THE LINEAR MULTIVARIABLE REGULATOR PROBLEM

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ABSTRACT

The problem is considered of regulating in the face of parameter uncertainty the output of a linear time-invariant system subjected to disturbance and reference signals. This problem has been solved by other researchers. In this paper a new and simpler algebraic solution is given.

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1. Introduction

This paper deals with the regulation of the linear multivariable system modeled by the equations

$$\dot{x}_1 = A_1x_1 + A_3x_2 + B_1u \tag{1}$$

$$\dot{x}_2 = A_2x_2 \tag{2}$$

$$y = C_1x_1 + C_2x_2 \tag{3}$$

$$z = D_1x_1 + D_2x_2. \tag{4}$$

Here x_1 is the plant state vector, u the control input, x_2 the vector of exogenous signals, y the vector of measurements available for control, and z the output to be regulated. The vectors u, x_1, x_2, y, and z belong to fixed finite-dimensional real linear spaces

$$\mathcal{U}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, \mathcal{Z} \tag{5}$$

respectively, and the time-invariant linear maps in (1) to (4) are defined on the appropriate spaces as follows:

$$A_i : \mathcal{X}_i \to \mathcal{X}_i, \quad C_i : \mathcal{X}_i \to \mathcal{Z}, \quad D_i : \mathcal{X}_i \to \mathcal{Z} \qquad (i=1,2)$$

$$A_3 : \mathcal{X}_2 \to \mathcal{X}_1, \quad B_1 : \mathcal{U} \to \mathcal{X}_1.$$

The vector A_3x_2 in (1) represents a plant disturbance, and the vector D_2x_2 in (4) represents a reference signal which the plant output $-D_1x_1$ is required to track. Equation (2) then models the class of disturbance and reference signals (e.g. steps, ramps, sinusoids).

Control action is to be provided by a compensator processing the measurements $y(\cdot)$, generating the control $u(\cdot)$, and modeled by

$$\dot{x}_c = A_c x_c + B_c y \tag{6}$$

$$u = F_c x_c + G_c y. \tag{7}$$

Here the compensator state vector x_c belongs to a finite-dimensional real linear space $\mathcal{X}_c$, and the linear maps A_c, B_c, F_c, G_c are time-invariant. It is convenient to consider a compensator formally as a 5-tuple

$$(\mathcal{X}_c, A_c, B_c, F_c, G_c)$$

where

$$A_c : \mathcal{X}_c \to \mathcal{X}_c, \; B_c : \mathcal{Y} \to \mathcal{X}_c$$

$$F_c : \mathcal{X}_c \to \mathcal{U}, \; G_c : \mathcal{Y} \to \mathcal{U}.$$

There are two control objectives: closed loop stability and output regulation. Consider a fixed compensator $(\mathcal{X}_c, A_c, B_c, F_c, G_c)$, and define the closed loop state vector, state space, and linear maps

$$x_L = \begin{bmatrix} x_1 \\ x_c \end{bmatrix}, \quad \mathcal{X}_L = \mathcal{X}_1 \oplus \mathcal{X}_c \tag{8a}$$

$$A_L = \begin{bmatrix} A_1 + B_1 G_c C_1 & B_1 F_c \\ B_c C_1 & A_c \end{bmatrix} : \mathcal{X}_L \to \mathcal{X}_L \tag{8b}$$

$$B_L = \begin{bmatrix} A_3 + B_1 G_c C_2 \\ B_c C_2 \end{bmatrix} : \mathcal{X}_2 \to \mathcal{X}_L \tag{8c}$$

$$D_L = [D_1 0] : \mathcal{X}_L \to \mathcal{Z}. \tag{8d}$$

From (1), (3), (4), (6), and (7) the closed loop is described by

$$\dot{x}_L = A_L x_L + B_L x_2 \tag{9}$$

$$z = D_L x_L + D_2 x_2 \ . \tag{10}$$

Closed loop stability means that A_L is stable, that is $\sigma(A_L) \subset \mathbb{C}^-$, and output regulation means that $z(t) \to 0$ as $t \to \infty$ for all $x_L(0)$ and $x_2(0)$. The compensator is called a synthesis if it provides closed loop stability and output regulation.

The spaces (5) are assumed to have fixed bases; so we regard $A_1, A_3, \ldots$ in (1) to (4) as linear maps or as real matrices, depending on the context. Similarly, in specifying a compensator, we shall suppose that a basis for $\mathcal{X}_c$ is specified; so we regard A_c, B_c, F_c, G_c also as real matrices.

Now consider a fixed synthesis $(\mathcal{X}_c, A_c, B_c, F_c, G_c)$. An n-dimensional data point $\mathcal{P} \in \mathbb{R}^n$ is a list of n numbers selected from among the elements of the plant matrices A_1, A_3, B_1 together with the compensator matrices A_c, B_c, F_c, G_c. A property of points in $\mathbb{R}^n$ is said to be stable at $\mathcal{P}$ if it holds throughout some open neighbourhood of $\mathcal{P}$. We say that the synthesis is structurally stable at $\mathcal{P}$ if closed loop stability and output regulation are both properties which are stable at $\mathcal{P}$. Clearly closed loop stability is a stable property (if A_L is stable it remains so under small perturbation), so the synthesis is structurally stable at $\mathcal{P}$ iff output regulation is a stable property at $\mathcal{P}$. The requirement of structural stability evidently reflects an uncertainty of some system parameters or the desire to achieve a degree of insensitivity to slow drift in certain parameters.

Our object in this paper is to solve two problems.

Problem 1 Find computable necessary and sufficient conditions (in terms of the given data A_1, A_3, B_1, A_2, C_1, C_2, D_1, D_2) for the existence of a synthesis. Give an algorithm to compute a synthesis when these conditions hold.

By a computable condition we mean one for which a verifying algorithm exists.

<u>Problem 2</u> This is Problem 1 with 'synthesis' replaced by 'structurally stable synthesis.'

These or similar problems have been treated by many researchers, among whom we mention S. P. Bhattacharyya [1,2], E. J. Davision [3,4], O. M. Grasselli [5], C. D. Johnson [6,7], P. C. Müller [8], J. B. Pearson [9], O. A. Sebakhy [10], H. W. Smith [11], W. M. Wonham [12,13], and P. C. Young [14]. In our view the algebraic solutions presented in this paper are simpler than previous solutions. With the exception of some technical facts, the treatment given here is self-contained.

2. Technical Preliminaries

<u>Notation</u> $\mathbb{R}$ (resp. $\mathbb{C}$) denotes the field of real (resp. complex) numbers. $\mathbb{C}^+$ (resp. $\mathbb{C}^-$) is the closed right-half (resp. open left-half) complex plane. We use the standard notation of linear algebra: if $A : \mathcal{X} \to \mathcal{X}$ is a linear transformation (map, for short), ImA is its image, Ker A its kernel, $\sigma(A)$ its complex spectrum, and $A|\mathcal{V}$ is the restriction of A to $\mathcal{V}$. The dimension of $\mathcal{X}$ is denoted by $d(\mathcal{X})$. For linear spaces $\mathcal{R}$ and $\mathcal{S}$, $\mathcal{R} \cong \mathcal{S}$ means $\mathcal{R}$ and $\mathcal{S}$ are isomorphic and $\mathrm{Hom}(\mathcal{R}, \mathcal{S})$ is the linear space of all maps $\mathcal{R} \to \mathcal{S}$. For maps M and N, $M \cong N$ means M and N are similar ($M=T^{-1}NT$ for some isomorphism T). While any linear space $\mathcal{X}$ is initially real, we shall introduce without comment its complexification. For example if $A : \mathcal{X} \to \mathcal{X}$ and $\lambda \in \sigma(A) \subset \mathbb{C}$ then Ker $(A-\lambda)$ is a complex subspace of the complexification of $\mathcal{X}$. $\mathbb{R}[s]$ (resp. $\mathbb{C}[s]$) is the ring of polynomials in s with coefficients in $\mathbb{R}$ (resp. $\mathbb{C}$). For polynomials $\alpha(s)$ and $\beta(s)$, $\alpha|\beta$ means α divides β. We

abbreviate degree to deg and greatest common divisor to gcd. Finally, for $n \geq 1$, $\underline{n}$ is the set $\{1,\ldots,n\}$.

