A TWO LEVELS ALGORITHM FOR TEARING

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ABSTRACT

This paper deals with tearing methods for the solution of a large scale system of linear algebraic equations. A modification algorithm is presented and evaluated with respect to other available techniques, namely Householder's Formula and Bennet's Algorithm. Then, an optimization problem related to the "best" way of tearing a given matrix A with a certain associated structure is taken into account and proven to be equivalent to the determination of a minimum essential set (MES) of an hypergraph H. Some algorithms for finding a MES in H derived from the minimum feedback vertex set problem algorithms are briefly described.

Then a particular way of applying the modification algorithm to a matrix rearranged according to the previously selected criterion is introduced and its complexity is compared with LU decomposition method.

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I. INTRODUCTION

Recently much effort has been devoted to tearing methods for the analysis of large scale electrical networks [1,2,3,4,5].

Tearing, usually referred to as diakoptics was introduced by Kron [6], and basically consists in breaking the original analysis problem into simple subproblems which can be solved either independently or according to a (possibly partial) ordering.

In [3], it has been pointed out that it is fundamental from the computational efficiency point of view to take into account the underlying structure of the system not only during the formulation phase but also in the solution phase.

This consideration leads to the study of efficient algorithms for tearing a system of algebraic equations

$$A \; x = b \tag{1.1}$$

since almost all the computer programs available solve any nonlinear dynamical networks by means of linearization and discretization techniques (e.g. [7,8]). The aim of this paper is to give a contribution to the development of such efficient algorithms.

Without any loss of generality, A and b will now onwards be assumed to take real values. If the matrix A is reducible, i.e., if A can be given a block triangular form by row and column reordering, the decomposition of the system in simple subsystems is straightforward. If this is not the case, a possible strategy is tearing someone of the nonzero entries of A so as to obtain a reducible matrix. Formally assume that A can be written as

$$x = x^o + \sum_{j=1}^{m} \bar{w}_j \, x^j,$$

where $\bar{w}$ is in turn the solution of

$$Dw = d,$$

where the matrix $D \in R^{m^2}$ and the vector $d \in R^m$ are easily computed as functions of the cut matrix C and of the m + 1 partial solutions obtained at the first level.

The aforementioned result is stated as Theorem 2.1 and proved in Sec. II. A few comments about the computational complexity of the method presented here as compared with the ones already available are given in Section III. Section IV deals with the problem of optimal tearing, i.e. with the problem of determining a cut matrix C such that a suitable measure of the overall computational effort involved by the method above is minimized. Under reasonable assumptions, it is proved that this problem is equivalent to a MES (minimal Essential Set) problem in an hypergraph H associated to A.

In Section V, a particular way of applying the results obtained in Section II to a matrix reordered according to the criterion given in Section IV is described and some computational remarks are introduced.

In Section VI some concluding remarks are given.

II. A MODIFICATION ALGORITHM

Consider a system of linear algebraic equations of the form

$$(B + C)\, x = b \qquad (2.1)$$

where $B \in \mathbb{R}^{n^2}$ is nonsingular, $C \in \mathbb{R}^{n^2}$ has rank m and is such that B + C

$$A = B + C, \tag{1.2}$$

where B is reducible and nonsingular. Then the classical tearing technique basically consists of two steps. At first, system (1.1) is analyzed assuming $A = B$; then, the so obtained result is modified to take into account the real structure of A, i.e. the "perturbation" due to the nonzero entries of the "cut matrix" C. The second step is not only fundamental in tearing but also in other important applications as piecewise analysis of a nonlinear resistive network where many systems, whose coefficient matrix is slightly changed, have to be solved sequentially [9,10]. In [6] as well as in [9], Householder's Formula is adopted. However, by means of Householder's Formula, the solution of the original problem is obtained via matrix inversion; this is, in general, a costly technique which furthermore fully destroys, with the original sparsity of A, any possibility of saving computer storage. In [10] these difficulties have been encompassed with a method related to Bennet's algorithm [11]. This algorithm consists in computing the LU factorization of A in terms of C and of the LU factorization of B.

In this paper a multi-steps two-levels modification technique is presented whose logical frame is as follows. The first level consists of $m + 1$ steps, where m is the rank of the cut matrix C; each step calls for the solution of a system of the form

$$Bx^j = b^j, \quad j = 0,1,\ldots,m;$$

where each vector $b^j \in R^n$ does depend upon the cut matrix C. At the second level, the solution x of system (1.1) (with A given by (1.2)) is computed as is nonsingular. Furthermore, let

$$C = HK' \tag{2.2}$$

where H, $K \in \mathbb{R}^{n\times m}$, and denote by e_i the i-th versor in $\mathbb{R}^m$.

<u>Theorem 1</u>

Let $x(\cdot):R^m \to R^n$ be a linear mapping defined as follows

$$w \mapsto x: \quad Bx = -Hw \tag{2.3}$$

and denote by x^o the solution of Bx = b and by $\bar{x}$ the solution of system (2.1). Then

$$\bar{x} = x^o + \sum_{i=1}^{m} x(e_i)\bar{w}_i \tag{2.4}$$

where, letting

$$Q \triangleq |K'x(e_1) \vdots K'x(e_2) \vdots \cdots \vdots K'x(e_m)|, \tag{2.5}$$

$\bar{w}$ is the solution of

$$(I - Q)w = K'x^o. \tag{2.6}$$

To prove Theorem 2.1, the following Lemmas will be used.

