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by

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ABSTRACT

In recent years a number of important results and unique features of the piecewise-linear analysis of nonlinear resistive networks has been derived. However, the applicability of the method relies on the fact that every nonlinear device is modeled by a piecewise-linear continuous function. In order to extend the applicability of piecewise-linear analysis to study more general nonlinear networks, three steps need to be carried out:

(i) the subdivision of the domain of the multi-dimensional nonlinear network function;

(ii) the interpolation of a piecewise-linear continuous function on the subdivided domain; and

(iii) the application of piecewise-linear analysis.

It turns out that the above three steps can be accomplished effectively by the use of simplices. Furthermore, with that, the difficulties encountered in the implementation of piecewise-linear analysis are greatly simplified.

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I. Introduction

In 1965, Katzenelson developed an efficient method for solving nonlinear resistive networks which contain uncoupled resistors represented by monotonically increasing, piecewise-linear, continuous functions [1]. During the past decade, the work has been greatly extended and generalized [2-8]. However there exists a fundamental limitation of the approach, that is, nonlinear resistors must be first approximated by continuous piecewise-linear functions.

The purpose of this paper is to develop a method which deals directly with the multi-dimensional nonlinear network function, thus extending the applicability of the piecewise-linear analysis to solve more general network problems. This method consists of two preliminary steps:

(i) the subdivision of the domain of the nonlinear network function; and

(ii) the interpolation of a piecewise-linear continuous function on the subdivided domain. Since the multi-dimensional piecewise-linearization is carried out with respect to the given nonlinear network function, the modeling process of nonlinear resistors is avoided.

In 1956, Stern first proposed the idea of using vertices of simplices for interpolation in the analysis of nonlinear resistive networks [9]. Iri extended Stern's work and furthermore, considered the problem of error estimation [10]. Ohtsuki and Yoshida based on the method above, employed multi-dimensional interpolation of transistor characteristics in applying Katzenelson's algorithm of piecewise-linear analysis [3]. Our present paper takes into account the latest advances in piecewise-linear analysis [4-8], and demonstrates the advantages of multi-dimensional interpolation by using simplices. These advantages are summarized as follows:

(i) The structure of simplices can be predetermined and necessary calculations can be easily made if the subdivision is carried out properly. A systematic procedure, called the simplicial subdivision, is presented in section II.

(ii) The interpolation of an affine function in a simplex is extremely simple. In addition, an affine function on a simplex can be obtained by making minor modifications of the affine functions on its adjoining simplices [11]. This property greatly improves the efficiency of the method.

(iii) The boundaries of a simplex are easily defined. Furthermore, there exists a one-to-one correspondence between boundaries and vertices of the simplex. Consequently, the most difficult part in the algorithm of the piecewise-linear analysis, namely, "boundary crossing," is easily done by replacing an "old" vertex by a new vertex. This procedure is called the replacement rule and is presented in section IV.

II. Simplicial Subdivision and Piecewise-Linear Interpolation

Let $\underset{\sim}{x}_0$, $\underset{\sim}{x}_1$,... and $\underset{\sim}{x}_n$ be (n+1) points in the n-dimensional space. A simplex $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$ is defined by

$$S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n) = \{\underset{\sim}{x} \mid \underset{\sim}{x} = \sum_{i=0}^{n} \mu_i \underset{\sim}{x}_i,\ 1 \geq \mu_i \geq 0, \qquad i = 1,2,\ldots,n \text{ and } \sum_{i=0}^{n} \mu_i = 1\} \tag{1}$$

In other words, $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$ is the convex combination of $\underset{\sim}{x}_0$, $\underset{\sim}{x}_1$,... and $\underset{\sim}{x}_n$ which are called the vertices of the simplex $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$ [12-19]. A simplex $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$ is said to be proper if and only if it cannot be contained in an n-dimensional hyperplane $\{\underset{\sim}{x} \mid \underset{\sim}{n}^T\underset{\sim}{x} = \text{constant}\}$. This has been proved to be equivalent to that the $(n+1) \times (n+1)$ matrix $\begin{Bmatrix} x_0 \cdots x_n \\ 1 \;\cdots\; 1 \end{Bmatrix}$ is nonsingular [12]. In this paper, this condition is assumed to hold. A simplex is considered "proper" unless explicitly stated. In a 2-dimensional space (n=2) the above condition simply asserts that x_0, x_1 and x_2 are not on a straight line, as shown in Fig. 1.

Corresponding to the (n+1) vertices, there are (n+1) boundaries. The boundary B_k corresponding to the vertex x_k is defined as

$$B_k = \{x \mid x \in S(x_0,\ldots,x_n) \text{ with } \mu_k = 0\}. \tag{2}$$

It is easy to see that B_k contains all the vertices except x_k. This one-to-one correspondence between vertices and boundaries is shown in Fig. 1 for the case n=2. The intersection of more than one boundary is called a corner. Thus a vertex is a corner which is the intersection of n boundaries.

In this paper, we assume that the solution of the nonlinear resistive network is bounded. The determination of such a bounded set, in which the approximate solution lies, has been considered by many authors [20-23]. The purpose of this section is twofold, namely: the derivation of a systematic method to subdivide the bounded set and the interpolation of a continuous piecewise-linear function on this set. Let the bounded set be contained in an n-dimensional rectangle $RL = \{x \mid a \leqq x \leqq b\}$, where $a < b$ to ensure that RL cannot be contained in an n-dimensional hyperplane.* The procedure of subdividing RL consists

*In this paper, we use the following notations:

$x \leqq y$ means $x_i \leqq y_i$, $i = k,2,\ldots,n$;
$x < y$ means $x_i < y_i$, $i = 1,2,\ldots,n$;
$x \leqq a$ means $x_i \leqq a$, $i = 1,2,\ldots,n$; and
$x \leq y$ means $x \leqq y$ and $x \neq y$.

of two steps, namely, the tessellation of RL into small rectangles and the subdivision of each small rectangle into simplices.

The first step is accomplished by the construction of a homeomorphism between RL and $C_p = \{z | 0 \leqq z \leqq p,\ p > 0$ and every component of p is an integer$\}$. The rectangle C_p contains $\prod_{i=1}^{n} p_i$ n-cubes. Let the transformation be defined by

$$z = T(x) = \begin{bmatrix} p_1/(b_1-a_1) & & 0 \\ & \ddots & \\ 0 & & p_n/(b_n-a_n) \end{bmatrix} (x-a) \qquad (3)$$

Accordingly, the rectangle RL is also divided into $\prod_{i=1}^{n} p_i$ rectangles. Each rectangle is mapped onto one and only one n-dimensional cube in C_p by $T(\cdot)$. This is illustrated in the following example.

Example 1:

Let $RL = \left\{x \middle| \begin{bmatrix} 5.5 \\ -1 \end{bmatrix} \leqq x \leqq \begin{bmatrix} 8.0 \\ 2 \end{bmatrix}\right\}$ and $C_p = \left\{z \middle| 0 \leqq z \leqq \begin{bmatrix} 4 \\ 3 \end{bmatrix}\right\}$. For any $x \in RL$, there is one and only one $z \in C_p$ defined by

$$z = \begin{bmatrix} 4/(8-5.5) & 0 \\ 0 & 3/(2-(-1)) \end{bmatrix} \left(x - \begin{bmatrix} 5.5 \\ -1 \end{bmatrix}\right)$$

The rectangle RL is divided into 12 small rectangles, as shown in Fig. 2.

Next, we wish to subdivide each small rectangle into simplices. In the z-space, the set of vertices of the cubes contained in C_p is defined by

$$V_p = \{I | 0 \leqq I \leqq p\ .\ \text{Every component of I is 0 or an integer,}\ p > 0\}. \qquad (4)$$

From (4), the set of vertices of the rectangles contained in RL in the x-space is $T^{-1}(V_p)$, i.e.,

$$V = \underset{\sim}{T}^{-1}(\underset{\sim}{V_p}) = \{\underset{\sim}{x} \mid \underset{\sim}{T}(\underset{\sim}{x}) \in \underset{\sim}{V_p}\} \tag{5}$$

The subdivision of cubes of $\underset{\sim}{C_p}$ into "non-overlapping" simplices is done by properly arranging the vertices of $\underset{\sim}{V_p}$ in a fixed order as shown by Kuhn [13,14].

<u>Lemma 1</u>:

Every $\underset{\sim}{z} \in \underset{\sim}{C_p}$ has a unique representation

$$\underset{\sim}{z} = \mu_0 \underset{\sim}{I}_0 + \dots + \mu_m \underset{\sim}{I}_m$$

Where

$$\mu_j > 0,\ \underset{\sim}{I}_j \in \underset{\sim}{V_p} \text{ for } j = 0,1,\dots,m\ (\leq n),\ \sum_{j=0}^{m} \mu_j = 1$$

and

$$\underset{\sim}{I}_0 \leq \underset{\sim}{I}_1 \leq \dots \leq \underset{\sim}{I}_m \leq \underset{\sim}{I}_0 + 1^*$$

Note that the point z is in the cube $C(\underset{\sim}{I}_0)$ in $\underset{\sim}{C_p}$, where

$$C(\underset{\sim}{I}_0) = \{\underset{\sim}{z} \mid \underset{\sim}{I}_0 \leq \underset{\sim}{z} \leq \underset{\sim}{I}_0 + 1\}.$$

In the case that m = n, z is an interior point of the simplex $S(\underset{\sim}{I}_0,\dots,\underset{\sim}{I}_n) \subset C(\underset{\sim}{I}_0)$. Otherwise, z lies on the boundary of a simplex. It should be emphasized that the simplex $S(\underset{\sim}{I}_0,\dots,\underset{\sim}{I}_n)$ is unique. The computation of μ_j's and $\underset{\sim}{I}_j$'s is illustrated by the following example.