We now recall some characterizations of stabilizability and detectability. For this consider a triple (C,A,B):

$$C : \mathcal{X} \to \mathcal{Y}, \quad A : \mathcal{X} \to \mathcal{X}, \quad B : \mathcal{U} \to \mathcal{X}.$$

Let

$$\mathcal{N} = \bigcap_{i \geq 0} \operatorname{Ker} (CA^i)$$

be the unobservable subspace of (C,A),

$$\langle A | \operatorname{Im} B \rangle = \sum_{i \geq 0} A^i \operatorname{Im} B$$

the controllable subspace of (A,B), and $\mathcal{X}^+(A)$ the unstable subspace of A. (See [13].) Then the pair (C,A) is detectable iff

$$\mathcal{N} \cap \mathcal{X}^+(A) = 0,$$

or equivalently

$$\operatorname{Ker} C \cap \operatorname{Ker}(A-\lambda) = 0 \quad (\lambda \in \mathbb{C}^+);$$

and the pair (A,B) is stabilizable iff

$$\mathcal{X}^+(A) \subset \langle A | \operatorname{Im} B \rangle ,$$

or equivalently

$$\mathcal{X} = \operatorname{Im}(A-\lambda) + \operatorname{Im} B \quad (\lambda \in \mathbb{C}^+).$$

Throughout this paper the following are <u>standing assumptions</u>:

$$\sigma(A_2) \subset \mathbb{C}^+ \tag{11}$$

$$\mathrm{Im}C_1 + \mathrm{Im}C_2 = \mathcal{Y} \tag{12}$$

$$\mathrm{Im}D_1 = \mathcal{Z} \tag{13}$$

$$(A_1, B_1) \text{ is stabilizable} \tag{14}$$

$$(C_1, A_1) \text{ is detectable.} \tag{15}$$

Assumption (11) involves no loss of generality, for any stable exogenous modes can be included in the plant description as they affect neither closed loop stability nor output regulation. Assumption (12) also involves no loss of generality, for if (12) does not hold initially we may redefine $\mathcal{Y}$ to be $\mathrm{Im}C_1 + \mathrm{Im}C_2$. Similarly we may assume that

$$\mathrm{Im}D_1 + \mathrm{Im}D_2 = \mathcal{Z}. \tag{16}$$

But a necessary condition for output regulation is clearly

$$\mathrm{Im}D_2 \subset \mathrm{Im}D_1 \ ; \tag{17}$$

so (13) follows from (16) and (17). Finally, we claim that (14) and (15) are necessary for closed loop stability. Indeed, if A_L is stable then

$$\mathcal{X}_L = \mathrm{Im}(A_L - \lambda) \quad (\lambda \in \mathbb{C}^+) \ ;$$

hence in particular, from (8b),

$$\mathcal{X}_1 = \mathrm{Im}(A_1 + B_1 G_c C_1 - \lambda) + \mathrm{Im}(B_1 F_c) \qquad (\lambda \in \mathbb{C}^+)$$

and

$$\mathrm{Ker}(B_cC_1) \cap \mathrm{Ker}(A_1+B_1G_cC_1-\lambda) = 0 \qquad (\lambda \in \mathbb{C}^+).$$

These conditions imply respectively

$$\mathcal{X}_1 = \mathrm{Im}(A_1-\lambda) + \mathrm{Im}B_1 \qquad (\lambda \in \mathbb{C}^+)$$

$$\mathrm{Ker}C_1 \cap \mathrm{Ker}(A_1-\lambda) = 0 \qquad (\lambda \in \mathbb{C}^+),$$

which are equivalent to (14) and (15). To summarize, then, (11) to (15) either involve no loss of generality or are necessary for the existence of a synthesis.

We next introduce the mathematical setting in which we shall solve Problems 1 and 2. For any linear space $\mathcal{R}$, define

$$\underline{\mathcal{R}} = \mathrm{Hom}(\mathcal{X}_2, \mathcal{R}).$$

For any map $A : \mathcal{X} \to \mathcal{X}$, define $\underline{A} : \underline{\mathcal{X}} \to \underline{\mathcal{X}}$ by

$$\underline{A}X = AX - XA_2 \qquad (X \in \underline{\mathcal{X}}).$$

Finally, for any map $C : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are distinct, define $\underline{C} : \underline{\mathcal{X}} \to \underline{\mathcal{Y}}$ by

$$\underline{C}X = CX \qquad (X \in \underline{\mathcal{X}}).$$

As an application of this notation we have the following very useful characterization of output regulation.

<u>Lemma 1</u>

Suppose A_L is stable. Then the output z in the system

$$\dot{x}_L = A_Lx_L + B_Lx_2$$

$$\dot{x}_2 = A_2x_2$$

$$z = D_L x_L + D_2 x_2$$

<u>is regulated iff</u>

$$\underline{D}_L \underline{A}_L^{-1} B_L = D_2 \; , \tag{18}$$

<u>or equivalently</u>

$$\begin{bmatrix} B_L \\ \\ D_2 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} \underline{A}_L \\ \\ \underline{D}_L \end{bmatrix} . \tag{19}$$

By (19) we mean of course that

$$B_L = \underline{A}_L X_L, \; D_2 = \underline{D}_L X_L$$

for some $X_L \in \underline{\mathcal{X}}_L$.

The closed loop transfer matrix in the above system is

$$D_L(s-A_L)^{-1}B_L + D_2.$$

If A_L is stable, output regulation is therefore equivalent to the condition

$$\lim_{s \to 0} s[D_L(s-A_L)^{-1}B_L + D_2](s-A_2)^{-1} = 0. \tag{20}$$

Thus conditions (18) and (20) are equivalent. The conciseness of (18) shows the power of the present algebraic approach. Notice that for constant exogenous signals, that is $A_2 = 0$, (18) and (20) both become

$$D_L A_L^{-1} B_L = D_2.$$

Lemma 1 is a restatement of Lemma 1 of [15]. We reprove it here for completeness.

