ANALYSIS OF MULTICOMMODITY COMMUNCATION NETS

by

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ANALYSIS OF MULTICOMMODITY COMMUNICATION NETS†

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Abstract

Two problems of analysis of multicommodity communication nets are dealt with in the paper:

a) checking the realizability of a given set of multicommodity requirements and providing a flow distribution for the realizable cases. And

b) finding the maximum value of the sum of multicommodity flows between a number of specified pairs of terminals.

Algorithms for solving both of these problems are presented based on the theory of nonlinear resistive networks. This network-theoretic approach eliminates the need for an exhaustive search and provides a way leading step-by-step to the optimal solution.

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1. Introduction

The problem of multi-commodity communication nets have been dealt with in a number of papers. However, only special cases in this field have actually been solved.

There exists an analogue of the max-flow - min-cut theorem for two commodity flows [1,2], and there are known some results for tree-structured communication nets, for a special class of "bi-path" networks [3], or for some integer branch capacity nets [9,10].

For the general case, however, the only available approach remains the linear program formulation, due to Ford and Fulkerson [4]. The original algorithm may be made more efficient [7] by applying the Dantzig-Wolfe decomposition principle. Its computational cost for normal networks remains, however, very large indeed.

The aim of this paper is to present another approach to analysis of multicommodity flow problems. It turns out that the method of analysis of one-commodity flows based on the theory of nonlinear resistive networks [5] may be extended to cover the case of multi-commodity flows.

The most important feature introduced by the network theoretical approach is that it is inherently capable of distributing the flow through the network in a way which minimizes some form of losses by adapting itself to the pecularities of the network topology, and by taking account of the potential ability of each edge and edge to participate in the transmission of flow.

On the computational side this approach eliminates the need for exhaustive research techniques and shows a systematic way leading step-by-step to solution.

In the paper, algorithms leading to solution of two main problems [8] of the multicommodity nets will be presented: (a) checking realizability of a given set of multi-commodity requirements and providing a flow distribution for realizable cases; (b) finding the maximum value of the sum of multi-commodity flows between specified pairs of terminals.

2. Max-Flow Min-Cut Theorem

For a given non-oriented communication net N let $G = (V,E)$ be the underlying linear graph, and suppose that each edge $[i,j] \in E$ has associated with it a non-negative real number $c[i,j] = c_{ij}$, the capacity of the edge $[i,j]$, which bounds from above the total amount of flow in either direction through this edge.

Before embarking on the more complex problem of multi-commodity flows, let us, in order to be to some extent self-contained, recall, with some modifications, the technique which can be applied when dealing with the one-commodity case.

Since the elements of linear network theory cannot provide the proper tools to describe the channel saturation property of a communication flow, it was proposed [5] to base the analysis of communication systems on the theory of nonlinear resistive networks.

In the case of a nonoriented communication net N the method requires to start with a network $\bar{N}$ isomorphic with N and to assign to an arbitrary element [j,k] of $\bar{N}$, corresponding to an edge with capacity $c_{j,k}$ of N, a nonlinear voltage-vs-current characteristic of the type shown in Fig. 1.

The characteristic has to satisfy the following conditions:

(a) It has to be confined to the strip between the straight lines $i = \pm c_{jk}$.

(b) Inside the strip the characteristic has a monotone character, i.e. it is a continuous curve going upwards and to the right with the border lines $i = \pm c_{jk}$ presenting its asymptotes.

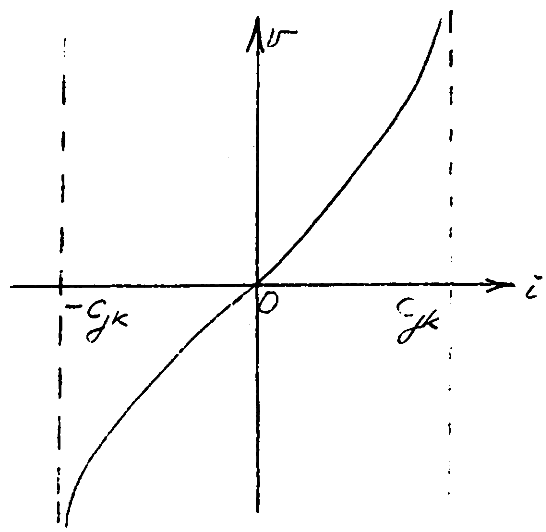

Fig. 1

It can readily be shown [5] that capacity of the min-cut between any pair of vertices (A,B) of the communication net N, and so the max-flow between these vertices may be found on such a network $\bar{N}$ simply by applying the following:

(i) The max-flow between any pair of vertices (A,B) of the communication net N is equal to the limit of the current between the corresponding vertices $\bar{A}$, $\bar{B}$ of the network $\bar{N}$, when the voltage applied between $\bar{A}$, $\bar{B}$ tends to infinity.