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* $\underset{\sim}{I}_0 + 1$ denotes a vector which is formed by adding unity to each component of $\underset{\sim}{I}_0$

Example 2:

Let $n = 6$. Let $\underset{\sim}{x} = [1.3 \quad 0.6 \quad 2.9 \quad 0.4 \quad 1.5 \quad 0.8]^T$. We first decompose $\underset{\sim}{x}$ into two parts: $\underset{\sim}{x} = \underset{\sim}{I}_0 + [0.3 \quad 0.6 \quad 0.9 \quad 0.4 \quad 0.5 \quad 0.8]^T$ where $\underset{\sim}{I}_0 = [1 \quad 0 \quad 2 \quad 0 \quad 1 \quad 0]^T$. The vector $[0.3 \quad 0.6 \quad 0.9 \quad 0.4 \quad 0.5 \quad 0.8]^T$ can be represented by

$$0.3\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} + 0.1\begin{bmatrix}0\\1\\1\\1\\1\\1\end{bmatrix} + 0.1\begin{bmatrix}0\\1\\1\\0\\1\\1\end{bmatrix} + 0.1\begin{bmatrix}0\\1\\1\\0\\0\\1\end{bmatrix} + 0.2\begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix} + 0.1\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} + 0.1\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}$$

Therefore

$$x = \begin{bmatrix}1.3\\0.6\\2.9\\0.4\\1.5\\0.8\end{bmatrix} = 0.3\begin{bmatrix}2\\1\\3\\1\\2\\1\end{bmatrix} + 0.1\begin{bmatrix}1\\1\\3\\1\\2\\1\end{bmatrix} + 0.1\begin{bmatrix}1\\1\\3\\0\\2\\1\end{bmatrix} + 0.1\begin{bmatrix}1\\1\\3\\0\\1\\1\end{bmatrix} + 0.2\begin{bmatrix}1\\0\\3\\0\\1\\1\end{bmatrix} + 0.1\begin{bmatrix}1\\0\\3\\0\\1\\0\end{bmatrix} + 0.1\begin{bmatrix}1\\0\\2\\0\\1\\0\end{bmatrix}$$

$$= \mu_6\underset{\sim}{I}_6 + \mu_5\underset{\sim}{I}_5 + \mu_4\underset{\sim}{I}_4 + \mu_3\underset{\sim}{I}_3 + \mu_2\underset{\sim}{I}_2 + \mu_1\underset{\sim}{I}_1 + \mu_0\underset{\sim}{I}_0.$$

The conditions of lemma 1 are fulfilled, namely,

(i) $\mu_0 > 0,\ \mu_1 > 0, \ldots, \mu_6 > 0$;

(ii) $\sum_{j=0}^{6} \mu_j = 1$; and

(iii) $I_0 \le I_1 \le I_2 \le I_3 \le I_4 \le I_5 \le I_6 = I_0 + 1$.

The result in Lemma 1 provides a canonical decomposition of the rectangle C_p into simplices. The decomposition for n = 2 is shown in Fig. 3. Each 2-cube (square) is divided into two 2-dimensional simplices. Each simplex contains three vertices. The decomposition of a unit 3-cube is shown in Fig. 4. A unit 3-cube is divided into six 3-dimensional simplices. Each simplex contains four vertices. In general, the n-dimensional simplices generated by the above method, called the simplicial subdivision, have the following properties:

(i) The union of these simplices is C_p.

(ii) They are "non-overlapping." More specifically, if the intersection of any two simplicies is nonempty, then it is either a bounday or a corner.

(iii) Every n-dimensional simplex contains I_0 and $I_0 + 1$ which define the cube $C(I_0)$ containing the simplex.

Consequently, RL is also divided into simplices which have the above properties. It should be mentioned that the homeomorphism T(·) preserves the ordering of vertices, since the matrix defined in (3) is a positive-definite diagonal matrix. In other words, $I_i \le I_j$ if and only if $x_i \le x_j$, where $I_i, I_j \in V_p$, $I_i = T(x_i)$ and $I_j = T(x_j)$. Thus, from Lemma 1, we obtain the following:

<u>Lemma 2</u>:

Every $x \in RL$ has a unique representation

$$x = \mu_0 x_0 + \dots + \mu_m x_m$$

where

$$\mu_j > 0,\ \underset{\sim}{x}_j \in V \text{ for } j = 0,1,\ldots,m\ (\leq n),\ \sum_{j=0}^{m} \mu_j = 1$$

and

$$\underset{\sim}{T}(\underset{\sim}{x}_0) \leq \underset{\sim}{T}(\underset{\sim}{x}_1) \leq \cdots \leq \underset{\sim}{T}(\underset{\sim}{x}_m) \leqq \underset{\sim}{T}(\underset{\sim}{x}_0) + 1$$

Having presented the simplicial subdivision of RL, we are ready to take a look of the piecewise-linear interpolation. Let the nonlinear equation be $\underset{\sim}{g}(\underset{\sim}{x}) = \underset{\sim}{y}^*$, where $\underset{\sim}{g}(\cdot)$ maps from R^n into self. Let $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$ be any simplex generated by the simplicial subdivision. An affine function approximating the given $\underset{\sim}{g}(\cdot)$ on $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$ can be defined by

$$\underset{\sim}{f}(\underset{\sim}{x}) = [\underset{\sim}{g}(\underset{\sim}{x}_0),\ \ldots,\ \underset{\sim}{g}(\underset{\sim}{x}_n)]\ \underset{\sim}{\mu} \tag{6}$$

for $\underset{\sim}{x} \in S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$ and $\underset{\sim}{\mu} = [\mu_0,\mu_1,\ldots,\mu_n]^T$ as in Eq. 1. Extending this interpolating procedure to all simplices, we have a piecewise-linear continuous function $\underset{\sim}{f}(\cdot)$ approximating $\underset{\sim}{g}(\cdot)$, which is defined on the rectangle RL. The continuity of $\underset{\sim}{f}(\cdot)$ is a direct consequence of the fact that the simplices do not overlap.

Suppose the function $g(\cdot)$ is continuous, then there exists $\varepsilon > 0$ such that $\|\underset{\sim}{g}(\underset{\sim}{x}) - \underset{\sim}{g}(\underset{\sim}{x}')\| \leqq \varepsilon$ for all $\underset{\sim}{x},\ \underset{\sim}{x}' \in S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$. This together with Eq. 1 leads to the following lemma.

<u>Lemma 3</u>:

If $\underset{\sim}{g}(\cdot)$ is continuous, then there exists $\varepsilon > 0$ such that

$$\|\underset{\sim}{f}(\underset{\sim}{x}) - \underset{\sim}{g}(\underset{\sim}{x})\| \leqq \varepsilon \quad \text{for all } x \in RL.$$

In this paper we shall adopt the following representation of Eq. (6),

$$\begin{bmatrix} f(x) \\ 1 \end{bmatrix} = \begin{bmatrix} g(x_0) & \cdots & g(x_n) \\ 1 & \cdots & 1 \end{bmatrix} \mu \triangleq G \mu \tag{7}$$

for $x \in S(x_0,\ldots,x_n) = \left\{ x \middle| \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 \cdots x_n \\ 1 \quad\; 1 \end{bmatrix} \mu \right\}$.

This is similar to the Wolfe secant formulation [12]. If $S(x_0,\ldots,x_n)$ is proper, Eq. (7) can also be written as

$$\begin{bmatrix} f(x) \\ 1 \end{bmatrix} = \begin{bmatrix} g(x_0) \cdots g(x_n) \\ 1 \qquad\quad 1 \end{bmatrix} \cdot \begin{bmatrix} x_0 \cdots x_n \\ 1 \quad\; 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ 1 \end{bmatrix} \triangleq J \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{8}$$

where J is the Jacobian matrix in the usual formulation of the piecewise-linear analysis. The geometric interpretation of (7) and (8) is shown in Fig. 5 for $n = 2$.

The advantages of this particular formulation in (7) are:

(i) There is no need to compute the Jacobian matrix J.

(ii) Since $x \in S(x_0,\ldots,x_n)$ if and only if the vector μ satisfies $1 \geqq \mu \geqq 0$, it is very easy to check whether an approximate solution is found in the simplex. More specifically, an approximate solution of the equation, $g(x) = y^*$, is found in $S(x_0,\ldots,x_n)$ if and only if the solution of Eq. (7) satisfies $1 \geqq \mu \geqq 0$.

III. <u>The Solution Curve in the Piecewise-Linear Analysis</u>

Consider an arbitrary nonlinear resistive network. Let the vector x represent the chosen network variables and the vector y the inputs to the network. It is well-known that a nonlinear resistive network can be described by [5,6]

$$g(x) = y \tag{9}$$

where $g(\cdot)$ is a continuous function from R^n into itself. In the piecewise-linear analysis, the continuous function $g(\cdot)$ is piecewise-linear, that is,

$$g(x) = J^{(m)}x + w^{(m)} = y, \quad m = 0,1,\ldots,\ell, \tag{10}$$

where $J^{(m)}$ is a constant Jacobian matrix and $w^{(m)}$ is a constant vector defined in the region R_m. The finite integer ℓ denotes that the total number of regions is finite. The piecewise-linear analysis amounts finding a continuous piecewise-linear curve from an initial point x^0 to a solution in the domain space such that the image of this curve is a straight line which connects $y^0 = g(x^0)$ and the given input y^* [4-8].

