \in \Omega\} \quad , \; y^1_{max} = f^1(x^*) \tag{26}$$

$$\bar{Y} = [y^1_{min}, y^1_{max}]. \tag{27}$$

Step 2: Set $y^1_j = y^1_{min} + (j-1)\Delta y^1_{min}$, $j = 1,2,3$. Set $n = 1$, $k = 3$.

Comment: k is the number of points at which evaluations have been performed and is an index used to avoid duplication of computations when the grid is refined.

Step 3: Compute $s(y^1_j)$, $s'(y^1_j)$ and $x(y^1_j) \in \arg\min\{f^2(x) | x \in \Omega,\ f^1(x) \leq y^1_j\}$ for $j = 2,3$.

(Error check)

Step 4: If $y^1_{n+1} - y^1_n \leq \Delta y^1_{min}$, go to step 11; else, compute the error bound

$$e \triangleq e(y^1_n, y^1_{n+1}, y^1_{n+2}) \tag{28}$$

for the intervals $[y^1_n, y^1_{n+1}]$, $[y^1_{n+1}, y^1_{n+2}]$.

Step 5: If $e \leq E$, go to step 6; else go to step 8.

(Monotonicity check)

Step 6: Compute the coefficients a,b,c of

$$q(y^1) \triangleq H'_{n,n+1}(y^1) = a(y^1)^2 + b(y^1) + c$$

Comment: Note that by construction $q(y^1_j) = s'(y^1_j) \leq 0$ for $j = n,n+1$. Hence $q(y^1) \leq 0$ for $y^1 \in [y^1_n, y^1_{n+1}]$ if any one of the following three conditions holds:

$$\text{(i)} \quad a \neq 0,\ y^1_n \leq -b/2a \leq y^1_{n+1},\ q(-b/2a) \leq 0 \tag{29a}$$

(ii) $a \neq 0$, $-b/2a \notin [y^1_n, y^1_{n+1}]$ (29b)

(iii) $a = 0$ (29c)

<u>Step 7</u>: If (29a) or (29b) or (29c) are satisfied, go to step 11; else, go to step 8.

(Grid Refinement)

<u>Step 8</u>: If $y^1_{n+2} - y^1_{n+1} > y^1_{n+1} - y^1_n$, set $\xi = \frac{1}{2}(y^1_{n+2}+y^1_{n+1})$, compute $s(\xi)$, $s'(\xi)$, $x(\xi)$, and go to step 9; else set $\xi = \frac{1}{2}(y^1_{n+1}+y^1_n)$, compute $s(\xi)$, $s'(\xi)$, $x(\xi)$, and go to step 10.

<u>Step 9</u>: Renumber as follows: $(y^1_j, x(y^1_j), s(y^1_j), s'(y^1_j)) \to (y^1_{j+1}, x(y^1_{j+1}), s(y^1_{j+1}), s'(y^1_{j+1}))$ for $j = n+2,\ldots,k$, set $y^1_{n+2} = \xi$ and go to step 4.

<u>Step 10</u>: Renumber as follows: $(y^1_j, x(y^1_j), s(y^1_j), s'(y^1_j)) \to (y^1_{j+1}, x(y^1_{j+1}), s(y^1_{j+1}), s'(y^1_{j+1}))$ for $j = n+1,\ldots,k$, set $y^1_{n+1} = \xi$, and go to step 4.

(Grid Continuation)

<u>Step 11</u>: Set $n = n + 1$.

<u>Step 12</u>: If $k \geq n + 2$, go to step 4; else go to step 13.

<u>Step 13</u>: If $y^1_{n+1} = y^1_{max} - \Delta y^1_{min}$, compute the polynomials $H_{j,j+1}(y^1)$ from the stored values of y^1_j, $s(y^1_j)$, $s'(y^1_j)$, $j = 1,2,\ldots,k$, and go to step 18; else, go to step 14.

<u>Step 14</u>: Set $\Delta\bar{y}^1 = (y^1_{n+1}-y^1_n)\theta[E/e]^{1/4}$.

<u>Step 15</u>: Set $\Delta y^1 = \max\{\Delta y^1_{min}, \min\{\Delta\bar{y}^1, \Delta y^1_{max}\}\}$.

<u>Step 16</u>: If $y^1_{n+1} = y^1_{max} - \Delta y^1_{min}$, set $y^1_{n+2} = y_{max}$, $s(y^1_{n+2}) = f^2(x^*)$, $s'(y^1_{n+2}) = 0$ and go to step 17; else set $y^1_{n+2} = \min\{y^1_{n+1} + \Delta y^1, y^1_{max} - \Delta y^1_{min}\}$, compute $x(y^1_{n+2})$, $s(y^1_{n+2})$, $s'(y^1_{n+2})$, set $k = k + 1$ and go to step 17.

<u>Step 17</u>: If $s(y^1_{n+1}) = s(y^1_n)$, go to step 11; else go to step 4.

<u>Step 18</u>: Plot Γ, print $\{x(y^1_j)\}^k_{j=1}$ and stop. ◻

Thus, in the two criteria case, the situation is quite straightforward: we present our results, the curve Γ, as a plot on a sheet of paper. The three criteria case is obviously more difficult, since there is no particularly convenient way for displaying a surface in three dimensional space. One approach, for the case $m = 3$, $y = (y^1,y^2,y^3)$, is to plot a set of parametrized curves, which are sections through Γ. For this purpose, we fix y^1 at a prescribed set of values, say $y^1_1, y^1_2, \ldots, y^1_N$, and plot the graphs Γ_i of the functions

$$s_i(y^2) = \min\{f^3(x) \mid x \in \Omega, f^2(x) \leq y^2, f^1(x) \leq y^1_i\} \tag{30}$$

which form a family of curves in the plane. The selection of the grid size for y^1 is now, probably, best fixed in advance, at a fairly large step size. It can later always be refined if it is deemed necessary to insert an extra piece of graph between two existing ones, so as to locally refine the picture.

As a final note, it should be pointed out that the approximation theory described in Sec. 1.3 of Appendix A of [14] can be utilized to decide on rules for economically truncating the infinite computation which, in principle, is required to solve a nonlinear programming problem and hence to obtain a point on Γ

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