<u>Lemma 2.1</u>

The solution of

$$Ax = b - Hw \tag{2.7}$$

is equal to $\bar{x}$ if and only if

$$w = Fw + K'x^o \tag{2.8}$$

where F is any m×m matrix such that

$$Fw = K'x(w), \ \forall w \in \mathbb{R}^m. \tag{2.9}$$

Proof. Let $\tilde{w}$ be a solution of (2.8) and $\hat{x}$ be the corresponding solution of (2.7), then

$$\hat{x} = x^o + x(\tilde{w})$$

furthermore

$$A\hat{x} = b - H(F\tilde{w} + K'x^o) = b - HK'(x(\tilde{w}) + x^o) = b - HK'\hat{x}$$

hence, in view of the nonsingularity of $A + HK'$, $\hat{x} = \bar{x}$. Conversely, if $\tilde{\tilde{w}}$ is such that

$$A\bar{x} = b - H\tilde{\tilde{w}} \tag{2.10}$$

then

$$H\tilde{\tilde{w}} = HK'\bar{x}$$

and, since H is rank m,

$$\tilde{\tilde{w}} = K'\bar{x}. \tag{2.11}$$

On the other hand, in view of (2.10)

$$\bar{x} = x(\tilde{\tilde{w}}) + x^o,$$

hence, from (2.9) and (2.11)

$$\tilde{\tilde{w}} = F\tilde{\tilde{w}} + K'x^o$$

i.e. $\tilde{\tilde{w}}$ is a solution of (2.8) ⌑

Lemma 2.2

The solution of equation (2.8) exists and is unique.

Proof. The nonsingularity of A + B implies that $\bar{x}$ exists and is unique. On the other hand, since A is nonsingular and H is rank m, then there exists a unique $\bar{\bar{w}}$ such that

$$A\bar{x} = b - H\bar{\bar{w}}.$$

Therefore, from Lemma 1, Lemma 2 follows. ⌑

Proof of Theorem 2.1

Let $w^1, w^2, \cdots, w^m$ be a m-tuple in $\mathbb{R}^m$, then,

$$\left| K'x(w^1) \vdots K'x(w^2) \vdots \cdots \vdots K'x(w^m) \right| = F \left| w^1 \vdots w^2 \vdots \cdots \vdots w^m \right|$$

so that, letting $w^i = e_i$, by (5) it follows

$$F = Q.$$

This and Lemma 2.2 guarantee that $\bar{w}$ exists and is unique. It can be written as:

$$\bar{w} = \sum_{i=1}^{m} e_i \bar{w}_i$$

In view of Lemma 2.1, $\bar{x}$ is the solution of

$$Ax = b - H\bar{w}$$

hence, by linearity, eq. (2.4) follows. ⌑

Theorem 2.1 induces straightforwardly the following Modification Algorithm.

STEP 0 Let $B \in \mathbb{R}^{n^2}$, $H \in \mathbb{R}^{n\times m}$, $K \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^n$, $b_o \triangleq - b$, $x(e_o) \triangleq x^o$, $|b_1 \vdots b_2 \vdots \cdots \vdots b_m| \triangleq H$, $i = 0$.

STEP 1 Compute the LU factorization of B.

STEP 2 Compute the solution $x(e_i)$ by forward elimination and back substitution of $Bx = - b_i$.

STEP 3 If $i = m$, go to STEP 4, otherwise $i = i + 1$ and go to STEP 2

STEP 4 Compute the LU factorization of

$$I_m - |K'x(e_1) \vdots \cdots \vdots K'x(e_m)| = I_m - Q$$

STEP 5 Compute the solution $\bar{w}$ by forward elimination and back substitution of $(I_m-Q)\ w = K'x(e_o)$.

STEP 6 Compute $\bar{x} = x(e_o) + \sum_{i=1}^{m} \bar{w}_i\ x(e_i)$.

STEP 7 End. ¤

Remarks 2.1. From a conceptual point of view MOD can be viewed as an application of what Mesarovic calls the Interaction Prediction Principle [12]. In fact, if system (2.2) is written as

$$Bx = b - HK'x = b - Hw$$

w can be interpreted as an interaction variable describing the effect of the modification occurred in B. The m versors $e_1,\ldots,e_m$ can be viewed as m independent estimates of the exact value of w. STEP 4 and 5 can be considered as the exact computation of the modification variable carried on by the "predictor" on the base of the informations achieved during the m previous stages of MOD and coded in Q.

III. COMPUTATIONAL REMARKS

In this section, the complexity of MOD algorithm is evaluated and compared with other available modification techniques, namely Householder's Formula and Bennett's algorithm.

Assumption 3.1. The number of multiplications required by a method is considered as its complexity measure. (Inversions are counted as multiplications.) ¤

Assumption 3.2. $n >> m$.

¤

Under these assumptions, STEP 1 of MOD requires

$$n + \sum_{k=1}^{n-1} \zeta_k(\gamma_k + 1) \tag{3.1}$$

operations, where $\zeta_k + 1$ is the number of nonzero elements in the first row and $\gamma_k + 1$ is the number of nonzero elements in the first column of the reduced matrix of order $n - k + 1$ during the k-th step of Gaussian Elimination performed in natural order on B[13]. STEP 2 requires $m + 1$ times the forward elimination and back substitution of the system $Bx = - b_i$ and then

$$(m + 1)\, n + (m + 1) \sum_{k=1}^{n-1} (\gamma_k + \zeta_k) \tag{3.2}$$

operations; STEP 4 requires

$$\frac{1}{3}m^3 - \frac{1}{3}m + mn \tag{3.3}$$

operations, being $I_m - Q$ in general full; STEP 5 requires

$$m^2 \tag{3.4}$$

operations and STEP 6 requires

$$mn \tag{3.5}$$

operations. In Table I, the above results are summarized and compared with Householder's formula and Bennett's algorithm, also in the case of B, H, K' full matrices.

It has to be noted that in the case of full matrices TA requires about half of the time required by Bennett's algorithm. In the general case of sparse matrices, TA requires less operations than Bennett's method if the following inequality holds:

$$n < \frac{(m+1)\sum_{k=1}^{n-1}(\gamma_k + \zeta_k)}{3m} \tag{3.6}$$

It has to be noted that in many cases the sparsity structure of B is such that $\sum_{k=1}^{n-1} (\gamma_k + \zeta_k) = \alpha n$, where α ranges from 3 to 20.