The continuous piecewise-linear curve, $L(x^0)$, is called the solution curve in the domain. The image of $L(x^0)$, $L(y^0)$, is called the solution curve in the range space. The computation of both solution curves is done by an iterative procedure. More specifically, a series of doublets, (x^i,y^i), $i = 0,1,2,\ldots$, is calculated such that

(i) $y^i = g(x^i)$;

(ii) $\{x|x = x^i + t(x^{i+1}-x^i),\ t \in [0,1]\}$ is in the region R_i, $i = 0,1,2,\ldots$; and

(iii) y^0, y^1, y^2 ... are on a straight line

A number of conditions has been derived, which guarantee that the sequence, y^0, y^1, y^2 ..., converges to y^* in a finite number of steps. Accordingly, the sequence in the domain, x^0, x^1, x^2 ..., converges to a point x^* which is a solution of (10). This is shown in Fig. 6. The main feature of this approach, which should be emphasized, is that the solution curve $L(x^0)$ enters a new region at each iteration.

The operations which are needed to carry out the analysis are summarized as follows:

(i) The determination of the boundary of the present region to be crossed. (When a solution cannot be found in the present region, the solution curve $L(\underset{\sim}{x}^0)$ traverses the region and reaches a boundary which is to be crossed.)

(ii) The identification of the new region into which the solution curve should enter. (This new region is uniquely defined when the solution curve reaches a single boundary.)

(iii) The formulation of the new equation to compute the next segments of the solution curves, $L(\underset{\sim}{x}^0)$ and $L(\underset{\sim}{y}^0)$.

When the simplicial subdivision is used to divide the domain into simplices and a continuous piecewise-linear function $\underset{\sim}{f}(\cdot)$ is interpolated on those simplices, the above three steps are greatly simplified as follows:

(i) The determination of the boundary B_k to be crossed is equivalent to the determination of the vertex $\underset{\sim}{x}_k$ to be deleted from the present simplex $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$. As discussed in the previous section, $\underset{\sim}{x}_k$ is opposite to B_k.

(ii) The computation of the vertex $\underset{\sim}{x}'_k$ which forms, together with the remaining n vertices, the new simplex $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_{k-1},\underset{\sim}{x}'_k,\underset{\sim}{x}_{k+1},\ldots,\underset{\sim}{x}_n)$. The solution curve is forced to enter this new simplex.

(iii) The new equation is simply formulated by replacing the (k+1)th column of the matrix $\underset{\sim}{G}$ in Eq. (7) by $\begin{bmatrix} \underset{\sim}{g}(\underset{\sim}{x}'_k) \\ 1 \end{bmatrix}$, that is,

$$\begin{bmatrix} \underset{\sim}{g}(\underset{\sim}{x}_0 & \cdots & \underset{\sim}{g}(\underset{\sim}{x}_{k-1}) & \underset{\sim}{g}(\underset{\sim}{x}'_k) & \underset{\sim}{g}(\underset{\sim}{x}_{k+1}) & \cdots & \underset{\sim}{g}(\underset{\sim}{x}_n) \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \qquad (11)$$

The procedure to replace $\underset{\sim}{x}_k$ by $\underset{\sim}{x}'_k$ is called the replacement rule and is presented in the next section.

IV. Replacement Rule

The derivation of the replacement rule relies on the fact that the generated simplices are "non-overlapping," i.e. if the intersection of two simplices is empty, then it is either a boundary of the simplices or a corner. Furthermore, for any given boundary of a simplex, there are two and only two simplices containing the boundary except that the boundary is itself a subset of the boundary of the rectangle RL. When a solution is not found in the present simplex $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$, the solution curve traverses $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$ and reaches either a boundary or a corner. Suppose the solution curve $L(\underset{\sim}{x}^0)$ reaches a boundary it is easy to see that the next simplex is uniquely defined because there are only two simplices which contain the boundary. In other words, if

(i) the solution curve is forced to enter a new simplex at each iteration; and

(ii) if the solution curve $L(\underset{\sim}{x}^0)$ reaches a boundary but not a corner, then the new simplex at each iteration is completely determined by the structure of the simplices.

Let the solution curve $L(\underset{\sim}{x}^0)$, traverse $S(\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n)$ and reach a point $\underset{\sim}{x}^i$ on the boundary $B_k = \left\{\underset{\sim}{x} \middle| \begin{bmatrix} \underset{\sim}{x} \\ 1 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{x}_0 & \cdots & \underset{\sim}{x}_n \\ 1 & \cdots & 1 \end{bmatrix} \underset{\sim}{\mu}\text{, the } \underline{k}\text{th component of } \mu,\ \mu_k\text{, is zero}\right\}$. The new region is determined by the boundary B_k and a new vertex $\underset{\sim}{x}'_k$ which is computed according to the following theorem:

Theorem 1:

Let $T(\underset{\sim}{x}_0) \leq T(\underset{\sim}{x}_1) \leq \cdots \leq T(\underset{\sim}{x}_n) = T(\underset{\sim}{x}_0) + 1$, where $\underset{\sim}{x}_j \in V$, $j = 0,1,\ldots,n$. Then the new vertex $\underset{\sim}{x}'_k$ is defined by

$$\underset{\sim}{x}_{k+1} + \underset{\sim}{x}_{k-1} - \underset{\sim}{x}_k, \quad k = 0,1,\ldots,n,$$

where $\underset{\sim}{x}_{n+1} = \underset{\sim}{x}_0$ and $\underset{\sim}{x}_{-1} = \underset{\sim}{x}_n$.

Before presenting the proof, we illustrate the meaning of this theorem by the following examples.

Example 3:

Consider a 3-dimensional cube. Let the old simplex be defined by $\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ as shown in Fig. 4. Suppose B_k is defined by $\begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$, i.e., $\begin{bmatrix}0\\1\\1\end{bmatrix}$ is to be deleted, the new vertex is $\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$. The integer k is 3. The new simplex is therefore defined by $\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Example 4:

Consider the simplex defined by $\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$. Let the vertex $\begin{bmatrix}0\\0\\0\end{bmatrix}$ be deleted. The new vertex is obtained as $\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix}$, and the new simplex is defined by $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\1\\1\end{bmatrix}$, as shown in Fig. 7.

Example 5:

Let $RL = \left\{ \underset{\sim}{x} \middle| \begin{bmatrix} 5.5 \\ -1 \end{bmatrix} \leqq x \leqq \begin{bmatrix} 8 \\ 2 \end{bmatrix} \right\}$ and $\underset{\sim}{C_p} = \{z|\ 0 \leqq \underset{\sim}{z} \leqq 4\}$ as shown in Fig. 8. Let the present simplex be defined by $\underset{\sim}{x}_0 = \begin{bmatrix} 5.5 + 2\cdot(\frac{8-5.5}{4}) \\ -1 + 2\cdot(\frac{2+1}{4}) \end{bmatrix}$,

$\underset{\sim}{x}_1 = \begin{bmatrix} 5.5 + 3\cdot(\frac{8-5.5}{4}) \\ -1 + 2\cdot(\frac{2+1}{4}) \end{bmatrix}$ and $\underset{\sim}{x}_2 = \begin{bmatrix} 5.5 + 3\cdot(\frac{8-5.5}{4}) \\ -1 + 3\cdot(\frac{2+1}{4}) \end{bmatrix}$. Suppose $\underset{\sim}{x}_1$ is deleted, the new vertex $\underset{\sim}{x}_1'$ is computed as $\underset{\sim}{x}_1' = \underset{\sim}{x}_2 + \underset{\sim}{x}_0 - \underset{\sim}{x}_1 = \begin{bmatrix} 5.5 + 2.(\frac{8-5.5}{4}) \\ -1 + 3\cdot(\frac{2+1}{4}) \end{bmatrix}$.

Proof of Theorem 1:

Since $\underset{\sim}{x}^i \in B_k$, it is uniquely represented by $\underset{\sim}{x}^i = \sum_{j=0}^{n} \mu_j^i \underset{\sim}{x}_j$ where $\mu_k^i = 0$ and $1 > \mu_j^i > 0$ for $j \neq k$. Let ε be an "arbitrarily small" positive number. The vector

$$\underset{\sim}{x}(\varepsilon) = \sum_{\substack{j=0 \\ j\neq(k-1),k \\ (k+1)}}^{n} \mu_j^i \underset{\sim}{x}_j + (\mu_{k-1}^i + \tfrac{\varepsilon}{2})\, \underset{\sim}{x}_{k-1} - \varepsilon\, \underset{\sim}{x}_k + (\mu_{k+1}^i + \tfrac{\varepsilon}{2})\, \underset{\sim}{x}_{k+1} \qquad (12)$$

defines a point in the new simplex. Note that $\underset{\sim}{x}(\varepsilon) \to \underset{\sim}{x}^i$ as $\varepsilon \to 0$. Thus $\underset{\sim}{x}(\varepsilon)$ is an interior point of the new simplex when ε is sufficiently small. Equation (12) can be rewritten as

$$\underset{\sim}{x}(\varepsilon) = \sum_{\substack{j=0 \\ j\neq(k-1),k \\ (k+1)}}^{n} \mu_j^i \underset{\sim}{x}_j + (\mu_{k-1}^i - \tfrac{\varepsilon}{2})\, \underset{\sim}{x}_{k-1} + \varepsilon\, (\underset{\sim}{x}_{k+1} + \underset{\sim}{x}_{k-1} - \underset{\sim}{x}_k) + (\mu_{k+1}^i - \tfrac{\varepsilon}{2})\, x_{k+1} \qquad (13)$$

Since $\underset{\sim}{T}(\cdot)$ is a homeomorphism, $\underset{\sim}{x}_k' = \underset{\sim}{x}_{k+1} + \underset{\sim}{x}_{k-1} - \underset{\sim}{x}_k$ is also a vertex of RL in V. The conditions of lemma 2 are fulfilled when ε is sufficiently small hence, the representation (13) is unique. Consequently the vertex to replace x_k is defined by $\underset{\sim}{x}_k' = \underset{\sim}{x}_{k+1} + \underset{\sim}{x}_{k-1} - \underset{\sim}{x}_k$. The same argument holds for k = 0 or n. The only difference is that the "new" simplex is in a new rectangle. This completes the proof.

