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THE MULTILAYER ROUTING PROBLEM:  
SOME NECESSARY AND SUFFICIENT CONDITIONS

by

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THE MULTILAYER ROUTING PROBLEM:  
SOME NECESSARY AND SUFFICIENT CONDITIONS<sup>†</sup>

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ABSTRACT

The multilayer routing problem is introduced and its relation to the single-row, single-layer routing problem is illustrated. An easily implementable sufficient condition on the routability of a net list over a single row of nodes is presented. The solution is given by a constructive forward marching procedure and the result is superior to that which was obtained by So [1]. The implementation algorithm is programmed on CDC 6400 computer. The nature of the optimum criterion relating to single-row routability is investigated and a necessary and sufficient condition is given to characterize the nature of optimality. Some necessary conditions are also presented which can be used to evaluate the sufficient condition and served as a lower bound for the channel capacity in the routing problem. The experimental results and necessary conditions together suggest that our sufficient condition is reasonably close to the optimum.

The more general routing problem is illustrated and possible future research areas are discussed.

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## 1. Introduction.

In this paper we deal with the problem of multilayer routing which was first introduced by H.C. So of the Bell Telephone Laboratories<sup>[1]</sup>. The problem is of central importance in the design of large and complex electronic systems due to the high density packaging requirement. The essence of the problem is to interconnect functional or circuit modules with hundreds or thousands of terminals by means of printed conductors which are layed on a multilayer board. It is assumed that the board has fixed geometries, i.e., each layer has fixed plated-through holes, uniformly spaced on a rectangular grid. Conductor pins (drilled-through holes to reach all layers) and vias (plated-through holes to be used for interconnection between layers) alternating on each row are provided. This is illustrated in Fig. 1. Thus functional or circuit modules with predetermined placement can be mounted, for example, on top of the multilayer board. The problem is then to interconnect terminals of the modules according to specifications by means of pins and vias of the multilayer board with printed conductors layed on each layer. It should be noted that, on each layer, conductor paths form a planar graph. Therefore the routability is of no problem in the absence of any constraint. However, due to various inherent physical and strategical constraints it is necessary to use multilayers; and to reach from one layer to another, we route conductor paths through vias.

There are various kinds of physical constraints associated with the problem, typically: the size of the multilayer board, the feasible number of layers, the minimum width of the conductor path, and the necessary separation between two adjacent parallel conductor paths. Therefore, a key question is whether a given problem can indeed be realized with a specified

multilayer board; and if it cannot be realized, whether there is an optimal realization for obtaining a maximum degree of interconnections. Or, the problem could be formulated in another way: for a given interconnection specification, what is an optimal design of a multilayer board to facilitate hundred percent routing? In this connection, So has made an important contribution. By reducing the multilayer problem to a single-row, single-layer problem, he was able to make an estimate on routability for any given problem. He also developed sufficient conditions and algorithms for routing which guarantee routability for the single-row, single-layer case. His results are however far from optimum; nevertheless, it should be pointed out that, prior to his work, all techniques have been empirical in nature and essentially based on cut and try [2-6].

In this paper we shall restrict most of our attention to the single-row, single-layer routing problem. We will present a sufficient condition as well as algorithms which lead to a significant improvement over So's work. As a matter of fact, our results yield realizations which include that of So's as a least optimum special case. We also have developed a number of necessary conditions which have lead us to believe that our sufficient condition is reasonably close to the optimum.

In Section 2, we present the general multilayer problem through an example and demonstrate how it can be reduced to a single-row, single-layer problem. We then introduce key definitions for the single-row, single-layer problem along with some physical and strategic constraints. The relation with the single-row, multilayer problem is next illustrated. In Section 3, we present our fundamental routing algorithm for the single-row, single-layer problem. The algorithm ensures routing for any problem

if we do not impose physical constraints on routing channel capacity. The algorithm is key to the results in Section 4 which gives a sufficient condition for routability with channel capacity constraints. In Section 5, a necessary and sufficient condition and a number of necessary conditions for routability on the single-row, single-layer problem are derived. In Section 6, we conclude the paper with a simple example on the single-row, multilayer problem and demonstrate how the sufficient and necessary conditions for the single-row, single-layer problem might be used.

## 2. The Multilayer Problem and Its Relation to the Single-Row, Single-Layer Problem.

Consider a backplane with a fixed rectangular array of pins and vias as shown in Fig. 2a. For convenience, we designate each pin or via according to its location, i.e., row and column. Thus  $b_5$  signifies that the pin is located at the intersection of  $b$ -th row and 5-th column. Suppose that the problem is to route a specified net list  $L = \{N_1, N_2, N_3\}$  where  $N_1 = \{a_1, b_5, e_9\}$ ,  $N_2 = \{c_1, c_3, d_5\}$  and  $N_3 = \{a_7, c_5, d_7, e_5\}$ . This implies that pins  $a_1$ ,  $b_5$ , and  $e_9$  in net  $N_1$  are to be interconnected, and so are the pins in  $N_2$  and  $N_3$ , respectively. A possible realization is shown in Fig. 2b. As shown in the figure, we have adopted a special strategy, that is, we depend on horizontal conductor paths to connect pins and vias which lie strictly in a row and we use only vertical conductor paths on a separate layer, as indicated by dotted lines, to connect pins or vias which lie on the same column. This scheme was called the unidirectional routing by So. It has the distinct advantage that it is strategically sound because it is amenable to a systematic study and it rules out the necessity of considering various other possible alternatives in routing. Furthermore, intuitively, the scheme

is basically economic and can handle many simple circuits interconnections with two layers only. Finally, it is seen that, with this, the general multirow, multilayer problem has essentially been reduced to a single-row, single-layer problem.

In this example, from the realization shown in Fig. 2b, we may consider that the problem has been reduced to seven simple single-row, single-layer problems with respect to rows a, b, c, d and e, and columns 2 and 8 as follows:

row a	$\{a_1, a_2\}, \{a_7, a_8\}$
row b	$\{b_2, b_5\}, \{b_3, b_7\}$
row c	$\{c_1, c_2, c_3\}, \{c_5, c_8\}$
row d	$\{d_2, d_5\}, \{d_7, d_8\}$
row e	$\{e_2, e_9\}, \{e_5, e_8\}$
column 2	$\{a_2, b_2, e_2\}, \{c_2, d_2\}$
column 8	$\{a_8, c_8, d_8, e_8\}$

Note that in the single-row specification, pins and vias are treated in the same way as individual nodes. Of course, in this simple example, the solutions for routing the single-row, single-layer problems are already given as shown. However, had we started with a realistic problem, the resulting single-row, single-layer problems would be more complicated. In order to acquire a better understanding of the single-row, single-layer problem, we need to introduce some definitions first. We shall do so again through an example.

Consider the example as shown in Fig. 3a where nodes  $a_1$  through  $a_9$  are evenly spaced on a row R. They represent either pins or vias of the multilayer board. A net  $N_i$  is defined as a set of nodes indicating common

potential, which are to be connected by conductor path on a plane. A net list  $L$  is a set of disjoint nets, which implies that no two nets share the same node. Thus in Fig. 3a,  $L = \{N_1, N_2, N_3, N_4\}$  and the four nets are  $N_1 = \{a_1, a_5\}$ ,  $N_2 = \{a_2, a_9\}$ ,  $N_3 = \{a_3, a_6\}$  and  $N_4 = \{a_4, a_7, a_8\}$ . For ease in discussion, we shall use a graph representation of a net list as shown in Fig. 3b. Thus Fig. 3a gives a physical realization of the net list in Fig. 3b. It should be emphasized that in Fig. 3b horizontal and vertical connecting lines are not conducting paths; they only prescribe the interconnection specifications. The advantage of using the graphical representation of a net list is that it gives, besides the interconnection specification a succinct exhibit of some key invariant properties such as density and cut number which are to be introduced later. In this paper we shall rely on the graph representation of a net list in defining some necessary terminologies.

We next describe the physical implementation of a given net list to fulfill certain constraints. In Fig. 3a we marked the space between  $R$  and the upper boundary the upper street  $U$ , and similarly, the lower street  $W$ . From the discussion in the Introduction, it is clear that there is an upper limit of the number of conductor paths parallel to  $R$  which can be routed in each street. This number is called the upper channel capacity for the upper street and the lower channel capacity for the lower street. In addition, the maximum number of parallel conductor paths which can be routed between two neighboring nodes is called the between-nodes channel capacity. Furthermore, for practical and strategic reasons we shall restrict the interconnection pattern in the same way as So. First, only rectilinear paths are allowed, i.e., paths contain only horizontal and vertical conductor segments as in Fig. 3a. Secondly we will not allow forward and backward zigzagging conductor paths



as in Fig. 4. (Upward and downward zigzagging is allowed and is used in  $N_4$  of Fig. 3a.)

A physical implementation of a net list, which satisfies the above constraints is called a realization of net list L over the nodes on single row R. Thus Fig. 3a is a realization of the net list L in Fig. 3b provided that the lower channel capacity is at least three and both the upper channel capacity and the between-nodes capacity are at least one. Let us define the unit interval (a,b) on R as the interval between two consecutive nodes a and b. For a given realization the maximum number of conductor paths in the upper street over all unit intervals is called the upper-street congestion,  $C(U)$ ; similarly we define the lower-street congestion,  $C(W)$  and the between-nodes congestion,  $C(V)$ . For the example in Fig. 3a,  $C(U) = C(V) = 1$  and  $C(W) = 3$ . It is important in practice to find realization which have small congestion numbers so that limits imposed by channel capacities can be met. Figure 5 shows an alternate realization of the net list in Fig. 3a with  $C(U) = C(W) = 2$  and  $C(V) = 1$ .

In practice, the upper channel capacity and the lower channel capacity of a multirow, multilayer board are usually the same; therefore, in order to use the channels most effectively, we would like to have physical realizations with street congestion  $C(U) \approx C(W)$ . In the remaining portion of this paper we shall assume  $C(U) = C(W)$  in deriving sufficient and necessary conditions, and we shall denote by M the street congestion, i.e.  $C(U) = C(W) \triangleq M$ . It is clear that street congestion for a realization depends not only on the routing but also on the structure of the given net list. Suppose that the street congestion for the best possible realization exceeds the channel capacity of the board, what should then be done. Obviously, we could use multilayers. To illustrate, let us assume that the channel

capacity for the example above is one. The problem needs to be solved is then to decompose the net list into two so that the problem can be realized with two layers. One possible solution for this problem is to have  $L_1 = \{N_1, N_4\}$  and  $L_2 = \{N_2, N_3\}$ , we then obtain a possible realization with two layers as shown in Fig. 6. Note that the street congestions for both layers are equal to one.

It is seen from this example, for a given single-row problem, we need in general to decompose the net list specification from a multilayer to that of the single-layers. Various methods of decomposition can be used; however, so far, we have not found an optimum method of decomposition. In Section 6, we shall give another example to illustrate the problem. In the next three sections, we will restrict our discussion to the single-row, single-layer case.

### 3. Sequential Routing Without Channel Capacity Constraints.

In this section we introduce a sequential routing method for the single-row, single-layer problem. For consistency, the method will always start from left to right in routing each net and we will show that the routing paths will never encounter blockage with the method. Our method depends in a critical way on the three types of nodes which are encountered in the process of routing. In realizing a given net list, nets are sequentially routed. Nodes associated with those nets which have not previously been routed are classified as type A. Nodes which have already been connected with conductor segments in the upper street are called type B; and, similarly, in the lower street, type C. These are illustrated in Fig. 7. In order to proceed further, we need to introduce formally the following definitions:

Definition 1: A net  $N$  is said to be over an interval  $[c,d]$ , if the

net contains at least two nodes  $n_i$  and  $n_j$ , i.e.  $n_i, n_j \in N$ , such that  $n_i \leq c < d \leq n_j$ . In the case where  $c = d$ , then  $n_i < c = d < n_j$ , we say that net  $N$  is over the node  $c$  or  $d$ .

Definition 2: A net  $N$  is said to be routed in the upper (lower) street,  $U(W)$  if the following holds: (i) the vertical conductor segment incident with each node in net  $N$  is routed in  $U(W)$ , (ii) over nodes of type A and C(B), the portions of conductor path realizing net  $N$  are in  $U(W)$ , and (iii) over nodes of type B(C), the portions of conductor path realizing net  $N$  are in  $W(U)$ .

Remark: It is necessary to use vertical conductor segments between nodes to interconnect horizontal conductor segments on different streets as shown in the example of Fig. 8.

Definiton 2 leads to a method of realization of a given net list. It is implicit from the definition that, in routing a current net  $N$  over an interval, the path depends on the status of all nodes in the interval at that moment. For example, in Fig. 8 the net  $N_1 = \{a_1, a_{10}\}$  is routed in  $U$ . Thus the vertical segments incident to  $a_1$  and  $a_{10}$  are in  $U$ . The path starts with  $a_1$  in  $U$ , switches to  $W$  over node  $a_2$  and then switches back to  $U$  over node  $a_3$ , etc., because  $a_2$  is of type B and  $a_3$  is of type C. The process continues in this fashion until every node has been reached.

It is intuitively clear that, for any given net list, once the routing order of the nets in the net list is chosen and every net is assigned to a specified street, all nets in the list can be routed. The topological structure pertaining to the net list is of no particular significance. We are now in a position to state the following theorem and routing algorithm.

Theorem 1 With no restriction on street congestions, any net list  $L$  can be successfully realized over a single row  $R$  such that (i) any net may

be preassigned to either street in the sense of Def. 2 and (ii) all nets can be routed in any predetermined order.

Routing Algorithm: Given a net list  $L$  with each net in the list assigned to be in a specific street  $X(X=U \text{ or } W)$  and all nets are ordered in the sequence to be routed. Let  $n$  be the total number of nets in  $L$  and  $N_i^X$  be a net in  $L$  to be routed in street  $X$ .

Step 0:  $i \leftarrow 0, N_0^X \leftarrow \phi$

Step 1: If  $i \leftarrow n$ , stop, end.  
Otherwise, set  $i \leftarrow i+1$ .

Step 2: Pick  $N_i^X$  from  $L$ , route  $N_i^X$  according to Definition 2.

Step 3: Go to Step 1.

Proof of Theorem 1: It suffices to show that, based on the routing algorithm, any net which has been previously routed will not block any conductor path of the nets to be routed. Let us assume that there is a blockage as shown in Fig. 9. Let  $N'$  be a net which has already been routed and assume that it blocks the conductor path of net  $N$  as shown, so it is impossible to route the net  $N = \{c,d\}$ . First, it is important to note that blockage can occur if and only if net  $N'$  is over node  $c$  and node  $d$  in opposite streets as shown. However, according to Definition 2, since both  $c$  and  $d$  are associated with nets which have not yet been routed when we route  $N'$ , so they both belong to type A nodes and thus the conductor path of  $N'$  over nodes  $c$  and  $d$  must be in the same streets according to Definition 2. Therefore, we have reached a contradiction; hence the theorem is proven.

Remark: For practical purpose, it is advisable to make minor local

modifications in the final realization. This is illustrated in Fig. 10.

In Fig. 10a, we show a net  $N$  which is routed in street  $U$  according to Definition

2. Fig. 10b illustrates the same net  $N$  after the modification. The places which modifications occur are at nodes  $a_1$  and  $a_9$ . We follow strictly the definition in routing  $N$  in Fig. 10a, and then make the modifications in Fig. 10b to simplify somewhat the routing. These modifications are strictly local and obviously do not affect routability in any way.

#### 4. Sufficient Condition for Routability.

In order to come up with sufficient condition for routability in terms of street congestion specifications, we need to introduce two pertinent properties of a given net list, namely: density and cut number. The first was originally introduced by So and entered in his sufficient condition for routability. However, density alone failed far short in telling the complete story of the structure of a net list.

Definition 3: The density for a unit interval  $(a,b)$ , denoted by  $d(a,b)$ , of a given net list is the number of nets over the unit interval  $(a,b)$ .

Stated in another way, if a vertical line is drawn between node  $a$  and node  $b$ , the number of intersections which the line makes with all nets over  $(a,b)$  is  $d(a,b)$ . Thus density is a function of the unit intervals. We denote by  $\rho$  the maximum of density over all unit intervals on  $R$ . Obviously  $\rho$  is intimately related to the street congestion that can be realized. It should be noted that both the density over a unit interval and the maximum density for a given net list can similarly be defined for any realization of a net list. Clearly they are invariant with respect to realizations of the net list. Figure 11 illustrates these concepts.

It is easy to see that we can state the following lemma pertaining

to the lower bound on street congestion.

Lemma 1: Let the congestions for the upper street and lower street be the same, and let  $M \triangleq C(U) = C(W)$ . For a given net list, the minimum street congestion for all possible realizations is equal to or greater than  $\rho'$  where  $\rho' = \lceil \frac{\rho}{2} \rceil$ ,  $\lceil \cdot \rceil$  denotes the smallest integer larger than or equal to the expression. Thus for  $\rho$  even,  $\rho' = \frac{\rho}{2}$  and for  $\rho$  odd,  $\rho' = \frac{\rho+1}{2}$ .

