LIKELIHOOD RATIOS AND TRANSFORMATION OF PROBABILITY

ASSOCIATED WITH TWO-PARAMETER WIENER PROCESSES

by

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Memorandum No. ERL-M571

1 December 1975

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Eugene Wong* and Moshe Zakai†

1. Introduction

Let X_t, $0 \le t \le 1$, be a standard Wiener process defined on a probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{P}_0)$. Let $\mathcal{P}$ be a probability measure on $(\Omega, \mathcal{F}_1)$ equivalent to $\mathcal{P}_0$. E and E_0 will denote expectation relative to $\mathcal{P}$ and $\mathcal{P}_0$ respectively. Let $\mathcal{F}_{xt}$ denote $\sigma(X_s, 0 \le s \le t)$. The following set of results are by now well known: [See e.g., 3]

(a) If $\frac{d\mathcal{P}}{d\mathcal{P}_0} = \exp\left\{\int_0^1 \phi_s dX_s - \frac{1}{2}\int_0^1 \phi_s^2 ds\right\}$ where ϕ is an $\{\mathcal{F}_t\}$ adapted process, then $W_t = X_t - \int_0^t \phi_s ds$ is a standard Wiener process with respect to $\{\Omega, \{\mathcal{F}_t\}, \mathcal{P}\}$.

(b) Under some additional conditions such as $\int_0^1 E\phi_s^2 ds < \infty$, the likelihood ratio is expressible as

$$L_t = E_0\left(\frac{d\mathcal{P}}{d\mathcal{P}_0}\middle|\mathcal{F}_{xt}\right) = \exp\left\{\int_0^t \hat{\phi}_s dX_s - \frac{1}{2}\int_0^t \hat{\phi}_s^2 ds\right\}$$

where $\hat{\phi}_t = E(\phi_t | \mathcal{F}_{xt})$.

(c) Even without the hypotheses of (a) and (b), the likelihood ratio (viz., the projection of $\frac{d\mathcal{P}}{d\mathcal{P}_0}$ on the σ-field generated by X_s, $0 \le s \le t$) is of the form

$$L_t = \exp\left\{\int_0^t v_s dX_s - \frac{1}{2}\int_0^t v_s^2 ds\right\}$$

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Research sponsored by the U.S. Army Research Office--Durham Grant DAHC04-75-G-0189.

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where v is an $\{\mathcal{F}_{xt}\}$ adapted process and $V_t = X_t - \int_0^t v_s ds$ is a standard Wiener process with respect to $(\Omega, \{\mathcal{F}_{xt}\}, \mathcal{P})$.

The purpose of this paper is to consider these and related problems for Wiener process with a two-dimensional parameter. An attempt in this direction was begun in [4] but the effort was only partly successful. It revealed the far more complex structure of the stochastic calculus in the two-parameter case, and a full elucidation of the form of the Radon-Nikodym derivative and likelihood ratio had to await the development of the calculus as presented in [6,7].

Let R_+^2 denote the positive quadrant of the plane. For two points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ we denote

$$\begin{array}{llll} a \prec b & \text{if} & a_1 \le b_1 & \text{and} \quad a_2 \le b_2 \\ a \prec\!\!\prec b & \text{if} & a_1 < b_1 & \text{and} \quad a_2 < b_2 \\ a \curlywedge b & \text{if} & a_1 \le b_1 & \text{and} \quad a_2 \ge b_2 \\ a \curlywedge\!\!\curlywedge b & \text{if} & a_1 < b_1 & \text{and} \quad a_2 > b_2 \end{array}$$

Furthermore, we shall adopt the notations

$$a \otimes b = (a_1, b_2)$$
$$a \wedge b = (\min(a_1, b_1), \min(a_2, b_2))$$
$$a \vee b = (\max(a_1, b_1), \max(a_2, b_2))$$

Observe that if $a \curlywedge b$ then $a \otimes b = a \wedge b$ and $b \otimes a = a \vee b$. Note also that $a \otimes b \otimes c = a \otimes c$. Finally, for a fixed point z_0 in R_+^2, R_{z_0} will denote the rectangle $\{z: z \prec z_0, z \in R_+^2\}$.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $\{\mathcal{F}_z, z \in R_{z_0}\}$ be a family of σ-subfields such that

$F_1)$ $z' \succ z \Rightarrow \mathcal{F}_{z'} \supset \mathcal{F}_z$

$F_2)$ $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$ where 0 denotes the origin

F_3) $\mathcal{F}_z = \bigcap_{z' \succ z} \mathcal{F}_{z'}$ for every z

F_4) $\mathcal{F}_{z \otimes z_0}$ and $\mathcal{F}_{z_0 \otimes z}$ are independent given $\mathcal{F}_z$

For each z, $\mathcal{F}_z^1$ will denote $\mathcal{F}_{z \otimes z_0}$ and $\mathcal{F}_z^2$ will denote $\mathcal{F}_{z_0 \otimes z}$.

Let $\{X_z, z \in R_{z_0}\}$ be a stochastic process defined on $(\Omega, \mathcal{F}, \mathcal{P})$ and adapted to $\{\mathcal{F}_z\}$ (i.e., for each z, X_z is $\mathcal{F}_z$-measurable). For b $\succ\succ$ a let (a,b) denote the rectangle $\{z: a \prec\prec z \prec b\}$ and X(a,b] the increment $X_b - X_{a \otimes b} - X_{b \otimes a} + X_a$.