Proof of Lemma 1

If we define

$$x_s = \begin{bmatrix} x_L \\ x_2 \end{bmatrix}, \quad \mathcal{X}_s = \mathcal{X}_L \oplus \mathcal{X}_2$$

$$A_s = \begin{bmatrix} A_L & B_L \\ 0 & A_2 \end{bmatrix} : \mathcal{X}_s \to \mathcal{X}_s$$

$$D_s = [D_L \; D_2] \quad : \mathcal{X}_s \to \mathcal{Z}$$

then the system is described simply by

$$\dot{x}_s = A_s x_s \; , \; z = D_s x_s \; .$$

So output regulation holds iff

$$\mathcal{X}_s^+(A_s) \subset \operatorname{Ker} D_s . \tag{21}$$

Since

$$\sigma(A_L) \cap \sigma(A_2) = \emptyset$$

$\underline{A}_L$ is invertible. Define

$$X_L = \underline{A}_L^{-1} B_L \in \underline{\mathcal{X}}_L \tag{22}$$

and

$$Q = \begin{bmatrix} I_L & -X_L \\ 0 & I_2 \end{bmatrix} : \mathcal{X}_s \to \mathcal{X}_s$$

where I_L (resp. I_2) is the identity on $\mathcal{X}_L$ (resp. $\mathcal{X}_2$). Then

$$A_L X_L - X_L A_2 = B_L$$

and so

$$Q^{-1}A_sQ = \begin{bmatrix} A_L & 0 \\ 0 & A_2 \end{bmatrix} .$$

Thus

$$\mathcal{X}_s^+(A_s) = Q \text{ Im} \begin{bmatrix} 0 \\ I_2 \end{bmatrix} = \text{Im} \begin{bmatrix} -X_L \\ I_2 \end{bmatrix} .$$

Hence from (21) output regulation holds iff

$$\underline{D}_L X_L = D_2;$$

or from (22)

$$\underline{D}_L \underline{A}_L^{-1} B_L = D_2. \qquad \blacksquare$$

Using (8) in (19) we obtain immediately the

Corollary

A compensator $(\mathcal{X}_c, A_c, B_c, F_c, G_c)$ which provides closed loop stability also provides output regulation iff

$$\begin{bmatrix} A_3+B_1G_cC_2 \\ B_cC_2 \\ D_2 \end{bmatrix} \in \text{Im} \begin{bmatrix} \underline{A}_1+\underline{B}_1\underline{G}_c\underline{C}_1 & \underline{B}_1\underline{F}_c \\ \underline{B}_c\underline{C}_1 & \underline{A}_c \\ \underline{D}_1 & 0 \end{bmatrix} . \tag{23}$$

We remark that if y = z, which is to say

$$\mathcal{Y} = \mathcal{Z}, \quad C_1 = D_1 \quad , \quad C_2 = D_2 ,$$

then (23) reduces to

$$\begin{bmatrix} A_3 \\ 0 \\ D_2 \end{bmatrix} \in \text{Im} \begin{bmatrix} \underline{A}_1 & \underline{B}_1\underline{F}_c \\ 0 & \underline{A}_c \\ \underline{D}_1 & 0 \end{bmatrix} . \tag{24}$$

As our final technical preliminary we condense the system description (1) to (4) by defining

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$$

$$A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} : \mathcal{X} \to \mathcal{X}, \ B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} : \mathcal{U} \to \mathcal{X}$$

$$C = [C_1 \ \ C_2] : \mathcal{X} \to \mathcal{Y}, \ D = [D_1 \ \ D_2] : \mathcal{X} \to \mathcal{Z}.$$

Then (1) to (4) become

$$\dot{x} = Ax + Bu, \ y = Cx, \ z = Dx.$$

3. Solution of Problem 1

Before solving Problem 1 we pose a simpler problem; namely, we consider pure gain controllers of the form

$$u = F_1x_1 + F_2x_2 \tag{25}$$

instead of dynamic compensators. Substituting (25) into (1) and rewriting (2) and (4) we obtain

$$\dot{x}_1 = (A_1+B_1F_1)x_1 + (A_3+B_1F_2)x_2 \tag{26a}$$

$$\dot{x}_2 = A_2x_2 \tag{26b}$$

$$z = D_1x_1 + D_2x_2 \ . \tag{26c}$$

<u>Problem 0</u> Find necessary and sufficient conditions for the existence of $F_1 : \mathcal{X}_1 \to \mathcal{U}$ and $F_2 : \mathcal{X}_2 \to \mathcal{U}$ so that $A_1 + B_1F_1$ is stable and the output z in (26) is regulated.

We shall call such a pair (F_1,F_2) a pure gain synthesis. Problem 0 is easily solved as follows.

Proposition 1 (Solution of Problem 0)

A pure gain synthesis exists iff

$$\begin{bmatrix} A_3 \\ D_2 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} \underline{A}_1 & \underline{B}_1 \\ \underline{D}_1 & 0 \end{bmatrix} . \tag{27}$$

Proof

(Necessity) Let (F_1,F_2) be a pure gain synthesis. Applying Lemma 1 with

$$A_L = A_1 + B_1F_1,\ B_L = A_3 + B_1F_2,\ D_L = D_1 \tag{28}$$

we find that

$$\begin{bmatrix} A_3+B_1F_2 \\ D_2 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} \underline{A}_1+\underline{B}_1\underline{F}_1 \\ \underline{D}_1 \end{bmatrix} \tag{29}$$

and hence

$$\begin{bmatrix} A_3+B_1F_2 \\ D_2 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} \underline{A}_1 & \underline{B}_1 \\ \underline{D}_1 & 0 \end{bmatrix} ,$$

which clearly implies (27).

(Sufficiency) We assume that (27) holds and shall construct suitable F_1 and F_2. First, select F_1 so that $A_1+B_1F_1$ is stable. From (27) then

$$\begin{bmatrix} A_3 \\ D_2 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} \underline{A}_1+\underline{B}_1\underline{F}_1 & \underline{B}_1 \\ \underline{D}_1 & 0 \end{bmatrix} ,$$

and hence F_2 exists such that (29) holds. Using (28) and (29) we conclude from Lemma 1 that (F_1,F_2) provides output regulation. ∎

We now proceed to solve Problem 1. However we shall make an additional assumption, namely

$$(C,A) \text{ is detectable.} \tag{30}$$

This can be justified in the following manner. Let

$$\mathcal{N} = \bigcap_{i \geq 0} \operatorname{Ker}(CA^i)$$

be the unobservable subspace of (C,A) and $\mathcal{X}^+(A)$ the unstable subspace of A. Since (C_1,A_1) is detectable, the undetectable subspace $\mathcal{N} \cap \mathcal{X}^+(A)$ of the pair (C,A) is independent of $\mathcal{X}_1$:

$$\mathcal{N} \cap \mathcal{X}^+(A) \cap \mathcal{X}_1 = 0.$$

Hence we may decompose $\mathcal{X}$ as

$$\mathcal{X} = \mathcal{X}_1 \oplus \bar{\mathcal{X}}_2 \oplus \tilde{\mathcal{X}}_2$$

where $\tilde{\mathcal{X}}_2 = \mathcal{N} \cap \mathcal{X}^+(A)$ and $\bar{\mathcal{X}}_2$ is any complement. Corresponding to this decomposition of $\mathcal{X}$, A, B, C, and D have representations of the form

$$\begin{bmatrix} A_1 & \bar{A}_3 & 0 \\ 0 & \bar{A}_2 & 0 \\ 0 & R & \tilde{A}_2 \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}$$

$$[C_1 \quad \bar{C}_2 \quad 0] \quad [D_1 \quad \bar{D}_2 \quad \tilde{D}_2]$$

respectively. Here the pair

$$\left([C_1\bar{C}_2], \begin{bmatrix} A_1 & \bar{A}_3 \\ 0 & \bar{A}_2 \end{bmatrix}\right)$$