IV. OPTIMAL DECOMPOSITION PROBLEM

This section deals with the important case where the system under consideration consists of a number of interconnected subsystems so that the unknown vector x can be "a priori" thought as "naturally" partitioned into a number of subvectors while an interaction pattern which is very

	Initial inversion or decomposition		Initial solution		Solution of the modified system	
	full	sparse	full	sparse	full	sparse
Householder	n^3	n^3	n^2	n^2	$4n^2$	$4n^2$
Bennett	$1/3\ n^3$	$\sum_{k=1}^{n-1} (\gamma_k\zeta_k+\zeta_k)$	n^2	$\sum_{k=1}^{n-1}(\gamma_k+\zeta_k)+n$	$(2m+1)n^2$	$(2m+1) \sum_{k=1}^{n-1} (\gamma_k+\zeta_k)+n$
MOD	$1/3\ n^3$	"	n^2	"	mn^2	$3mn + m \sum_{k=1}^{n-1} (\gamma_k+\zeta_k)$

Table I. Comparison of Householder's Formula, Bennett's Algorithm and MOD.

strong among the elements of the same subvector and relatively weak among elements of different subvectors.

In this situation, it is generally possible, in view of Theorem 2.1, to decompose the original problem into a number of simpler subproblems the solutions of which when suitably combined, give a solution of the original problem. Such a decomposition can actually be done in more than a single way, so that an obviously important task is to find, among all the possible decompositions, at least one of those which are optimal in some specific sense. One such problem is, in general, extremely difficult to solve and it is even hard to find situations where the computational effort needed to solve it is worth to be paid. However, it is quite reasonable, when searching for an optimal decomposition, to restrain the attention to some suitable subclass of all possible decompositions, thus resulting in a computationally feasible and economically efficient procedure. The herein adopted approach consists in restraining the search for an optimal decomposition only to those decompositions which retain, in a sense, the natural structure of the given system, which is supposed to originally consist of a number of well identified interacting subsystems.

In order to illustrate the decomposition procedure and specifically state the corresponding optimization problem it is necessary to introduce some further definitions and notations.

Let Q_k be the set of the first k integers, Σ_k be the class of all ordered sets of k elements, $\mathcal{P}_k$ and Π_{kh} be the classes of all possible permutations from Σ_k to Σ_k and partitions from Σ_k to Σ_h, $h \leq k$, respectively.

Definition 4.1

A partition $\pi(\cdot) \in \Pi_{kh}$ is said to be regular if

$$\pi(Q_k) = \{\{1,2,\ldots,j_1\}, \{j_1+1,j_1+2,\ldots,j_2\},\ldots,\{j_{h-1}+1,j_{h-1}+2,\ldots,k\}\}$$

for some $j_1, j_2, \ldots, j_{h-1} \in Q_k$, $0 \triangleq j_o < j_1 < j_2 <\cdots< j_{h-1} < j_h \triangleq k$.

⌑

Remark 4.1

Any regular partition $\bar{\pi}(\cdot) \in \Pi_{nq}$ induces in an obvious way a corresponding partition $\tilde{\pi}(\cdot)$ on R^{n^2}; formally:

$$\tilde{\pi}(\cdot): \quad A \mapsto \tilde{A}$$

$$\tilde{a}_{rs} = \begin{vmatrix} a_{j_{r-1}+1,\, j_{s-1}+1} & \cdots & a_{j_{r-1}+1,\, j_s} \\ \vdots & & \\ a_{j_r, j_{s-1}+1} & \cdots & a_{j_r, j_s} \end{vmatrix}, \quad \forall r,s \in Q_q.$$

⌑

Definition 4.2

A matrix $A \in R^{n^2}$ is block lower triangular with respect to a regular partition $\bar{\pi}(\cdot) \in \Pi_{nq}$ if $\tilde{A}$ is lower triangular; i.e. if $\tilde{a}_{rs} = 0$ whenever $r < s$; $r, s \in Q_q$.

⌑

Proposition 4.1

For any $\pi(\cdot) \in \Pi_{kh}$ there exist a unique regular partition $\bar{\pi}(\cdot) \in \Pi_{kh}$ and a unique permutation $p(\cdot) \in \mathcal{P}_k$ such that $\pi(\cdot) = \bar{\pi}(p(\cdot))$.

⌑

Remark 4.2

Proposition 4.1 says that any partition is a regular partition of a permutation. The partition $\bar{\pi}(\cdot)$ and the permutation $p(\cdot)$ will henceforth be referred to as the regular partition and the permutation defined by $\pi(\cdot)$.

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<u>Remark 4.3</u>

Any permutation $p(\cdot) \in \mathcal{P}_n$ induces in an obvious way a corresponding (symmetric) permutation $\hat{p}(\cdot)$ on R^{n^2}; formally:

$$\hat{p}(\cdot): \quad A \mapsto EAE'$$

where $E \in R^{n^2}$ is a unimodular matrix defined as follows. Let $\{p_1, p_2, \ldots, p_n\} \triangleq p(Q_n)$, then $e_{ij} = \delta_{p_i, j}$, $\forall i,j \in Q_n$, where $\delta_{r,s}$ is the Kronecker function.

¤

<u>Definition 4.3</u>

For any $\pi(\cdot) \in \Pi_{nq}$, let $T_\pi(\cdot) \triangleq \tilde{\pi}(\hat{p}(\cdot))$ where $\tilde{\pi}(\cdot)$ and $\hat{p}(\cdot)$ are induced by the regular partition $\bar{\pi}(\cdot)$ and the premutation $p(\cdot)$ defined by $\pi(\cdot)$. ¤

<u>Definition 4.4</u>

A matrix $A \in R^{n^2}$ is block reducible relative to $\pi(\cdot) \in \Pi_{nq}$ if there exists $p^o(\cdot) \in \mathcal{P}_q$ such that $T_{p^o(\pi)}(A)$ is lower triangular. ¤

<u>Remark 4.4</u>

Consider the problem of finding x such that $Ax = b$, where $A \in R^{n^2}$ and $b \in R^n$ are given (Basic Problem). Assume also that the system under consideration consists of q interconnected subsystems and that this kind of structural information can be specified by means of a regular partition $\bar{\pi}(\cdot) \in \Pi_{nq}$. If A is block reducible relative to $\bar{\pi}(\cdot)$, then splitting the Basic Problem into an equivalent set of q (partially ordered) subproblems is an almost trivial task. If this is not the case, a conceivable approach, in view of Theorem 2.1 consists in looking for a minimum rank matrix C such that A − C is block reducible relative to a partition $\pi(\cdot) \in \Pi_{n\mu}$, $\mu \leq n$, the largest element of which has a cardinality significantly less than n. This kind of approach is followed in the sequel, where an optimal decomposition problem is formally stated and solved. ¤

Given $A \in R^{n^2}$ and $S \subset Q_n$, the reduction of A to $Q_n - S$ is denoted by A_S. Formally, let $\pi(\cdot) \in \Pi_{n2}$ be such that $\pi(Q_n) = \{Q_n - S, S\}$; furthermore, let $\tilde{\tilde{A}} \triangleq T_\pi(A)$; then, $A_S \triangleq \tilde{\tilde{a}}_{11}$.