<u>Definition</u>. $\{X_z, \mathcal{F}_z, z \in R_{z_0}\}$ is said to be:

M_1) a <u>martingale</u> if $E\{X_{z'} | \mathcal{F}_z\} = X_z$ almost surely

M_2) a <u>weak martingale</u> if $E\{X(z,z'] | \mathcal{F}_z\} = 0$

M_3) a <u>strong martingale</u> if $E\{X(z,z'] | \mathcal{F}_z^1 \vee \mathcal{F}_z^2\} = 0$

M_4) an <u>adapted i-martingale</u> of $E\{X(z,z'] | \mathcal{F}_z^i\} = 0$, i = 1,2

M_5) a <u>Wiener process</u> if $\{X_z, \mathcal{F}_z, z \in R_{z_0}\}$ is a strong martingale, and X is a Gaussian process with $EX_z = 0$ and

$$EX(A)X(B) = \text{Area}(A \cap B) \quad \text{for all rectangles A and B .}$$

We note that if X satisfies condition M_4 it is said to be an i-martingale whether or not it is $\{\mathcal{F}_z\}$ adapted. In (1)-(5) the conditions are to hold for all z and all z' $\succ$ z. With these definitions, we can easily verify that a process is a martingale if and only if it is both an adapted 1-martingale and an adapted 2-martingale. A strong martingale is also a martingale, and an adapted one or two martingale is also a weak martingale. We owe most of these definitions to [1]. In the appendices, a summary of the principal results concerning the stochastic calculus for a Wiener process is presented. These results in a more general form and in greater detail can be found in [6,7].

Let $(\Omega, \{\mathcal{F}_z\})$ be a measurable space on which two probability measures $\mathcal{P}$ and $\mathcal{P}_0$ are defined. Let $\{X_z, \mathcal{F}_z, z \in R_{z_0}\}$ be a Wiener process under $\mathcal{P}_0$ and let $\mathcal{F}_{xz}$ denote the σ-field generated by $\{X_\zeta, \zeta \prec z\}$. We shall attempt to answer the following questions:

(a) Suppose that $\mathcal{P}$ and $\mathcal{P}_0$ are equivalent and

$$\frac{d\mathcal{P}}{d\mathcal{P}_0} = \exp\left\{\int_{R_{z_0}} \phi_\zeta dX_\zeta - \frac{1}{2}\int_{R_{z_0}} \phi_\zeta^2 dX_\zeta\right\},$$

how does X behave under $\mathcal{P}$?

(b) With whatever additional assumptions which might be necessary, is it possible to obtain an explicit expression for the likelihood ratio

$$L_z = E_0\left\{\frac{d\mathcal{P}}{d\mathcal{P}_0}\middle|\mathcal{F}_{xz}\right\} \ ?$$

(c) If we do not assume that $\mathcal{P}$ and $\mathcal{P}_0$ are equivalent, but only that their restrictions on $\mathcal{F}_{xz}$ are equivalent, can the general form of the Radon-Nikodym derivative on $\mathcal{F}_{xz}$ be found?

We believe that these questions are answered with reasonable completeness by the results of this paper. We are satisfied that the form of these results is quite general, even if the conditions under which they are proved may not be the best possible, The order of our presentation will be as follows: In section 2 we shall obtain a series of formulas which provide an answer to (c), and in section 3 a generalization to the exponential formula for Wiener processes. In section 4 we shall give an interpretation for these formulas in terms of some conditional moments of the process X under the

$\mathcal{P}$-measure. Finally, in section 5 an application of these results to the following hypothesis testing problem which arises in signal detection will be considered:

> H_0: The observation $\{\xi_z,\ z \in R_{z_0}\}$ is a white Gaussian noise.
>
> H: The observation is of the form $\xi_z = \theta_z + \eta_z$ where η is a white Gaussian noise and θ is a random signal.

It will be shown that in this case the likelihood ratio is expressible in terms of $\hat{\theta}_z = E(\theta_z|\mathcal{F}_{xz})$ and $\rho(z,z') = \operatorname{cov}(\theta_z\theta_{z'}|\mathcal{F}_{xz\vee z'})$.

2. Likelihood Ratio Formulas on Increasing Paths

Let $(\Omega,\mathcal{F})$ be a measurable space and $\{X_z,\ z \in R_{z_0}\}$ a family of measurable functions. Let $\mathcal{F}_{xz} = \sigma(X_\zeta,\ \zeta \in R_z)$ and assume $\mathcal{F}_{xz_0} = \mathcal{F}$. Let $\mathcal{P}$ and $\mathcal{P}_0$ be two equivalent probability measures on $(\Omega,\mathcal{F})$ such that under $\mathcal{P}_0$, X is a Wiener process. Denote the likelihood ratio by

$$L_z = E_0\left\{\frac{d\mathcal{P}}{d\mathcal{P}_0}\middle|\mathcal{F}_{xz}\right\} . \tag{2.1}$$

Then L is a positive $(\{\mathcal{F}_{xz}\},\mathcal{P}_0)$ martingale. In addition, we shall assume

$$E_0 L_z^2 < \infty , \qquad \forall z \in R_{z_0} \tag{2.2}$$

so that we can invoke the representation theorem of [5] and write L in the form

$$L_z = 1 + \int_{R_z} \alpha_\zeta\, dX_\zeta + \int_{R_z\times R_z} \beta_{\zeta,\zeta'}\, dX_\zeta\, dX_{\zeta'} . \tag{2.3}$$

Whence it follows that L can be chosen to be almost surely sample-continuous.