Proof:

Assume that we could achieve a realization with street congestion  $M = C(U) = C(W) < \rho'$ , say  $\rho'-1$ . Since the maximum density is  $\rho$ , there exists at least a unit interval  $(a,b)$  with  $d(a,b) = \rho$ . At this interval we can route at most  $\rho'-1$  nets in  $U$  and  $W$  thus a total of  $2(\rho'-1)$  nets. Thus, if  $\rho$  is even,  $2\rho'-2 = \rho-2 < \rho$ ; and, if  $\rho$  is odd, then  $2\rho'-2 = \rho-1 < \rho$ . In both cases we have at least one net over  $(a,b)$  which cannot be routed with a street congestion of  $\rho'-1$ . Therefore, we conclude that the street congestion is at least  $\rho'$ .

Remark: In this lemma we have established a lower bound on the street congestion for the realization of any net list. One may hope that we could always achieve a realization with street congestion equal to  $\rho'$ . This is, however, not always possible as illustrated by the example shown in Fig. 12. In Fig. 12a, the net list is shown with  $\rho = 4$ . The realization given in Fig. 12b yields a street congestion of  $M = C(U) = C(W) = 3$  which is larger than  $\rho' = 2$ . As a matter of fact, it can be shown later in Section 5 that there exists no realization with street congestion equal to  $\rho' = 2$ . The crux of the matter is that density alone does not prescribe all the structural properties of a net list. What we need to obtain is not just an estimated lower bound but are some sufficient conditions which ensure

routability. In this connection we have found that there exists another key property of the structure of a net list: the cut number.

Definition 4: The cut number  $c$  of a node  $i$ ,  $c(i)$ , is the number of nets over node  $i$ , not counting the net in which node  $i$  belongs. The cut number of a net is the maximum cut number of the nodes which are associated with the net.

Figure 11 illustrates the above concepts. It is easy to see that, like density, the cut number is invariant with respect to any realizations of a net list.

From the definitions of density and cut number, we can state the following useful properties of a net list: The density at any unit interval  $(a,b)$  differs with the cut number at node  $a$  or node  $b$  by at most one. The density at adjacent unit intervals differ at most by one. The cut number at adjacent nodes differ at most by one. These properties will be useful when we consider in detail the structural properties of a net list.

With this, we are ready to present the following theorem which gives the sufficient condition for realization of a net list.

Theorem 2. A net list  $L$  over  $R$  is realizable with  $C(U) = C(W) = M \geq \rho'$  if the following holds: for every unit interval  $(a,b)$  with density  $d(a,b) = I$ ,  $I \geq M+1$ , there exists at least  $2(I-M)$  nets over  $(a,b)$  such that each of them has cut number less than  $I$ .

Remark: This theorem suggests a constructive forward scanning procedure to determine a suitable street congestion by examining all intervals with density greater than  $\rho'$ . First, if the condition stated in Theorem 2 is satisfied with  $M = \rho'$ , the net list can indeed be realized in an optimum way with street congestion equal to  $\rho'$ . If not, we increase  $M$  by one each time until the sufficient condition is met.

Remark: By definition the cut number at any node is at most  $\rho-1$ .

Thus, if  $M = \rho-1$ , Theorem 2 is always satisfied; hence, the least optimum realization follows. It is also easy to show that the between-nodes congestion is at most equal to street congestion minus one, or  $M-2$ . This case corresponds to what was given in So [1].

In the following we give the assignment algorithm based on Theorem 2.

Assignment Algorithm:

Step 0:  $M \leftarrow \rho'$

Step 1:  $I \leftarrow M+1$

Step 2: Let  $(a,b)$  be the leftmost unsearched unit interval with density equal to  $I$ .

Step 3: (i) There are less than  $2(I-M)$  nets over  $(a,b)$  with cut number less than  $I$ , then go to Step 5. (ii) There are  $2(I-M)$  or more nets over  $(a,b)$  with cut number less than  $I$ .

(a) There are  $I-M$  nets over  $(a,b)$  with cut number less than  $I$  already assigned in both  $U$ , and  $W$ . Then go to Step 4.

(b) There are  $I-M$  nets over  $(a,b)$  with cut number less than  $I$  already assigned in  $U$  or  $W$  with  $I-M-1$  in the other street. Then assign a net with cut number less than  $I$  which has not yet been assigned to that other street. Go to Step 4.

(c) There are  $I-M-1$  nets over  $(a,b)$  with cut number less than  $I$  in both  $U$  and  $W$ . Pick two nets over the unit interval with cut number of the nets less than  $I$  which have not yet been assigned. Assign one net to each street.

Step 4: If there exists no more unsearched unit interval with density equal



to  $I$ ; then if  $I = \rho$ , go to Step 6, otherwise set  $I \leftarrow I+1$  and go to Step 2. If there remains any unsearched unit interval with density equal to  $I$ , go to Step 2.

Step 5: Wrong exit, street congestion exceeds  $M$  according to the sufficient conditions. Set  $M \leftarrow M+1$ , go to Step 1.

Step 6: All necessary nets are assigned, stop, end.

It should be mentioned that the above algorithm is strictly for the purpose of assigning nets in sequence to the appropriate streets. After an assignment is made, the net must then be routed according to the routing algorithm described in the previous section. Thus in realizing a given net list, we use the assignment algorithm in conjunction with the routing algorithm. After the assignment and routing algorithms terminate, there may be nets left over. These remaining nets can be routed in any street at any order using the routing algorithm again to complete the realization.

Example: Let us illustrate the assignment algorithm and the routing algorithm with an example. The graph representation of the net list is shown in Fig. 13a. Since  $\rho = 4$ ,  $\rho' = 2$ , we need only to examine unit intervals with density greater than 2. They are  $(a_3, a_4)$ ,  $(a_5, a_6)$  and  $(a_7, a_8)$  which have density three and  $(a_4, a_5)$  which has a density four. From left to right, we check first  $(a_3, a_4)$ . Note that nets  $N_1$  and  $N_3$  have cut number less than 3, thus we assign  $N_1$  to  $U$  and  $N_3$  to  $W$ . These are routed as shown by the solid lines in Fig. 13b. Examination of unit intervals  $(a_5, a_6)$  and  $(a_7, a_8)$  yield the same assigned nets. We therefore increase  $I$  by one and examine the interval  $(a_4, a_5)$ , we assign next  $N_2$  to  $U$  and  $N_4$  to  $W$  and obtain the routing immediately as shown by the dotted lines in Fig. 13b. At this stage the assignment algorithm is terminated and the only remaining net is  $N_5$ . We

may assign it arbitrarily. In Fig. 13c, we route  $N_5$  in street W and complete the net list realization. It is seen that the street congestion is equal to  $\rho' = 2$ .

Proof of Theorem 2:

We claim that by combining the assignment algorithm and the routing algorithm, we can achieve 100% routability with a street congestion M. The proof contains three steps:

(i) Obviously, we do not need to examine those unit intervals with density equal to or less than M, because in the worst case all M nets over these intervals can appear in one street.

(ii) For every unit interval (a,b) with density equal to  $I > M$ , from Theorem 2 we know that there are at least  $2(I-M)$  nets over (a,b) with cut number less than I. Over each such unit interval (a,b) we assign  $I-M$  nets in U and  $I-M$  nets in W. If we can show that in actual routing the nets will not switch in the unit interval (a,b). Then we will have routing with street congestion M. This is due to the fact that, for any such unit interval, there are at most  $I - 2(I-M) = 2M-I$  remaining nets after the assignment which need to be routed. In the worst case, if all these remaining nets are routed in the same street, then  $(2M-I) + (I-M) = M$  which is the street congestion sought. Note, from Theorem 1, we know that routing without blockage is guaranteed, thus it is only the street congestion which we need to check.

(iii) To complete the proof, we need to show that the assigned nets for each unit interval (a,b) with  $d(a,b) = I > M$  when routed do not switch from one street to the other over the unit interval (a,b) and over any other unit intervals (x,y) with  $d(x,y) \geq I$ . To understand the following

arguments which constitute the proof we must bear in mind several important concepts. We must understand precisely the nature of our routing algorithm which depends on the type of nodes a net encounters. Furthermore, in the assignment algorithm, it is stated that we assign only those nets which have cut number less than  $I$ . Thus the cut number pertaining to nodes which belong to these assigned nets must be less than  $I$ . Since over any unit interval with density  $I$ , the cut number of nodes which are associated with the unit interval can be either  $I$  or  $I-1$ , we can therefore rule out immediately the consideration of unit intervals  $(x,y)$  with  $d(x,y) > I$ . This is due to the fact that for these intervals, the cut number at node  $x$  or  $y$  is at least  $I$ , and therefore these nodes must be of type A. From the routing algorithm, the paths associated with nets over any unit interval  $(x,y)$  with both  $x$  and  $y$  belong to type A stay in the same street without switching. Therefore, we need to consider only those unit intervals with density equal to  $I$ .

There are several cases to be analyzed depending upon the type of nodes which are associated with the unit interval under consideration (i.e. at the time of routing). Clearly, if the two nodes are of type A, according to our routing algorithm, the net to be routed does not switch over the interval. In the case where two nodes are of type B(C), as shown in Fig. 14a and 14b (the case where two nodes are of type C is symmetrical to that of type B). The cut number of at least one of the nodes is  $I$ , which is impossible to occur because by the assignment algorithm, all the nodes associated with nets assigned have cut number of at most  $I-1$  at this stage. Therefore, we are left with only those cases in which the two nodes are of different types. There are four of these cases as shown in Fig. 14 which need to be considered. The other possibilities have been ruled out because of symmetry and because

of the fact that in our algorithms routing and checking are always from left to right.

In the cases shown in Figs. 14c, 14d and 14f, since  $d(a,b) = I$  the cut number at node a must also be I from the definition of the cut number, which contradicts the statement that the cut number of the nets already routed is less than I. Therefore only Fig. 14e needs to be considered. If we run into such a situation as in Fig. 14e, we find that a net has already been assigned and routed in the upper street as shown, thus by Step 3-b of assignment algorithm we need only to assign another net in the lower street. Following our routing algorithm, since node a is of type A and node b is of type B, we can route the net over (a,b) in the lower street without switching. This completes the proof of Theorem 2.

The assignment and routing algorithms have been programmed on CDC 6400. In the Appendix I we give several examples together with a table which summarizes the result. The algorithms seem to be rather efficient. It is possible to give the following estimate on the computing time in the worst case. Let n be total number of unit intervals and let  $n_i$  be the density of those unit intervals, which exceed  $\rho'$ . In general, there are about  $n/2$  such unit intervals, and an average number for  $n_i$  is  $(\rho+\rho')/2$ . In the worst case we have  $\rho \approx n/2$ . In our algorithm, we only examine the nets over unit intervals with density greater than  $\rho'$ ; therefore, the total computing time is proportional to  $\sum_i n_i \approx \left(\frac{\rho+\rho'}{2}\right) \left(\frac{n}{2}\right) \approx \frac{3n^2}{16}$ .

##### 5. Necessary and Sufficient Condition on Single-Row, Single-Layer Routing.

In the previous section we have taken advantage of the two key properties of the net list structure in deriving the sufficient condition for routability with channel capacity constraints, namely: density and cut number. The assignment algorithm and the routing algorithm together yield physical

realizations for any given net list. Two logical questions to ask now are how good is the sufficient condition and do our algorithms lead to optimum or near optimum realizations. In Fig. 15 we show an example in which according to our sufficient condition and algorithms, the street congestion  $M$  is equal to 3, yet there exists another realization with  $M=2$ . Thus we know that our result does not give the optimum solution in general. In order to be able to evaluate our sufficient condition we need to look into the structure of a net list in a more detailed fashion. We need to develop necessary and sufficient condition on routability. It turns out that density and cut number together are not enough to characterize the intricate relations among various nets of a given net list. There exists other structural properties of a net list which must be taken into account. First let us introduce the following definitions:

Definition 5: We denote by  $S_k$  a closed interval  $[x,y]$  with the property that the cut number for both nodes  $x$  and  $y$  is  $k-1$  and for any other node in  $(x,y)$  is greater than or equal to  $k$ . Furthermore, for consistency, we denote the nodes in the interval  $[x,y]$  by the sequence  $(x, a_1, a_2, \dots, a_p, y)$ .

Definition 6: We denote by  $\bar{\eta}(S_k)$  the set of nets with the property that each net  $N$  has at least two nodes  $a, b$  such that  $a \leq x < y \leq b$  and the net  $N$  has no nodes in the open interval  $(x,y)$ .

Definition 7: We denote by  $\tilde{\eta}(S_k)$  the set of nets with the property that each net has at least one node in the open interval  $(x,y)$ .

Definition 8: We denote by  $\Sigma_k$  a set of disjoint intervals  $S_k^{(i)}$ , i.e.,

$$\Sigma_k = \bigcup_i S_k^{(i)},$$

Definition 9: For any  $S_k^{(i)} \in \Sigma_k$ , we define

$$H_k^{(i)} \triangleq \bar{\eta}(S_k^{(i)}) \text{ and for } \Sigma_k \text{ with } |\Sigma_k| \geq 2$$

$$H_k^{(i|j,\ell,\dots,m)} \triangleq \bar{\eta}(S_k^{(i)}) \cap \tilde{\eta}(S_k^{(j)}) \cap \tilde{\eta}(S_k^{(\ell)}) \cap \dots \cap \tilde{\eta}(S_k^{(m)})$$

$$\text{for } S_k^{(i)}, S_k^{(j)}, S_k^{(\ell)}, \dots, S_k^{(m)} \in \Sigma_k$$

Definition 10: Given a realization of a net list, we say that a net  $N \in \bar{\eta}(S_k)$  is routed in street  $U(W)$  completely over the interval  $S_k = [x,y]$  if over all  $z \in \{x, a_1 \dots a_p, y\}$ , the conductor path of  $N$  is in street  $U(W)$  and does not switch in  $[x,y]$ .

The above definitions are illustrated with the example in Fig. 16.

We are now in a position to derive some necessary conditions. First let us state the following lemma.

Lemma 2: Given a net list  $L$ , if there exists any  $S_k$  with  $k > \rho'$  such that  $|\bar{\eta}(S_k)| < 2(k-K)$  then, for all realizations, the street congestion,  $M > K$ .

Remark: The lemma above gives us a simple procedure of establishing a lower bound of street congestion. In Fig. 17, we see that  $S_3 = [a_3, a_8]$  and  $\bar{\eta}(S_3) = \{N_3\}$ . Thus  $|\bar{\eta}(S_3)| = 1 < 2(3-K)$ , the largest integer for  $K$  is 2. Hence according to lemma 2, the street congestion  $M > 2$ . Using the sufficient condition of Section 4, we find  $M = 3$ . We may therefore conclude that  $M = 3$  is the optimum street congestion for the given net list.

Proof: With reference to Fig. 18a consider node  $a_p$ . Assume that over node  $a_p$ , excluding the net which is incident with  $a_p$ , there are  $h_1 + g_1$  nets routed in street  $U$  and  $h_2 + g_2$  nets routed in street  $W$ . Let the  $h_1 + h_2$  nets belong to  $\bar{\eta}(S_k)$  and the  $g_1 + g_2$  nets belong to  $\tilde{\eta}(S_k)$ . Since  $a_p$  is next

to node  $y$  and  $c(y) = k-1$ ,  $c(a_p) = k$ ; we have  $h_1 + h_2 + g_1 + g_2 = k$ . If  $g_1 = 0$ , then at node  $a_p$  the lower street congestion is at least  $k - h_1$ . Assume that  $g_1 \neq 0$ , if we make a vertical cut at node  $a_p$ , among the nets in  $\tilde{n}(S_k)$  crossing this cut in street  $U$ , let  $N_1$  be the one of farthest distance away from  $a_p$ . This net  $N_1$  has a node  $b_1 \in [a_1, a_p]$  because  $N_1 \in \tilde{n}(S_k)$ . Note, as shown in Fig. 18b, at node  $b_1$ , there are the same  $h_1$  nets in street  $U$  which were over node  $a_p$ , and there are the same  $h_2$  nets in street  $W$  which were over node  $a_p$ . Because otherwise net  $N_1$  would intersect with any of the nets. Now, let us assume that there are  $g_1^{b_1}$  nets which are in  $\tilde{n}(S_k)$  and are in street  $U$  over node  $b_1$ . If  $g_1^{b_1} = 0$ , then the lower street congestion is at least  $k - h_1$ . Assume that  $g_1^{b_1} \neq 0$ , then over node  $b_1$  there is a net  $N_2 \in \tilde{n}(S_k)$  which has the same property as  $N_1$  with respect to  $a_p$ . Since the number of nodes in  $[a_1, a_p]$  is finite, we can find a node  $s \in [a_1, a_p]$  such that  $g_1^s = 0$ . This implies that the lower street congestion at node  $s$  is at least  $k - h_1$ . Using the same argument we can state that there exists a node  $t \in [a_1, a_p]$  such that the upper street congestion at  $t$  is at least  $k - h_2$ . Therefore the street congestion for all possible realizations must satisfy the following inequality:

$$M \geq \frac{\max_{h_1+h_2=|\tilde{n}(S_k)|} (k-h_1, k-h_2)}{2} \geq k - \frac{|\tilde{n}(S_k)|}{2} > k - (k-K) = K.$$

In view of Lemma 2, we can immediately state that a necessary condition for  $M = K$  is that, for every  $S_k$ ,  $k > \rho'$ , we have  $|\tilde{n}(S_k)| \geq 2(k-K)$ . We state

this as a corollary.