Given $\pi(\cdot) = \{\pi_1(\cdot), \pi_2(\cdot), \ldots, \pi_q(\cdot)\} \in \Pi_{nq}$ and $S \in Q_n$, with $|S| = m$, let $\pi_S(\cdot) \in \Pi_{\gamma q}$, $\gamma \triangleq n - m$, be such that

$$\pi_S(Q_n - S) = \{\pi_1(Q_n)\theta S,\ \pi_2(Q_n)\theta S, \ldots,\ \pi_q(Q_n)\theta S\},$$

where

$$S_1 \theta S_2 \triangleq S_1 - (S_1 \cap S_2)$$

For any pair of sets S_1 and S_2.

<u>Definition 4.5</u>

For any $S \subset Q_n$, the matrix $A \in R^{n^2}$ is block S-reducible relative to $\pi(\cdot) \in \Pi_{nq}$ if there exists $p^*(\cdot) \in \mathcal{P}_q$ such that $T_{p^*(\pi_S)}(A_S)$ is lower triangular. ¤

<u>Proposition 4.2</u>

Let $A \in R^{n^2}$, $S \subset Q_n$ and $\pi(\cdot) \in \Pi_{nq}$. If A is block S-reducible relative to $\pi(\cdot)$, then there exists $\pi^*(\cdot) \in \Pi_{n2}$ such that $\tilde{\tilde{a}}_{11}$ is block lower triangular with respect to the regular partition defined by $p^*(\pi_S(\cdot))$, where $\tilde{\tilde{A}} = T_{\pi^*}(A)$. ¤

Example 4.1. Let $A \in \mathbb{R}^{8\times 8}$ and $\pi(8) = \{\{1,2,3,4\}, \{5,6\}, \{7,8\}\}$ (Fig. 1). Let $S = \{2,6\}$. A is block S-reducible. In fact, $\pi_S(6) = \{\{2,3,4\}, \{5\}, \{7,8\}\}$ and there exists a permutation $p^*(\pi_S) = \{\{5\}, \{7,8\}, \{2,3,4\}\}$ such that $T_{p^*(\pi_S)}(A_S)$ is lower triangular (Fig. 2). ¤

An optimization problem which, in view of Theorem 4.1, Remark 4.4 and Proposition 4.2, is of obvious interest can now be formally stated as follows.

Optimal Decomposition Problem (ODP)

Given $A \in R^{n^2}$ and $\pi(\cdot) \in \Pi_{nq}$ find $S \subset Q_n$ of minimal cardinality such that A is block S-reducible relative to $\pi(\cdot)$. ¤

In order to solve the problem above, it is quite natural to restate it in graph-theoretical terms.

A directed hypergraph $H = (X,Y)$ is constituted by a node set X and an arc set Y the elements of which are ordered pairs of nonempty subsets of X. Let $\Omega = \{\omega_1,\omega_2,\ldots,\omega_t\}$, $t > 1$, be an ordered subset of X. If, for each $\omega_i \in \Omega$, there exists a pair Z_-^i and Z_+^i of subsets of X such that

$$\omega_i \in Z_-^i \cap Z_+^i$$

$$\eta_i \triangleq (Z_+^i, Z_-^{i+1}) \in Y, \ \eta_i \neq \eta_j, \quad \forall i,j \in Q_t, \ i \neq j,$$

where

$$Z_-^{t+1} \triangleq Z_-^1,$$

then Ω is a cycle of H.

A directed hypergraph without cycles is said to be acyclic.

Given a directed hypergraph $H = (X,Y)$ and a subset Z of X, the section hypergraph of H with respect to Z is an hypergraph $H_Z = (X - Z, Y_Z)$, where

$$Y_Z \triangleq \{(\tilde{X}_i,\tilde{X}_j) | \tilde{X}_i = X_i \theta Z,\ \tilde{X}_j = X_j \theta Z, (X_i,X_j) \in Y\}.$$

Any subset S of X is an essential set of H if H_S is acyclic. Any essential set of minimum cardinality is said to be a minimum essential set. Its cardinality is said to be the index of H.

Given a matrix $A \in R^{n^2}$ and a partition $\pi(\cdot) = \{\pi_1(\cdot),\pi_2(\cdot),\ldots,\pi_q(\cdot)\} \in \Pi_{nq}$, let $H(A,\pi)$ be the direct hypergraph relative to A and $\pi(\cdot)$ defined as follows: $H(A,\pi) = (X,Y)$ where $X = Q_n$, $Y = Y_{ex} \cup Y_{in}$ where

$$Y_{ex} = \{(r,s) | r \in \pi_j(Q_n),\ s \in \pi_i(Q_n),\ i,\ j \in Q_q,\ i \neq j,\ a_{rs} \neq 0\}$$

and $$Y_{in} = \{y^j_{in} | y^j_{in} = (\pi_j(Q_n),\ \pi_j(Q_n))\ \forall j \in Q_q\}$$

The arcs in Y_{ex} are called external arcs, the arcs in Y_{in} are called internal arcs. It has to be noted that $|Y_{in}| = q$.