The square-integrability condition of L is made necessary by the fact that unlike the one-parameter case the stochastic-integral representation for Wiener-martingales has been proved only for square-integrable martingales and not for martingales in general. Because of this, it is not yet clear whether all Radon-Nikodym derivatives on a Wiener space are sample continuous. However, we believe that the square-integrability condition (2.2) can be weakened and that the form that we will derive is valid for all continuous likelihood ratios.

Equation (2.3-1) can be put in the form

$$L_z = 1 + \int_{R_z} L_{\zeta'\otimes z}\, u(z,\zeta')\, dX_{\zeta'} \tag{2.4-1}$$

with

$$u(z,\zeta') = \frac{1}{L_{\zeta'\otimes z}} [\alpha_{\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta')\beta_{\zeta,\zeta'} dX_\zeta] . \tag{2.5-1}$$

Alternatively, (2.4-1) and (2.5-1) can be recast into the form

$$L_z = 1 + \int_{R_z} L_{z\otimes\zeta}\, \tilde{u}(z,\zeta)\, dX_\zeta \tag{2.4-2}$$

$$\tilde{u}(z,\zeta) = \frac{1}{L_{z\otimes\zeta}} [\alpha_\zeta + \int_{R_z} I(\zeta \curlywedge \zeta')\beta_{\zeta,\zeta'} dX_{\zeta'}] . \tag{2.5-2}$$

We recognize (2.4-1) as a representation of L as a 1-martingale, and (2.4-2) a representation as a 2-martingale. Since $L_z > 0$ almost surely, we can now apply the differentiation formula (B.2) of the appendix to $\ell n\ L_z$ and get

$$\ell n\ L_z = \int_{R_z} u(z,\zeta')dX_{\zeta'} - \frac{1}{2}\int_{R_z} u^2(z,\zeta')d\zeta'$$
$$= \int_{R_z} \tilde{u}(z,\zeta)dX_\zeta - \frac{1}{2}\int_{R_z} \tilde{u}^2(z,\zeta)d\zeta .$$

It follows that we have

$$L_z = \exp\left\{\int_{R_z} u(z,\zeta')dX_{\zeta'} - \frac{1}{2}\int_{R_z} u^2(z,\zeta')d\zeta'\right\} \tag{2.6-1}$$

$$= \exp\left\{\int_{R_z} \tilde{u}(z,\zeta)dX_\zeta - \frac{1}{2}\int_{R_z} \tilde{u}^2(z,\zeta)d\zeta\right\} . \tag{2.6-2}$$

Equation (2.7) is reminiscent of the exponential formula in one dimension, and indeed it is precisely that. We note from (2.5-1) that

$$u(z,\zeta') = u(\zeta'\otimes z,\zeta')$$

so that the exponent in (2.6-1) is a semimartingale on horizontal lines. Thus, (2.6-1) can be considered a representation of L as a positive martingale on horizontal paths, and (2.6-2) as a representation on a vertical path. Thus, the similarity of (2.6) to the exponential formula for one-parameter Wiener processes comes as no surprise. Indeed, the representation (2.6) can be generalized to any increasing path.

Let Γ be an increasing path connecting the origin and z_0. For any point $z \in R_{z_0}$, z_Γ will denote the smallest point on Γ greater or equal to z. We say $\{\phi_z, z \in R_{z_0}\}$ is $\mathcal{F}_\Gamma$-adapted if for each z ϕ_z is $\mathcal{F}_{z_\Gamma}$-measurable. In appendix A, stochastic integrals for $\mathcal{F}_\Gamma$-adapted integrands have been defined. Using this definition, we can rewrite (2.3) for $z \in \Gamma$ as

$$L_z = 1 + \int_{R_z} L_{\zeta_\Gamma}\, u_\Gamma(\zeta)\, dX_\zeta \tag{2.7}$$

where (c.f. (B.10))

$$(2.8) \qquad u_\Gamma(\zeta) = (L_{\zeta_\Gamma})^{-1}\Big[\alpha_\zeta + \int_{\zeta\otimes\zeta' \in D_1^\Gamma} \beta_{\zeta',\zeta}\, I(\zeta' \curlywedge \zeta)\, dX_{\zeta'} + \int_{\zeta'\otimes\zeta \in D_2^\Gamma} \beta_{\zeta,\zeta'}\, I(\zeta \curlywedge \zeta')\, dX_{\zeta'}\Big] .$$

Observe that only one of the two integrals in the definition of u_Γ is non-zero. For $\zeta \in D_1^\Gamma$, $\zeta' \otimes \zeta$ cannot be in D_2^Γ, and for $\zeta \in D_2^\Gamma$, $\zeta \otimes \zeta'$ cannot be in D_1^Γ. So defined $u_\Gamma(\zeta)$ is $\mathcal{F}_{\zeta_\Gamma}$-measurable, and an application of the one-dimensional differentiation rule to the path Γ yields

$$(2.9) \qquad L_z = \exp\left\{\int_{R_z} u_\Gamma(\zeta)dX_\zeta - \frac{1}{2}\int_{R_z} u_\Gamma^2(\zeta)d\zeta\right\}$$

for all $z \in \Gamma$.

Theorem 2.1. Let $(\Omega, \mathcal{F}, \mathcal{P}_0)$ be a probability space and $\{X_z, z \in R_{z_0}\}$ a Wiener process. Let $\mathcal{F}_{xz}$ denote the σ-field generated by $\{X_\zeta, \zeta \prec z\}$ and assume $\mathcal{F} = \mathcal{F}_{xz_0}$.