Corollary: Given a net list L, a necessary condition for  $M=K$  is that, for every  $S_k$ ,  $k > \rho'$ , we have  $|\bar{\eta}(S_k)| \geq 2(k-K)$ .

In the following, we give a more stringent necessary condition to ensure routability with street congestion  $M=K$ .

Theorem 3: Given a net list L, if for every  $S_k$ ,  $k > \rho'$ ,  $|\bar{\eta}(S_k)| \geq 2(k-K)$ , then a necessary condition for  $M = K$  is that there are at least  $k - K$  nets in  $\bar{\eta}(S_k)$  routed over  $S_k$  completely in both street U and street W, respectively.

Remark: Theorem 3 puts more restriction on Lemma 2 in that each of the  $k-K$  nets assigned in U(W) has to be routed completely in U(W) over every interval  $S_k$ . Referring to Fig. 15, we see that  $S_3^{(1)} = [a_3, a_8]$ ,  $S_3^{(2)} = [a_{11}, a_{14}]$ ,  $S_4^{(1)} = [a_4, a_5]$ ,  $S_4^{(2)} = [a_6, a_7]$ ,  $S_4^{(3)} = [a_{12}, a_{13}]$  and  $\bar{\eta}(S_3^{(1)}) = \{N_1, N_2\}$ ,  $\bar{\eta}(S_3^{(2)}) = \{N_6, N_7\}$ ,  $\bar{\eta}(S_4^{(1)}) = \{N_1, N_2, N_3, N_4\}$ ,  $\bar{\eta}(S_4^{(2)}) = \{N_1, N_2, N_3, N_5\}$ ,  $\bar{\eta}(S_4^{(3)}) = \{N_1, N_6, N_7, N_8\}$ . If we route in street U(W) the portions of  $N_1(N_2)$  which are over  $S_3^{(1)}$ , the portions of  $N_6(N_7)$  which are over  $S_3^{(2)}$ , the portions of  $N_3(N_4)$  which are over  $S_4^{(1)}$ , the portions of  $N_3(N_5)$  which are over  $S_4^{(2)}$  plus the portions of  $N_1(N_8)$  which are over  $S_4^{(3)}$  as shown in Fig. 15c with the remaining portions of all the nets above routed last, we then have a realization with  $M=2$ . On the other hand, in Fig. 15b, because there is no net in  $\bar{\eta}(S_3^{(2)})$  which is routed completely in U over  $S_3^{(2)}$ , by Theorem 3, we have  $M > 2$ .

Proof: Assume that there is a realization with  $M=K$  for the following situation: There exists an interval  $[a,b] \subset (x,y)$ , where  $S_k = [x,y]$  and  $K > \rho'$  such that there are at most  $k - K - 1$  nets in  $\bar{\eta}(S_k)$  over  $[a,b]$  completely in street U. Then it follows that there are at least  $k - K + 1$  nets in  $\bar{\eta}(S_k)$  over  $[a,b]$  completely in street W. (See Fig. 19 for illustration.)



Now we can use a similar argument as in the proof of Lemma 2 to arrive at  $M \geq K+1$ , which is a contradiction. Thus  $M > K$ .

Theorem 4: Given a net list  $L$ , if for every  $S_k$ ,  $k > \rho'$ ,  $|\bar{\eta}(S_k)| \geq 2(k-K)$ , and there are at least  $k-K$  nets in  $\bar{\eta}(S_k)$  routed over  $S_k$  completely in both street  $U$  and street  $W$ , respectively, then street congestion is  $M=K$ .

Proof: From Theorem 1 we know that routing is of no problem, thus we need not concern about street congestion for unit intervals with density less than or equal to  $K$ . The only places we need to consider are unit intervals with density greater than  $K$ . If we can show that for any unit interval  $(a,b)$  with density  $I > K$ , the street congestion is less than or equal to  $K$  in that unit interval, then we have proven the theorem.

To show this, since  $d(a,b) = I > K$ , then, by definition, there is an interval  $S_I \supset (a,b)$ , and by the assumption, there are at least  $I-K$  nets in  $\bar{\eta}(S_I)$  routed over  $S_I$  completely in both street  $U$  and street  $W$ , respectively. Since  $S_I \supset (a,b)$ , so there are at least  $I-K$  nets in  $\bar{\eta}(S_I)$  routed over  $(a,b)$  completely in both street  $U$  and street  $W$ , respectively. Since  $d(a,b) = I$ , there are at most  $I-2(I-K)$  nets left unrouted over  $(a,b)$ . In the worst case, all of the  $I-2(I-K)$  are routed in one street over  $(a,b)$ , say in street  $U$ , then in street  $U$  the street congestion is  $(I-K) + I - 2(I-K) = K$ , so the street congestion holds.

Remark: Theorem 3 and Theorem 4 set up the criterion for optimum street congestion of a given net list. If we can achieve a realization such that for any  $S_k$ ,  $k > K \geq \rho'$ , there are at least  $k-K$  nets routed over  $S_k$  completely in both street  $U$  and street  $W$ , respectively, and  $K$  is the smallest such number satisfying the condition, then street congestion  $M=K$  is the optimum.

Combining these two theorems, we can state the following:

**Theorem 5:** Given a net list L, then street congestion  $M=K \geq \rho'$  if and only if for every  $S_k$ ,  $k > K \geq \rho'$ ,  $|\bar{\eta}(S_k)| \geq 2(k-K)$  and there are at least  $k-K$  nets in  $\bar{\eta}(S_k)$  routed over  $S_k$  completely in both street U and street W, respectively.

We can now easily arrive at some interesting necessary conditions on the behavior of  $S_k$ 's interacting with each other based on the criterion set forth in Theorem 5. In the following we will present three lemmas relating to the interactions of different  $S_k$ 's. The proofs of Lemma 3, 4 and 5 are given in the Appendix II. First let us illustrate some of the concepts with an example.

Consider the net list representation of Fig. 20. The following can be immediately obtained:

$$\rho = 7, \rho' = 4$$

$$S_5^{(1)} = [a_5, a_{12}], S_5^{(2)} = [a_{17}, a_{24}]$$

$$H_5^1 \triangleq \bar{\eta}(S_5^{(1)}) = \{N_1, N_3\}, H_5^2 = \bar{\eta}(S_5^{(2)}) = \{N_7, N_{11}\}$$

$$H_5^{(1|2)} \triangleq \bar{\eta}(S_5^{(1)}) \cap \bar{\eta}(S_5^{(2)}) = \{N_1, N_3\},$$

$$H_5^{(2|1)} \triangleq \bar{\eta}(S_5^{(2)}) \cap \bar{\eta}(S_5^{(1)}) = \{N_7\}.$$

We notice that Lemma 2 is satisfied with  $M \geq 4$ . Suppose that we want to see whether  $M=4$  can be realized. By Theorem 3, we know that  $N_1(N_3)$  has to be routed in street U(W) completely over  $S_5^{(1)}$  and  $N_7(N_{11})$  has to be routed in street U(W) completely over  $S_5^{(2)}$ . However, this is impossible because both nets  $N_1$  and  $N_3$  have nodes in  $S_5^{(2)}$ , which means that both nets

must switch somewhere before  $a_{17}$  is reached in order to also satisfy the condition that nets  $N_7$  and  $N_{11}$  are routed completely in street U and street W over  $S_5^{(2)}$ , respectively. But  $N_7$  has a node in  $S_5^{(1)}$ , which means that in order to satisfy Lemma 3,  $N_7$  will intersect  $N_1$ . This suggests that  $M > 4$ . Clearly the information in  $H_5^{(1|2)}$  and  $H_5^{(2|1)}$  must be included.

Lemma 3: Given a net list L, if, for some nonempty  $\Sigma_k$  with  $k > \rho'$ , there exist an  $S_k^{(\ell)} \in \Sigma_k$  and a nonempty  $\Sigma'_k \subset \Sigma_k - \{S_k^{(\ell)}\}$  such that for each  $S_k^{(i)} \in \Sigma'_k$ ,

$$|H_k^{(i)} - H_k^{(i|\ell)}| \leq k - K - 1$$

Then a necessary condition for  $M=K$  is

$$|H_k^{(\ell)} - \bigcup_{S_k^{(i)} \in \Sigma'_k} H_k^{(\ell|i)}| \geq 2(k-K).$$

Remark: For the example above in Fig. 20, we note that

$$|H_5^{(1)} - H_5^{(1|2)}| = 0 \leq 5 - 4 - 1$$

$$|H_5^{(2)} - H_5^{(2|1)}| = 1 < 2(5-4) = 2.$$

Therefore, for this example, the necessary condition for  $M=4$  is not satisfied.

Lemma 4: Given a net list L, if, for some nonempty  $\Sigma_k$  with  $k > \rho'$ , there exist an  $S_k^{(\ell)}$  and a nonempty  $\Sigma'_k \subset \Sigma_k - \{S_k^{(\ell)}\}$  such that for each  $S_k^{(i)} \in \Sigma'_k$

$$k - K \leq |H_k^{(i)} - H_k^{(i|\ell)}| < 2(k-K)$$

Then a necessary condition for  $M=K$  is

$$|H_k^{(\ell)} - \bigcup_{S_k^{(i)} \in \Sigma'_k} H_k^{(\ell|i)}| \geq k - K$$

Remark: For the example above in Fig. 20, suppose that we check  $S_5^{(2)}$  first, we know that

$$k - M = 1 = |H_5^{(2)} - H_5^{(2|1)}| = 1 < 2(5-4) = 2$$

However,

$$|H_5^{(1)} - H_5^{(1|2)}| = 0 < 5 - 4 = 1$$

Thus  $M=4$  is not a feasible street congestion.

Lemma 5: Given a net list  $L$ , if some  $\Sigma_k$  has an element  $S_k^{(\ell)}$  and two nonempty subsets

$$\Sigma'_k \subset \Sigma_k - \{S_k^{(\ell)}\}$$

$$\Sigma''_k \subset \Sigma_k - \{S_k^{(\ell)}\}$$

such that for each  $S_k^{(i)} \in \Sigma'_k$  and  $S_k^{(j)} \in \Sigma''_k$

$$|H_k^{(i)} - H_k^{(i|j,\ell)}| < 2(k-K)$$

and  $|H_k^{(j)} - H_k^{(j|i,\ell)}| < 2(k-K)$

then a necessary condition for  $M=K$  is

$$|H_k^{(\ell)} - \bigcup_{s_k^{(i)} \in \Sigma_k'} \bigcup_{s_k^{(j)} \in \Sigma_k''} H_k^{(\ell|i,j)}| \geq 2(k-K)$$

To conclude, we have developed a set of necessary conditions based on Theorem 5 which can be used to check how good is the sufficient condition in Section 4. In Appendix I we give a number of examples. These examples indicate that in most cases the sufficient condition of Section 4 is very close in giving the optimal solution. In those cases for which the necessary conditions yield a street congestion which can be realized with our sufficient condition, the result is of course optimum.

## 6. Conclusion

In Section 2 we discussed briefly the decomposition of a single-row net list from multilayer to single layers. Let us illustrate here with an example, taking advantage of what we have learned on single-layer routing in the last two sections.

In Fig. 21a, we have the graph representation of a net list for a single-row, two-layer problem. Fig. 21b illustrates one possible decomposition. If we use the necessary condition to check the street congestion, we find that, for the second layer, we need a street congestion of at least three. Fig. 21c gives another possible decomposition. Here, we use the sufficient condition and find that a street congestion of two can be realized in both layers. Thus if the specified channel capacity for the problem is two, the decomposition in Fig. 21b is not acceptable, while that in Fig. 21c is good. In [1], So gave a decomposition method for the single-row, multilayer problem. We have found that much more can be said on this problem. The basic idea is to decompose a given net list to achieve minimal but equal street congestion at all levels.

To summarize, in this paper, we have made an extensive study of the single-row, single-layer routing problem. A very good sufficient condition which can be easily implemented and several necessary conditions based on a necessary and sufficient condition on routability with channel capacity constraints have been derived. Two useful algorithms, one on routing and the other on assignment, have been obtained to implement our sufficient condition for physical realizations. These algorithms have been programmed and tested with many examples.

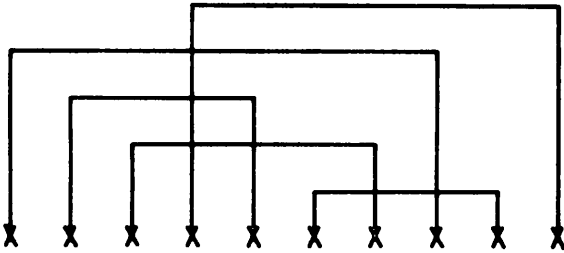
The general problem of multirow and multilayer routing has been alluded to in Sections 1, 2 and 6 and its relation to the single-row, single-layer problem has been illustrated with examples. It should be noted that the multirow, multilayer problem is a very general and interesting problem. Much more work needs to be done in order to answer such questions as routability, optimal street congestion and minimal layers as well as to develop routing methods and algorithms.

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APPENDIX I

Example 1



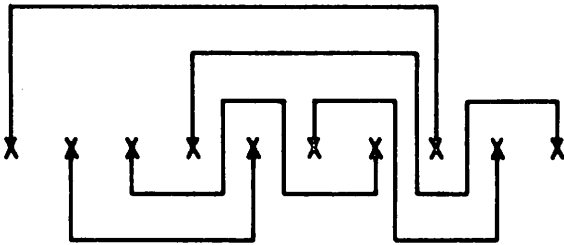
(a) NET LIST (5 NETS) OVER R (10 NODES)

5 NETS, 10 NODES

MAXIMUM DENSITY = 4

STREET CONGESTION = 3

COMPUTING TIME =  $2.5 \times 10^{-3}$  SEC.



(b) NET LIST ROUTABLE WITH STREET CONGESTION 3

$$S_3 = [a_3, a_8]$$

$$|\bar{n}(S_3)| = 1 < 2$$

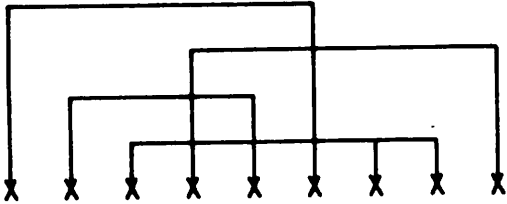
$$\Rightarrow M \geq 3 \text{ (BY LEMMA 2)}$$

$$M = 3 \text{ (BY THEOREM 2)}$$

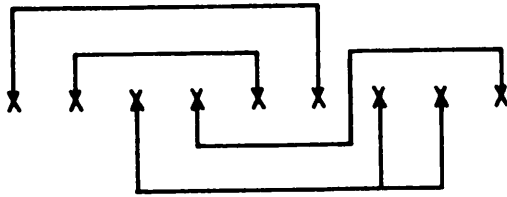
$$\Rightarrow \text{OPTIMAL SOLUTION}$$



Example 2



(a) NET LIST (4 NETS) OVER R (9 NODES)



(b) NET LIST ROUTABLE WITH STREET CONGESTION 2

4 NETS, 9 NODES

MAXIMUM DENSITY = 4

STREET CONGESTION = 2

COMPUTING TIME =  $2.3 \times 10^{-3}$  SEC.

$M = 2$  (BY THEOREM 2)

$M \geq 2$  (BY LEMMA 1)

$\Rightarrow$  OPTIMAL SOLUTION

Example 3

9 NETS, 20 NODES

MAXIMUM DENSITY = 5

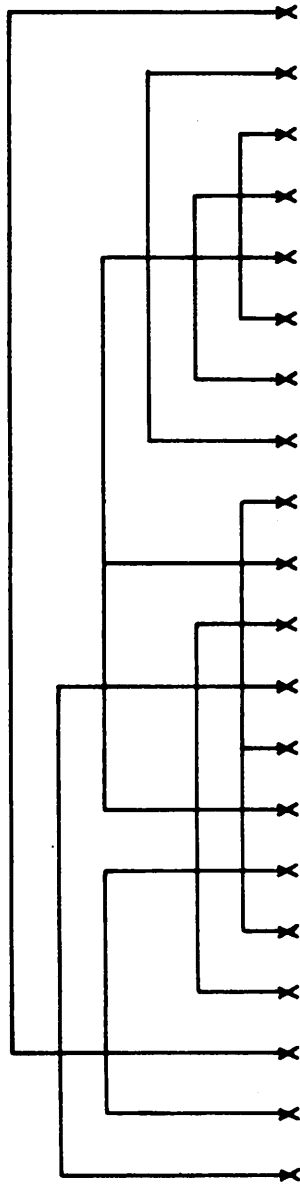
STREET CONGESTION = 3

COMPUTING TIME =  $4.4 \times 10^{-3}$  SEC.