Example 4.2. The hypergraph $H(A,\pi)$ associated to A, π of Fig. 1 is shown in Fig. 3. ¤

Lemma 4.1

For any $A \in R^{n^2}$, $\pi(\cdot) \in \Pi_{nq}$ and $p(\cdot) \in \mathcal{P}_q$, $H(A,\pi)$ is isomorphic to $H(A,p(\pi))$. ¤

Lemma 4.2

For any $A \in R^{n^2}$, $\pi(\cdot) \in \Pi_{nq}$, let $\bar{\pi}(\cdot)$ and $p(\cdot)$ be the regular partition and the permutation defined by $\pi(\cdot)$; i.e.: $\pi(\cdot) = \bar{\pi}(p(\cdot))$. Then $H(A,\pi)$ is isomorphic to $H(\hat{p}(A),\bar{\pi})$ where $\hat{p}(\cdot)$ is the (symmetric) permutation on R^{n^2} induced by $p(\cdot)$.

¤

Lemma 4.3

For any $A \in R^{n^2}$ and $\pi(\cdot) \in \Pi_{nq}$, $H(A,\pi)$ is acyclic if and only if A is block reducible relative to $\pi(\cdot)$.

Proof

If A is block reducible relative to $\pi(\cdot)$, then (see Definition 4.4) there exists $p^o(\cdot) \in \mathcal{P}_q$ such that $T_{p^o(\pi)}(A)$ is lower triangular. Thus, if $\pi(\cdot)$ and $p(\cdot)$ are the regular partition and the permutation defined by $p^o(\pi(\cdot))$, the hypergraph $H(\hat{p}(A),\bar{\pi})$ is acyclic since none of its external arcs is going from $\bar{\pi}_i(Q_n)$ to $\bar{\pi}_j(Q_n)$, whatever $i, j \in Q_q$ may be, with $i > j$. Since, in view of Lemma 4.2, $H(\hat{p}(A),\bar{\pi})$ is isomorphic to $H(A,p^o(\pi))$ and, in view of Lemma 4.1, $H(A,p^o(\pi))$ is isomorphic to $H(A,\pi)$, the conclusion can be drawn that $H(A,\pi)$ is acyclic.

Conversely, if $H(A,\pi)$ is acyclic, then there exists $p^\infty(\cdot) \in \mathcal{P}_q$ such that, letting $\pi^\infty(\cdot) \triangleq p^\infty(\pi(\cdot))$, $H(A,\pi^\infty)$ has no external arcs going from $\pi^\infty_i(Q_n)$ to $\pi^\infty_j(Q_n)$, whatever $i, j \in Q_q$ may be, with $i > j$. This means that $T_{p^\infty(\pi)}(A)$ is lower triangular, hence A is block reducible relative to $\pi(\cdot)$.

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Theorem 4.1

For any $A \in R^{n^2}$, $S \subset Q_n$ and $\pi(\cdot) \in \Pi_{nq}$, A is block S-reducible relative to $\pi(\cdot)$ if and only if S is an essential set of $H(A,\pi)$.

Proof. If S is an essential set of $H(A,\pi)$, $H_S(A,\pi)$ is acyclic. Let A_S be the reduction of A to $Q_n - S$. By definition of section hypergraph and of $H(A_S,\pi_S)$, $H(A_S,\pi_S)$ is isomorphic to $H_S(A,\pi)$. In view of Lemma 4.3, A_S is block reducible relative to $\pi(\cdot)$ and, according to Proposition 4.2, A is block S-reducible relative to $\pi(\cdot)$.

Conversely, if A is block S-reducible relative to $\pi(\cdot)$, according to Definition 4.5, there exists $p^*(\cdot) \in \mathcal{P}_q$ such that $T_{p^*(\pi_S)}(A_S)$ is lower triangular. According to Definition 4.5 and Lemma 4.3, $H(A_S,\pi_S)$ is acyclic. Being $H_S(A,\pi)$ isomorphic to $H(A_S,\pi_S)$, S is an essential set of H. ¤

Corollary 4.1. Given $A \in \mathbb{R}^{n^2}$ and $\pi(\cdot) \in \Pi_{nq}$, ODP is equivalent to the determination of a minimum essential set of $H(A,\pi)$.

It has to be noted that the problem of finding a minimum essential set in $H(A,\pi)$ can be easily proven to be hard.† In fact, it can be considered a generalization of the minimum feedback vertex set problem which is well known to be hard [14]. The minimum feedback vertex set problem has been investigated in a number of papers and some satisfactory approaches have been developed. In particular, preliminary simplifications [15,16,17], branch and bound techniques [15,16,17,18] and near optimal algorithms [17, 18] have been devised.

In the sequel, some definitions are introduced so that the extension of almost all the results obtained in relation with the minimum feedback

†A problem is said to be hard (NP-complete) if it belongs to a class of well-known combinatorial problems (covering, sequencing, knapsack, 0-1 integer programming, Hamiltonian circuit, etc.) which are equivalent, in the sense that no algorithm terminating with a number of steps bounded by a polynomial in the dimension of the problem (length of the input) exists for their solution. Moreover, it has been shown that a polynomial bounded algorithm for one of them yields polynomial bounded algorithms for all. This result strongly suggests that these problems will remain "intractable" perpetually [14].

vertex set problem to the problem of determining a minimum essential set of $H(A,\pi)$ is possible. This extension is possible due to the very particular structure of the hypergraph associated to A.

Definition 4.6. The elimination of $x \in \pi_j(Q_n)$ from $H(A,\pi)$ is accomplished: (i) forming the section hypergraph $H_x(A,\pi)$, (ii) adding a set of new edges $Y^x_{ex} = \{(r,s) \mid (r,s) \notin Y_{ex}, (r,x) \in Y_{ex}, (x,s) \in Y_{ex} \forall s \in \pi_j(Q_n)\}$.

¤

Definition 4.7. Let $x \in \pi_j(Q_n)$ be a node of $H(A,\pi)$:

(i) The external out-degree of x, $d^+_{ex}(x)$ is

$$d^+_{ex}(x) = |Y^+_{ex}(x)| \text{ where } Y^+_{ex}(x) = \{y \in Y_{ex} \mid y = (x,z), z \in Q_n\}$$

(ii) The external in-degree of x, $d^-_{ex}(x)$ is

$$d^-_{ex}(x) = |Y^-_{ex}(x)| \text{ where } Y^-_{ex}(x) = \{y \in Y_{ex} \mid y = (z,x), z \in Q_n\}$$

¤

A self-loop is an edge $y = (r,s)$ in Y_{ex} such that $r \equiv s$.