3. Distribution of Flow in the Net

Notice that the wording of (i) does not refer to any particular shape of the characteristics of the elements of $\bar{N}$, apart from requiring them to satisfy the conditions (a) and (b) of Section 1.

However, although the maximum current between any pair of vertices does not depend on the exact form of the current voltage relations of the elements, the distribution of the flow of current throughout the network can be very much influenced by the form of this dependence.

The problem of proper distribution turns out to be of special importance for the multi-commodity case. As is well known, not all combinations of multi-commodity flows can be accommodated in a net even if the sum of the flows of all the commodities is less than the maximum possible [8]. In order to respond to all such critical demands, the optimal flow distribution has to be evidently, very carefully looked for.

In trying to get some results for optimal flow distribution in communication net N by studying behaviour of the corresponding network $\bar{N}$, let us notice that the nonoriented character of N requires for the edge characteristics $v = v(i)$ in addition to (a) and (b) to satisfy the following:

(c) The characteristics $v_k = v_k(i_k)$, $k = 1,2,\ldots,e$ of all the e elements of $\bar{N}$ have to pass through the origin of coordinates and to be symmetric with respect to it; i.e.

$$v_k(0) = 0 \quad \text{and} \quad v_k(-i_k) = -v_k(i_k), \quad k = 1,2,\ldots,e .$$

In addition we shall require that

(d) The derivative $r_k(i_k) = dv_k(i_k)/di_k$ exists and does not vanish for any i_k in the open interval $-c_k < i_k < c_k$ for $k = 1,2,\ldots,e$.

We shall refer to its inverse:

$$g_k(i_k) = \frac{1}{r_k(i_k)} = \left(\frac{dv_k(i_k)}{di_k}\right)^{-1} \tag{1}$$

as to the differential conductance of the element k.

The initial value $g_k(0)$ of the differential conductance of the edge k without current is some positive real number. Because of the asymptotic character of the lines $i_k = \pm c_k$ there is

$$\lim_{i_k \to \pm c_k} g_k(i_k) = 0 \quad . \tag{2}$$

This dependence of differential conductance from current and its vanishing when the magnitude of current approaches its upper bound presents one of the most important features of this method.

In order that the modeling of N by means of $\bar{N}$ operates properly, we shall require that the actual differential conductance provide an indication of capability of the edge to accept additional current. More specifically, we require that:

(e) The differential conductance $g_k(i_k)$ be in the interval $0 \leq i_k < c_k$ a continuous monotonically decreasing function of the current i_k (tending to zero when $i_k \to c_k$). Notice that because of condition (c), $g_k(i_k)$ has in the interval $-c_k < i_k \leq 0$ a monotonically increasing character.

4. <u>One-Commodity Flow</u>

Suppose that an excitation current i enters $\bar{N}$ at a vertex 1, and leaves it at vertex 2. Suppose that the network has (n+1) vertices and e edges. Let us assign some arbitrary orientations to the edges of $\bar{N}$ and let A be the $(n \times e)$ incidence matrix of $\bar{N}$.

In order to find distribution of current flow in $\bar{N}$, i.e. the $(e \times 1)$ vector I_e of edge currents due to excitation current i, we shall iteratively compute ΔI_e caused by small increments Δi and sum them up, while the excitation increases from zero up to its final value i.

For a given vector $I_e = (i_1, i_2, \ldots, i_e)'$ of edge currents, the conductance of the edge k may be - for small current changes - taken to be equal to the differential conductance $g_k(i_k)$. Thus, for the given distribution vector I_e the diagonal branch conductance matrix takes on the form:

$$G = \mathrm{diag}[g_1(i_1), g_2(i_2), \ldots, g_e(i_e)] \tag{3}$$

or in short,

$$G = G(I_e) \ , \tag{4}$$

where $G(\cdot)$ presents a vector-to-matrix operator defined by (3) and (4).

Let V and I denote the $n \times 1$ vectors of the node voltages with respect to the reference and of excitation currents entering $\bar{N}$ at the n other than reference nodes, respectively.

The computation starts with $I^o = 0$, $V^o = 0$, $I_e^o = 0$.

With an increment Δi of excitation current between the vertices 1 and 2, there is

$$\Delta I = (\Delta i, -\Delta i, 0, 0, \ldots, 0)' \tag{5}$$

and the increment of ΔI_e may be computed by executing the following steps:

1. The node-to-datum short circuit admittance matrix of $\bar{N}(0)$, i.e. of the network $\bar{N}$ at zero state,

$$Y_n^o = Y_n(0) = AG(0)A' \ . \tag{6}$$

2. The open-circuit impedance matrix

$$Z(0) = Y_n^{-1}(0) \; . \tag{7}$$

3. The increment of the vector of node-to-reference voltages

$$\Delta V = Z(0)\Delta I = [z^1(0) - z^2(0)]\Delta i \tag{8}$$

Here $z^i(0)$ denotes the column i of the matrix $Z(0)$.