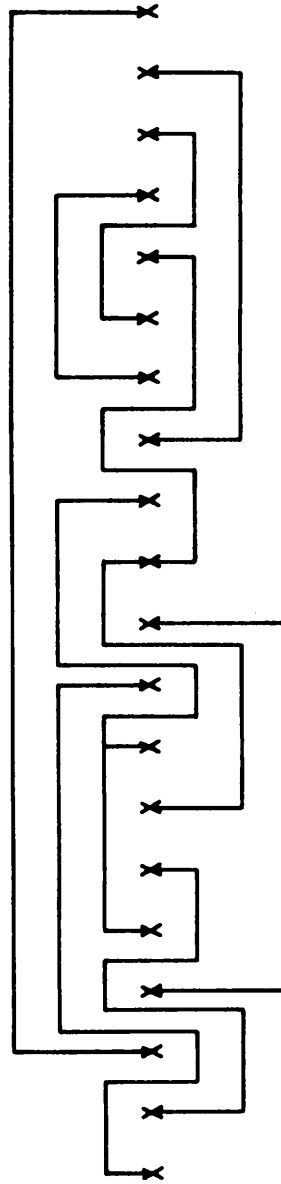
$M = 3$  (BY THEOREM 2)

$M > 3$  (BY LEMMA 1)

$\Rightarrow$  OPTIMAL SOLUTION



(a) NET LIST (9 NETS) OVER R (20 NODES)



(b) NET LIST ROUTABLE WITH STREET CONGESTION 3

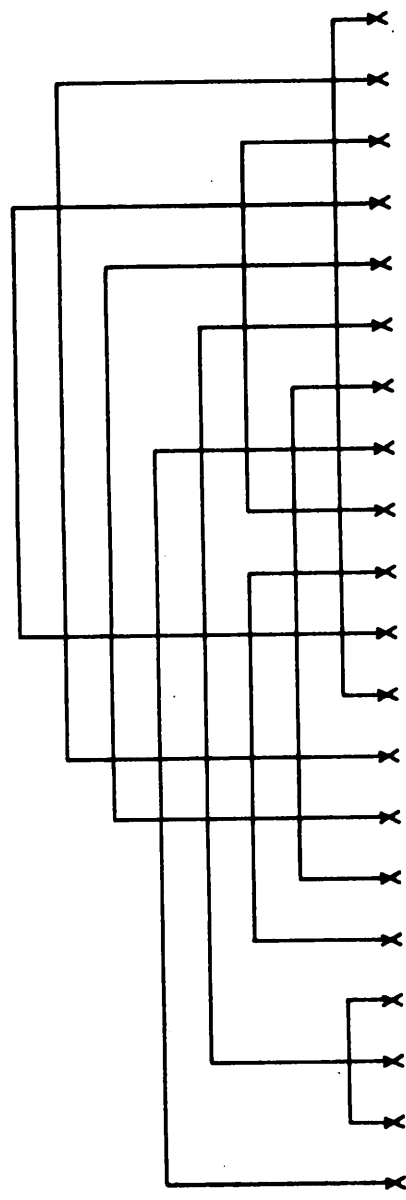
Example 4

10 NETS, 20 NODES

MAXIMUM DENSITY = 8

STREET CONGESTION = 5

COMPUTING TIME =  $4.6 \times 10^{-3}$  SEC.



(a) NET LIST (10 NETS) OVER R (20 NODES)

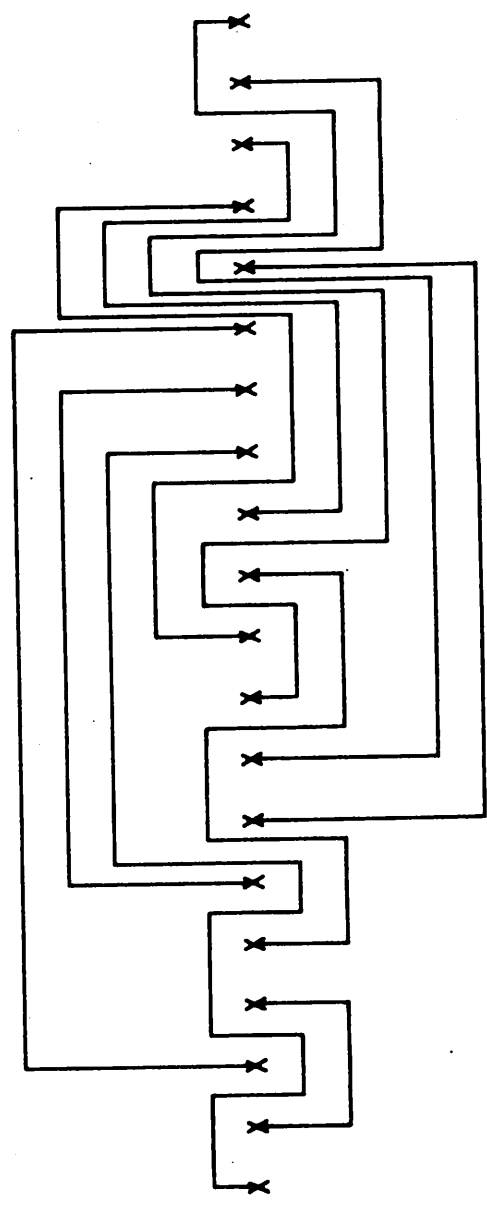
$$S_5 = [a_7, a_{16}]$$

$$|\bar{n}(S_5)| = 1 < 2$$

$\Rightarrow M > 5$  (BY LEMMA 2)

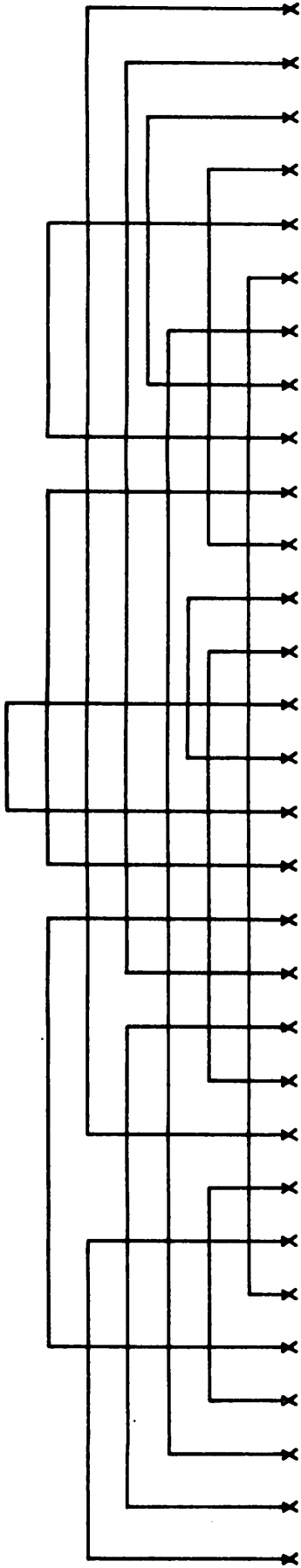
$M = 5$  (BY THEOREM 2)

$\Rightarrow$  OPTIMAL SOLUTION

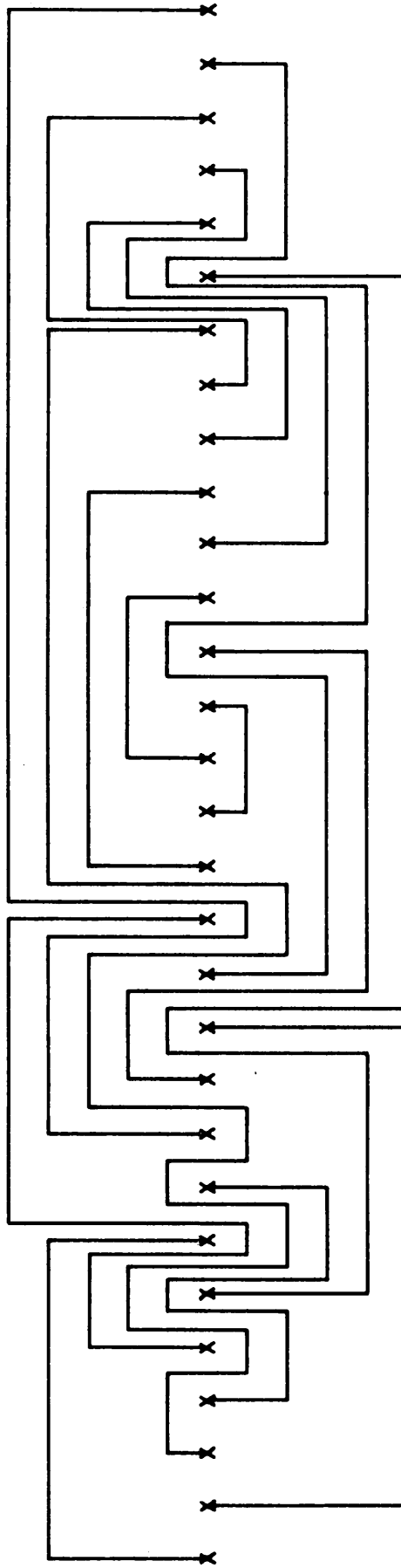


(b) NET LIST ROUTABLE WITH STREET CONGESTION 5

Example 5



(a) NET LIST (15 NETS) OVER R (30 NODES)



(b) NET LIST ROUTABLE WITH STREET CONGESTION 5

15 NETS, 30 NODES

MAXIMUM DENSITY = 8

STREET CONGESTION = 5

COMPUTING TIME =  $6.7 \times 10^{-3}$  SEC.

$$S_5 = [a_9, a_{26}]$$

$$\bar{m}(S_5) = 1 < 2$$

$$\Rightarrow M > 5 \quad (\text{BY LEMMA 2})$$

$$M = 5 \quad (\text{BY THEOREM 2})$$

$\Rightarrow$  OPTIMAL SOLUTION

# of Nets	# of Nodes	Max. Density	Street Congestion (Implementation) Result	Street Congestion (Hing So)	Computing time ( $10^{-3}$ sec.)	Our Results Optimum?
4	9	4	2	3	2.3	yes.
5	10	4	3	3	2.5	yes.
6	14	4	3	3	3.2	yes.
9	20	5	3	4	4.4	yes.
10	20	8	5	7	4.6	yes.
15	30	8	5	7	6.7	yes.
20	40	10	5	9	8.5	yes.
25	50	14	8	13	11.0	yes.
30	60	12	8	11	14.2	yes.

TABLE OF SOME IMPLEMENTATION RESULTS AND  
COMPARISON TO HING SO'S PREDICTION

APPENDIX II

Proof of Lemma 3: We first show that for the case where  $|\Sigma'_k| = 1$ , and  $\bigcup_{S_k^{(i)} \in \Sigma'_k} H_k^{(\ell|i)} = H_k^{(\ell|i)}$ , the conclusion that  $|H_k^{(\ell)} - H_k^{(\ell|i)}| \geq$

$2(k-M)$  is true. We then show that in the general case where  $|\Sigma'_k| > 1$  the conclusion is also true. To show the first step where  $|\Sigma'_k| = 1$ , we know from the assumption that there are less than or equal to  $k - K - 1$  nets that are in  $H_k^{(i)} - H_k^{(i|\ell)} = \bar{\eta}(S_k^{(i)}) - \bar{\eta}(S_k^{(i)}) \cap \tilde{\eta}(S_k^{(\ell)})$ . To satisfy Theorem 3, there is at least one net  $N_1 \in \bar{\eta}(S_k^{(i)}) \cap \tilde{\eta}(S_k^{(\ell)})$  that is over  $S_k^{(i)}$  completely in, say, street U and there is at least one net  $N_2 \in \bar{\eta}(S_k^{(i)}) \cap \tilde{\eta}(S_k^{(\ell)})$  that is over  $S_k^{(i)}$  completely in street W. (See Fig. A-II-1.) Let's suppose that

$$|H_k^{(\ell)} - H_k^{(\ell|i)}| < 2(k-K), \text{ i.e.}$$

$$|\bar{\eta}(S_k^{(\ell)}) - \bar{\eta}(S_k^{(\ell)}) \cap \tilde{\eta}(S_k^{(i)})| < 2(k-K).$$

Then to satisfy Theorem 3, there is at least one net  $N_3 \in \bar{\eta}(S_k^{(\ell)}) \cap \tilde{\eta}(S_k^{(i)})$  such that  $N_3$  is completely over  $S_k^{(\ell)}$  in street U(W). This is impossible because then  $N_3$  intersects  $N_1(N_2)$ . Thus we conclude that  $|H_k^{(\ell)} - H_k^{(\ell|i)}| \geq 2(k-K)$ .

For the general case where  $|\Sigma'_k| > 1$ , we know that, by the above argument, for each

$$S_k^{(i)} \in \Sigma'_k, \text{ we have}$$

$$|H_k^{(\ell)} - H_k^{(\ell|i)}| \geq 2(k-K), \text{ thus we have}$$

$$\begin{aligned}
& \left| \bigcap_{S_k^{(i)} \in \Sigma'_k} H_k^{(\ell)} - H_k^{(\ell|i)} \right| \geq 2(k-K) \\
& \parallel \\
& \left| H_k^{(\ell)} - \bigcup_{S_k^{(i)} \in \Sigma'_k} H_k^{(\ell|i)} \right|.
\end{aligned}$$

Proof of Lemma 4: We prove this in similar steps as in Lemma 3. We first look at the case where  $|\Sigma'_k| = 1$ . Since  $k-K \leq |H_k^{(i)} - H_k^{(i|\ell)}| < 2(k-K)$ , to satisfy Theorem 3, there is a net  $N_1 \in \bar{\eta}(S_k^{(i)}) \cap \tilde{\eta}(S_k^{(\ell)})$  which is routed over  $S_k^{(i)}$  completely in street U or W, say in street W. Let's assume that  $|H_k^{(\ell)} - H_k^{(\ell|i)}| < k-K$ . To satisfy Theorem 3 again, there is at least one net  $N_2 \in \bar{\eta}(S_k^{(\ell)}) \cap \tilde{\eta}(S_k^{(i)})$  which is routed over  $S_k^{(\ell)}$  completely in street U and there is at least a net  $N_3 \in \bar{\eta}(S_k^{(\ell)}) \cap \tilde{\eta}(S_k^{(i)})$  which is routed over  $S_k^{(\ell)}$  completely in street W. (See Fig. A-II-2.) The above situation is impossible because then  $N_3$  intersects  $N_1$ . So we conclude that  $|H_k^{(\ell)} - H_k^{(\ell|i)}| \geq k-K$ .

For the case where  $|\Sigma'_k| > 1$ , it is easy to see that for each  $S_k^{(i)} \in \Sigma'_k$ , we have  $|H_k^{(\ell)} - H_k^{(\ell|i)}| \geq k-K$  hence we have

$$\begin{aligned}
& \left| \bigcap_{S_k^{(i)} \in \Sigma'_k} (H_k^{(\ell)} - H_k^{(\ell|i)}) \right| \geq k-K \\
& \parallel \\
& \left| H_k^{(\ell)} - \bigcup_{S_k^{(i)} \in \Sigma'_k} H_k^{(\ell|i)} \right|.
\end{aligned}$$

Proof of Lemma 5: We first consider how 3 different  $S_k^{(i)}$ 's may interact with each other (i.e.  $|\Sigma'_k| = 1$  and  $|\Sigma''_k| = 1$ ), then we consider the general case.

For the case where  $|\Sigma'_k| = 1$  and  $|\Sigma''_k| = 1$ , in order to satisfy Theorem 3, and  $|H_k^{(i)} - H_k^{(i|j,\ell)}| < 2(k-K)$   $|H_k^{(j)} - H_k^{(j|i,\ell)}| < 2(k-K)$  for  $S_k^{(i)} \in \Sigma'_k$ ,  $S_k^{(j)} \in \Sigma''_k$ , we have that there is at least a net  $N_1 \in \bar{\eta}(S_k^{(i)}) \cap \tilde{\eta}(S_k^{(\ell)}) \cap \tilde{\eta}(S_k^{(j)})$  routed completely over  $S_k^{(i)}$  in street, say, U. There is at least a net  $N_2 \in \bar{\eta}(S_k^{(j)}) \cap \tilde{\eta}(S_k^{(i)}) \cap \tilde{\eta}(S_k^{(\ell)})$  which is routed completely over  $S_k^{(j)}$  in street W. (If  $N_2$  were in street U, then  $N_2$  intersects  $N_1$ .)