V. The Algorithm

With the replacement rule, the new region in which the solution curve enters is easily determined. The equation for the new region is

$$\begin{bmatrix} \underset{\sim}{g}(\underset{\sim}{x}_0) & \cdots & \underset{\sim}{g}(\underset{\sim}{x}_{k-1}) & \underset{\sim}{g}(\underset{\sim}{x}_k') & \underset{\sim}{g}(\underset{\sim}{x}_{k+1}) & \cdots & \underset{\sim}{g}(\underset{\sim}{x}_n) \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{\mu} \\ 1 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{y}^* \\ 1 \end{bmatrix} \qquad (14)$$

where $\underset{\sim}{y}^*$ denotes the input. Let the solution of (14) be $\hat{\underset{\sim}{\mu}}$. If all the components of $\hat{\underset{\sim}{\mu}}$ are larger than or equal to zero, a solution $\underset{\sim}{x}^*$ is found in this simplex,

$$\underset{\sim}{x}^* = [\underset{\sim}{x}_0 \cdots \underset{\sim}{x}_{k-1}\ \underset{\sim}{x}_k'\ \underset{\sim}{x}_{k+1} \cdots \underset{\sim}{x}_n]\ \hat{\underset{\sim}{\mu}} \qquad (15)$$

The semgent of the solution curve $L(\underset{\sim}{x}^0)$ in this simplex is simply obtained by connecting $\underset{\sim}{x}^i$ and $\underset{\sim}{x}^*$, i.e.,

$$\underset{\sim}{x}(t) = \underset{\sim}{x}^i + t(\underset{\sim}{x}^* - \underset{\sim}{x}^i) \quad 1 \geqq t \geqq 0. \qquad (16)$$

In the case that $\hat{\underset{\sim}{\mu}}$ defines a point outside of the simplex, two steps need to be taken to continue the tracing of the solution curve in the domain. The first step is the determination of the sign of the $\underline{k}$th component of $\hat{\underset{\sim}{\mu}}$, $\hat{\mu}_k$. If $\hat{\mu}_k > 0$, then

$$\underset{\sim}{\mu}(t) = \underset{\sim}{\mu}^i + t(\hat{\underset{\sim}{\mu}} - \underset{\sim}{\mu}^i), \quad t > 0,$$

defines the segment of $L(\underset{\sim}{x}^0)$ entering the simplex. If $\hat{\mu}_k < 0$, then

$$\underset{\sim}{\mu}(t) = \underset{\sim}{\mu}^i - t(\underset{\sim}{\mu}-\underset{\sim}{\mu}^i)$$

defines the segment of $L(\underset{\sim}{x}^0)$ entering the simplex. Let SGN $(\hat{\mu}_k) = 1$ if $\hat{\mu}_k > 0$ and SGN $(\hat{\mu}_k) = -1$ if $\hat{\mu}_k < 0$. The solution curve entering the simplex is then defined by $\underset{\sim}{\mu}(t)$.

$$\underset{\sim}{\mu}(t) = \underset{\sim}{\mu}^i + \text{SGN}\ (\hat{\mu}_k) \cdot t \cdot (\hat{\underset{\sim}{\mu}}-\underset{\sim}{\mu}^i). \tag{17}$$

The second step is the computation of $\lambda^i > 0$ such that $\underset{\sim}{\mu}(\lambda^i)$ defines a point on the next boundary of the simplex. This is accomplished by checking the components of $\underset{\sim}{\mu}(t)$. At least one component of $\underset{\sim}{\mu}(t)$ will approach zero when the value of t increases from zero. This fact is derived from the following lemma.

<u>Lemma 4</u>:

At least one component of $(\hat{\underset{\sim}{\mu}}-\underset{\sim}{\mu}^i)$ is positive (negative) if $\hat{\underset{\sim}{\mu}} \neq \underset{\sim}{\mu}^i$. Thus

$\lambda^i > 0$ is computed by

$$\lambda^i = \min_j \left\{ \frac{-\mu_j^i}{\text{SGN}(\hat{\mu}_k)\cdot(\hat{\mu}_j-\mu_j^i)} > 0 \right\}$$

The vector $\underset{\sim}{\mu}(\lambda^i)$ defines a point on a boundary of the simplex, namely:

$$\underset{\sim}{x}^{i+1} = [\underset{\sim}{x}_0 \;\cdots\; \underset{\sim}{x}_{k-1}\; \underset{\sim}{x}'_k\; \underset{\sim}{x}_{k+1} \;\cdots\; \underset{\sim}{x}_n]\; \underset{\sim}{\mu}(\lambda^i)$$

The solution curve $L(\underset{\sim}{x}^0)$ in this simplex is then defined by

$$\{\underset{\sim}{x} \mid \underset{\sim}{x} = \underset{\sim}{x}^i + t(\underset{\sim}{x}^{i+1}-\underset{\sim}{x}^i),\ 1 \geq t \geq 0\}$$

The computation of λ^i also determines an index which identifies the vertex to be deleted from the simplex. The procedure discussed above is repeated until a solution is found.

The following algorithm summarizes the above discussion.

ALGORITHM

Step 1: Choose $\underset{\sim}{x}_0$ and

$\underset{\sim}{x}_i = \underset{\sim}{x}_{i-1} + \underset{\sim}{E}_i$, $i = 1,2,\ldots,n$, where

$E_i = [0,\ldots,0,e_i,0,\ldots,0]^T$ and $e_i > 0$ is the ith component of $\underset{\sim}{E}_i$.

Step 2: Let $\underset{\sim}{\mu}^0 = \frac{1}{n+1}[1,\ldots,1]^T$, i.e.,

$\underset{\sim}{x}^0 = \frac{1}{n+1}\sum_{i=0}^{n} \underset{\sim}{x}_i$ which is the center of the initial simplex.

Set $i = 0$ and SGN $(\hat{\mu}_k^0) = 1$.

Step 3: Compute $\underset{\sim}{\hat{\mu}}$ according to the equation

$$\begin{bmatrix} \underset{\sim}{g}(\underset{\sim}{x}_0) & \cdots & \underset{\sim}{g}(\underset{\sim}{x}_n) \\ 1 & \cdots & 1 \end{bmatrix} \underset{\sim}{\hat{\mu}}_i = \begin{bmatrix} \underset{\sim}{y}^* \\ 1 \end{bmatrix}$$

If very component of $\underset{\sim}{\hat{\mu}}^i$ is nonnegative, a solution

$x^* = [\underset{\sim}{x}_0,\ldots,\underset{\sim}{x}_n]\ \underset{\sim}{\hat{\mu}}^i$ is found. STOP

Step 4: Otherwise, compute λ^i from

$\underset{\sim}{\mu}(t) = \underset{\sim}{\mu}^i + \text{SGN}\,(\hat{\mu}_k^i)\cdot t\cdot(\underset{\sim}{\hat{\mu}}^i - \underset{\sim}{\mu}^i)$, such that

(i) $0 \leqq \underset{\sim}{\mu}(t) \leqq 1$ for $0 \leqq t \leqq \lambda^i$

(ii) there is one and only one index k satisfying

$\mu(\lambda^i)_k = 0$

(iii) $1 > \mu(\lambda^i)_j > 0$ for $j \neq k$.

Step 5: Replace x_k by $(\underset{\sim}{x}_{k+1} + \underset{\sim}{x}_{k-1} - \underset{\sim}{x}_k)$

If this new vertex is outside of the interested range (RL) the algorithm is terminated. Otherwise let $i = i+1$ and go to Step 3.

The convergence of Algorithm I depends completely on the continuous piecewise-linear function $\underset{\sim}{f}(\cdot)$ interpolated on the rectangle RL. Let B(RL) denote the boundary of $RL = \{\underset{\sim}{x} | \underset{\sim}{a} \leq \underset{\sim}{x} \leq \underset{\sim}{b}\}$. Then the algorithm will locate a solution in a finite number of steps under the following conditions:

(i) The Jacobian matrix, $\underset{\sim}{G} = \begin{bmatrix} \underset{\sim}{g}(\underset{\sim}{x}_0) & \cdots & \underset{\sim}{g}(\underset{\sim}{x}_n) \\ 1 & \cdots & 1 \end{bmatrix}$ is nonsingular at each iteration;

(ii) The solution curve in the range does not go back to $\underset{\sim}{y}^0 = \underset{\sim}{f}(\underset{\sim}{x}^0)$;

(iii) The solution curve in the domain never reaches a corner;

(iv) The solution curve does not reach B(RL).