4. The increment of the vector of edge voltages

$$\Delta V_e = A'\Delta V = \Delta i A'[z^1(0) - z^2(0)] \tag{9}$$

5. The increment of the vector of edge currents:

$$\Delta I_e = G(0)\Delta V_e = \Delta i G(0)A'[z^1(0) - z^2(0)] \; . \tag{10}$$

Note that if one of the terminals of excitation is the reference node, then in ΔI in (5) there is just one nonzero element and between parentheses in (8), (9) and (10) there is just one column of $Z(0)$.

From the first iteration of the algorithm it follows that application of the excitation vector:

$$I^1 = I^0 + \Delta I = \Delta I \tag{11}$$

results in distribution vector of edge currents:

$$I_e^1 = I_e^0 + \Delta I_e = \Delta I_e \; . \tag{12}$$

For the next iteration a new increment Δi is applied, and the computation is repeated with the state of $\bar{N}$ defined by the edge currents vector I_e^1.

<u>Remark</u>: 1. Normally it will happen that some elements of the vector ΔV_e in (9) and ΔI_e in (10) will be positive and others negative or zero. Because of the symmetric character of the characteristics, there is $g_k(-i_k) = g_k(i_k)$ and the computation is not affected by the possible

difference in sign of the elements. Of course, if we prefer to have all elements nonnegative, we may change the orientation of some of the edges and this will result in: $\Delta V_e \geq 0$ and $\Delta I_e \geq 0$.

It may happen that some edge current, say i_k, passes through its stationary value at some value of the excitation current. If this happens, Δv_k and Δi_k change their sign in the next iteration, the change of differential conductance $g_k(i_k)$ is reversed and the procedure adapts itself automatically to the new situation. Because of the symmetric character of the characteristics no special attention is required to check if, due to these changes, the sign of the total edge current was changed during the computation or it remains invariant.

The procedure goes on and it terminates if either the required excitation i is achieved, or if at some iteration j the admittance matrix Y_n^j appears to be singular and its inversion required in step (2) cannot be executed.

Singularity of Y_n^j indicates that the network $\bar{N}$ is disconnected, with the vertices 1 and 2 belonging to different parts of $\bar{N}$. Thus at iteration j the differential conductances of the edges belonging to some cut separating the vertices 1 and 2 appear to be zero. Since $|Y_n^k| \neq 0$ for $k < j$, the value i^j of the excitation current at iteration j approximates the max-flow or min-cut of the network.

The exact position of min-cut can readily be located if we look for the zero diagonal elements of the diagonal branch conductance matrix $G(I_e^{j-1})$ corresponding to the vector of edge currents I_e^{j-1} in the $(j-1)$ iteration. The edges discovered in this way evidently contain the branches belonging to the min-cut (and may be some additional branches as well).

5. Optimal Characteristic of Network Elements

As has been mentioned in Section 2, the proper distribution of flow in the channels of a multi-commodity communication net is of paramount importance for its operation.

Since the procedures presented in this paper are sequential in character, leading in a finite number of steps to the required solution, it is important to provide an optimality criterion for all the intermediate stages of computation.

As it happens, the Kirchhoff distribution of currents throughout $\bar{N}$ shows a possible approach to the problem.

Suppose that at some stage of the procedure the vector of the edge currents turns out to be $I_e = (i_1, i_2, \ldots, i_e)$ so that at this stage the conductances of the edges of the network $\bar{N}$ have to be taken equal to $g_k(i_k)$, $k = 1,2,\ldots,e$.

Application of the current excitation increment Δi between the vertices 1,2 of $\bar{N}$ results in an increment $\Delta I = (\Delta i_1, \Delta i_2, \ldots, \Delta i_e)$ of the edge currents. As is well known [6] for a resistive network with external current sources, among all those distributions of edge currents, which obey the Kirchhoff current law, the distribution which satisfies Kirchhoff voltage law corresponds to the minimum of absorbed power.

Since the distribution of currents in network $\bar{N}$ satisfies both the Kirchhoff laws, the loss of power corresponding to ΔI_e:

$$P = \sum_{k=1}^{e} \frac{(\Delta i_k)^2}{g_k(i_k)} \tag{13}$$

presents the minimum power loss for all the possible distributions of the external current Δi throughout $\bar{N}$ obeying the Kirchhoff current law.