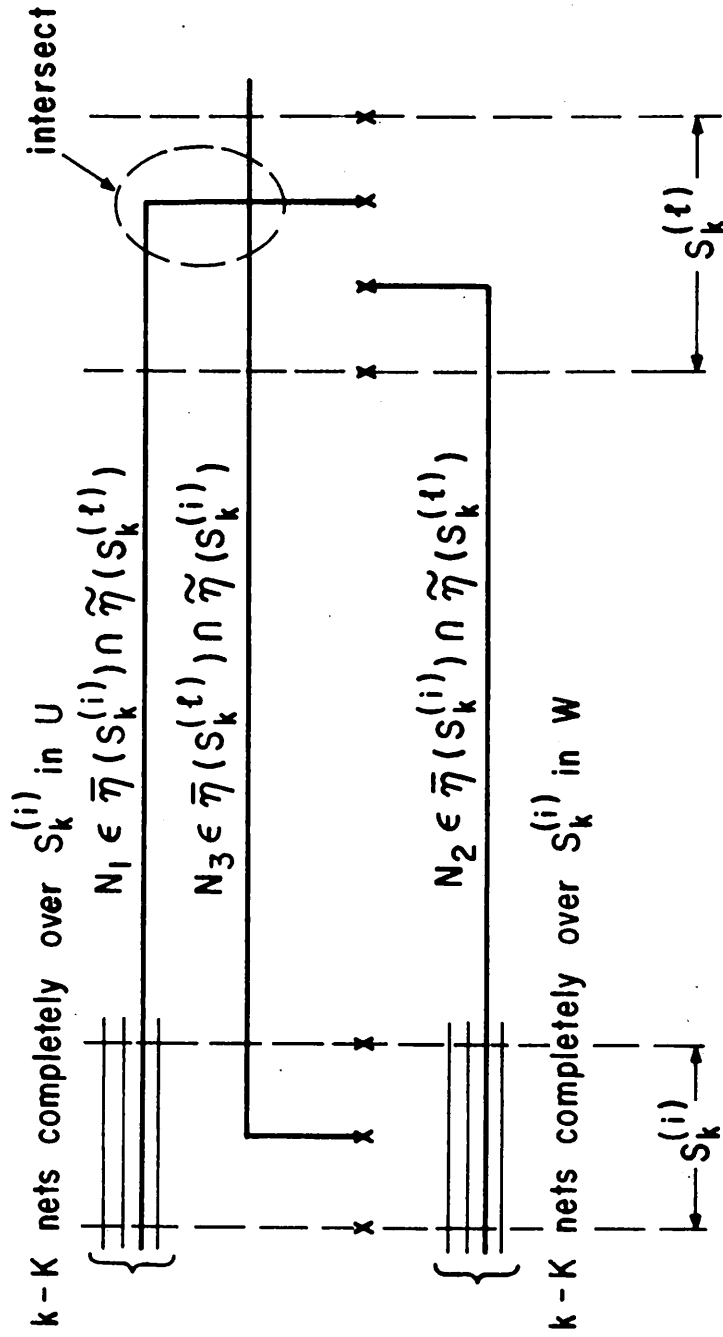
Let's assume that  $|H_k^{(\ell)} - H_k^{(\ell|i,j)}| < 2(k-K)$ . Then to satisfy Theorem 3, there is at least a net  $N_3 \in \bar{\eta}(S_k^{(\ell)}) \cap \tilde{\eta}(S_k^{(i)}) \cap \tilde{\eta}(S_k^{(j)})$  such that  $N_3$  is routed completely over  $S_k^{(\ell)}$  in street U(W). (See Fig. A-II-3.)

The above situation is impossible because  $N_3$  intersects  $N_1(N_2)$  in street U(W). Hence we conclude that  $|H_k^{(\ell)} - H_k^{(\ell|i,j)}| \geq 2(k-K)$ .

In general case where  $|\Sigma'_k| \geq 1$  and  $|\Sigma''_k| \geq 1$ , we have

$$\begin{aligned}
 & \left| \bigcap_{S_k^{(i)} \in \Sigma'_k} \bigcap_{S_k^{(j)} \in \Sigma''_k} (H_k^{(\ell)} - H_k^{(\ell|i,j)}) \right| \geq 2(k-K) \\
 & \parallel \\
 & \left| H_k^{(\ell)} - \bigcup_{S_k^{(i)} \in \Sigma'_k} \bigcup_{S_k^{(j)} \in \Sigma''_k} H_k^{(\ell|i,j)} \right| .
 \end{aligned}$$





$$\bar{\eta}(S_k^{(i)}) - \bar{\eta}(S_k^{(i)}) \cap \tilde{\eta}(S_k^{(t)}) \leq k-K-1$$

Fig. A-II-1.

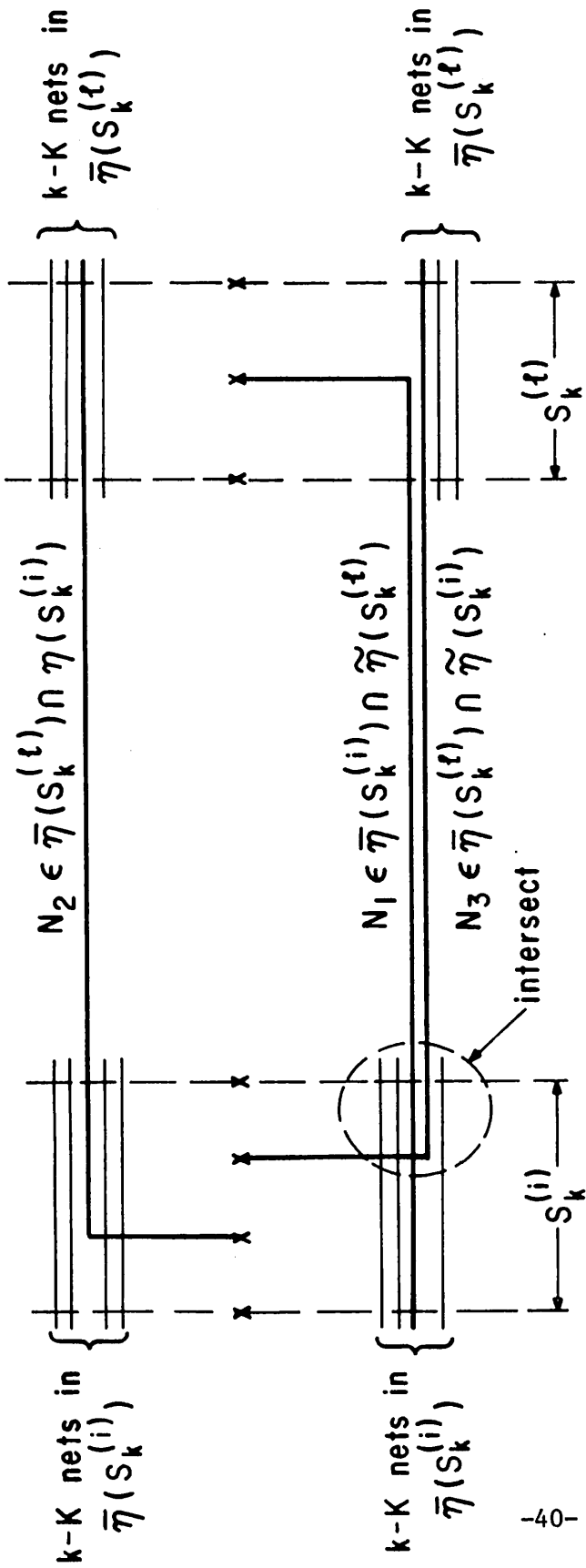


Fig. A-II-2.

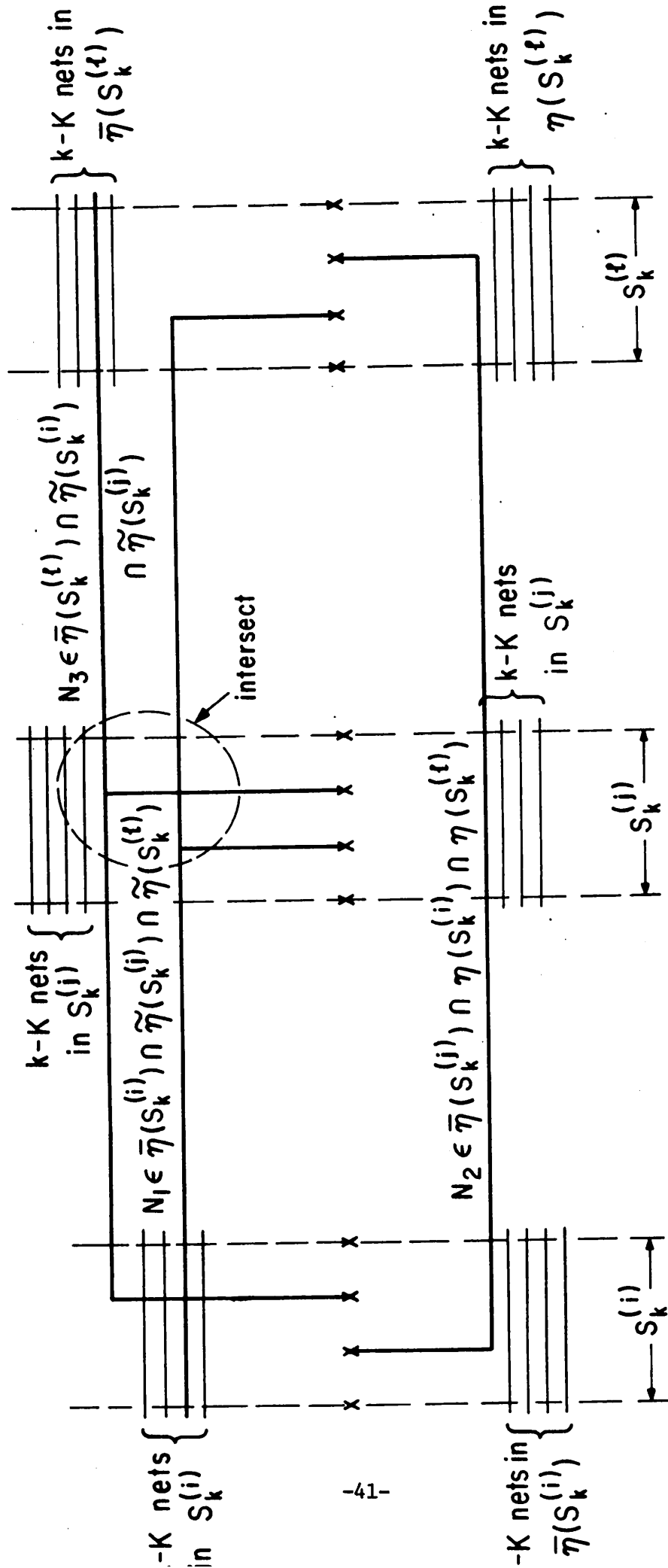


FIG. A-II-3.

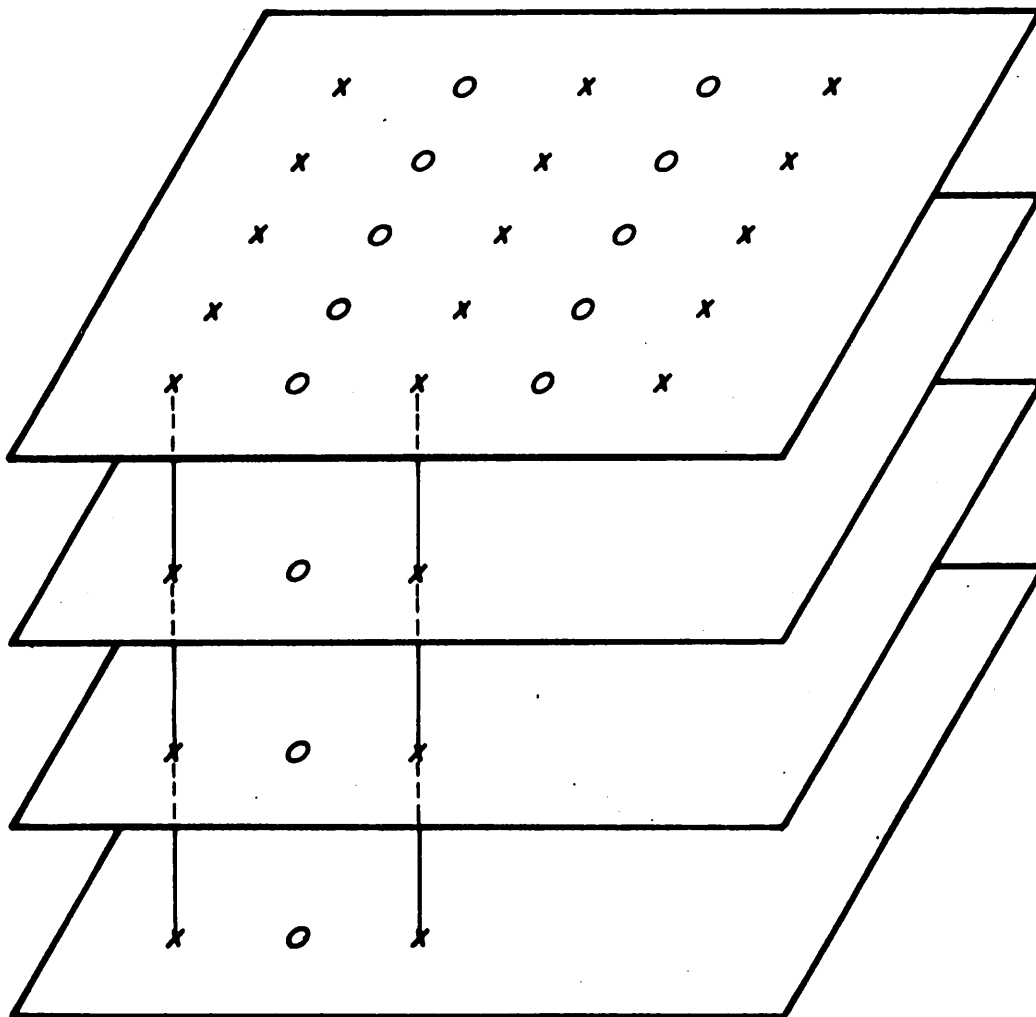
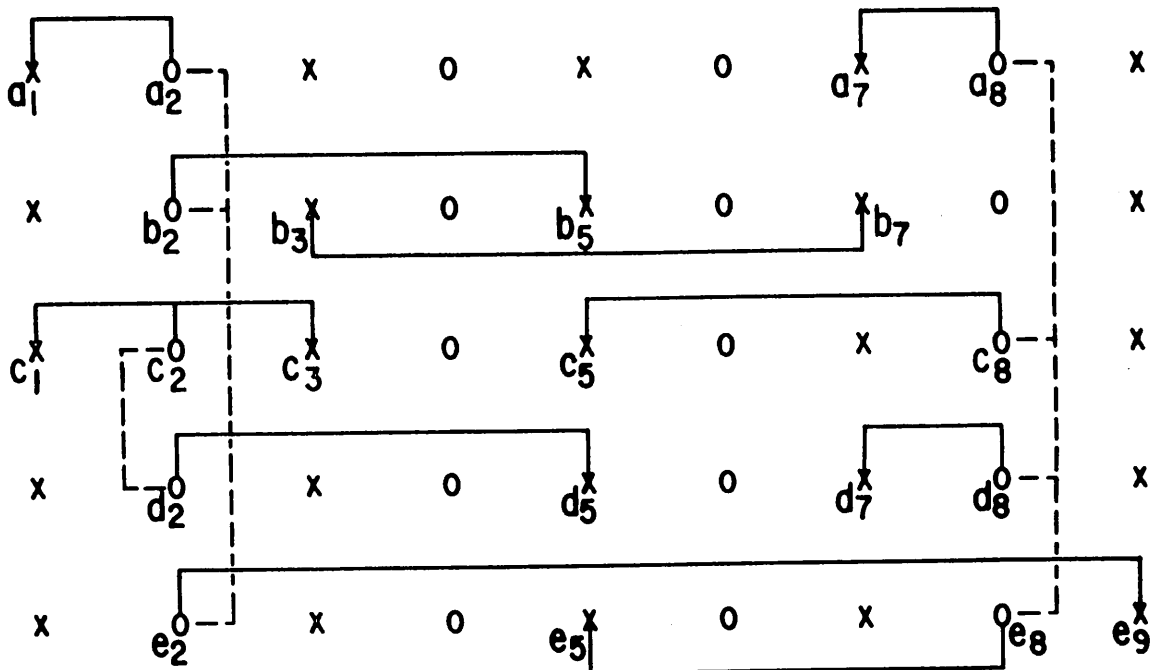


Fig. 1. PINS AND VIAS ALTERNATING ON EACH ROW.

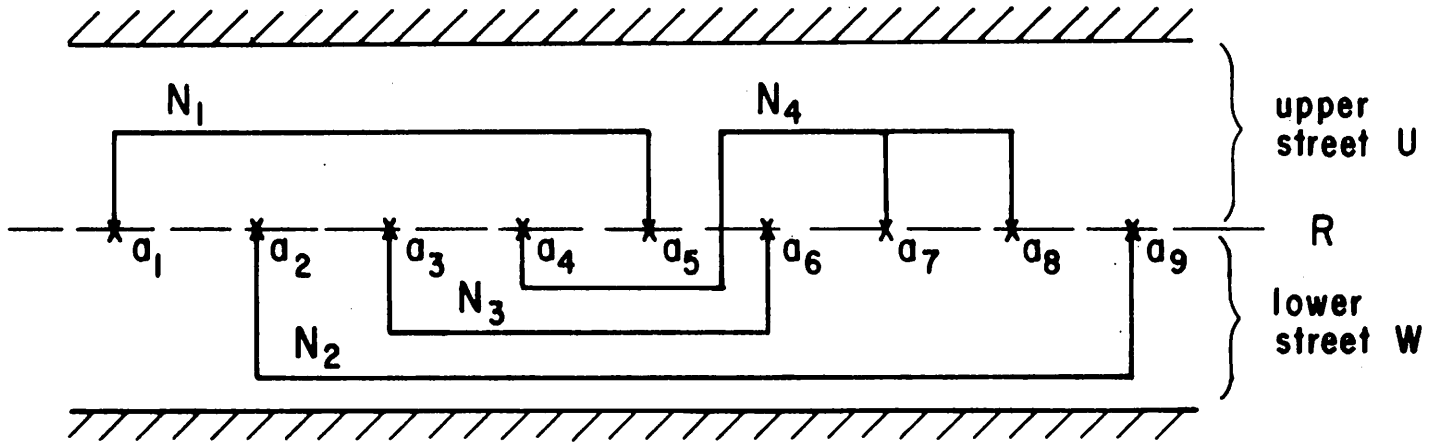
	1	2	3	4	5	6	7	8	9
a	x	0	x	0	x	0	x	0	x
b	x	0	x	0	x	0	x	0	x
c	x	0	x	0	x	0	x	0	x
d	x	0	x	0	x	0	x	0	x
e	x	0	x	0	x	0	x	0	x

(a) A BACKPLANE WITH 5 ROWS AND 9 COLUMNS.