Conditions (i), (ii) and (iii) guarantee that the solution curve in the domain is well determined at each iteration. Condition (ii) and (iii) further exclude the possibility that the solution curve in the domain becomes cyclic. Condition (iv) asserts that the solution curve stays in RL before a solution is found. Two possible situations which violate conditions (iv) are shown in Figs. 9 and 10. If the image of B(RL) is not the boundary of $\underset{\sim}{f}$(RL), then the solution curve might reach B(RL), and hence, the algorithm is terminated without locating a solution as shown in Fig. 9. The same problem might occur if $L(\underset{\sim}{y}^0,\underset{\sim}{y}^*)$ is not contained in $\underset{\sim}{f}$(RL) as shown in Fig. 10. The above difficulties arise from that the interior of $\underset{\sim}{f}$(B(RL)) is not convex.

Finally, it should be pointed out that in the first iteration, the condition $\underset{\sim}{x}_0 \leq \underset{\sim}{x}_1 \leq \cdots \leq \underset{\sim}{x}_n$ is satisfied, i.e. the vertices obey a ordering property. In step 5, $\underset{\sim}{x}_k$ is replaced by $(x_{k+1}+x_{k-1}-x_k)$ without rearranging the vertices. This is true even in the case that k = 0 or n. The reason is that Theorem 1 is applicable to a "circular" list of vertices. This property is very important from a computational point of view. If a rearrangement of vertices is necessary, the method of matrix modification cannot be applied [6,11].

Example 6: To illustrate the Algorithm we consider the tunnel diode circuit, as shown in Fig. 11.

The network equation describing the circuit is

$$2e_1 - e_2 + i_2(e_1) - 1 = y_1 = 0$$

$$-e_1 + 2e_2 - i_1(1-e_2) = y_2 = 0$$

where e_1 and e_2 are node voltages as shown in the figure and are the components of the vector $\underset{\sim}{x}$. The tunnel diodes are represented by

$$i(v) = \frac{29}{12}v - \frac{7}{8}v^2 + \frac{1}{12}v^3,$$

as shown in Fig. 12. Let

$$\underset{\sim}{x}_0 = \begin{pmatrix} -0.5 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{x}_1 = \underset{\sim}{x}_0 + \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{x}_2 = \underset{\sim}{x}_1 + \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$$

The equation to be solved at the first iteration is

$$\begin{bmatrix} \underset{\sim}{g}(\underset{\sim}{x}_0) & \underset{\sim}{g}(\underset{\sim}{x}_1) & \underset{\sim}{g}(\underset{\sim}{x}_2) \\ 1 & 1 & 1 \end{bmatrix} \underset{\sim}{\mu} = \begin{bmatrix} -3.4375 & -1 & -1.5 \\ -1.125 & 1.625 & 0 \\ 1 & 1 & 1 \end{bmatrix} \underset{\sim}{\mu} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The solution is $\underset{\sim}{\mu} = \begin{bmatrix} -0.657 \\ 0.455 \\ 1.202 \end{bmatrix}$

Since $\underset{\sim}{\mu}_0$ is the only component which satisfies

$$\frac{-\mu_0^0}{\mathrm{SGN}(\hat{\mu}_0)\cdot(\hat{\mu}_0-\mu_0^0)} > 0$$

The vertex $\begin{bmatrix} -0.5 \\ 0 \end{bmatrix}$ is replaced by

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

The new simplex is defined by

$$\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.5 \end{bmatrix},$$

and the new equation is

$$\begin{bmatrix} 0.5 & -1 & -1.5 \\ -0.5 & 1.625 & 0 \\ 1 & 1 & 1 \end{bmatrix} \underset{\sim}{\mu} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The solution is $\underset{\sim}{\mu} = \begin{bmatrix} 0.8125 \\ -0.25 \\ 0.4375 \end{bmatrix}$

It is easy to see that the vertex $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has to be replaced by $\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$.

The new simplex is then defined by

$$\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

and the new equation is

$$\begin{bmatrix} 0.5 & 0 & -1.5 \\ -0.5 & 1.5 & 0 \\ 1 & 1 & 1 \end{bmatrix} \underset{\sim}{\mu} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

The solution is $\underset{\sim}{\mu} = \begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix} > 0$

The approximate solution is found to be

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} .$$

Substituting $\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$ into the network equation, we find

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.03 \\ -0.03 \end{bmatrix}$$

whereas the actual input is $y^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

VI. Conclusion

In this paper, the method of simplicial subdivision and its application to piecewise-linear analysis are presented. The method of subdivision provides a systematic way to tessellate an n-dimensional rectangle into simplices. Interpolation is then applied to each simplex. When applied to piecewise-linear analysis, the transition from a simplex to another is extremely simple and is determined by the structure of the simplices.

In fact, the procedure is equivalent to the deletion of a vertex from the present simplex and the identification of a vertex to form the new simplex. Consequently, the most difficult part of the piecewise-linear analysis has been overcome. Most of the theorems and techniques in piecewise-linear analysis can be applied to this new formulation.

Further study is needed in order to make the method more effective. Investigation on the determination of the bounded set in order to select RL for a given equation is important. Another area to explore is the possibility of changing the size of the simplicies in tracing the solution curve.

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Figure Captions

Fig. 1. A two dimensional proper simplex $S(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)$ and its boundaries.

Fig. 2. Tessellation of RL into small rectangles and the corresponding Tessellation of $\tilde{C}_p$ into unit cubes.

Fig. 3. Canonical decomposition of RL (2-dimensional) into simplices.

Fig. 4. Canonical decomposition of a 3-dimensional cube into simplices.

Fig. 5. Relations among the various spaces as given in Eqs. (7) and (8).

Fig. 6. The solution curve in the domain, $L(\tilde{x}^0)$, enters a new region at each iteration and converges to a solution $\tilde{x}^*$ as $\tilde{y}^*$ is reached.

Fig. 7. Geometrical interpretation of "Replacement Rule" (Examples 3 and 4).

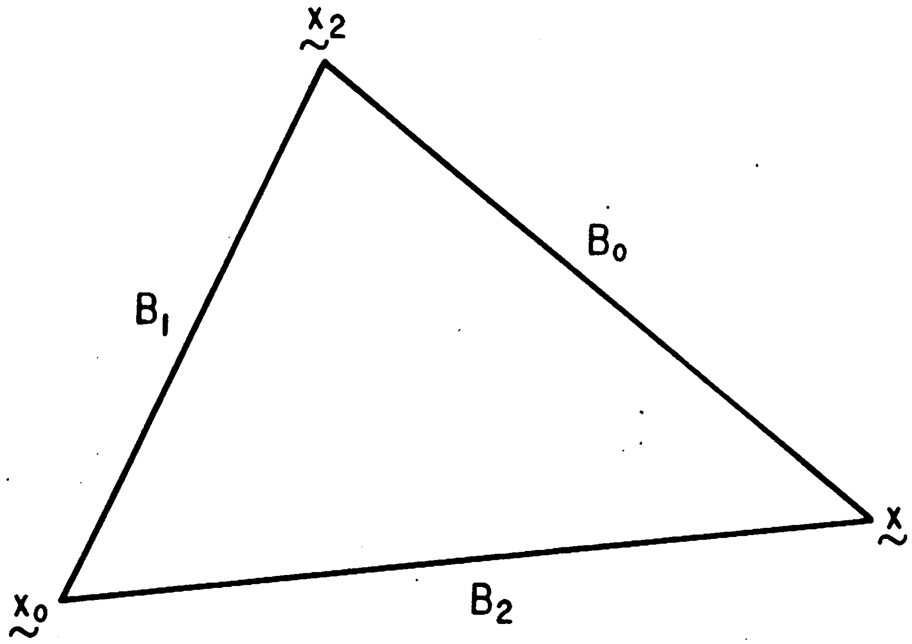
Fig. 8. Illustration of Theorem 1 (Example 5).

Fig. 9. The solution curve $L(\tilde{x}^0)$ reaches the boundary of RL.

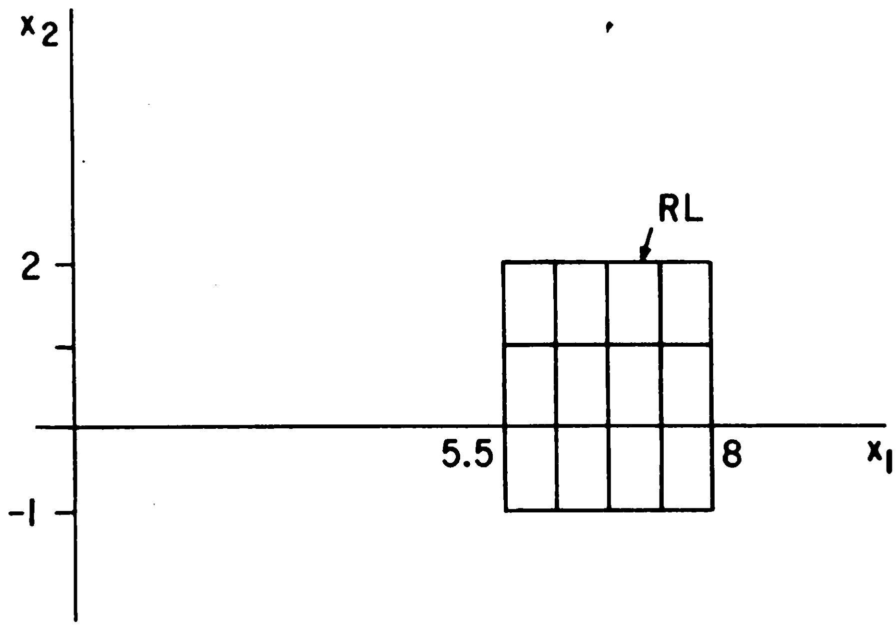
Fig. 10. The line segment $L(\tilde{y}^0, \tilde{y}^*)$ is not contained in $\tilde{f}(RL)$.