Proposition 4.3. The following local transformations of $H(A,\pi)$ are index preserving:

R1. Elimination of x when $\min(d^-_{ex}(x), d^+_{ex}(x)) \leq 1$ [15].

R2. Deletion of all the edges $\in Y_{ex}$ incident at x except those forming doublets†, if after removing those edges in the doublets, $\min(d^-_{ex}(x), d^+_{ex}(x)) = 0$. [15].

R3. Deletion of $(y,x) \in Y_{ex}$ if $(y,z) \in Y_{ex}$ whenever $(x,z) \in Y_{ex}$.

†A doublet is a cycle formed with arcs in Y_{ex} of length 2.

Likewise for $(x,y) \in Y_{ex}$. [16].

¤

Proposition 4.4. If $(x,x) \in Y_{ex}$, $H_x(A,\pi)$ has an index which is one less than the original. [15]. ¤

The previous Propositions form a basis for efficient branch and bound techniques. [15,16]. Moreover, the preliminary reduction performed by means of the rules described in Proposition 4.3 and 4.4 can often determine a minimum essential set [17]. As an example, consider the hypergraph in Fig. 3. By means of R1 and Proposition 4.4 repetitively applied, the minimum essential set $S = \{2,6\}$ is eventually obtained.

An alternative approach consists in applying, instead of branch and bound techniques when the preliminary reductions fail, near optimal algorithms in order to find non optimum but "good" solutions.

Proposition 4.5. The following algorithm [18] can find a minimal essential set of $H(A,\pi)$, i.e., an essential set S of $H(A,\pi)$ such that no proper subset of S is also an essential set of $H(A,\pi)$:

STEP 0. Set $H_o \triangleq (X_o,Y_o) = H(A,\pi)$, $n = |X_o|$, $i = 0$, $S = \phi$.

STEP 1. If there exists an edge $(x_i,x_i) \in Y_{iex}$, go to STEP 2, else, go to STEP 4.

STEP 2. Form the section hypergraph of H_i w.r.t. x_i, H_{ix_i} and set $H_{i+1} = H_{ix_i}$. $S = S \cup \{x_i\}$, go to STEP 4.

STEP 3. Eliminate any x_i in H_i and set H_{i+1} equal to the obtained hypergraph.

STEP 4. $i = i + 1$, if $i = n$, go to STEP 5, else go to STEP 1.

STEP 5. End. ¤

V. THE TEARING ALGORITHM

In this section, a particular way of applying MOD to a system $\tilde{\tilde{A}}x = b$, where $\tilde{\tilde{A}}$ is defined in Proposition 4.2 and is assumed to be obtained by means of any decomposition algorithm, is described and some computational remarks are given.

Let $\tilde{\tilde{a}}_{11}$ be as in Proposition 4.2; then

$$\tilde{\tilde{a}}_{11} = \begin{vmatrix} \tilde{\tilde{a}}''_{11} & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \tilde{\tilde{a}}''_{21} & \tilde{\tilde{a}}''_{22} & & \underset{\sim}{0} \\ \vdots & \vdots & \ddots & \\ \tilde{\tilde{a}}''_{q1} & \cdots & & \tilde{\tilde{a}}''_{qq} \end{vmatrix} ;$$

let the dimension of $\tilde{\tilde{a}}''_{kk}$ be p_k, $k = 1, \ldots, q$; let x be partitioned according to $\pi^*(\cdot)$ into $\begin{vmatrix} x^1 \\ x^2 \end{vmatrix}$; let x^1 be partitioned according to $p^*(\pi_S(\cdot))$ into $\begin{vmatrix} x^1_1 \\ \vdots \\ x^1_q \end{vmatrix}$ and b be partitioned into $\begin{vmatrix} b^1 \\ b^2 \end{vmatrix}$, where b^1 is partitioned into $\begin{vmatrix} b^1_1 \\ \vdots \\ b^1_q \end{vmatrix}$ as well. In view of MOD, the following algorithm can be applied:

TE (TEaring) Algorithm.

STEP 0. Let $C \triangleq |C_1 \vdots C_2|$ where $C_1 \triangleq \underset{\sim}{0}$ and $C_2 \triangleq \begin{vmatrix} \tilde{\tilde{a}}_{12} \\ \tilde{\tilde{a}}_{22} - Im \end{vmatrix}$;

$$B \triangleq \tilde{\tilde{A}} - C = \begin{vmatrix} \tilde{\tilde{a}}_{11} & \underset{\sim}{O} \\ \tilde{\tilde{a}}_{21} & I_m \end{vmatrix}, \quad x(e_o) \triangleq x^o, \quad {}^o b \triangleq b, \quad \left| {}^1b \vdots {}^2b \vdots \ldots \vdots {}^m b \right| \triangleq C_2, \quad i = 0.$$

STEP 1. Compute the LU factorization of $\tilde{\tilde{a}}''_{kk}$, $k = 1, \ldots, q$.

STEP 2. $k = 0$.

STEP 3. $k = k + 1$. Compute the solution $x^1_k(e_i)$ of the following system by forward elimination and back substitution

$$\tilde{\tilde{a}}''_{kk} \, x^1_k = {}^i b^1_k - \sum_{p=1}^{k-1} \tilde{\tilde{a}}''_{kp} \, x^1_p(e_i)$$

STEP 4. Compute $x^2(e_i) = {}^i b_2 - \tilde{\tilde{a}}_{21} \, x^1(e_i)$.

STEP 5. If $k = q$, go to STEP 6, else, go to STEP 3.

STEP 6. $i = i + 1$. If $i = m + 1$ go to STEP 7, else go to STEP 2.

STEP 7. Compute the LU factorization of

$$I_m - \left| x^1(e_1) \vdots \cdots \vdots x^1(e_m) \right| = I_m - Q$$

STEP 8. Compute the solution $\bar{x}^2$ by forward elimination and back substitution of

$$(I_m - Q) \, x^2 = x^2(e_o)$$

STEP 9. Compute $\bar{x}^1 = x^1(e_o) + \sum_{p=1}^{m} \bar{x}^2_p \, x^1(e_p)$

STEP 10. End.