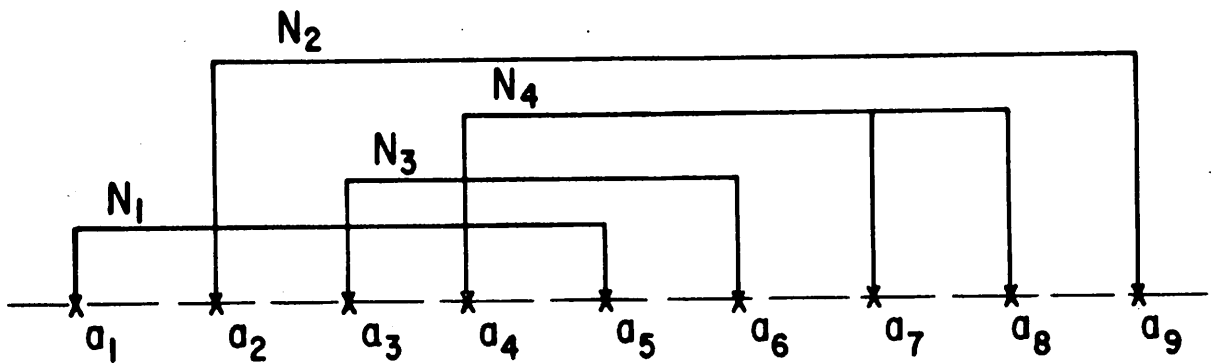


(b) 3 NETS DEFINED ON A BACKPLANE.

Fig. 2. A MULTILAYER, MULTIROW PROBLEM REDUCED TO 7 SINGLE-ROW, SINGLE-LAYER PROBLEMS.



(a) A REALIZATION OF A NET LIST IN A GIVEN SPACE



(b) A GRAPH REPRESENTATION OF A NET LIST.

Fig. 3. Realization and Representation of a Net List.

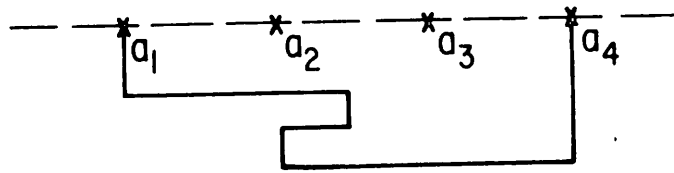


Fig. 4. FORWARD AND BACKWARD ZIGZAGGING NOT ALLOWED IN ROUTING.

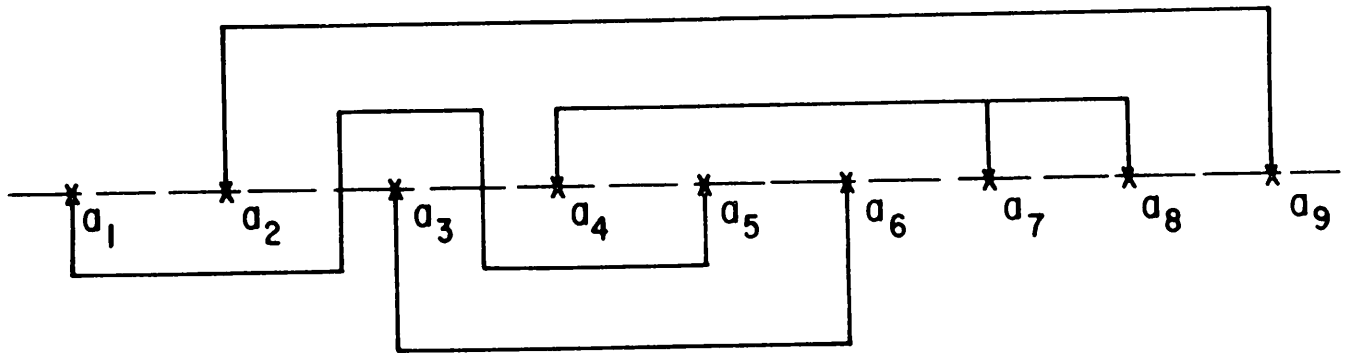


Fig. 5. ALTERNATE REALIZATION OF NET LIST IN Fig. 3.

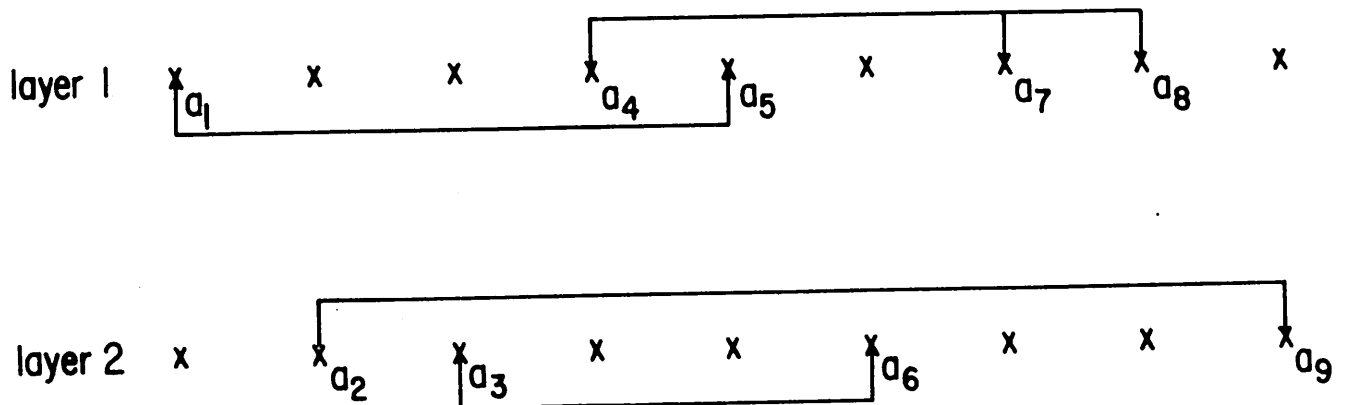
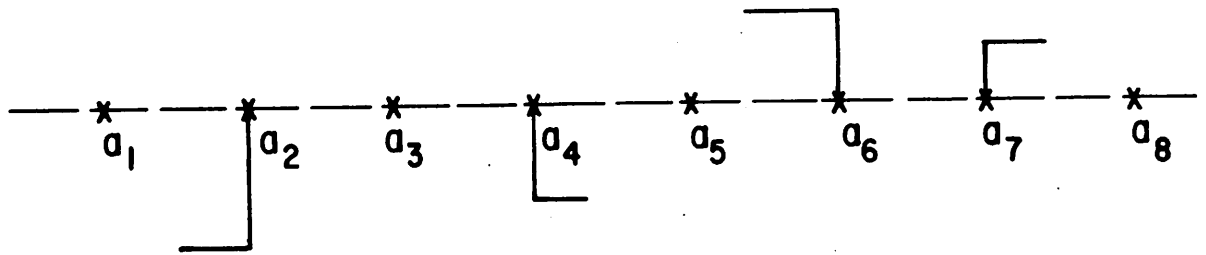


Fig. 6. A TWO LAYER DECOMPOSITION OF THE NET LIST IN FIG 5.



Type A nodes :  $a_1, a_3, a_5$  and  $a_8$

Type B nodes :  $a_6, a_7$

Type C nodes :  $a_2, a_4$

Fig. 7. Three Different Types of Nodes Displayed.

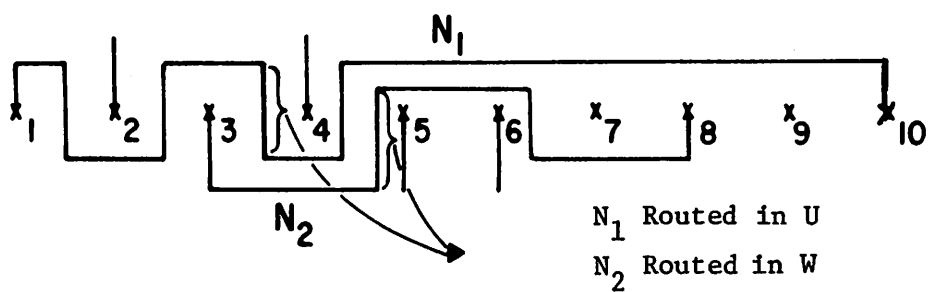


Fig. 8 Vertical Conductor Segments to Interconnect Horizontal Conductor Segments in Different Streets.



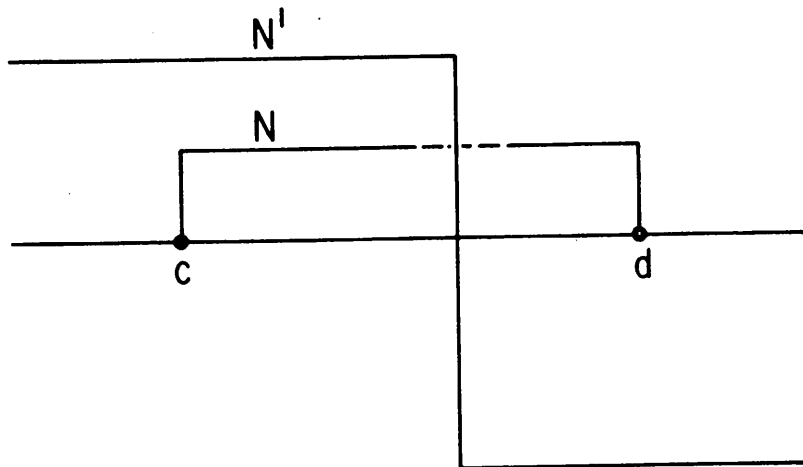
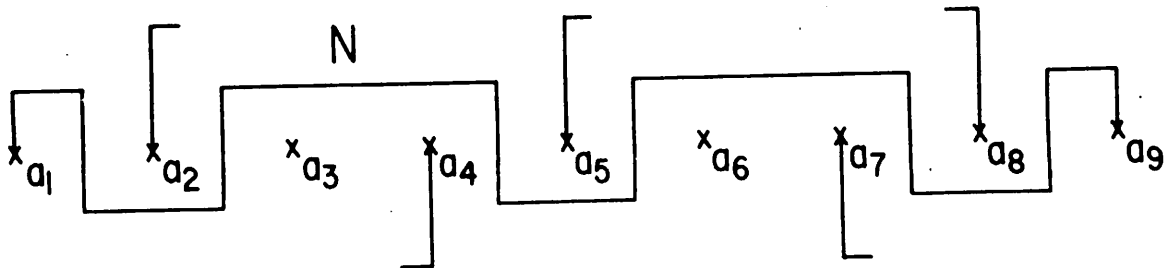
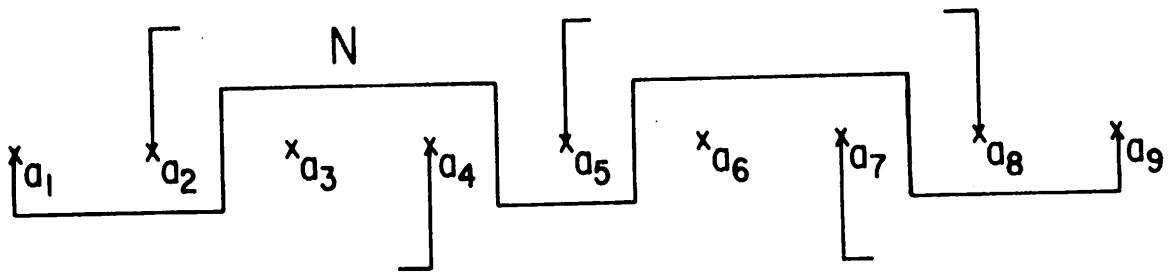


Fig. 9. NET N INTERSECTS NET N' SOMEWHERE IN INTERVAL (c,d).

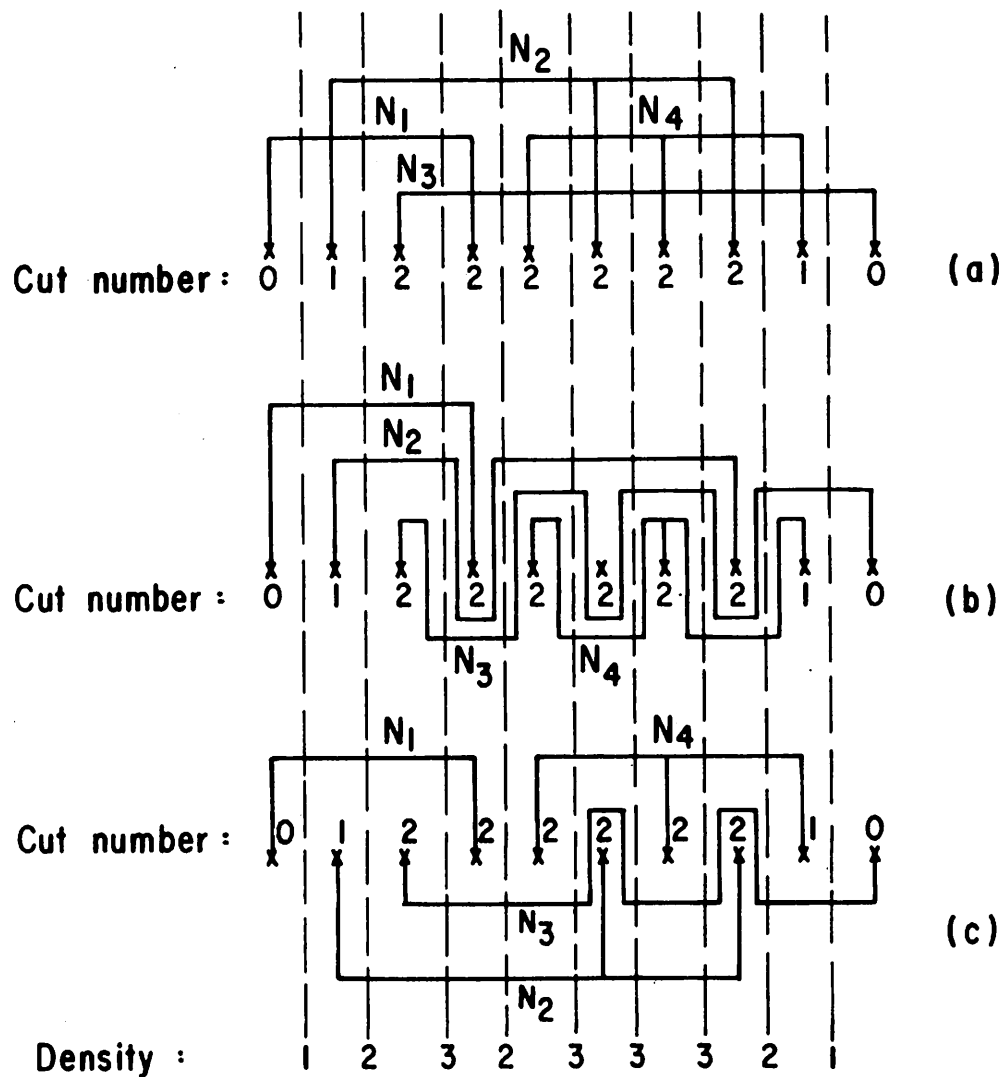


(a) REALIZATION OF N USING THE ROUTING ALGORITHM



(b) LOCAL MODIFICATIONS AT  $a_1$  and  $a_9$  HAVE BEEN MADE.

FIG. 10. LOCAL MODIFICATION OF DEFINITION 2.

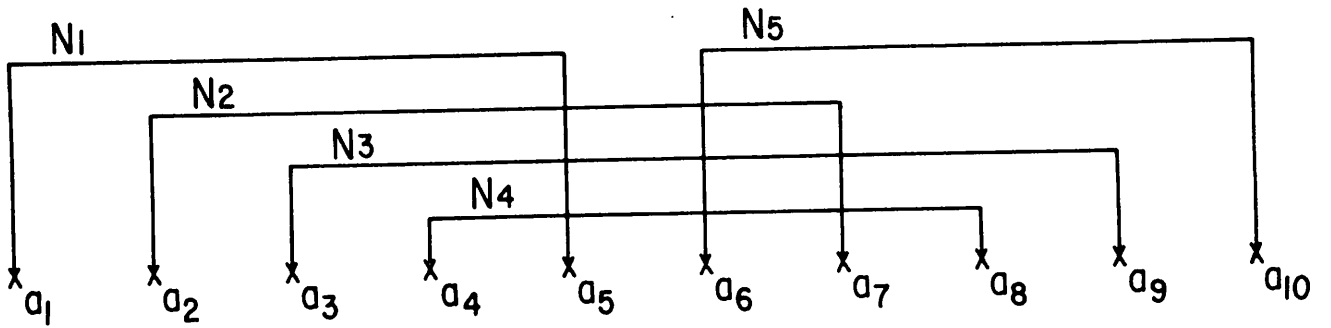


Max. Density = 3, both density and cut number are invariant.