is detectable and

$$A_2 \cong \begin{bmatrix} \bar{A}_2 & 0 \\ R & \tilde{A}_2 \end{bmatrix}.$$

These representations correspond to the system

$$\dot{x}_1 = A_1x_1 + \bar{A}_3\bar{x}_2 + B_1u \tag{31a}$$

$$\dot{\bar{x}}_2 = \bar{A}_2\bar{x}_2 \tag{31b}$$

$$\dot{\tilde{x}}_2 = \tilde{A}_2\tilde{x}_2 + R\bar{x}_2 \tag{31c}$$

$$y = C_1x_1 + \bar{C}_2\bar{x}_2 \tag{31d}$$

$$z = D_1x_1 + \bar{D}_2\bar{x}_2 + \tilde{D}_2\tilde{x}_2. \tag{31e}$$

It is readily apparent from (31) that a necessary condition for output regulation is $\tilde{D}_2 = 0$; that is

$$\mathcal{N} \cap \mathcal{X}^+(A) \subset \text{Ker } D. \tag{32}$$

Conversely, if (32) is assumed then in (31) $\tilde{x}_2$ is a superfluous exogenous signal: it is decoupled from the plant, the measurements y, and the output z.

To summarize, (32) is necessary for the existence of a synthesis; so we assume (32). The undetectability of (C,A) corresponds to a redundant description of the exogenous signal; so we assume (30). Since (30) trivially implies (32), we need only assume (30).

<u>Theorem 1</u> (Solution of Problem 1)

<u>Assume (30). A synthesis exists iff</u>

$$\begin{bmatrix} A_3 \\ D_2 \end{bmatrix} \in \mathrm{Im} \begin{bmatrix} \underline{A}_1 & \underline{B}_1 \\ \underline{D}_1 & 0 \end{bmatrix}. \tag{27bis}$$

We observe that a synthesis exists iff a pure gain synthesis exists. For the system at hand, (27) apparently corresponds to the 'steady-state invertibility condition' of [3] and to the 'decomposability condition' of [12]. The proof of Theorem 1 is in three parts: first we prove necessity of (27), then present a synthesis algorithm, and finally show that the algorithm does indeed yield a synthesis.

<u>Proof of Theorem 1</u> (Necessity)

If $(\mathcal{X}_c, A_c, B_c, F_c, G_c)$ is a synthesis then by the Corollary to Lemma 1 (23) holds. Hence in particular

$$\begin{bmatrix} A_3+B_1G_cC_2 \\ D_2 \end{bmatrix} \in \mathrm{Im} \begin{bmatrix} \underline{A}_1+\underline{B}_1\underline{G}_c\underline{C}_1 & \underline{B}_1\underline{F}_c \\ \underline{D}_1 & 0 \end{bmatrix}$$

$$\subset \mathrm{Im} \begin{bmatrix} \underline{A}_1 & \underline{B}_1 \\ \underline{D}_1 & 0 \end{bmatrix} .$$

This implies (27). ∎

In view of assumption (30) an obvious synthesis procedure is the following: Use an observer to generate an estimate $x_c = \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix}$ of the state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of the system

$$\dot{x} = Ax + Bu, \quad y = Cx.$$

Then apply the control $u = F_1x_{c1} + F_2x_{c2}$ where (F_1, F_2) is a pure gain synthesis. This is accomplished by the

Synthesis Algorithm (SA)

Step 1. Let $\mathcal{X}_c = \mathcal{X}$ and select $B_c : \mathcal{Y} \to \mathcal{X}_c$ so that $A-B_cC$ is stable.

Step 2. Select $F_1 : \mathcal{X}_1 \to \mathcal{U}$ so that $A_1+B_1F_1$ is stable.

Step 3. Select $F_2 : \mathcal{X}_2 \to \mathcal{U}$ so that

$$\begin{bmatrix} A_3+B_1F_2 \\ D_2 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} \underline{A}_1+\underline{B}_1\underline{F}_1 \\ \underline{D}_1 \end{bmatrix} . \tag{29bis}$$

Step 4. Set $F_c = [F_1,F_2]$, $A_c = A-B_cC+BF_c$, $G_c = 0$.

Proof of Theorem 1 (Sufficiency)

Obviously Steps 1 and 2 of SA are possible, and if (27) holds we can choose F_2 to satisfy (29) just as we did in the proof of Proposition 1. So it remains to show that $(\mathcal{X}_c,A_c,B_c,F_c,0)$ is a synthesis.

Writing

$$B_c = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} : \mathcal{Y} \to \mathcal{X}_1 \oplus \mathcal{X}_2$$

we have

$$A_c = \begin{bmatrix} A_1-B_{c1}C_1+B_1F_1 & A_3-B_{c1}C_2+B_1F_2 \\ -B_{c2}C_1 & A_2-B_{c2}C_2 \end{bmatrix} . \tag{33}$$

Hence

$$A_L = \begin{bmatrix} A_1 & B_1F_c \\ B_cC_1 & A_c \end{bmatrix}$$

$$\cong \begin{bmatrix} A_1+B_1F_1 & B_1F_1 & B_1F_2 \\ 0 & A_1-B_{c1}C_1 & A_3-B_{c1}C_2 \\ 0 & -B_{c2}C_1 & A_2-B_{c2}C_2 \end{bmatrix}$$

$$= \begin{bmatrix} A_1+B_1F_1 & B_1F_c \\ 0 & A-B_cC \end{bmatrix} \quad ;$$

thus A_L is stable.

To show that output regulation holds let

$$X_1 = (\underline{A}_1+\underline{B}_1\underline{F}_1)^{-1}\ (A_3+B_1F_2) \in \underline{\mathcal{X}}_1$$

and

$$X_c = \begin{bmatrix} X_1 \\ -I \end{bmatrix} \in \underline{\mathcal{X}}_c \quad . \tag{34}$$

It is easily checked using (29) and (33) that

$$A_3 = \underline{A}_1X_1 + \underline{B}_1\underline{F}_cX_c \tag{35a}$$

$$B_cC_2 = \underline{B}_c\underline{C}_1X_1 + \underline{A}_cX_c \tag{35b}$$

$$D_2 = \underline{D}_1X_1 \ . \tag{35c}$$

Thus (23) holds. Output regulation now follows from the Corollary to Lemma 1. ■

A synthesis as computed by SA employs a full order dynamic observer of the state x. Such a synthesis may be inefficient in the sense of employing more integrators than is necessary. A reduced order synthesis may be obtained by using either a minimal order observer of the state x or a minimal order observer of Fx where $F = [F_1F_2]$ is a pure gain synthesis.

A synthesis procedure of the latter type (see [16] and [17]) amounts to choosing $\mathcal{X}_c$ of minimal dimension such that there exist maps

$$H : \mathcal{X}_c \to \mathcal{X}_c, \; K : \mathcal{Y} \to \mathcal{X}_c, \; T : \mathcal{X} \to \mathcal{X}_c$$

$$F_c : \mathcal{X}_c \to \mathcal{U}, \; G_c : \mathcal{Y} \to \mathcal{U}$$

with the properties

H is stable

$TA - HT = KC$

$F_cT + G_cC = F.$

It is routine to verify that a synthesis is then $(\mathcal{X}_c, A_c, B_c, F_c, G_c)$ where

$$A_c = H + TBF_c, \; B_c = K + TBG_c.$$

4. The Structure of a Feedback Synthesis

We shall say that a synthesis is of feedback type if the compensator processes the regulated output z; that is, if $y = z$. Our object now is to point out a basic feature of a feedback synthesis as obtained by SA.