Returning to the communication net N we recall that the magnitude of the current $|i_k|$ is bounded by capacity c_k of the edge k, so that the difference $\varepsilon_k(i_k) = c_k - |i_k|$ presents the residual capacity of the edge k which still remains at our disposal.

There is $\Delta\varepsilon_k(i_k) = -\Delta|i_k|$ and if the characteristics of the elements would be chosen so that

$$g_k(i_k) = \varepsilon_k^2(i_k) \tag{14}$$

then the current distribution throughout the network $\bar{N}$ would - according to (13) - minimize the expression

$$P = \sum_{k=1}^{e} \left[\frac{\Delta\varepsilon_k(i_k)}{\varepsilon_k(i_k)}\right]^2 \tag{15}$$

The ratio $\frac{\Delta\varepsilon_k(i_k)}{\varepsilon_k(i_k)}$ presents the relative depletion of residual capacity of the edge k by the flow increment $\Delta\varepsilon_k(i_k)$.

Thus, it follows from the above discussion, that assigning to the elements of $\bar{N}$, v-i characteristics defined by (14), results in such a distribution of flow throughout the net that the sum of the squares of relative capacity depletions, shortly SSRCD, taken over all edges of the net will be minimum over all such distributions which obey Kirchhoff current law.

In order to translate the condition (14) into the voltage-current relation of a typical edge characteristic, we can write (omitting the subscript for simplicity):

$$\frac{dv}{di} = \frac{1}{g(i)} = \frac{1}{\varepsilon^2(i)} = \frac{1}{[c - |i|]^2} \tag{16}$$

For the positive branch of the characteristic, i.e, for $0 \leq i < c$, and assuming $v(0) = 0$, we get

$\frac{dv}{di} = \frac{1}{(c-i)^2}$, which with $v(0) = 0$ renders

$$v = \frac{1}{c-i} - \frac{1}{c} = \frac{i}{c(c-i)} \ .$$

For the negative branch: $-c < i \leq 0$, there is

$$\frac{dv}{di} = \frac{1}{(c+i)^2} \quad \text{and} \quad v = -\frac{1}{c+i} + \frac{1}{c} = \frac{i}{c(c+i)} \ .$$

Thus the $v - i$ relation for the whole open interval $(-c,c)$ turns out to be

$$v = \frac{i}{c(c - |i|)} \tag{17}$$

as shown in Fig. 2.

Such an h-characteristic (hyperbolic in each of the two quadrants) satisfies for each i in the open interval $(-c,c)$ the condition (16), so that for each current the differential conductance of such an element is equal to the square of its residual capacity.

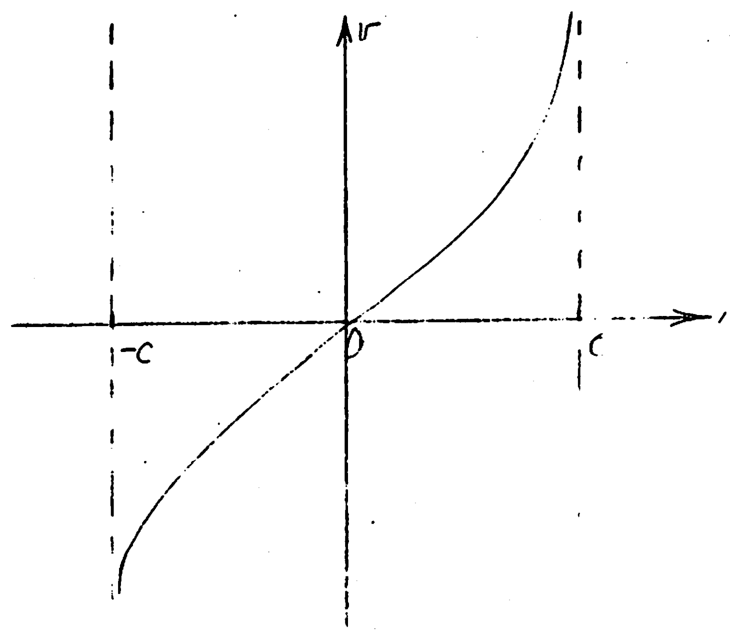

Fig. 2

We shall adopt henceforth as the criterion for optimality of the flow distribution in the net the condition of achieving the minimum of SSRCD.

Thus, the above discussion shows, that the optimal flow distribution of some commodity in a communication net N may be read out from the current distribution in the edges of the resistive network $\bar{N}$ with the h-shaped characteristics (17) assigned to all of its elements.

It may be noticed that in place of the logarithmic characteristic suggested in [5] introduction of the above optimality criterion leads to the modified characteristic (17).

As pointed out in Section 2, this change is of no consequence if the maximum of one-commodity flow is all that is looked for. However, for multi-commodity cases, the problem of proper characteristic shape cannot be left aside.