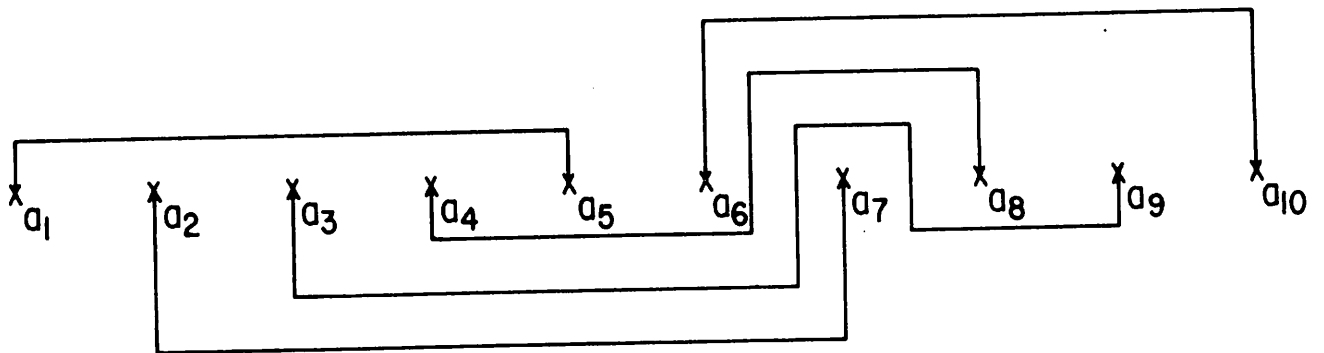
cut number of net 1 = 2  
 cut number of net 2 = 2  
 cut number of net 3 = 2  
 cut number of net 4 = 2

(a) A net list with 4 nets over a single row with 10 nodes  
 (b) Realization 1  
 (c) Realization 2

Fig. 11.

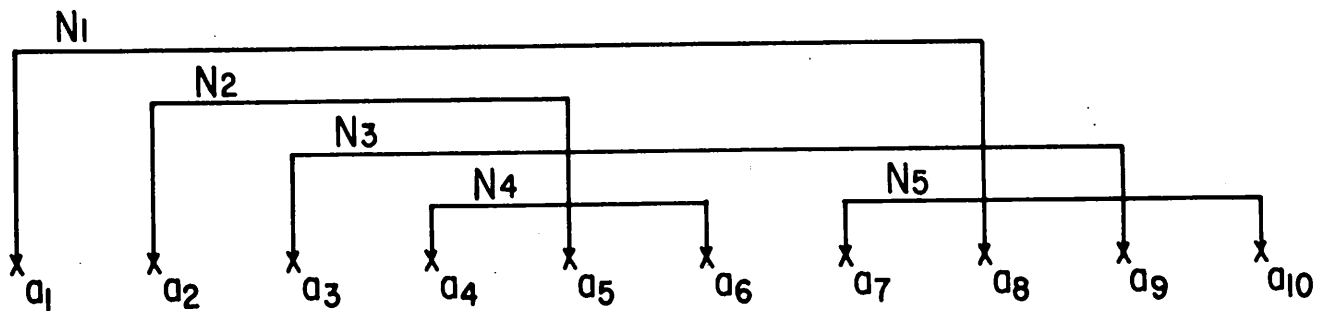


(a) A NET LIST WITH 5 NETS:  $\rho = 4, \rho' = 2$ .

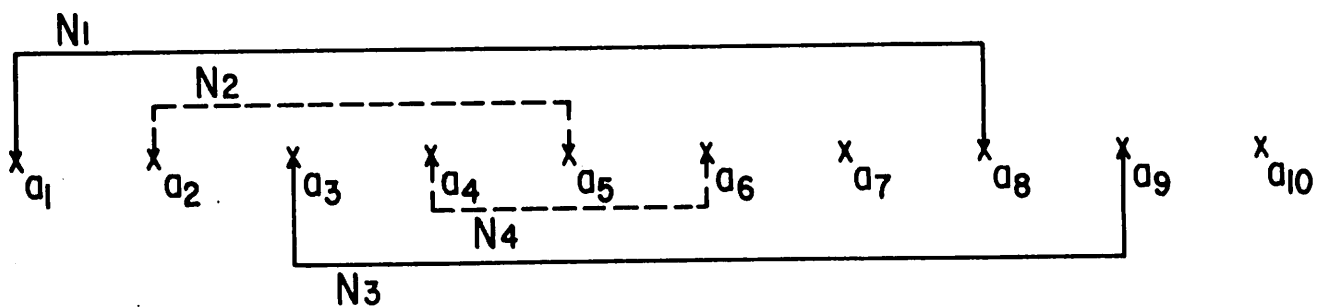


(b) One realization which yields  $M = C(U) = C(W) = 3 > \rho' = 2$ .

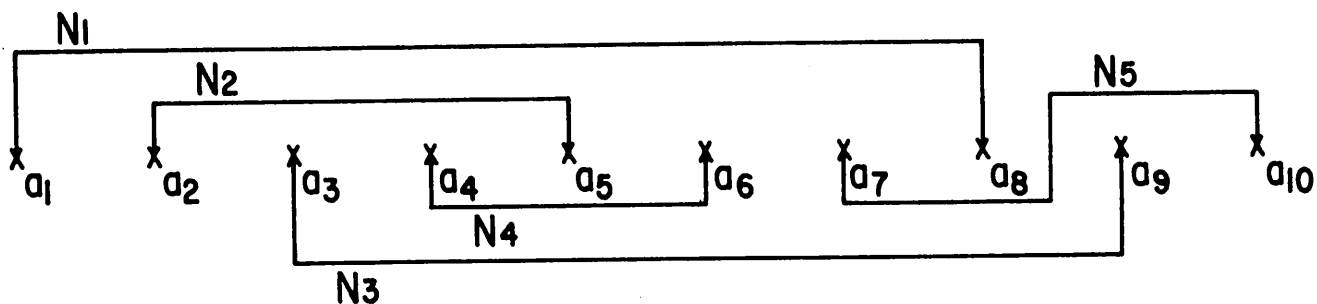
Fig. 12.



(a) A GRAPH REPRESENTATION OF A NET LIST WITH  $\rho = 4$  and  $\rho' = 2$ .

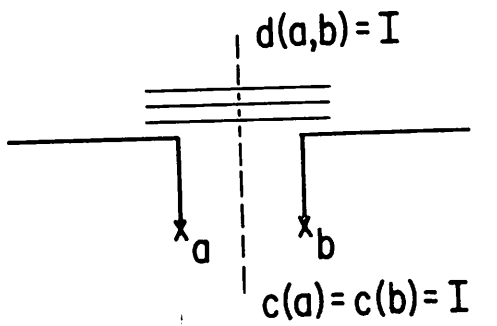


(b) ASSIGNMENT OF NETS INTO STREETS.

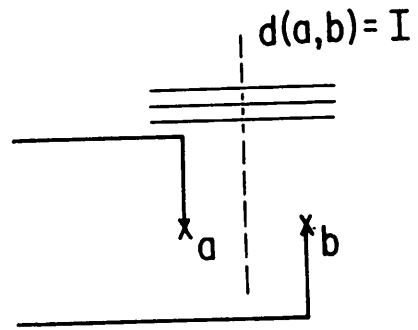


(c) A COMPLETE ROUTING WITH  $M = 2$ .

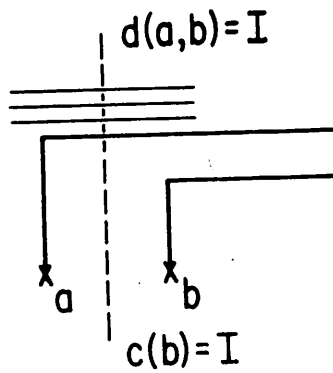
Fig. 13.



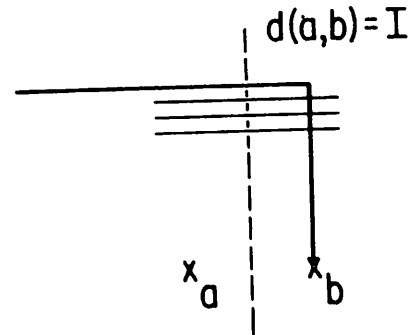
(a) TWO NODES ARE OF TYPE B.



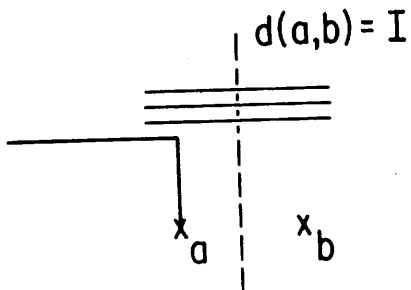
(d) NODE a IS TYPE B.  
NODE b IS TYPE C.



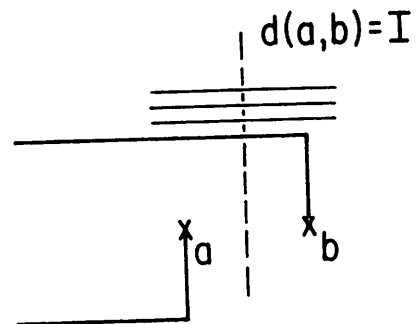
(b) TWO NODES ARE OF TYPE B.



(e) NODE a IS TYPE A.  
NODE b IS TYPE B.

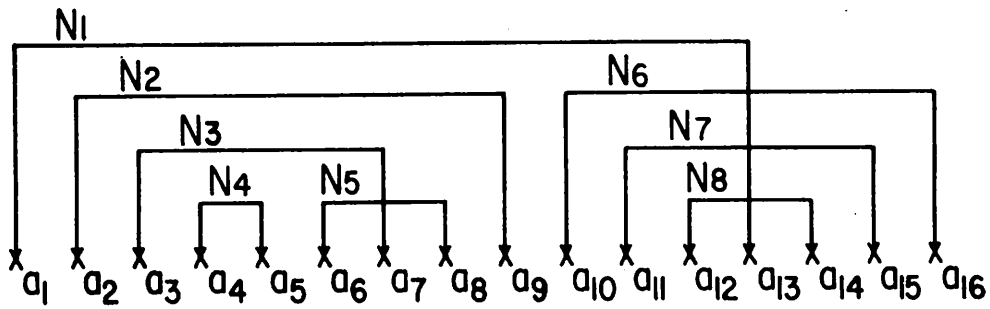


(c) NODE a IS TYPE B.  
NODE b IS TYPE A.

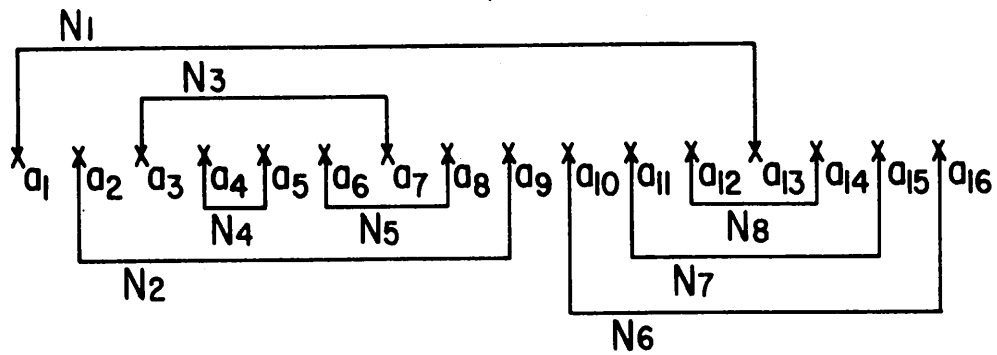


(f) NODE a IS TYPE C.  
NODE b IS TYPE B.

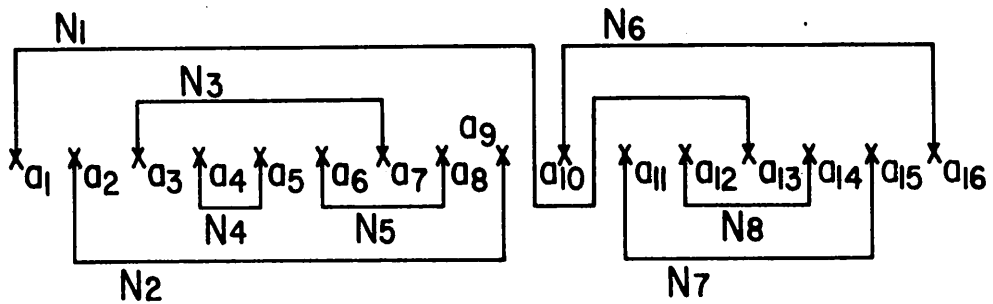
Fig. 14.



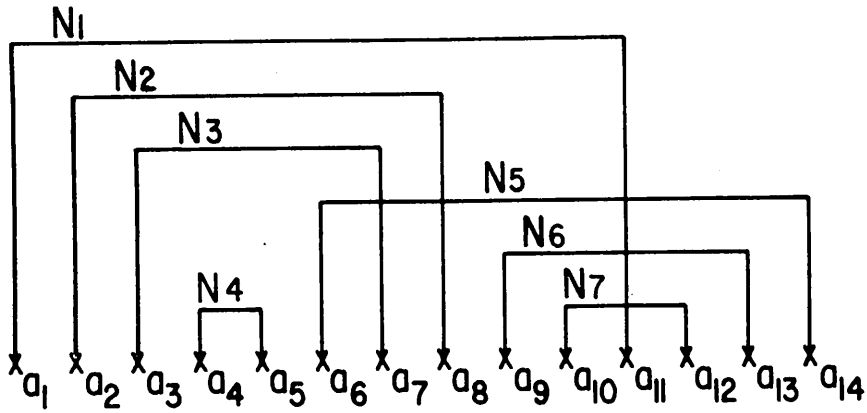
(a) A NET LIST WITH 8 NETS,  $\rho = 4$



(b) A REALIZATION OF (a) USING SUFFICIENT CONDITION,  $M = 3$ .



(c) A REALIZATION OF (a) WITH  $M = 2$ .



(a) A NET LIST WITH 7 NETS.

$$s_1 = [a_1, a_{14}], s_2 = [a_2, a_{13}], s_3^{(1)} = [a_3, a_8]$$

$$s_3^{(2)} = [a_9, a_{12}], s_4^{(1)} = [a_4, a_5], s_4^{(2)} = [a_6, a_7], s_4^{(3)} = [a_{10}, a_{11}]$$

$$\bar{n}(s_1) = \phi, \tilde{n}(s_1) = \{N_1, N_2, \dots, N_7\}$$

$$\bar{n}(s_2) = \phi, \tilde{n}(s_2) = \tilde{n}(s_1)$$

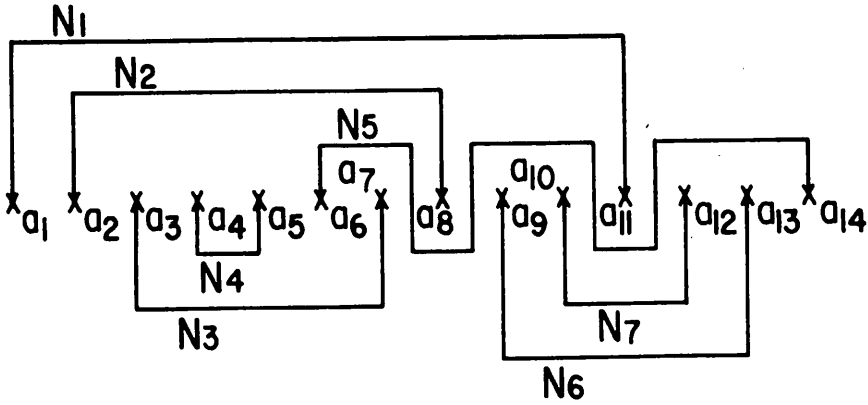
$$\bar{n}(s_3^{(1)}) = \{N_1, N_2\}, \tilde{n}(s_3^{(1)}) = \{N_3, N_4, N_5\}$$

$$\bar{n}(s_3^{(2)}) = \{N_5, N_6\}, \tilde{n}(s_3^{(2)}) = \{N_1, N_7\}$$

$$\Sigma_1 = \{s_1\}, \Sigma_2 = \{s_2\}, \Sigma_3 = \{s_3^{(1)}, s_3^{(2)}\}, \Sigma_4 = \{s_4^{(1)}, s_4^{(2)}, s_4^{(3)}\}$$

$$H_3^{(1|2)} = \{N_1\}, H_3^{(2|1)} = \{N_5\}$$

Fig. 16(a).