Proposition 2

Assume (27), (30), and $y = z$, and consider a synthesis $(\mathcal{X}_c, A_c, B_c, F_c, G_c)$ obtained by SA. There is a monomorphism* $V : \mathcal{X}_2 \to \mathcal{X}_c$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X}_c & \xrightarrow{A_c} & \mathcal{X}_c \\ \uparrow V & & \uparrow V \\ \mathcal{X}_2 & \xrightarrow{A_2} & \mathcal{X}_2 \end{array} \qquad (36)$$

*A monomorphism is an injective morphism, i.e. a one-to-one linear transformation.

The interpretation of (36) is that A_c incorporates a copy of A_2; precisely,

$$A_c|\mathrm{Im}V \cong A_2.$$

The use of a copy of A_2 in A_c is explicit in the controllers of Johnson [7] and Davison [3].

Proof of Proposition 2

Using the notation introduced in the proof of Theorem 1 (Sufficiency), if $C_1 = D_1$ and $C_2 = D_2$ we find from (35b) and (35c) that $\underline{A}_c X_c = 0$. Since X_c is injective (see (34)) it suffices to take $V = X_c$. ∎

The above controller feature is not a result of using SA. Indeed, every feedback synthesis has this feature.

Proposition 3

Assume (27), (30), and $y = z$. For any synthesis $(\mathcal{X}_c, A_c, B_c, F_c, G_c)$ there is a monomorphism $V : \mathcal{X}_2 \to \mathcal{X}_c$ such that (36) commutes.

Proof

Let $(\mathcal{X}_c, A_c, B_c, F_c, G_c)$ be any synthesis. From the Corollary to Lemma 1 we know that (24) holds; that is, there exist $X_1 \in \underline{\mathcal{X}}_1$ and $V \in \underline{\mathcal{X}}_c$ such that

$$A_3 = A_1X_1 - X_1A_2 + B_1F_cV \qquad (37a)$$

$$0 = A_cV - VA_2 \qquad (37b)$$

$$D_2 = D_1X_1 \ . \qquad (37c)$$

It remains to show that V is injective.

For a proof by contradiction suppose that Ker V $\neq 0$. From (37b)

$$\text{Ker}(VA^i) \supset \text{Ker } V \quad (i \geq 0);$$

hence (V, A_2) is not observable. So there exist $\lambda \in \sigma(A_2)$ and $x_2 \in \mathcal{X}_2$, $x_2 \neq 0$, such that

$$Vx_2 = 0, \; A_2x_2 = \lambda x_2. \tag{38}$$

Set $x_1 = -X_1x_2 \in \mathcal{X}_1$. Then from (37a) and (38)

$$(A_1-\lambda)x_1 + A_3x_2 = (A_1-\lambda)x_1 + (A_1X_1-X_1A_2)x_2 = 0;$$

and from (37c)

$$D_1x_1 + D_2x_2 = 0.$$

Consequently

$$0 \neq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{Ker } D \cap \text{Ker}(A-\lambda),$$

which contradicts (30). ∎

Proposition 3 is not true if the assumption $y = z$ is dropped, as the following example shows.

Example

Consider a first order stable plant whose output is to follow a step reference signal:

$$\dot{x}_1 = -x_1+u$$

$$\dot{x}_2 = 0$$

$$z = -x_1+x_2.$$

Suppose the reference signal is available for measurement:

$$y = x_2.$$

It is easily checked that (27) and (30) hold. A synthesis is obtained by the feedforward control u = y: a dynamic compensator is not necessary. The closed loop transfer function is s/(s+1). Thus the feedforward connection has, without dynamics, provided the necessary closed loop zero at s = 0 to cancel the reference signal pole at s = 0. Synthesis by feedforward is considered more generally by Davison [18].

5. Solution of Problem 2

Problem 2 is solved by Theorems 2a and 2b.

Theorem 2a

A synthesis which is structurally stable at A_3 exists only if

$$\mathcal{X}_1 = \underline{A}_1 \operatorname{Ker} \underline{D}_1 + \operatorname{Im}\underline{B}_1. \tag{39}$$

It is not difficult to show that (39) is equivalent to the condition

$$\mathcal{X}_1 = (A_1-\lambda) \operatorname{Ker} D_1 + \operatorname{Im}B_1 \qquad (\lambda \in \sigma(A_2))$$

which in turn is equivalent to

$$\operatorname{Im} \begin{bmatrix} A_1-\lambda & B_1 \\ D_1 & 0 \end{bmatrix} = \mathcal{X}_1 \oplus \mathcal{Z} \quad (\lambda \in \sigma(A_2)).$$

This latter condition is the one which arises in the work of Davison [4] and Wonham [13].

Proof of Theorem 2a

If $(\mathcal{X}_c, A_c, B_c, F_c, G_c)$ is a synthesis which is structurally stable at A_3 then, by the Corollary to Lemma 1, (23) is a property which is stable at A_3. From (23) we have

$$\begin{bmatrix} A_3+B_1G_cC_2 \\ D_2 \end{bmatrix} \in \text{Im} \begin{bmatrix} \underline{A}_1+\underline{B}_1\underline{G}_c\underline{C}_1 & \underline{B}_1\underline{F}_c \\ \underline{D}_1 & 0 \end{bmatrix}$$

$$\subset \text{Im} \begin{bmatrix} \underline{A}_1 & \underline{B}_1 \\ \underline{D}_1 & 0 \end{bmatrix}. \tag{40}$$

Letting $D_1^{\dagger} : \mathcal{Z} \to \mathcal{X}_1$ be any right inverse of D_1 we find from (40) that

$$\begin{bmatrix} A_3+B_1G_cC_2-\underline{A}_1(D_1^{\dagger}D_2) \\ 0 \end{bmatrix} \in \text{Im} \begin{bmatrix} \underline{A}_1 & \underline{B}_1 \\ \underline{D}_1 & 0 \end{bmatrix};$$

equivalently

$$A_3+B_1G_cC_2-\underline{A}_1(D_1^{\dagger}D_2) \in \underline{A}_1\text{Ker } \underline{D}_1+\text{Im}\underline{B}_1. \tag{41}$$

Now clearly (41) is a property which is stable at $A_3 \in \underline{\mathcal{X}}_1$ only if (39) holds. ∎

Our object now is to prove a converse of Theorem 2a; that is, to show that (39) is a sufficient condition. For this, however, we need an additional assumption. Recall from [15] the definition that z <u>is readable from</u> y if there is a map $Q : \mathcal{Y} \to \mathcal{Z}$ such that $z = Qy$, which is to say $D_1 = QC_1$ and $D_2 = QC_2$. It was shown in [15] (Theorem 1) that a necessary condition for structural stability (at a suitable data point) is that z be readable from y. Hence we here assume this.