6. Multi-Commodity Flow with a Given Requirement Vector

Suppose there are p commodities which have to be transmitted simultaneously through the communication net N, with the commodity ℓ to be transmitted between the (ordered) pair of terminal vertices (s_ℓ, t_ℓ) which we shall refer to as source and sink, respectively. We shall not assume that $\{s_\ell, t_\ell\} \cap \{s_m, t_m\} = \phi$ for $\ell \neq m$, however $\{s_\ell, t_\ell\} \neq \{s_m, t_m\}$ for $\ell \neq m$, i.e. the pairs of terminals for different commodities cannot have more than one terminal in common. Thus if there are in N, $(n+1)$ vertices, then:

$$p \leq \frac{1}{2}n(n+1) \ . \tag{18}$$

Note that in undirected nets any distribution for the (s_ℓ, t_ℓ) case renders the solution for the (t_ℓ, s_ℓ) case, if we reverse the direction of the corresponding flow in each edge of the net with its magnitude remaining invariant.

Suppose there are in N, e edges, and let $\{c_\ell\}$, $\ell = 1,2,\ldots,e$ be the set of edge capacities.

Let $\bar{N}$ be as above, a directed network, isomorphic with N, with an appropriate h-characteristic assigned to each of its elements, and let A be its incidence matrix.

Suppose, there is a requirement for a set $\{r_\ell\}$, $\ell = 1,2,\ldots,p$, of amounts of the p commodities to be sent through the net.

Let us visualize the pairs of terminals (s_ℓ,t_ℓ), $\ell = 1,2,\ldots,p$, connected by additional edges: (t_ℓ,s_ℓ), representing the current sources (with vanishingly small conductances) through which the multi-commodity flows have to be injected into the net.

Let T be the (n×p) incidence matrix relating those additional edges to the vertices of the net, and let $Z^1 = (Y_n)^{-1}$ present, like in Section 3, the open-circuit impedance matrix of $\bar{N}$ in zero state.

It may be readily recognized that any column ℓ, $\ell = 1,2,\ldots,p$ of the matrix

$$B^1 = Z^1T \qquad (19)$$

presents, analogously to (8) the vector of node-to-reference voltages of the (linear) network $\bar{N}^1$ when just a current $i_\ell = 1$ enters this network through the edge (t_ℓ,s_ℓ) with the other additional edges being currentless.

Thus, premultiplication of B by A' renders, like in (9), a matrix

$$M^1 = A'B^1 = A'Z^1T \qquad (20)$$

whose columns present edge voltages corresponding to different commodity flows.

In order to get current distribution throughout the network, we have, like in (10) to premultiply M^o by the branch conductance matrix G^o. However, the results of Section 4 suggest, that if

$$C = \mathrm{diag}(c_1, c_2, \ldots, c_e) \tag{21}$$

is the diagonal branch capacitance matrix of N, then the recommended matrix G^o may be obtained as

$$G^1 = C^2 \; . \tag{22}$$

Thus denoting by $q^{\ell 1}$ the column ℓ of the matrix

$$Q^1 = C^2 M^1 = C^2 A' Z^1 T \; , \tag{23}$$

we realize that for a vanishingly small, positive real number $\Delta\rho$, the optimal distribution of the flow $\Delta\rho$ of any commodity ℓ through the communication net N, i.e., the distribution which minimizes the SSRCD is presented by $\Delta\rho q^{\ell 1}$ for $\ell = 1,2,\ldots,p$.

When going over to simultaneous multicommodity flows, we have to take into account that the flows of different commodities, when passing through the same edge never cancel, so that the sum of the magnitudes of all the flows passing through any edge is bounded by its capacitance.

Let the set of requirements $F = \{r_\ell\}$ be ordered and arranged in form of a diagonal matrix

$$R = \mathrm{diag}(r_1, r_2, \ldots, r_p) \; . \tag{24}$$

The method we shall apply in studying the possibility of accommodating $\{r_\ell\}$ through the net N will be:

(a) presenting R as a sum of, say, m increments:

$$\left.\begin{aligned} \Delta R_k &= \mu_k R \\ \sum_{k=1}^{m} \mu_k &= 1 \end{aligned}\right\} \tag{25}$$

with μ_k, $k = 1,2,\ldots,m$ being a small, positive, real number;

(b) trying sequentially to distribute each increment ΔR_k, $k = 1,2,\ldots,m$ in the optimal fashion through the net N, through which the flows ΔR_i, $i = 1,2,\ldots,k-1$ have already been distributed in the previous $k-1$ steps.*

Starting with the network $\bar{N}$ in the zero state - call its linear approximation $\bar{N}^1$ - we find Q^1, (23), by the steps described above.