(b) A REALIZATION OF (a).

$N_1, N_2 \in \bar{n}(S_3^{(1)})$  are routed in street U completely over  $S_3^{(1)}$ .

$N_6 \in \bar{n}(S_3^{(2)})$  is routed in street W completely over  $S_3^{(2)}$ .

Fig. 16.

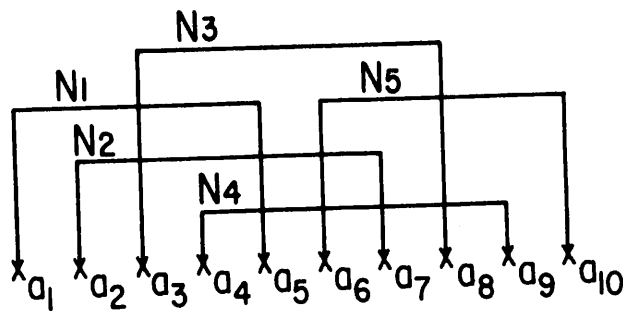
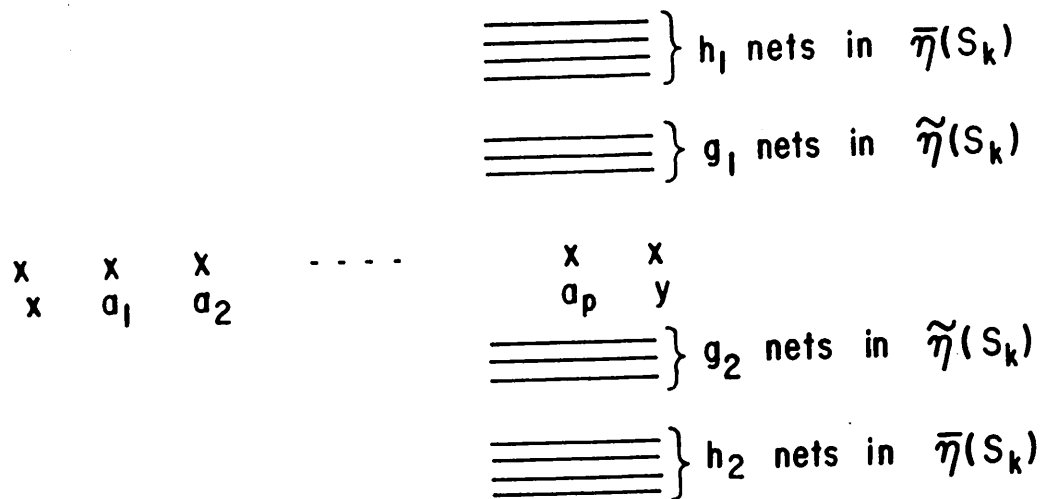


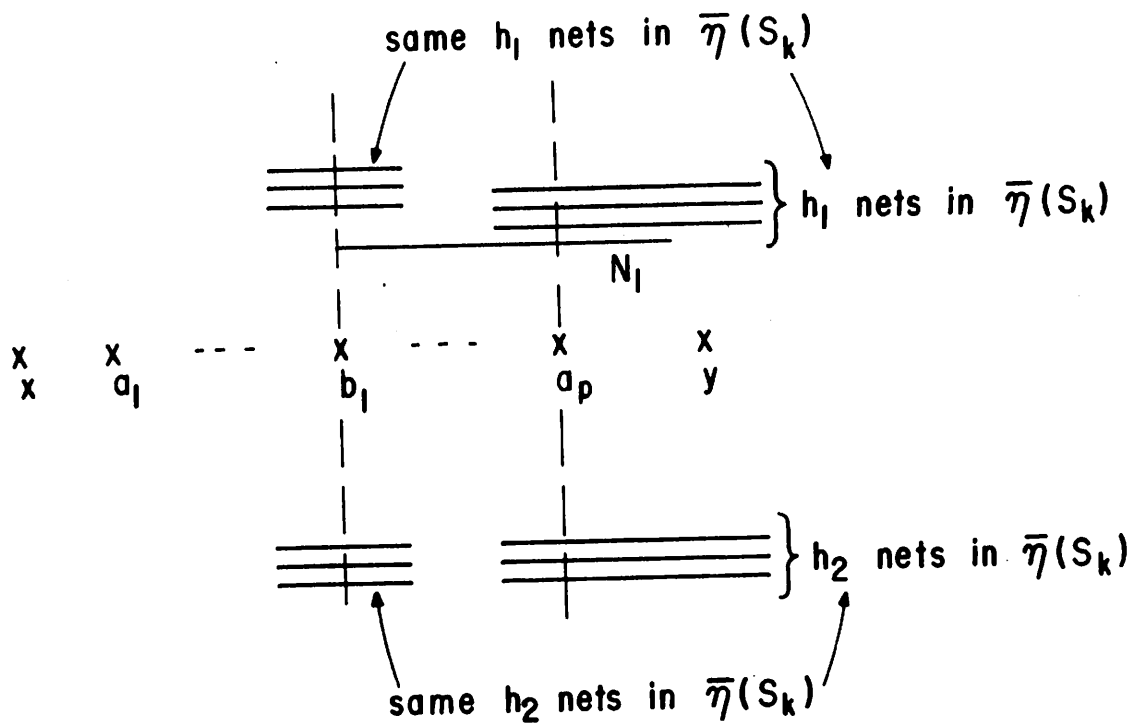
Fig. 17.





$$h_1 + h_2 + g_1 + g_2 = k$$

(a)



(b)

Fig. 18.

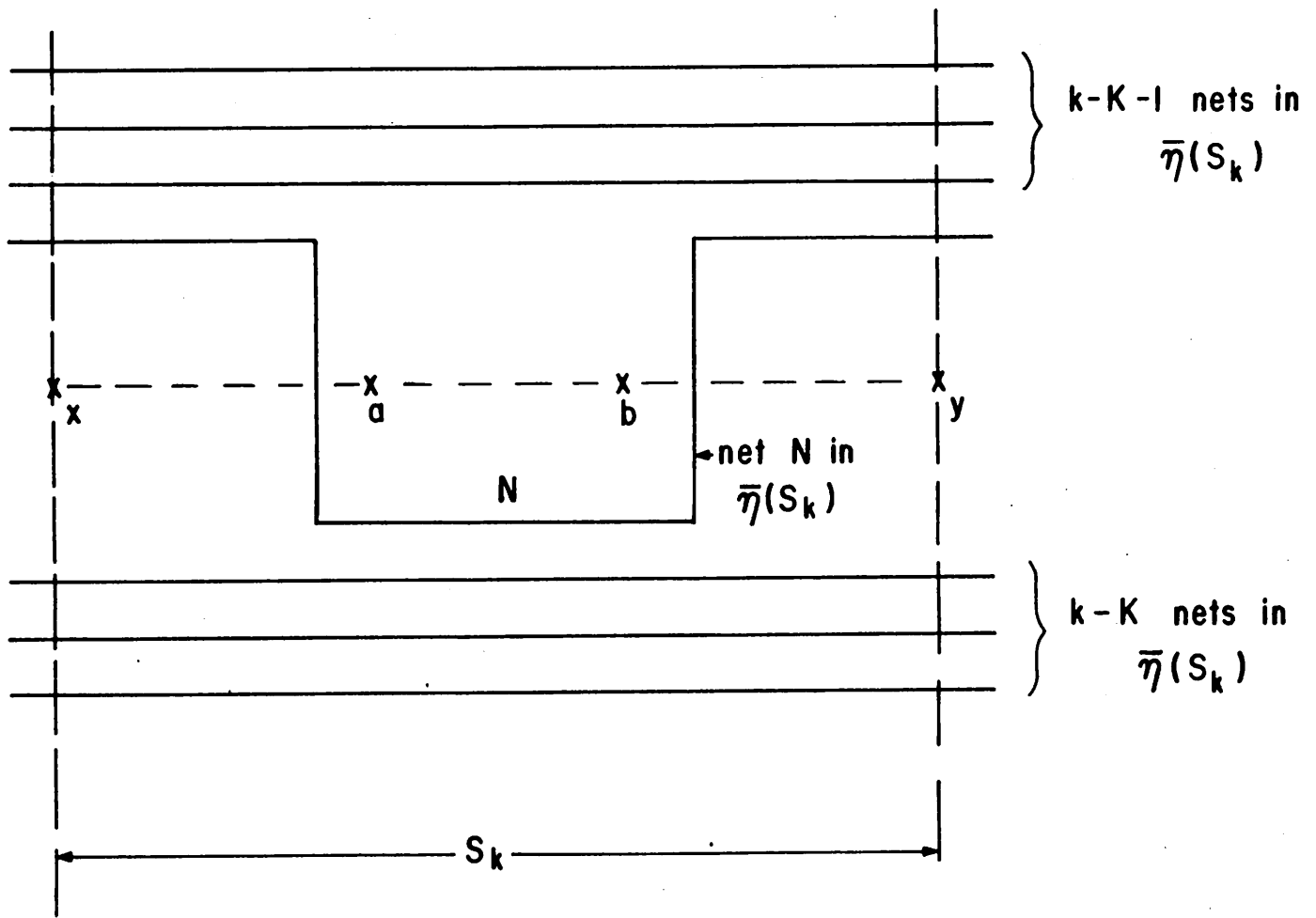


Fig. 19.

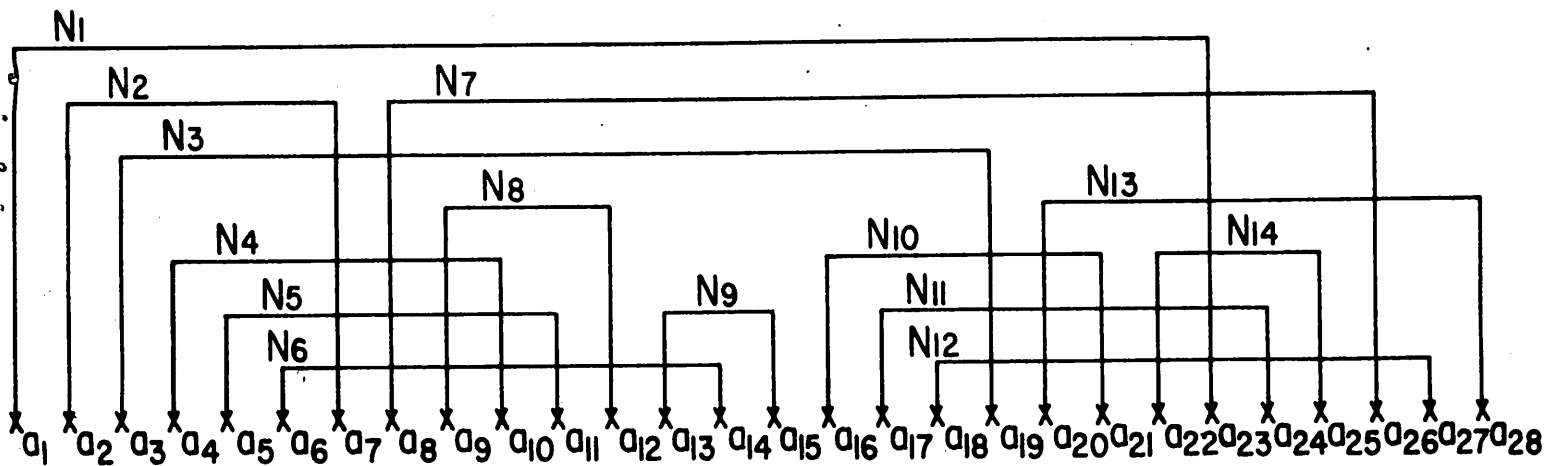
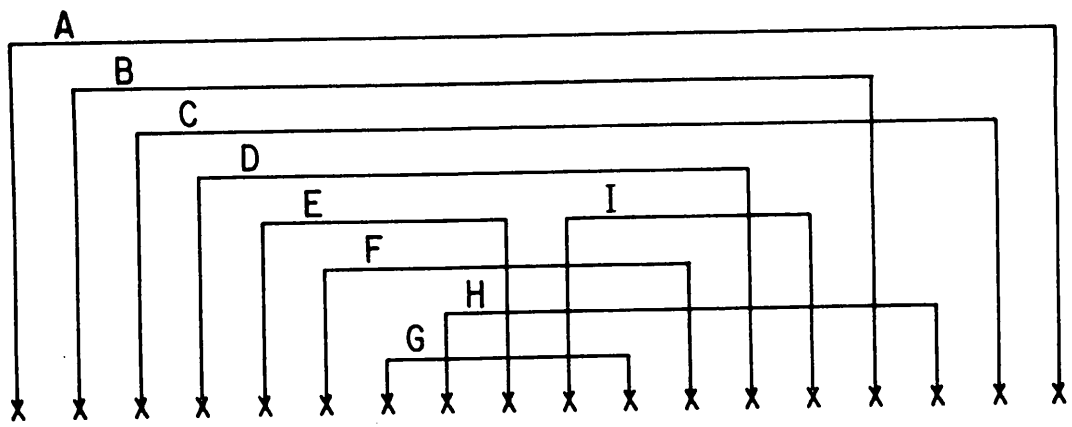
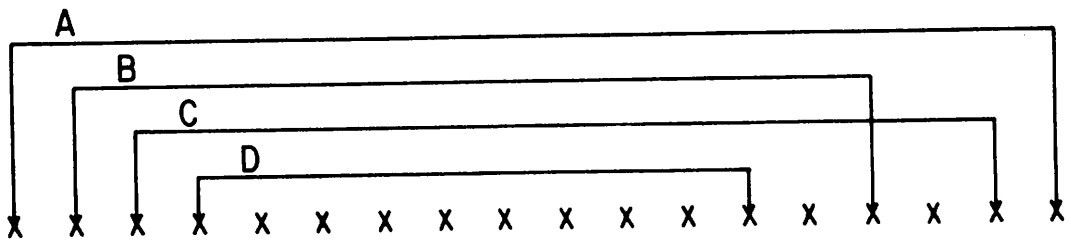


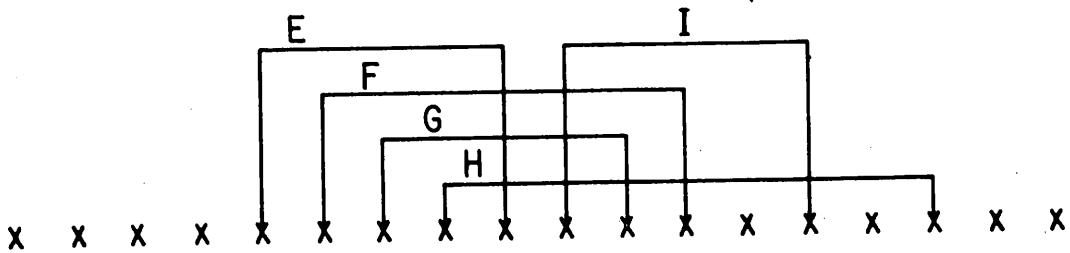
Fig. 20. A NET LIST WITH 14 NETS,  $\rho = 7$ .



(a) A NET LIST WITH  $\rho = 8$ .

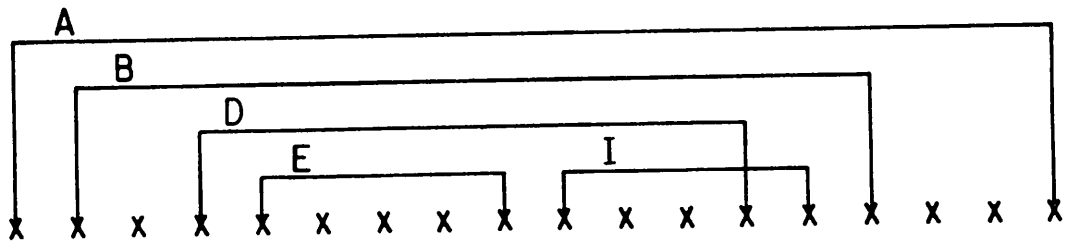


(i) 1st layer decomposition with  $\rho_1 = 4$

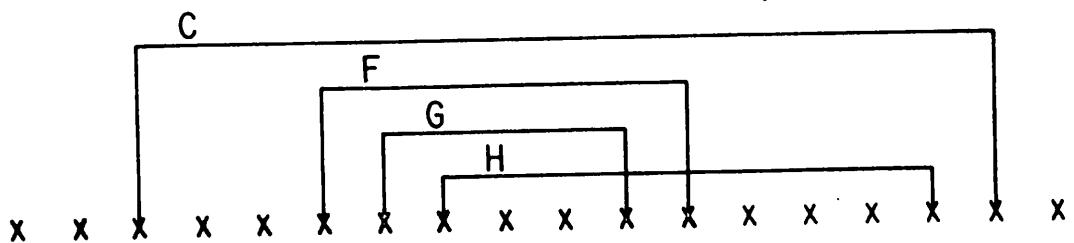


(ii) 2nd layer decomposition with  $\rho_2 = 4$

(b) A NET LIST DECOMPOSITION TO 2 LAYERS.



(i) 1st layer decomposition with  $\rho_1 = 4$



(ii) 2nd layer decomposition with  $\rho_2 = 4$

(c) ANOTHER DECOMPOSITION OF (a).

Fig. 21.