If such Q exists we can imbed $\mathcal{Z}$ in $\mathcal{Y}$: write

$$\mathcal{Y} = \mathcal{W} \oplus \mathcal{Z}$$

for a suitable linear space $\mathcal{W}$. Then

$$C_1 = \begin{bmatrix} E_1 \\ D_1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} E_2 \\ D_2 \end{bmatrix}$$

for suitable maps $E_i : \mathcal{X}_i \to \mathcal{W}$ (i=1,2), and

$$y = \begin{bmatrix} w \\ z \end{bmatrix}$$

where $w = E_1x_1 + E_2x_2 \in \mathcal{W}$. Here Q is the natural projection $\mathcal{W} \oplus \mathcal{Z} \to \mathcal{Z}$. Now for a compensator $(\mathcal{X}_c, A_c, B_c, F_c, G_c)$ define

$$B_{cw} = B_c|\mathcal{W}, \quad B_{cz} = B_c|\mathcal{Z}$$

$$G_{cw} = G_c|\mathcal{W}, \quad G_{cz} = G_c|\mathcal{Z}.$$

Then the overall system equations are

$$\dot{x}_1 = A_1x_1 + A_3x_2 + B_1u$$

$$\dot{x}_2 = A_2x_2$$

$$w = E_1x_1 + E_2x_2$$

$$z = D_1x_1 + D_2x_2$$

$$\dot{x}_c = A_cx_c + B_{cw}w + B_{cz}z$$

$$u = F_cx_c + G_{cw}w + G_{cz}z.$$

The compensator is now formally a 7-tuple

$(\mathcal{X}_c, A_c, B_{cw}, B_{cz}, F_c, G_{cw}, G_{cz})$.

Theorem 2b

Assume that z is readable from y and that (39) holds. Then there is a synthesis in which $B_{cw} = 0$, $G_{cw} = 0$, and $G_{cz} = 0$ and which is structurally stable at $(A_1, A_3, B_1, B_{cz}, F_c)$.

Notice that the data point $(A_1, A_3, B_1, B_{cz}, F_c)$ includes the plant data (A_1, A_3, B_1) together with the nonzero compensator data excluding A_c: small arbitrary perturbations in A_c cannot be permitted if output regulation is to be maintained. Notice also that the compensator is of feedback type processing only the output z ($B_{cw}=0$, $G_{cw}=0$).

The format of the proof of Theorem 2b is the same as the proof of Theorem 1 (Sufficiency): first we give a synthesis procedure and then show that the resulting compensator has the required properties. For the synthesis procedure we need some notation.

Let

$$\mathcal{X}_2 = \bigoplus_{i=1}^{k} \mathcal{X}_{2i}$$

be a rational canonical decomposition (rcd) of $\mathcal{X}_2$ relative to A_2. Thus $\mathcal{X}_{2i}$ is A_2-invariant ($i \in \underline{k}$), $A_{2i} \triangleq A_2 | \mathcal{X}_{2i}$ is cyclic ($i \in \underline{k}$), the minimal polynomial (mp) of A_{2i} divides that of $A_{2,i+1}$ ($i \in \underline{k-1}$), and the mp of A_{2k} is the same as that of A_2. Let $q = d(\mathcal{Z})$ and define

$$\mathcal{X}_{2e} = \mathcal{X}_{2k} \oplus \dots \oplus \mathcal{X}_{2k} \quad \text{(q-fold direct sum)}$$

and

$$A_{2e} : \mathcal{X}_{2e} \to \mathcal{X}_{2e}, \quad A_{2e} | \mathcal{X}_{2k} = A_{2k}.$$

Thus A_{2e} is the q-fold direct sum of the largest cyclic component of A_2.

Now if a is any one of the subscripts 1,...,k,e and $\mathcal{R}$ is any linear space, define

$$\underline{\mathcal{R}}_a = \text{Hom}(\mathcal{X}_{2a},\mathcal{R}).$$

Similarly if A : $\mathcal{X} \to \mathcal{X}$ define $\underline{A}_a : \underline{\mathcal{X}}_a \to \underline{\mathcal{X}}_a$ by

$$\underline{A}_a X_a = AX_a - X_a A_{2a} \quad (X_a \in \underline{\mathcal{X}}_a),$$

and if C : $\mathcal{X} \to \mathcal{Y}$ define $\underline{C}_a : \underline{\mathcal{X}}_a \to \underline{\mathcal{Y}}_a$ by

$$\underline{C}_a X_a = CX_a \qquad (X_a \in \underline{\mathcal{X}}_a).$$

Structurally Stable Synthesis Algorithm (SSSA)

Step 1. Define $\mathcal{X}_e = \mathcal{X}_1 \oplus \mathcal{X}_{2e}$ and select $A_{3e} : \mathcal{X}_{2e} \to \mathcal{X}_1$ so that (D_e,A_e) is detectable. Here

$$A_e = \begin{bmatrix} A_1 & A_{3e} \\ 0 & A_{2e} \end{bmatrix} : \mathcal{X}_e \to \mathcal{X}_e$$

$$D_e = [D_1 \ 0] : \mathcal{X}_e \to \mathcal{Z}.$$

The next four steps consist in obtaining a synthesis via SA for the system

$$\dot{x}_1 = A_1 x_1 + A_{3e} x_{2e} + B_1 u \tag{42a}$$

$$\dot{x}_{2e} = A_{2e} x_{2e} \tag{42b}$$

$$y = z = D_1 x_1. \tag{42c}$$

Step 2. Let $\mathcal{X}_c = \mathcal{X}_e$ and select $B_{cz} : \mathcal{Z} \to \mathcal{X}_c$ so that $A_e - B_{cz}D_e$ is stable.

STep 3. Select $F_1 : \mathcal{X}_1 \to \mathcal{U}$ so that $A_1+B_1F_1$ is stable.

Step 4. Select $F_{2e} : \mathcal{X}_{2e} \to \mathcal{U}$ so that

$$\begin{bmatrix} A_{3e}+B_1F_{2e} \\ 0 \end{bmatrix} \in \operatorname{Im} \begin{bmatrix} \underline{A}_{1e}+\underline{B}_{1e}\underline{F}_{1e} \\ \underline{D}_{1e} \end{bmatrix} \quad . \tag{43}$$

Step 5. Set $F_c = [F_1F_{2e}]$, $A_c = A_e-B_cD_e+B_eF_c$, $B_{cw} = 0$, $G_{cw} = 0$, $G_{cz} = 0$. Here

$$B_e = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} : \mathcal{U} \to \mathcal{X}_e .$$

Before proceeding we require some technical facts. We first recall the notion of a generic property (see [13]). For any field $\mathbb{K}$ a property Π of points in $\mathbb{K}^n$ is <u>generic</u> if the set of points where Π fails lies in a proper algebraic variety in $\mathbb{K}^n$. Suppose Π is generic on $\mathbb{C}^n$; that is, Π fails only on a proper variety in $\mathbb{C}^n$. Then Π is generic when restricted to $\mathbb{R}^n$. To see this let $\mathcal{P} = (p_1,\ldots,p_n)$ be a representative point in $\mathbb{C}^n$. If Π is generic on $\mathbb{C}^n$ there is a nonzero polynomial

$$\phi(s_1,\ldots,s_n) \in \mathbb{C}[s_1,\ldots,s_n]$$

such that Π fails only at points $\mathcal{P} \in \mathbb{C}^n$ where $\phi(p_1,\ldots,p_n) = 0$. Factor ϕ as

$$\phi(s_1,\ldots,s_n) = \phi_1(s_1,\ldots,s_n) + i\ \phi_2(s_1,\ldots,s_n)$$

where $\phi_j \in \mathbb{R}[s_1,\ldots,s_n]$ $(j=1,2)$. Now ϕ_1 and ϕ_2 are not both identically zero; hence

$$\psi = \phi_1^2 + \phi_2^2 \in \mathbb{R}[s_1,\ldots,s_n]$$

is not identically zero. Now Π fails at $\mathcal{P} \in \mathbb{R}^n$ only if $\psi(p_1,\ldots,p_n) = 0$. Hence Π is generic on $\mathbb{R}^n$.