Taking the product

$$J_e^1 = \Delta J_e^1 = Q^1 \Delta R_1 \tag{26}$$

we realize that each column ℓ of ΔJ_e^1, $\ell = 1,2,\ldots,p$ presents the vector of incremental currents flowing in the edges of $\bar{N}^1$ with just the excitation current Δr_ℓ^1 applied between the vertices (s_ℓ, t_ℓ) of the network.

When all the p-commodities are flowing simultaneously through $\bar{N}$, then in order to evaluate the magnitude of the overall flow passing through the edges of the network, we introduce the symbol $\mathrm{mod}(J_e^1)$ to denote the matrix whose elements are the moduli of the elements of ΔJ_e^1.

If 1_p denotes a $(p \times 1)$ vector with all elements equal to unity, then

$$I_e^1 = \mathrm{mod}(J_e^1) \cdot 1_p \tag{27}$$

presents the vector of the magnitudes of the overall flow in the edges of $\bar{N}$.

This computation ends the first iteration.

The second iteration starts with the current I_e^1 flowing in the edges.

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* This algorithm is based on assumption, that the problem is <u>convex</u>.

Let $\text{diag } I_e^1$ denote the diagonal matrix obtained by placing the elements of I_e^1 along the main diagonal with their order preserved. The recommended branch conductance matrix of the linear network $\bar{N}^2$ approximating $\bar{N}$ at the beginning of the second iteration may be written as

$$G^2 = (C - \text{diag } I_e^1)^2 \; . \qquad (28)$$

In general if at the end of iteration $k-1$ the vector of the overall flow in the edges of $\bar{N}$ is I_e^{k-1} then $\bar{N}$ behaves for small excitations like the linear network $\bar{N}^k$.

Starting with the diagonal residual branch capacitance matrix

$$C^k = C - \text{diag } I_e^{k-1} \qquad (29)$$

the iteration k goes through the steps:

$$G^k = (C^k)^2 \qquad (30)$$

$$Y_n^k = AG^kA' \qquad (31)$$

$$Z^k = (Y_n^k)^{-1} \qquad (32)$$

$$M^k = A'Z^kT \qquad (33)$$

$$Q^k = G^kM^k \qquad (34)$$

$$\Delta J_e^k = Q^k \Delta R_k \; . \qquad (35)$$

The flow in $\bar{N}$ is defined by

$$J_e^k = J_e^{k-1} + \Delta J_e^k \qquad (36)$$

being equivalent to the overall flow

$$I_e^k = \text{mod } J_e^k 1_p \; . \qquad (37)$$

Notice that:

The computation of the edge currents for each commodity following the steps (29) - (36) may result in the elements of J_e^k and ΔJ_e^k being positive, negative or zero. In fact if the current $(J_e^k)_{s\ell}$ of some commodity ℓ, in an edge s at some iteration k passes through its stationary value, then the increments

$$(\Delta J_e^{k-1})_{s\ell} \text{ and } (\Delta J_e^{k+1})_{s\ell}$$

computed in step (35) differ in sign. In such a case the algorithm automatically adapts itself to the new situation and the magnitude of the flow of commodity s in edge ℓ, found in (36) passes through its extremum.

It is important to note that the operator mod is applied only in (37) when the overall depletion of the edge capacities defining the next state of the network is evaluated.

The computation may finish in one of two ways.

First, it may happen that at some iteration h+1 the matrix Y_n^{h+1} turns out to be singular (or nearly so), so that its inverse Z^{h+1} cannot be evaluated. This fact indicates that the edge capacities of some cut set separating the points s_ℓ and t_ℓ for some pair $q \in \{1,2,\ldots,p\}$ (or a number of such pairs) have been depleted and the network $\bar{N}^h$ is not connected any more.

This cut-set, call it Cs, may be identified by looking for the vanishing elements of the matrix G^{h+1} in (30). The set of these elements may contain besides Cs some additional elements which can readily be eliminated by inspection.

The maximal set of requirements proportional to the original one which can be accommodated by N is:

$$R_{max} = \left(\sum_{j=1}^{h} \mu_j \right) R \quad . \tag{38}$$

As a second alternative, the algorithm may successfully continue through all the iterations, required by (25), which shows that the whole set of requirements may be accommodated by N.

In each case the distribution of the flow throughout the edges of N for all the commodities may be read out from the matrix J_e obtained in executing step (36) for the last time, with the element $(J_e)_{s\ell}$ corresponding to the total flow of commodity ℓ in edge s.

The diagonal matrix C - diag I_e with I_e obtained in the last execution of step (37) presents the residual edge capacities, "unused" in accommodating the requirements presented to the net.

7. The Maximum Sum of Multi-Commodity Flows

Let us suppose that there are p communication sources applied to a given net N between given pairs (s_ℓ, t_ℓ), $\ell = 1,2,\ldots,p$ of vertices of N.