¤

<u>Remark 5.1</u>. H and K in MOD are equal respectively to C_2 and $\begin{vmatrix} 0 \\ \tilde{I}_m \end{vmatrix}$ in TE.

¤

In order to compare the performances of TE with respect to LU decomposition method the number of multiplications needed in TE is evaluated.

Let $\varepsilon_i^k + 1$ be the number of nonzero elements in the first row and $\xi_i^k + 1$ be the number of nonzero elements in the first column of the reduced matrix of order $p_k - i + 1$ during the i-th step of Gaussian Elimination performed in natural order on $\tilde{\tilde{a}}''_{kk}$. Let N be the total number of nonzero elements in $\tilde{\tilde{a}}_{21}$ and $\tilde{a}''_{r1}$, $r < 1$. TE requires

$$\sum_{k=1}^{q} \sum_{i=1}^{p_k} (\varepsilon_i^k \xi_i^k + \varepsilon_i^k) \text{ multiplications} \tag{5.1}$$

in STEP 1

$$n - m + (m + 1) \left(\sum_{k=1}^{q} \left(\sum_{i=1}^{p_k} (\varepsilon_i^k + \xi_i^k) + p_k\right) + N\right) \text{ multiplications} \tag{5.2}$$

in STEP 3.

$$m^3/3 - m/3 \text{ multiplications} \tag{5.3}$$

in STEP 7.

$$m^2 \text{ multiplications} \tag{5.4}$$

in STEP 8.

Globally,

$$\sum_{k=1}^{q} \sum_{i=1}^{p_k} (\varepsilon_i^k \xi_i^k + \varepsilon_i^k) + (m + 1) \left(\sum_{k=1}^{q} \left(\sum_{i=1}^{p_k} (\varepsilon_i^k + \xi_i^k) + p_k \right) + N \right) +$$

$$+ m^3/3 + m^2 - m/3 + m\,(n - m) \text{ multiplications} \tag{5.5}$$

LU decomposition requires [13]

$$\sum_{i=1}^{n-1} (\zeta_i \gamma_i + 2\zeta_i + \gamma_i) + n \text{ multiplications} \tag{5.6}$$

where ζ_i and γ_i have been defined in Section II.

<u>Assumption 5.1</u> The elimination orderings in LU decomposition and in TA are the same. Moreover, rows and columns have been rearranged so that the chosen elimination ordering is correspondent to the natural ordering.

$\square$

In view of Assumption 5.1, let $\eta_i^k = \zeta_i - \varepsilon_i^k$ and $\theta_i^k = \gamma_i - \xi_i^k$; then (5.6) can be rewritten as

$$\sum_{k=1}^{q} \sum_{i=1}^{p_k} (\varepsilon_i^k \xi_i^k + \varepsilon_i^k) + \sum_{k=1}^{q} \left(\sum_{i=1}^{p_k} (\varepsilon_i^k + \xi_i^k) + p_k \right) + \sum_{k=1}^{q} \sum_{i=1}^{p_k} (\varepsilon_i^k \theta_i^k + \eta_i^k \xi_i^k +$$

$$+ \eta_i^k \theta_i^k + 2\,\eta_i^k + \theta_i^k) + \sum_{k=n-m+1}^{n-1} (\zeta_k \gamma_k + 2\zeta_k + \gamma_k) + m. \tag{5.7}$$

TE requires less operations than LU decomposition if the following inequality holds:

$$m\left(\sum_{k=1}^{q} \left(\sum_{i=1}^{p_k} (\varepsilon_i^k + \xi_i^k) + p_k \right) + N \right) \quad m^3/3 + m(n - \tfrac{1}{3}) < \sum_{k=1}^{q} \sum_{i=1}^{p_k} (\varepsilon_i^k \theta_i^k +$$

$$+ \eta_i^k \xi_i^k + \eta_i^k \theta_i^k + 2\eta_i^k + \theta_i^k) + \sum_{k=n-m+1}^{n-1} (\zeta_k \gamma_k + 2\zeta_k + \gamma_k) + m \tag{5.8}$$

By inspection, it is possible to claim that TE tends to overcome when m decreases and η_i^k, θ_i^k increase w.r.t. N. A precise evaluation can be given if some particular cases are investigated. In Table II, some comparison are given.

For all the cases considered in Table II, the following assumptions hold: (i) $p_k = p$, $k = 1,\ldots,q$ and (ii) $\tilde{a}_{12}$, $\tilde{a}_{21}$ $\tilde{a}_{22}$ and $\tilde{a}''_{rs}$, $r \leq s$, $s = 1, \ldots, q$ are full matrices.

Remark 5.2. If the structure of the system is repetitive, i.e., if the matrices on the main diagonal of $\tilde{\tilde{a}}_{11}$ are equal, TE performs only once STEP 1, saving in this way many operations, while LU decomposition method cannot exploit this structural property. ⌑

Remark 5.3. TA does not generate any fill-in in $\tilde{\tilde{a}}''_{rs}$, $r \leq s$, and in $\tilde{\tilde{a}}_{21}$. ⌑

Remark 5.4. At the k-th step of TA only a square matrix of dimension p_k and a rectangular matrix of dimensions $p_k \times p_r$ (where $p_r = \max_{j=1,\ldots,k-1} p_j$) must be retained in the fast memory. This feature makes TA suitable for the analysis of large systems with small computers. ⌑

Remark 5.5. STEP 1 of TA can be accomplished by parallel computation. ⌑

VI. CONCLUSIONS

In this paper, tearing methods for solving large scale systems of linear algebraic equations have been discussed. In particular, a modification algorithm has been presented and proven to be more efficient than the most used techniques, in almost all the applications. Then, the problem of determining what are the "best" elements to be torn in the original given

Table II. Comparison of LU decomposition (Gaussian Elimination) and TA

Assumptions	Operations	
	TA	LU
$p >> q, m$	$\approx q/3\ p^3$	$\approx (q^2/4 + q/12)p^3$
$q >> m, p$	$\approx \frac{1}{2}\ p^2(m + 1)q^2$	$\approx (p^3/4 + (2m+1)p^2/4)q^2$
$q \approx p = \bar{p} >> m$	$\approx \frac{1}{2}\ (m + 4/3)\bar{p}^4$	$\approx \frac{1}{4}\ \bar{p}^5$

matrix A has been taken into account. In particular, the reduction of A into a block inferior triangular matrix with as few as possible "modifications" has been assumed as goal to achieve. Under reasonable assumptions, the problem of determining the optimal rearranging of A according to the selected goal, has been proven to be equivalent to a minimum essential set in an hypergraph associated to A. Some preliminary reduction rules and a near optimal algorithm derived straightforwardly from the minimum feedback vertex set problem reduction rules and near optimal algorithms, have been introduced. A particular way of applying the previously presented modification algorithm to a matrix rearranged according to the chosen criterion has been discussed and its performances compared with LU decomposition method.