(a) Suppose $\mathcal{P}$ is a probability measure equivalent to $\mathcal{P}_0$ such that the likelihood ratio

$$L_z = E_0\left\{\frac{d\mathcal{P}}{d\mathcal{P}_0}\middle|\mathcal{F}_{xz}\right\}$$

is $\mathcal{P}_0$-square-integrable (i.e., $E_0 L_z^2 < \infty$, $\forall z \prec z_0$). Then for any increasing path Γ there exists an $\mathcal{F}_\Gamma$-adapted process u_Γ so that for all $z \in \Gamma$

$$(2.7) \qquad L_z = 1 + \int_{R_z} u_\Gamma(\zeta)\, L_{\zeta_\Gamma}\, dX_\zeta$$

and

$$(2.9) \qquad L_z = \exp\left\{\int_{R_z} u_\Gamma(\zeta)dX_\zeta - \frac{1}{2}\int_{R_z} u_\Gamma^2(\zeta)d\zeta\right\}.$$

(b) Conversely, let Γ be an increasing path and u_Γ an $\mathcal{F}_\Gamma$-adapted process satisfying

$$\int_{R_{z_0}} u_\Gamma^2(\alpha)d\zeta < \infty \text{ almost surely } \mathcal{P}_0 .$$

Define for $z \in \Gamma$

$$L_z = \exp\left\{\int_{R_z} u_\Gamma(\zeta)dX_\zeta - \frac{1}{2}\int_{R_z} u_\Gamma^2(\zeta)d\zeta\right\} .$$

Suppose that $E_0 L_{z_0} = 1$. Then $\frac{d\mathcal{P}}{d\mathcal{P}_0} = L_{z_0}$ defines a probability measure $\mathcal{P}$ and $E_0(L_{z_0}|\mathcal{F}_{xz}) = L_z$.

<u>Proof</u>: (a) Since L_z is a $\mathcal{P}_0$-square-integrable $\mathcal{F}_{xz}$-martingale, we can write as in (2.3)

$$L_z = 1 + \int_{R_z} \alpha_\zeta \, dX_\zeta + \int_{R_z \times R_z} \beta_{\zeta,\zeta'} \, dX_\zeta \, dX_{\zeta'} .$$

Define u_Γ by (2.8). Then (2.7) follows. An application of the one-parameter differentiation formula to $\ln L_z$ on Γ yields (2.9).

(b) Conversely, if $\int_{R_z} u_\Gamma^2(\zeta)d\zeta < \infty$ almost surely ($\mathcal{P}_0$) then $M_z = \int_{R_z} u_\Gamma(\zeta)dX_\zeta$ is well-defined as a local martingale on Γ with

$$\langle M,M\rangle_z = \int_{R_z} u_\Gamma^2(\zeta)d\zeta .$$

Hence, $L_z = e^{M_z - \frac{1}{2}\langle M,M\rangle_z}$ defines a probability measure if $EL_{z_0} = 1$. Since

$$L_z = 1 + \int_{R_z} L_{\zeta_\Gamma} u_\Gamma(\zeta)dX_\zeta$$

it follows that $E_0(L_{z_0}|\mathcal{F}_{xz}) = L_z$, almost surely. □

Let $Y_z = \int_{R_z} \alpha(\zeta)\left(dX_\zeta - u_\Gamma(\zeta)d\zeta\right)$, where α is a bounded deterministic function. Then, under $\mathcal{P}_0$, Y_z can be considered a semimartingale on Γ, and the one-parameter differentiation rule (B.8) yields

$$L_z Y_z = \int_{R_z} L_{\zeta_\Gamma}[\alpha(\zeta) + Y_{\zeta_\Gamma} u_\Gamma(\zeta)]dX_\zeta$$

so that $L_z Y_z$ is a $\mathcal{P}_0$-martingale on Γ. Therefore Y_z is a $\mathcal{P}$-martingale on Γ. This gives us the interpretation

$$E\left[\left(dX_\zeta - u_\Gamma(\zeta)d\zeta\right) | \mathcal{F}_{xz_\Gamma}\right] = 0$$

or

(2.10) $$u_\Gamma(\zeta)d\zeta = E\left[dX_\zeta | \mathcal{F}_{xz_\Gamma}\right] .$$

Specializing to horizontal and vertical paths yields an interpretation for the functions u and $\tilde{u}$ in (2.6) as follows.

(2.11-1) $$u(z,\zeta')d\zeta' = E\left[dX_{\zeta'} | \mathcal{F}_{x,\zeta'\otimes z}\right]$$

(2.11-2) $$\tilde{u}(z,\zeta)d\zeta = E\left[dX_\zeta | \mathcal{F}_{x,z\otimes\zeta}\right] .$$

A more precise statement of (2.10) or (2.11) can be made as follows: For a fixed Γ define a <u>Γ-martingale</u> Y by the property

$$E\left\{Y(z,z'] | \mathcal{F}_{z_\Gamma}\right\} = 0 \quad \text{for all} \quad z' \succ\!\!\succ z \quad .$$

This generalizes the concept of i-martingale (adapted or non-adapted). Now, a precise statement of (2.10) or (2.11) is given by

Theorem 2.2. Let u_Γ, X, $\mathcal{P}$ and $\mathcal{P}_0$, be as in Theorem 2.1. Then

$$Y_z = X_z - \int_{R_z} u_\Gamma(\zeta)d\zeta$$

is a Γ-martingale with respect to $\mathcal{P}$.