Let $\mathcal{F} = \{F_j\}$ be the family of sets $F_j = \{r_\ell^j\}$, $\ell = 1,2,\ldots,p$ of feasible requirements which can be accommodated through N and put $S_j = \sum_{\ell=1}^{p} r_\ell^j$. In what follows we shall often refer to the requirement r_ℓ as to communication source applied to the net.

Let $F^* \in \mathcal{F}$ be the set (or one of such sets) for which the sum S^* of requirements reaches the possible maximum, i.e.

$$S^* = \max_{F_j \in \mathcal{F}} S_j \quad . \tag{39}$$

Our intention will be to find S^* through defining F^* and the corresponding distribution of multicommodity flows in N.

Once more let the requirement sets F_j be presented in ordered form by the diagonal matrices R_j, (24), with R^* corresponding to F^*.

We shall start with the network $\bar{N}$ in the zero state and sequentially build up the looked for R^* as the sum of diagonal matrix increments:

$$R^* = \sum_{i=1}^{M} \Delta R_i^* \qquad (40)$$

with each R_k^*, $k = 1,2,\ldots,M$ being optimally distributed throughout the edges of $\bar{N}$ carrying already the flows corresponding to $R_{k-1}^* = \sum_{i=1}^{k-1} \Delta R_i^*$.

This time, however, there is an additional degree of freedom since we are free to choose the increment ΔR_k in an arbitrary fashion constrained only by the condition $\Delta R_k \geq 0$, $k = 1,2,\ldots,M$. The discussion which follows leads to the proper choice of these increments.

Suppose, like in the previous section, that at the end of iteration $(k-1)$ the matrix of the commodity flows J_e^{k-1} and the overall flow vector I_e^{k-1} in the network $\bar{N}$ under the excitation $R_{k-1}^* = \sum_{i=1}^{k-1} \Delta R_i^*$ are known.

The various operators related to the linear network $\bar{N}^k$ approximating the behaviour of $\bar{N}$ under these conditions can be found by performing the steps (29) - (34) of iteration k.

It follows from the previous discussion that any column ℓ, $\ell = 1,2,\ldots,p$, of the matrix Q^k presents the vector of edge currents of $\bar{N}^k$ when the excitation applied to $\bar{N}^k$ consists just of $r_\ell = 1$ with $r_j = 0$ for $j \neq \ell$.

In order to be able to express the following results in a concise form, let us agree to some shortcuts in writing.

If $A = (a_{pq})$ and $B = (b_{pq})$ are two matrices of the same order $(m \times n)$, then let us introduce a symbol $\square$ for an operation on A and B, and a matrix operator sign defined by:

$$A \square B = (a_{ij} \cdot b_{ij}) \tag{41}$$

and

$$\text{sign } A = (s(a_{ij})) \quad , \tag{42}$$

where $s(a_{ij}) = 1$ for $a_{ij} \geq 0$ and $s(a_{ij}) = -1$ for $a_{ij} < 0$.

Thus $A \square B$ presents actually a direct product of two matrices in which the multiplication is performed elementwise, and all elements of the matrix sign A are ± 1, the -1 being reserved only for the places where $a_{ij} < 0$.

Let now $\Delta\rho$ be a vanishingly small, positive, real number, and let us try to define the changes of the magnitudes of the edge currents of $\bar{N}$ when in addition to the excitation R^*_{k-1} there appears a $\Delta\rho$ current source of each of the p commodities, one at a time.

It may readily be recognized that the p columns of the matrix:

$$\Delta\rho \cdot T^k = \text{mod}(J_e^{k-1} + \Delta\rho Q^k) - \text{mod}(J_e^{k-1}) \tag{43}$$

present exactly the looked for magnitude changes. Namely the column ℓ, $\ell = 1,2,\ldots,p$ of T^k corresponds to the case when the additional excitation is: $\Delta r_\ell = 1$, and $\Delta r_j = 0$ for $j \neq \ell$.

Applying the symbols (41) and (42), we can rewrite (43) in a concise form

$$\Delta\rho T^k = \Delta\rho Q^k \square \text{ sign } J_e^{k-1} , \tag{44}$$

which leads to:

$$T^k = Q^k \,\square\, \text{sign}\, J_e^{k-1} \tag{45}$$

The meaning of (44) is, of course, that the change in the current magnitude is positive and equal to the current increment, if either the current is zero, or both the current and its increment are of the same sign. Otherwise this change is negative.

At this stage of discussion, we have to define the method of choosing, for the network N under the excitation R^*_{k-1} the optimal increment of excitation, ΔR^*_k.