Next we require

<u>Lemma 2</u>

<u>Let</u> $A : \mathcal{X} \to \mathcal{X}$, $\bar{A} : \bar{\mathcal{X}} \to \bar{\mathcal{X}}$ <u>be maps with invariant factors</u> $\alpha_i(s)$ $(i \in \mathbf{m})$, $\bar{\alpha}_i(s)$ $(i \in \bar{\mathbf{m}})$ <u>respectively</u>. <u>Define</u> $L : \mathrm{Hom}(\bar{\mathcal{X}}, \mathcal{X}) \to \mathrm{Hom}(\bar{\mathcal{X}}, \mathcal{X})$ <u>by</u>

$$LX = AX - X\bar{A}.$$

<u>Then</u>

(a) $d(\mathrm{Ker}\ L) = \sum_{i,j} \deg \gcd(\alpha_i, \bar{\alpha}_j)$.

(b) <u>There exists a monomorphism</u> $V \in \mathrm{Ker}\ L$ <u>iff</u>

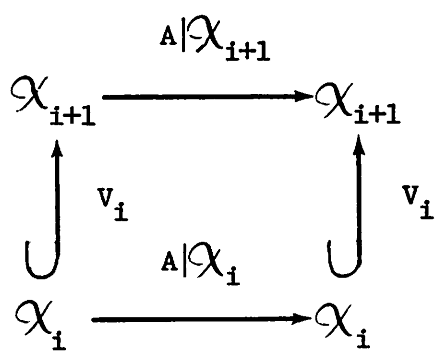
$$\bar{\alpha}_i \mid \alpha_{m-\bar{m}+i} \quad (i \in \bar{\mathbf{m}}).$$

Part (a) is immediate from Theorem 1, p. 219 of [19]. Part (b) can be derived from this Theorem, however it has been proved by Corfmat and Morse ([20], Lemma 1).

As a simple application of Lemma 2b we find that if

$$\mathcal{X} = \bigoplus_{i=1}^{m} \mathcal{X}_i$$

is a rcd of $\mathcal{X}$ relative to A, then for each $i \in \underline{m-1}$ there is a monomorphism $V_i : \mathcal{X}_i \to \mathcal{X}_{i+1}$ such that the following diagram commutes:



Proof of Theorem 2b

We first show that when (39) holds SSSA can be carried out, and second that the resulting compensator provides the required structural stability. As the proof is fairly long it is divided into four steps.

(i) We shall prove that Step 1 of SSSA is possible. Since (D_1, A_1) is detectable

$$d[(A_1-\lambda)\text{Ker } D_1] = d(\text{Ker } D_1) = d(\mathcal{X}_1)-q \quad (\lambda \in \sigma(A_{2e})). \tag{44}$$

Furthermore (D_e, A_e) is detectable iff

$$(A_{3e}, A_{2e}) \text{ is observable} \tag{45}$$

and for each $\lambda \in \sigma(A_{2e})$

$$A_{3e}\text{Ker}(A_{2e}-\lambda) \cap (A_1-\lambda)\text{Ker } D_1 = 0. \tag{46}$$

Now by construction A_{2e} has q cyclic components in a rcd. And since D_1 is surjective, $q \leq d(\mathcal{X}_1)$. These two facts show that (45) is a generic property of A_{3e}. Similarly, (44) together with the fact

$$d[\text{Ker}(A_{2e}-\lambda)] = q \quad (\lambda \in \sigma(A_{2e}))$$

shows that (46) is a generic property of complex A_{3e} and hence of real A_{3e} for each $\lambda \in \sigma(A_{2e})$. Since the conjunction of a finite number of generic properties is generic, we find that

$$(D_e, A_e) \text{ detectable}$$

is a generic property of A_{3e}. Hence Step 1 of SSSA is accomplished by 'almost any' $A_{3e} : \mathcal{X}_{2e} \to \mathcal{X}_1$.

(ii) Obviously Steps 2 and 3 are now possible, so we show that Step 4 is.

From (39) we have

$$\underline{\mathcal{X}}_{1k} = \underline{A}_{1k} \text{ Ker } \underline{D}_{1k} + \text{Im}\underline{B}_{1k}$$

and hence

$$\underline{\mathcal{X}}_{1e} = \underline{A}_{1e} \text{ Ker } \underline{D}_{1e} + \text{Im}\underline{B}_{1e} \;;$$

equivalently

$$\underline{\mathcal{X}}_{1e} = (\underline{A}_{1e}+\underline{B}_{1e}\underline{F}_{1e}) \text{ Ker } \underline{D}_{1e} + \text{Im}\underline{B}_{1e}.$$

Thus there exists $F_{2e} \in \underline{\mathcal{U}}_e$ such that

$$A_{3e} + B_1F_{2e} \in (\underline{A}_{1e}+\underline{B}_{1e}\underline{F}_{1e})\text{Ker } \underline{D}_{1e},$$

which is equivalent to (43).

We have now shown that SSSA can be carried out. Furthermore we know from the proof of Theorem 1 that

$$A_L = \begin{bmatrix} A_1 & B_1F_c \\ B_{cz}D_1 & A_c \end{bmatrix} \tag{47}$$

is stable. So it remains to show that output regulation is a property which is stable at (A_1,A_3,B_1,B_{cz},F_c).

(iii) We claim that

$$\text{Ker } \underline{B}_{cze} = 0 \tag{48}$$

and

$$\text{Im}\underline{A}_{ce} \cap \text{Im}\underline{B}_{cze} = 0. \tag{49}$$

To establish this recall that $(\mathcal{X}_c, A_c, 0, B_{cz}, F_c, 0, 0)$ is a synthesis for system (42) and that (D_e, A_e) is detectable. Hence Proposition 2 implies the existence of a monomorphism $V_e : \mathcal{X}_{2e} \to \mathcal{X}_c$ such that

$$A_c V_e = V_e A_{2e}. \tag{50}$$

Since A_{2e} has exactly q invariant factors each of which is the mp of A_2, we conclude from (50) and Lemma 2b that the mp of A_2 divides at least q invariant factors of A_c,

Since A_L is stable

$$\underline{\mathcal{X}}_{Le} = \text{Im}\underline{A}_{Le}\ . \tag{51}$$

From (47) this implies that

$$\underline{\mathcal{X}}_{ce} = \text{Im}\underline{A}_{ce} + \text{Im}\underline{B}_{cze}$$

which in turn implies

$$\underline{\mathcal{X}}_{ck} = \text{Im}\underline{A}_{ck} + \text{Im}\underline{B}_{czk}. \tag{52}$$

Now let $\{\alpha_{ci}(s)\}$ be the invariant factors of A_c and $\alpha_2(s)$ the mp of A_2. Applying Lemma 2a we have

$$d(\text{Ker } \underline{A}_{ck}) = \sum_i \deg \gcd(\alpha_{ci}, \alpha_2).$$

Then, since α_2 divides at least q α_{ci}'s,

$$d(\text{Ker } \underline{A}_{ck}) \geq q \cdot \deg \alpha_2 = q \cdot d(\mathcal{X}_{ck}). \tag{53}$$

However

$$d(\mathrm{Im}\underline{B}_{czk}) \leq d(\underline{\mathcal{Z}}_k)$$

$$= d(\mathcal{Z})\cdot d(\mathcal{X}_{2k})$$

$$= q\cdot d(\mathcal{X}_{2k}). \tag{54}$$

So from (52), (53), and (54)

$$\mathrm{Im}\ \underline{B}_{czk} \cong \underline{\mathcal{Z}}_k \tag{55}$$

and

$$\mathrm{Im}\ \underline{A}_{ck} \cap \mathrm{Im}\ \underline{B}_{czk} = 0. \tag{56}$$

Now (55) implies that Ker $\underline{B}_{czk} = 0$, which implies (48); and (49) follows from (56). This proves the claim.