Fig. 11. Tunnel diode circuit

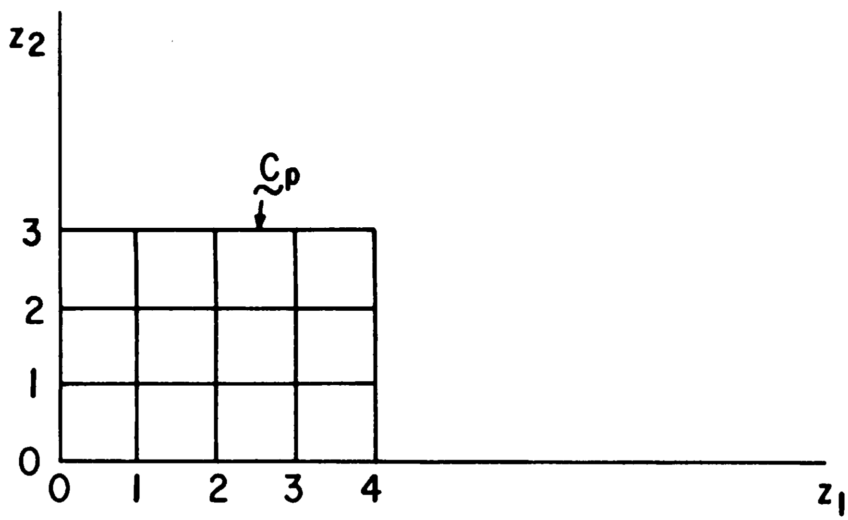
Fig. 12. Model of a tunnel diode.



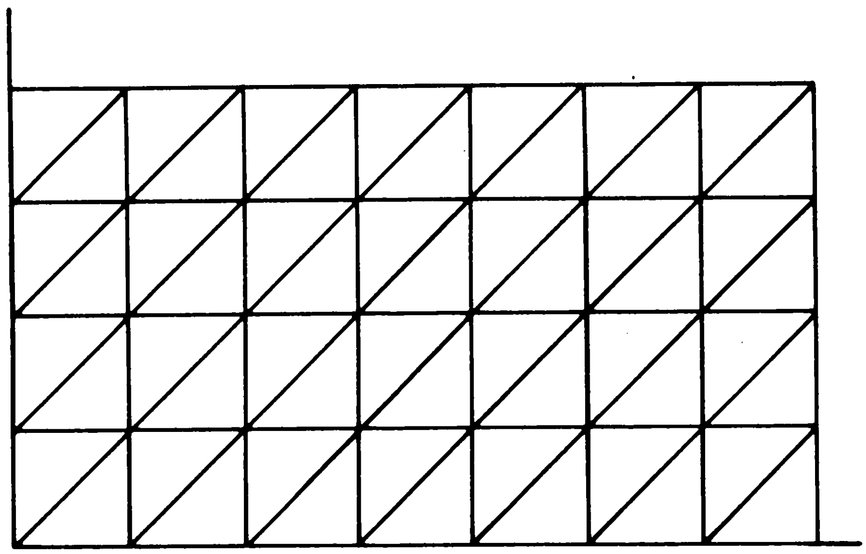
x2
B0
B1
x1
x0
B2

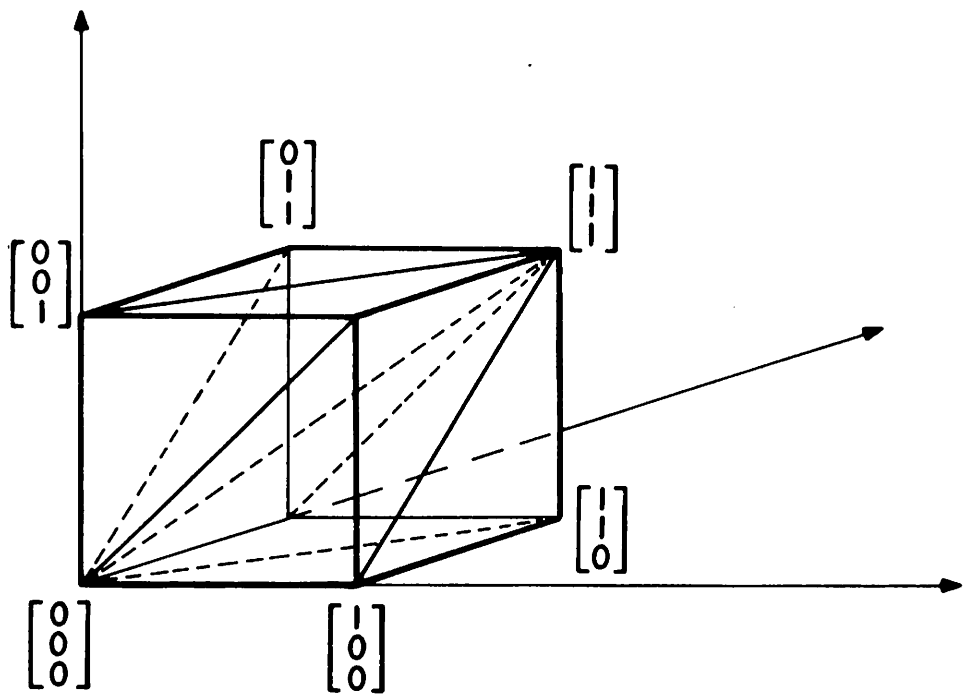


x_2
RL
2
5.5
8
x_1
-1



z_2
$\underset{\sim}{C}_p$
3
2
1
0
0
1
2
3
4
z_1

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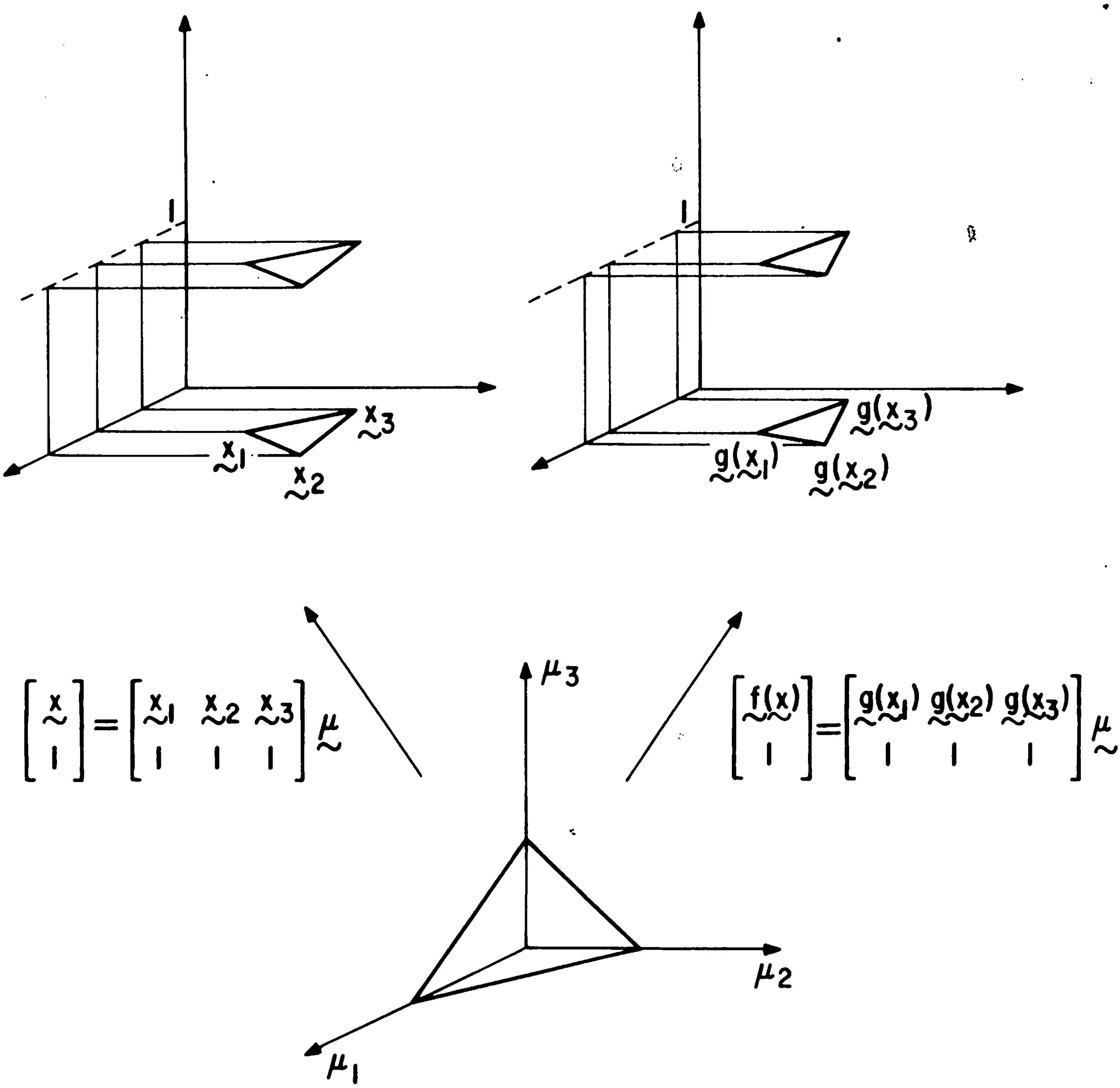
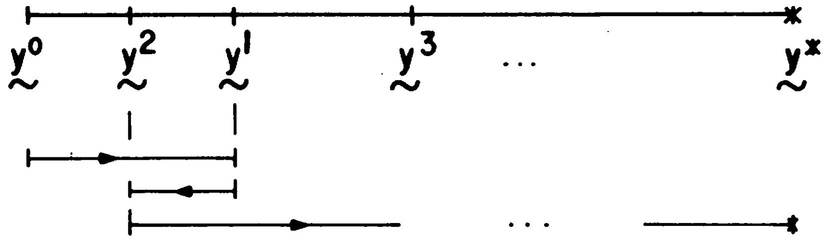
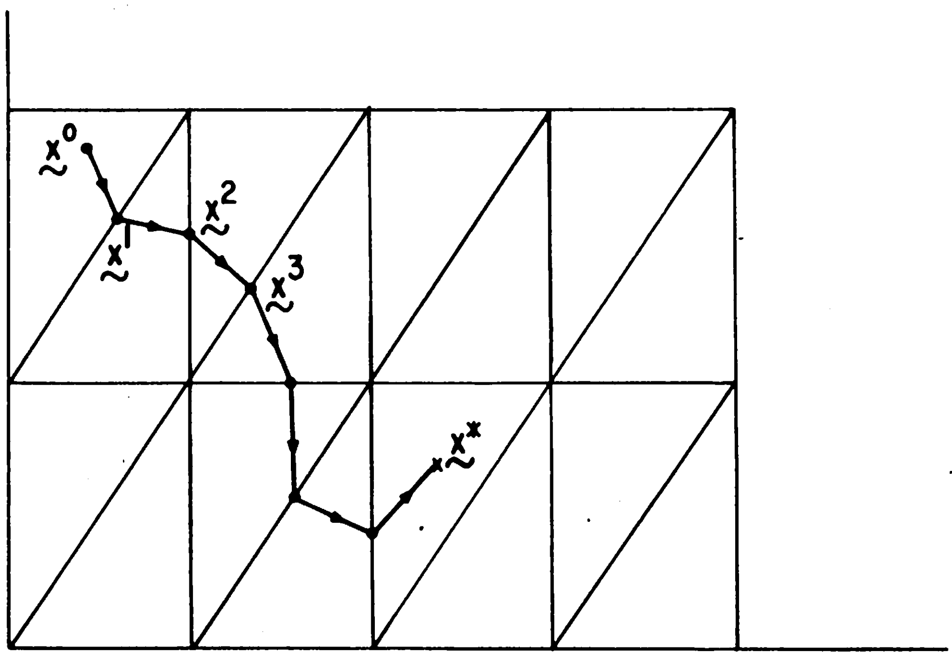
[1 0 0]