To this end, consider the family of incremental requirement sets $K = \{\Delta F\}$, where $\Delta F = \{\Delta r_1, \Delta r_2, \ldots, \Delta r_p\}$, $\Delta r_i \geq 0$, $i = 1,2,\ldots,p$ such that

$$\sum_{i=1}^{p} \Delta r_i = \Delta\rho = \text{const.} \tag{46}$$

Each ΔF corresponds to a diagonal excitation matrix $\Delta R = \text{diag}(\Delta r_1, \Delta r_2, \ldots, \Delta r_p)$ which, when applied to $\bar{N}$ under the given excitation R^*_{k-1} renders the incremental multicommodity flow distribution matrix $\Delta J_e^k = Q^k \Delta R$ (35).

The reasoning which lead to (44) shows that the changes of the magnitudes of multicommodity flows in the edges of $\bar{N}$ are provided by elements of the matrix

$$T^k \Delta R \tag{47}$$

and the changes of the magnitudes of the equivalent, total edge currents are given by the vector equal to the sum of columns of (47), i.e. by the vector h:

$$h = T^k \Delta R \, 1_p \, , \tag{48}$$

where 1_p stands for the p-vector, each element of which is equal to unity.

Since the change of the overall flow magnitude in an edge is equal to the depletion of its capacity, the vector h, found in (48), provides as well the edge capacity depletions under the additional excitation ΔR.

Thus, with C^k, (29), standing for the diagonal matrix of edge capacities in iteration k, the vector d of relative capacity depletions is:

$$d = (C^k)^{-1} h \ . \tag{49}$$

Applying (48), (45) and (34) to (49), we get:

$$d = C^k (M^k \,\square\, \text{sign}\, J_e^{k-1}) \Delta R \cdot 1p \ . \tag{50}$$

Let us now define as the optimal member of the family K such a requirement set ΔF_k^* which leads to the maximum element of the vector <u>d</u> being minimal, i.e., to minimization of the maximum relative capacity depletion for all the edges of N.*)

Let us denote by W^k the $(e \times p)$ multicommodity depletion matrix:

$$W^k = (w_{ij}^k) = C^k (M^k \,\square\, \text{sign}\, J_e^{k-1}) \ . \tag{51}$$

The problem of finding the optimal increment ΔT_k^* may be cast in the form of the following linear program.

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* The algorithm which follows is based on a conjecture that a finite sequence of such choices of incremental requirement sets will ultimately sum up to the looked for optimum requirement set F^*.

For a given $\Delta\rho > 0$ find

$$\Delta r_1, \Delta r_2, \ldots, \Delta r_p, \quad v \geq 0$$

such that v is minimum subject to

$$\left.\begin{array}{l} w_{11}^k \Delta r_1 + w_{12}^k \Delta r_2 + \ldots + w_{1p}^k \Delta r_p - v \leq 0 \\ w_{21}^k \Delta r_1 + w_{22}^k \Delta r_2 + \ldots + w_{2p}^k \Delta r_p - v \leq 0 \\ \cdot \\ \cdot \\ \cdot \\ w_{e1}^k \Delta r_1 + w_{e2}^k \Delta r_2 + \ldots + w_{ep}^k \Delta r_p - v \leq 0 \\ \quad \Delta r_1 + \quad \Delta r_2 + \ldots + \Delta r_p \qquad\qquad = \Delta\rho \end{array}\right\} \qquad (52)$$

This program looks for $\Delta F_k = \{\Delta r_1, \Delta r_2, \ldots, \Delta r_p\} \varepsilon K$, such that the upper bound on relative capacity depletion in the edges of N, presented by v be minimal.

This F_k is the looked for optimum ΔF_k^*. It may be presented in the diagonal form ΔR_k required by the algorithm. Thus the overall sequence of steps for iteration k is: (29) - (34), (51), (52), (35) - (37).

The computation for the maximum sum terminates when at some iteration $M+1$ the matrix Y_n^{M+1} turns out to be singular (or almost singular) indicating that the edge capacities of some set separating the points s_ℓ and t_ℓ for every pair $\ell = 1,2,\ldots,p$ in $\bar{N}^k$ are zero. Thus the overall flow vector I_e^M obtained in the last iteration of (37) presents the looked for maximum sum of the p flows which can be transmitted through the net.

8. Conclusion

The algorithms presented in the paper have not been streamlined with respect to minimizing the complexity of computation.

An immediate step to reduce the computational load is possible if we notice that the matrices Y_n^k which have to be inverted in the step (32) are only slightly modified from iteration to iteration. Thus the Householder's, Bennett's or similar method can be used to take advantage of this fact.

Another point which deserves special attention in this respect is the magnitude of the increment of the requirement vector which we apply at each iteration. Good judgement in modifying the magnitude of this vector according to the changing state of the net at each iteration is apt to reduce appreciably the required computer time.

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