Proof. Fix two points $z \prec z'$, and for $\{\alpha: \alpha \succ z \text{ and } \alpha \in \Gamma\}$ define

$$M_\alpha = \int_{R_\alpha} I(z \prec \zeta \prec z')[dX_\zeta - u_\Gamma(\zeta)d\zeta] .$$

Since M_α is a $\mathcal{P}$-martingale on the portion of Γ from z_Γ to z_0 we have

$$E(M_{z_0}|\mathcal{F}_{z_\Gamma}) = M_{z_\Gamma} .$$

Since $M_{z_0} = Y(z,z']$ and $M_{z_\Gamma} = 0$, the desired result follows. □

Before proceeding to the derivation of a two-dimensional exponential formula for L_z, consider the special case where $u_\Gamma(\zeta) = \phi_\zeta$ is independent of path. In that case the formula (2.9) becomes

$$L_z = \exp\left\{\int_{R_z} \phi_\zeta dX_\zeta - \frac{1}{2}\int_{R_z} \phi_\zeta^2 d\zeta\right\}$$

which being path independent is already a full-fledged two-dimensional exponential formula. Needless to say, the condition that u_Γ be independent of path is a severe one and the circumstances under which this obtains will become apparent in the next section.

3. A Two-Dimensional Exponential Formula

The exponential formulas for the likelihood ratio given by (2.6) and (2.9) are two dimensional in form, but clearly one-dimensional in spirit.

Our next objective is to derive a formula which is inherently two-dimensional. The starting point is (2.5-1) and (2.4-2). Observe that (2.4-2) is in the form of (B.5-2) so that (B.6-1) applies. It yields

(3.1) $$L_{\zeta'\otimes z} = L_{\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta') L_{\zeta'\otimes\zeta} \tilde{u}(\zeta'\otimes\zeta,\zeta)\, dX_\zeta \; .$$

If we denote

(3.2) $$Y_{z,\zeta'} = \alpha_{\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta') \beta_{\zeta,\zeta'}\, dX_\zeta$$

then (2.5-1) acquires the form

(3.3) $$u(z,\zeta') = \left(\frac{1}{L_{\zeta'\otimes z}}\right) Y_{z,\zeta'} \; .$$

For a fixed ζ', $L_{\zeta'\otimes z}$ and $Y_{z,\zeta'}$ are 2-martingales, and we can apply (B.2-2) to get

$$\begin{aligned} u(z,\zeta') = \left(\frac{\alpha_{\zeta'}}{L_{\zeta'}}\right) &+ \int_{R_z} I(\zeta \curlywedge \zeta') \frac{1}{L_{\zeta'\otimes\zeta}} \beta_{\zeta,\zeta'}\, dX_\zeta \\ &- \int_{R_z} I(\zeta \curlywedge \zeta') \frac{Y_{z\otimes\zeta,\zeta'}}{L^2_{\zeta'\otimes\zeta}} L_{\zeta'\otimes\zeta}\, \tilde{u}(\zeta'\otimes\zeta,\zeta)\, dX_\zeta \\ &- \int_{R_z} I(\zeta \curlywedge \zeta') \frac{1}{L_{\zeta'\otimes\zeta}} \tilde{u}(\zeta'\otimes\zeta,\zeta') \beta_{\zeta,\zeta'}\, d\zeta \\ &+ \int_{R_z} I(\zeta \curlywedge \zeta') \left[\frac{Y_{z\otimes\zeta,\zeta'}}{L^3_{\zeta'\otimes\zeta}}\right] L^2_{\zeta'\otimes\zeta}\, \tilde{u}^2(\zeta'\otimes\zeta,\zeta)\, d\zeta \; . \end{aligned}$$

Observe that because of the term $I(\zeta \curlywedge \zeta')$ in the integrals, we have

$$u(z,\zeta') = u(\zeta'\otimes z,\zeta')$$

so that

$$\frac{Y_{z\otimes\zeta,\zeta'}}{L_{\zeta'\otimes\zeta}} = u(z\otimes\zeta,\zeta') = u(\zeta'\otimes\zeta,\zeta') \; .$$

Then it follows that we can write for $z \succ \zeta'$

$$(3.4\text{-}1) \qquad u(z,\zeta') = \theta_{\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta')\rho(\zeta,\zeta')[dX_\zeta - \tilde{u}(\zeta'\otimes\zeta,\zeta)d\zeta]$$

where

$$(3.5) \qquad \theta_{\zeta'} = u(\zeta',\zeta') = \left(\frac{\alpha_{\zeta'}}{L_{\zeta'}}\right)$$

and

$$(3.6) \qquad \rho(\zeta,\zeta') = \frac{\beta(\zeta,\zeta')}{L_{\zeta'\otimes\zeta}} - u(\zeta'\otimes\zeta,\zeta')\tilde{u}(\zeta'\otimes\zeta,\zeta) \ .$$

By symmetry we can also write

$$(3.4\text{-}2) \qquad \tilde{u}(z,\zeta) = \theta_\zeta + \int_{R_z} I(\zeta \curlywedge \zeta')\rho(\zeta,\zeta')dX_{\zeta'} - \int_{R_z} I(\zeta \curlywedge \zeta')\rho(\zeta,\zeta')u(\zeta'\otimes\zeta,\zeta')d\zeta' \ .$$

Equation (3.4-1) yields

$$(3.7) \qquad u^2(z,\zeta') = \theta^2_{\zeta'} + 2\int_{R_z} I(\zeta \curlywedge \zeta')u(\zeta'\otimes\zeta,\zeta')\rho(\zeta,\zeta')[dX_\zeta - \tilde{u}(\zeta'\otimes\zeta,\zeta)d\zeta] + \int_{R_z} I(\zeta \curlywedge \zeta')\rho^2(\zeta,\zeta')d\zeta \ .$$