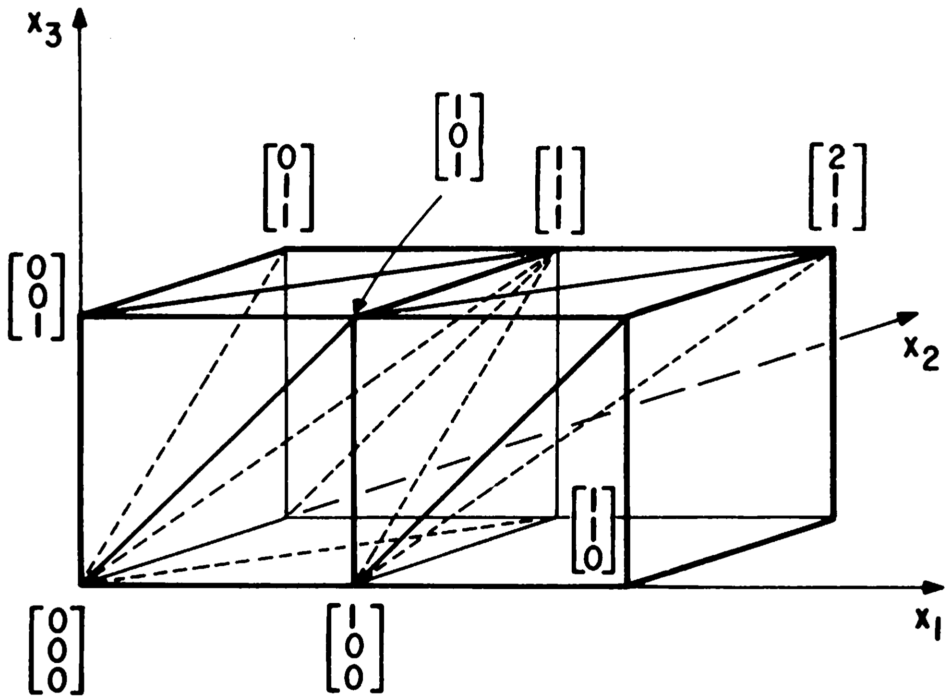

1
x1
x2
x3
1
g(x1)
g(x2)
g(x3)
[x; 1] = [x1 x2 x3; 1 1 1] μ
[f(x); 1] = [g(x1) g(x2) g(x3); 1 1 1] μ
μ3
μ2
μ1



Range



Domain



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x2
x1
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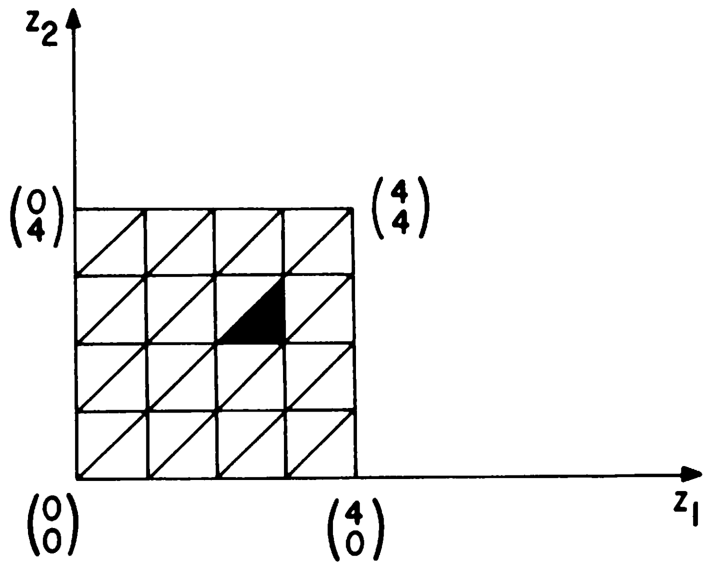
[1 1 1]
[2 1 1]
[1 1 0]
[0 0 0]
[1 0 0]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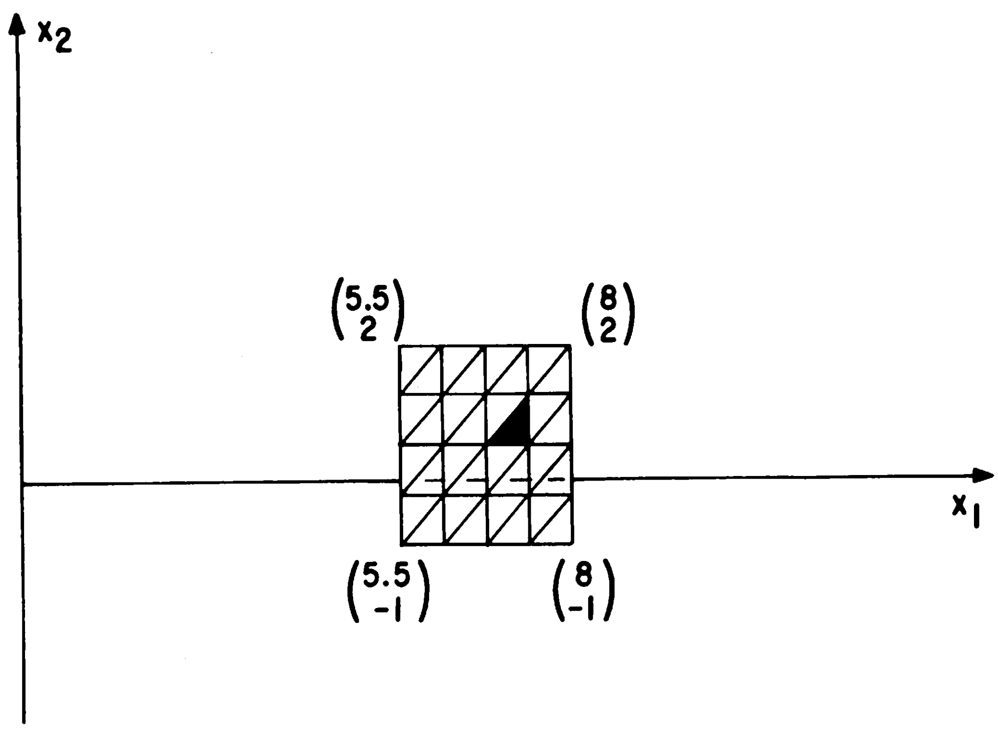
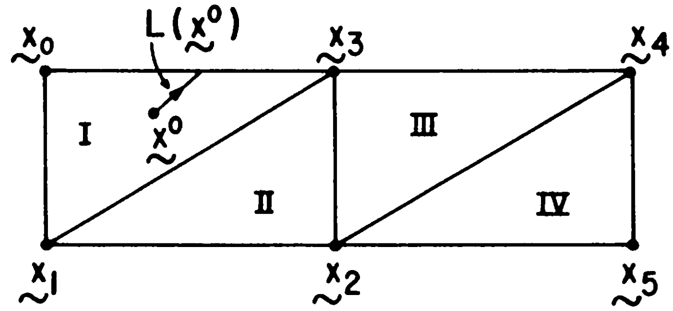
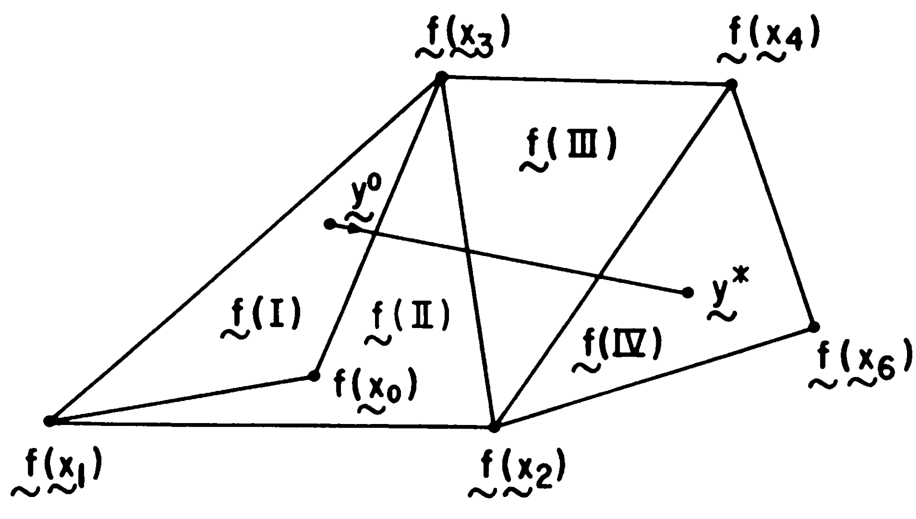
(4 4)
(0 0)
(4 0)
z1