(iv) Returning to (51) we know in view of (47) that for any $R_{1e} \in \underline{\mathcal{X}}_{1e}$ there exist $X_{1e} \in \underline{\mathcal{X}}_{1e}$ and $X_{ce} \in \underline{\mathcal{X}}_{ce}$ such that

$$R_{1e} = \underline{A}_{1e}X_{1e} + \underline{B}_{1e}\underline{F}_{ce}X_{ce}$$

$$0 = \underline{B}_{cze}\underline{D}_{1e}X_{1e} + \underline{A}_{ce}X_{ce}.$$

But from (48) and (49) this implies

$$\underline{\mathcal{X}}_{1e} = \underline{A}_{1e}\ \mathrm{Ker}\ \underline{D}_{1e} + \underline{B}_{1e}\underline{F}_{ce}\ \mathrm{Ker}\ \underline{A}_{ce}$$

from which there follows

$$\underline{\mathcal{X}}_{1k} = \underline{A}_{1k}\ \mathrm{Ker}\ \underline{D}_{1k} + \underline{B}_{1k}\underline{F}_{ck}\ \mathrm{Ker}\ \underline{A}_{ck}. \tag{57}$$

Now for each $i \in \underline{k}$ A_{2i} is imbedded in A_{2k} in the sense that $A_{2k}V_i \doteq V_iA_{2i}$

for some monomorphism $V_i : \mathcal{X}_{2i} \to \mathcal{X}_{2k}$. A brief computation using this fact and (57) yields

$$\underline{\mathcal{X}}_{1i} = \underline{A}_{1i} \text{ Ker } \underline{D}_{1i} + \underline{B}_{1i}\underline{F}_{ci} \text{ Ker } \underline{A}_{ci} \qquad (i \in \underline{k})$$

and hence

$$\underline{\mathcal{X}}_1 = \underline{A}_1 \text{ Ker } \underline{D}_1 + \underline{B}_1 F_c \text{ Ker } \underline{A}_c . \tag{58}$$

We conclude the proof by showing that (24) is stable at $(A_1, A_3, B_1, B_{cz}, F_c)$. Clearly (58) is stable at this data point, so it suffices to show that (24) follows from (58).

If $D_1^{\dagger}$ is any right inverse of D_1 then (24) is equivalent to

$$\begin{bmatrix} A_3 - \underline{A}_1(D_1^{\dagger} D_2) \\ 0 \\ 0 \end{bmatrix} \in \text{Im} \begin{bmatrix} \underline{A}_1 & \underline{B}_1 \underline{F}_c \\ 0 & \underline{A}_c \\ \underline{D}_1 & 0 \end{bmatrix} ,$$

and this is equivalent to

$$A_3 - \underline{A}_1(D_1^{\dagger} D_2) \in \underline{A}_1 \text{ Ker } \underline{D}_1 + \underline{B}_1 \underline{F}_c \text{ Ker } \underline{A}_c .$$

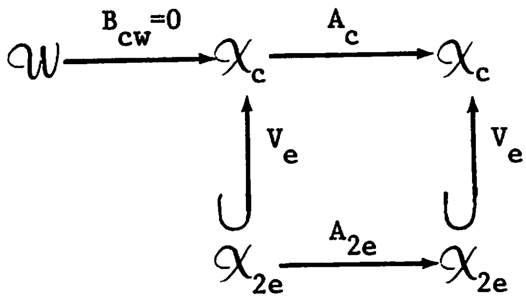
But this follows from (58). ∎

6. The Structure of a Structurally Stable Synthesis

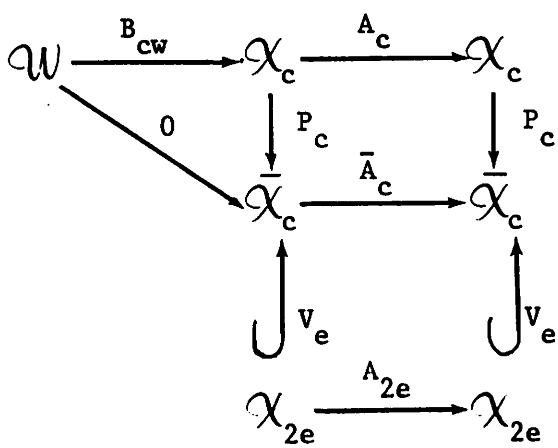
We observed in Section 4 that a feedback synthesis incorporates in A_c a copy of A_2. For the structurally stable feedback synthesis obtained by SSSA a stronger statement is true: A_c incorporates a q-fold reduplication of the maximal cyclic component of A_2. More precisely, from (50) we have

Proposition 4

Assume that z is readable from y and that (39) holds, and consider a structurally stable synthesis computed by SSSA. There is a monomorphism $V_e : \mathcal{X}_{2e} \to \mathcal{X}_c$ such that the following diagram commutes:



To complete the parallel of this section with Section 4 we state without proof the following counterpart of Proposition 3: Assume z is readable from y and (39) holds. Let $(\mathcal{X}_c, A_c, B_{cw}, B_{cz}, F_c, G_{cw}, G_{cz})$ be a structurally stable synthesis. Then there is an A_c-invariant subspace $\mathcal{R}_c \subset \mathcal{X}_c$ and a monomorphism $V_e : \mathcal{X}_{2e} \to \bar{\mathcal{X}}_c = \mathcal{X}_c / \mathcal{R}_c$ such that the following diagram commutes.



(59)

Here P_c is the canonical projection and $\bar{A}_c$ the induced map in the factor space. We have stated this result informally, omitting the data point at which the synthesis is structurally stable. For a precise statement and proof the reader is referred to [15], Proposition 3 and Theorem 2.

7. Concluding Remarks

It is expected that the algebraic approach employed in this paper would prove useful in a regulator problem for linear decentralized systems.

The synthesis theory presented in this paper deals with systems in state-space form. Uncertainty about the system is then taken to be uncertainty about parameters in the matrices in the state-space description. There is an implicit assumption here that the state-space description is derived from physical laws rather than from a realization of an input-output impulse response. This is because the function (suitably defined) which maps an impulse response to its state-space realization is not continuous in the natural topologies, and hence 'slight uncertainty' about the impulse response need not correspond to 'slight uncertainty' about the state-space description. An important open problem therefore is a synthesis theory for systems modeled by input-output maps.

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The present algebraic approach arises naturally from Lemma 1. This lemma is due to W. M. Wonham. The criticism, expressed in the Concluding Remark, against posing the regulator problem in the state-space setting was raised by G. Zames. The research reported here was conducted while the author held a Postdoctorate Fellowship from the National Research Council of Canada.

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