Putting (3.4-1) and (3.7) into (2.6-1) yeilds

$$(3.8) \qquad L_z = \exp\left\{\int_{R_z} \theta_{\zeta'}dX_{\zeta'} - \frac{1}{2}\int_{R_z} \theta^2_{\zeta'}d\zeta' - \frac{1}{2}\int_{R_z\times R_z} \rho^2(\zeta,\zeta')d\zeta d\zeta' + \int_{R_z\times R_z} \rho(\zeta,\zeta')[dX_\zeta - \tilde{u}(\zeta'\otimes\zeta,\zeta)d\zeta][dX_{\zeta'} - u(\zeta'\otimes\zeta,\zeta')d\zeta']\right\}$$

which is the two-dimensional exponential formula that we have sought.

Given ρ and θ, (3.4) can be viewed as a pair of linear integral equations with unknowns u and $\tilde{u}$. Indeed, if we set $u(a\otimes b,a) = h(a,b)$ and

$\tilde{u}(a\otimes b,b) = \tilde{h}(a,b)$, (3.4) can be rewritten in the form

$$h(a,b) = h_0(a,b) + \int_{R_{a\otimes b}\times R_{a\otimes b}} G_a(\zeta,\zeta')h(\zeta',\zeta)d\zeta d\zeta'$$

$$\tilde{h}(a,b) = \tilde{h}_0(a,b) + \int_{R_{a\otimes b}\times R_{a\otimes b}} \tilde{G}_b(\zeta,\zeta')\tilde{h}(\zeta',\zeta)d\zeta d\zeta' .$$

If ρ is bounded then so are G and $\tilde{G}$, in which case Picard iteration converges, and the existence and uniqueness of h and $\tilde{h}$ are not in question. Therefore, if ρ is assumed to be bounded then (3.8) can be viewed as an expression of L_z in terms of θ and ρ.

Summarizing, we have the following:

<u>Theorem 3.1</u>. Let $\{L_z, z \in R_{z_0}\}$ be an almost surely positive square-integrable martingale defined on $(\Omega,\mathcal{F},\mathcal{P}_0)$ where $\mathcal{F}$ is generated by a Wiener process $\{X_z, z \in R_{z_0}\}$. Let $L_0 = 1$. Then, there exist functions θ, ρ, u and $\tilde{u}$ satisfying (3.4) such that L_z can be expressed by (3.8). Further, L satisfies

$$(3.9)\quad L_z = 1 + \int_{R_z} L_\zeta\theta_\zeta dX_\zeta + \int_{R_z\times R_z} L_{\zeta'\otimes\zeta}[\rho(\zeta,\zeta') + u(\zeta'\otimes\zeta,\zeta')\tilde{u}(\zeta'\otimes\zeta,\zeta)]dX_\zeta dX_{\zeta'},$$

Conversely, let θ_z be an $\mathcal{F}_{xz}$-measurable function defined for $z \in R_{z_0}$ and $\rho(z,z')$ be an $\mathcal{F}_{xz\vee z'}$-measurable function defined for all $z, z' \in R_{z_0}$ such that z z'. Suppose that (3.4) has unique solutions for u and $\tilde{u}$, and when θ, ρ, u and $\tilde{u}$ are substituted into (3.8), it yields an L_z satisfying $E_0 L_{z_0} = 1$. Then, L_z is a positive martingale which is the unique solution to (3.4).

<u>Corollary</u>. Let $\{X_z, z \in R_{z_0}\}$ be a Wiener process defined on $(\Omega,\mathcal{F},\mathcal{P}_0)$ and denote $\mathcal{F}_{xz} = \sigma(X_\zeta, \zeta \prec z)$. Let $\mathcal{P}$ be a probability measure on $(\Omega,\mathcal{F})$ such that the restrictions of $\mathcal{P}$ and $\mathcal{P}_0$ to $\mathcal{F}_{xz_0}$ are equivalent. Suppose that the likelihood ratio

$$L_z = E_0\left(\frac{d\mathcal{P}^x}{d\mathcal{P}_0^x}\Big|\mathcal{F}_{xz}\right)$$

is $\mathcal{P}_0$ square-integrable. Then, it satisfies (3.8) and (3.9).

Proof. The fact that L_z satisfies (3.8) has already been proved by the steps leading to (3.8). To obtain (3.9), we return to (2.3) and use (3.5) and (3.6) to identify α and β. Finally, to go from (3.8) to (3.9), we rewrite (3.8) using (3.4) to get back to (2.6-1), viz.,

$$L_z = \exp\left\{\int_{R_z} u(z,\zeta')dX_{\zeta'} - \frac{1}{2}\int_{R_z} u^2(z,\zeta')d\zeta'\right\} .$$

If $E_0 L_{z_0} = 1$, this implies (2.4-1), i.e.,

$$L_z = 1 + \int_{R_z} L_{\zeta'\otimes z} u(z,\zeta')dX_{\zeta'} .$$

Now, we can use (3.1) and (3.4-1), whence (3.9) follows. □

We observe that if $\rho \equiv 0$ then (3.8) degenerates into the form given at the end of section 2. In that case $u(z,\zeta) = \tilde{u}(z,\zeta) = \theta_\zeta$, and u_Γ is indeed independent of the path. This situation arises when and only when L_z satisfies the equation

$$L_z = L + \int_{R_z} \theta_\zeta L_\zeta dX_\zeta + \int_{R_z\times R_z} \theta_\zeta \theta_{\zeta'} L_{\zeta\vee\zeta'} dX_\zeta dX_{\zeta'}$$