x2
(5.5 2)
(8 2)
x1
(5.5 -1)
(8 -1)

Domain RL

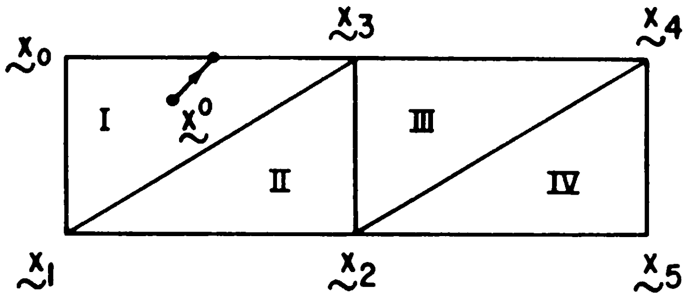


Range

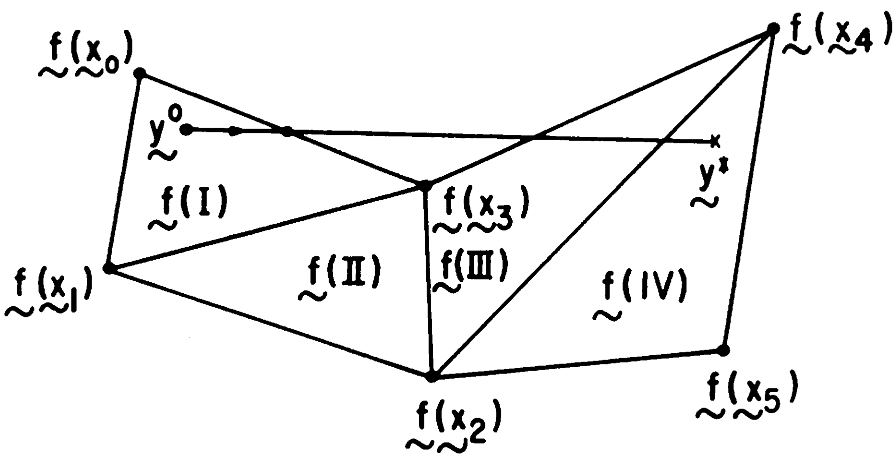


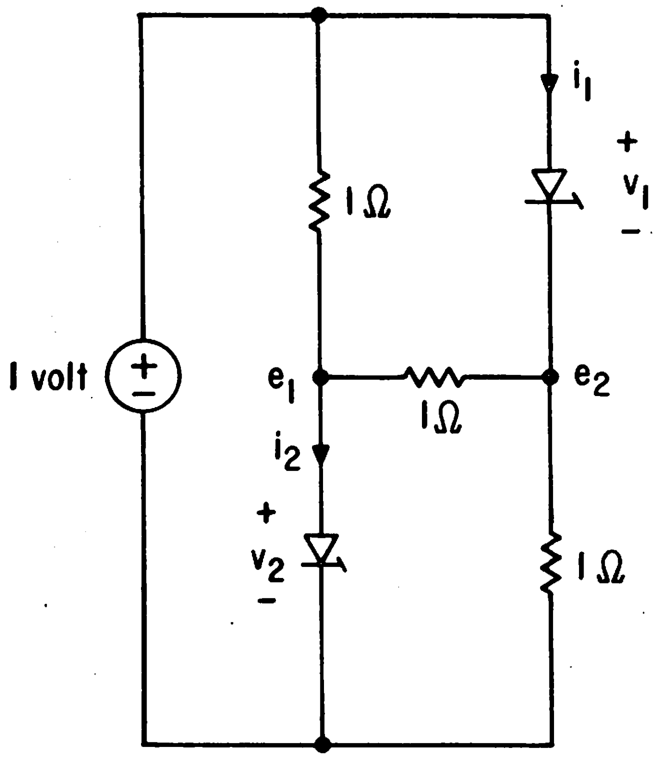
$f(\mathrm{I}) \subset f(\mathrm{II})$

Domain

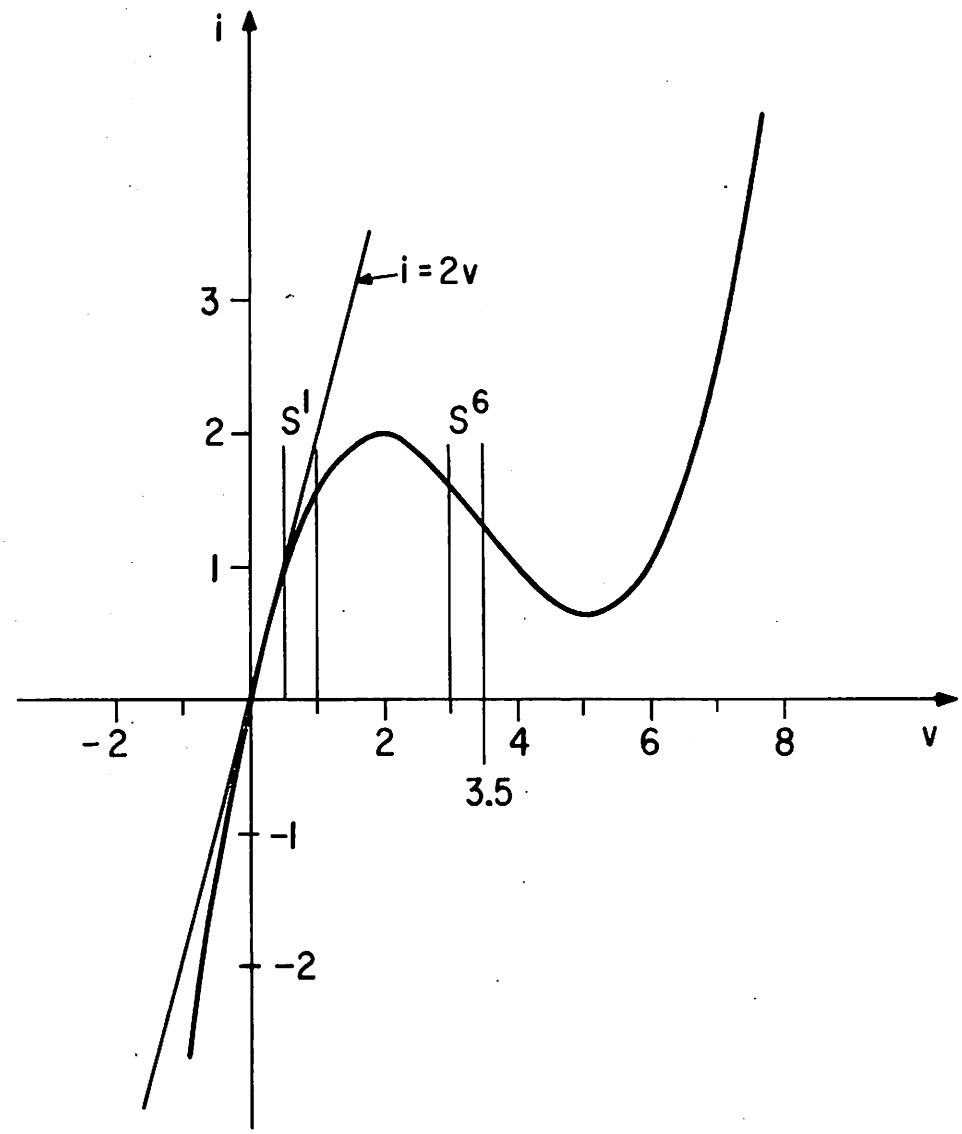


Range





i1
+
v1
−
1 Ω
1 volt
e1
e2
1Ω
i2
+
v2
−
1 Ω



i
i = 2v
3
2
1
S1
S6
-2
2
4
6
8
v
3.5
-1
-2