Further work could be done in defining new criteria to be followed in rearranging A and in deriving efficient algorithms to solve the related optimization problems.

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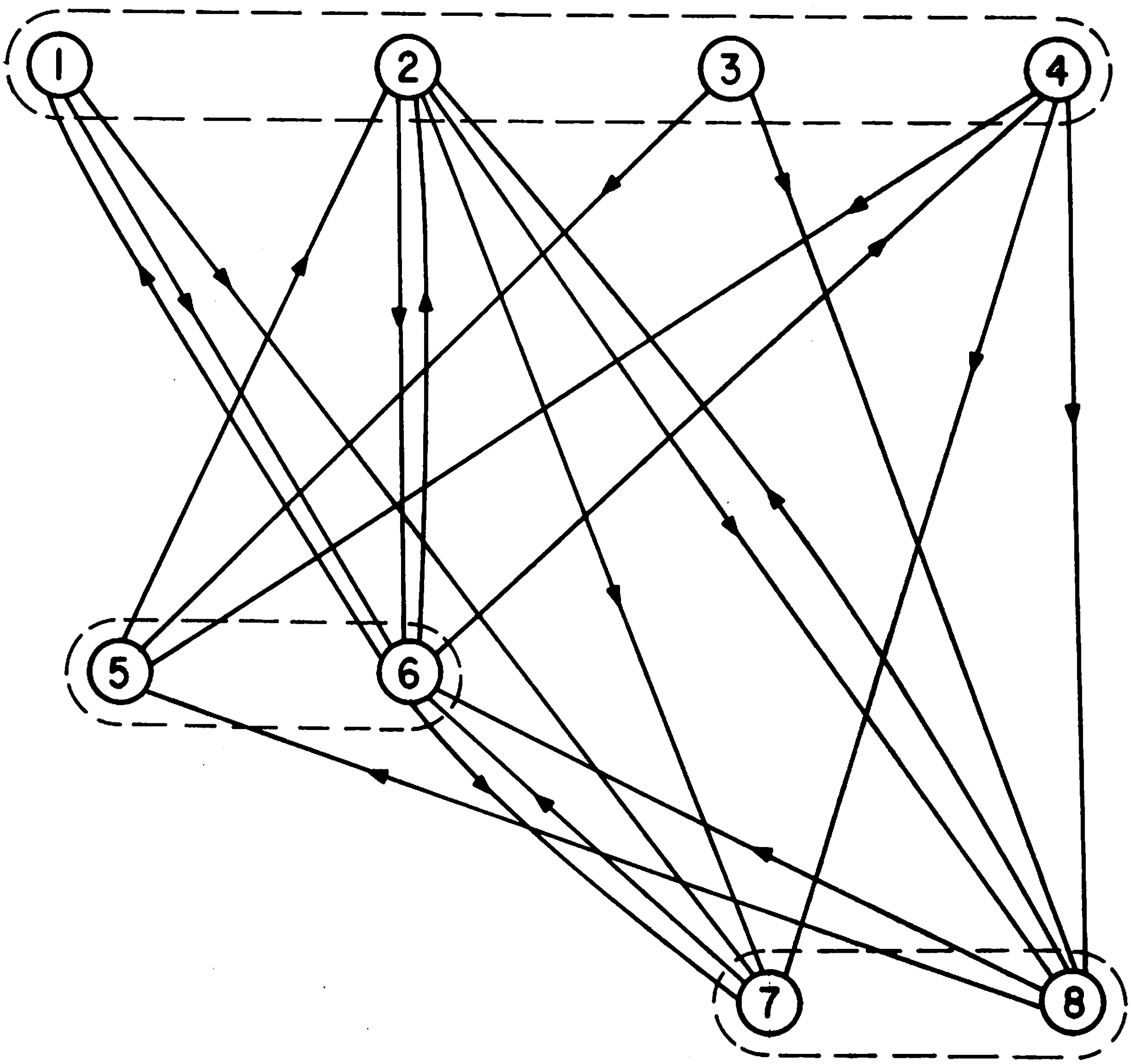
CAPTIONS

Fig. 1. A matrix A and its partition in blocks.

Fig. 2. $\tilde{\tilde{A}}$ generated by $p^*(\pi_S(6))$.

Fig. 3. The hypergraph associated to A, π of Fig. 1.



$y_{in}^{1} = (\{1,2,3,4\}, \{1,2,3,4\})$

$y_{in}^{2} = (\{5,6\}, \{5,6\})$

$y_{in}^{3} = (\{7,8\}, \{7,8\})$

Figure 3

$A =$

	1	2	3	4	5	6	7	8
1	X	X	X	X		X	X	
2	X	X	X	X		X	X	X
3	X	X	X	X	X			X
4	X	X	X	X	X		X	X
5		X			X	X		
6	X	X		X	X	X	X	
7						X	X	X
8		X			X	X	X	X

Figure 1

$\tilde{\tilde{A}} =$

	5	7	8	1	3	4	2	6
5	X						X	X
7		X	X					X
8	X	X	X				X	X
1		X		X	X	X	X	X
3	X		X	X	X	X	X	
4	X	X	X	X	X	X	X	
2		X	X	X	X	X	X	X
6	X	X		X		X	X	X

Figure 2