4. Interpretation of the Functions θ and ρ

The interpretation of θ comes immediately from those of u and $\tilde{u}$ and the relationship $\theta(\zeta) = u(\zeta,\zeta) = \tilde{u}(\zeta,\zeta)$. We have from (2.11)

(4.1) $$\theta(\zeta)d\zeta = E\left(dX_\zeta|\mathcal{F}_{x\zeta}\right) .$$

The interpretation of ρ is more obscure. A hint as to what it should be comes from comparing (3.9) with eq. (4.12) of [4]. (In the latter equation the factor $\frac{1}{2}$ is due to a slightly different definition of the stochastic integral of the second type.) These equations are similar, and the comparison suggests that while u and $\tilde{u}$ are conditional expectations of dX given σ-fields of various kinds, ρ should be the covariance of such conditional expectations. Specifically, we should have

$$(4.2\text{-}1) \qquad u(\zeta'\otimes\zeta,\zeta')d\zeta' = E\left(dX_{\zeta'} \middle| \mathcal{F}_{x,\zeta'\otimes\zeta}\right)$$

$$(4.2\text{-}2) \qquad \tilde{u}(\zeta'\otimes\zeta,\zeta)d\zeta = E\left(dX_{\zeta} \middle| \mathcal{F}_{x,\zeta'\otimes\zeta}\right)$$

$$(4.3) \qquad \rho(\zeta,\zeta')d\zeta d\zeta' = E\left[\left(dX_{\zeta}-\tilde{u}(\zeta'\otimes\zeta,\zeta)\right)\left(dX_{\zeta'}-u(\zeta'\otimes\zeta,\zeta)\right) \middle| \mathcal{F}_{x,\zeta'\otimes\zeta}\right]$$

for all ζ, ζ' in R_{z_0} such that $\zeta \curlywedge \zeta'$. We note that because $\zeta \curlywedge \zeta'$, $\zeta'\otimes\zeta$ can be replaced by $\zeta \vee \zeta'$ as is done in [4].

To verify (4.3) precisely, we must show that if

$$(4.4) \qquad Y_z = \int_{R_z\times R_z} f(\zeta,\zeta')\Big\{[dX_{\zeta}-\tilde{u}(\zeta'\otimes\zeta,\zeta)d\zeta][dX_{\zeta'}-u(\zeta'\otimes\zeta,\zeta')d\zeta'] - \rho(\zeta,\zeta')d\zeta d\zeta'\Big\}$$

where f is any bounded deterministic function, then Y is a weak martingale with respect to $(\{\mathcal{F}_{xz}\},\mathcal{P})$, or equivalently, Y_zL_z is a weak martingale with respect to $(\{\mathcal{F}_{xz}\},\mathcal{P}_0)$. To do this we follow the procedure of appendix B, by first writing Y_z and L_z in the form of (B.5-1) and then representing the integrands as stochastic integrals of the form (B.4-1).

Define

$$(4.5\text{-}1) \qquad v(z,\zeta') = \int_{R_z} I(\zeta \curlywedge \zeta')f(\zeta,\zeta')[dX_{\zeta}-\tilde{u}(\zeta'\otimes\zeta,\zeta)d\zeta]$$

$$(4.6\text{-}1) \qquad w(z,\zeta') = \int_{R_z} I(\zeta \curlywedge \zeta')f(\zeta,\zeta')[u(\zeta'\otimes\zeta,\zeta')\big(dX_\zeta - \tilde{u}(\zeta'\otimes\zeta,\zeta)d\zeta\big) - \rho(\zeta,\zeta')d\zeta] \; .$$

Then,

$$(4.7\text{-}1) \qquad Y_z = \int_{R_z} [v(z,\zeta')dX_{\zeta'} + w(z,\zeta')d\zeta'] \; .$$

Similarly, we can also write

$$(4.7\text{-}2) \qquad Y_z = \int_{R_z} [\tilde{v}(z,\zeta)dX_\zeta + \tilde{w}(z,\zeta)d\zeta]$$

with

$$(4.5\text{-}2) \qquad \tilde{v}(z,\zeta) = \int_{R_z} I(\zeta \curlywedge \zeta')f(\zeta,\zeta')[dX_{\zeta'} - u(\zeta'\otimes\zeta,\zeta')d\zeta']$$

$$(4.6\text{-}2) \qquad \tilde{w}(z,\zeta) = \int_{R_z} I(\zeta \curlywedge \zeta')f(\zeta,\zeta')\Big\{\tilde{u}(\zeta'\otimes\zeta,\zeta)[dX_{\zeta'} - u(\zeta'\otimes\zeta,\zeta')d\zeta'] - \rho(\zeta,\zeta')d\zeta'\Big\} \; .$$

Using (2.4.1) and applying the differentiation rule for 1-semimartingale, we get

$$(4.8) \qquad L_z Y_z = \int_{R_z} L_{\zeta'\otimes z}[v(z,\zeta') + u(z,\zeta')Y_{\zeta'\otimes z}]dX_{\zeta'} + \int_{R_z} L_{\zeta'\otimes z}[w(z,\zeta') + u(z,\zeta')v(z,\zeta')]d\zeta' \; .$$

From (4.5-1), (4.6-1) and (3.4-1), we get

$$w(z,\zeta') + u(z,\zeta')v(z,\zeta') = \int_{R_z} I(\zeta \curlywedge \zeta')[v(\zeta'\otimes\zeta,\zeta')\rho(\zeta,\zeta') + 2f(\zeta,\zeta')u(\zeta'\otimes\zeta,\zeta')][dX_\zeta - \tilde{u}(\zeta'\otimes\zeta,\zeta)d\zeta] \; .$$

It follows from (3.1) that

$$L_{\zeta'\otimes z}[w(z,\zeta') + u(z,\zeta')v(z,\zeta')]$$
$$= \int_{R_z} I(\zeta \curlywedge \zeta') L_{\zeta'\otimes\zeta}[\tilde{u}(uv+w) + (v\rho+2fu)]dX_\zeta$$

where the arguments of the functions in the integrand are (ζ,ζ') for f and ρ, $(\zeta'\otimes\zeta,\zeta')$ for u, v and w, and $(\zeta'\otimes\zeta,\zeta)$ for $\tilde{u}$. Thus, (4.8) can now be written as

$$L_zY_z = \int_{R_z} L_{\zeta'\otimes z}[v(z,\zeta') + u(z,\zeta')Y_{\zeta'\otimes z}]dX_{\zeta'} + \int_{R_z} G(z,\zeta)dX_\zeta \quad .$$

Symmetry dictates that $G(z,\zeta)$ must be such that

(4.9)
$$L_zY_z = \int_{R_z} L_{\zeta'\otimes z}[v(z,\zeta') + u(z,\zeta')Y_{\zeta'\otimes z}]dX_{\zeta'}$$
$$+ \int_{R_z} L_{z\otimes\zeta}[\tilde{v}(z,\zeta) + \tilde{u}(z,\zeta)Y_{z\otimes\zeta}]dX_\zeta$$

which is clearly a weak martingale with respect to $\mathcal{P}_0$.

5. Random Signal in Additive White Gaussian Noise

The following situation often arises in signal processing problems. The observation is represented by a process ξ_z of

$$\xi_z = \theta_z + \eta_z$$

where θ is a random process representing the signal and η is a white Gaussian noise. To deal with such a model, we can integrate both sides of the equation and get

(5.1)
$$X_z = \int_{R_z} \theta_\zeta d\zeta + W_z \,, \qquad z \in R_{z_0}$$

where X represents the observed process and W is a Wiener process. Let

$(\Omega,\mathcal{F},\mathcal{P})$ be the probability space in which the processes X, θ and W are defined. For problems in signal detection and filtering it is useful to introduce a probability measure $\mathcal{P}_0$ on $(\Omega,\mathcal{F})$ with respect to which X itself is a Wiener process.

Lemma. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space and let $\{\mathcal{F}_z,\ z\in R_{z_0}\}$ be a family of σ-fields such that θ_z is $\mathcal{F}_z$-measurable for each z and $\{W_z,\ z\in R_{z_0}\}$ is a standard Wiener process with respect to $\{\mathcal{F}_z\}$. Define

$$V_z = \exp\left\{-\int_{R_z}\theta_\zeta dW_\zeta - \frac{1}{2}\int_{R_z}\theta_\zeta^2 d\zeta\right\} \tag{5.2}$$

and assume that $|\theta_\zeta(\omega)| \le c$ for almost all (ζ,ω). Then for $\alpha \ge 1$, we have

$$1 \le EV_z^\alpha \le \exp[(\frac{\alpha^2-\alpha}{2})c^2\text{Area}(R_z)] . \tag{5.3}$$

Proof. Using the differentiation rule (B.2-1), we can write

$$V_z^\alpha = 1 - \int_{R_z} V_{\zeta\otimes z}^\alpha\theta_\zeta dW_\zeta + \frac{1}{2}(\alpha^2-\alpha)\int_{R_z}\theta_\zeta^2 V_{\zeta\otimes z}^\alpha d\zeta \quad .$$

If we set

$$\begin{aligned} V_{nz} &= V_z \quad \text{if} \quad \sup_{\zeta\in R_z}(V_{\zeta\otimes z}) \le n \\ &= 0 \quad \text{otherwise} \end{aligned}$$

then

$$V_{nz}^\alpha \le 1 - \int_{R_z} V_{n,\zeta\otimes z}^\alpha\theta_\zeta dW_\zeta + \frac{1}{2}(\alpha^2-\alpha)\int_{R_z}\theta_\zeta^2 V_{n,\zeta\otimes z}^\alpha d\zeta$$

and

$$EV_{nz}^\alpha \le 1 + \frac{1}{2}(\alpha^2-\alpha)c^2\int_{R_z} EV_{n,\zeta\otimes z}^\alpha d\zeta$$

or

$$EV^{\alpha}_{n,st} \leq 1 + \frac{c^2}{2}(\alpha^2-\alpha)t \int_0^s EV^{\alpha}_{n,\sigma t} d\sigma$$

and

$$EV^{\alpha}_{n,st} \leq \exp \frac{c^2}{2}(\alpha^2-\alpha)ts$$

and the right hand side of (5.3) follows from Fatou's lemma.

Since $\int_{R_z} E[V^{\alpha}_{\zeta\otimes z}\theta_\zeta]^2 d\zeta < \infty$, the stochastic integral $\int_{R_z} V^{\alpha}_{\zeta\otimes z}\theta_\zeta dW_\zeta$ has zero mean so that

$$EV^{\alpha}_z = 1 + \frac{1}{2}(\alpha^2-\alpha) \int_{R_z} E(\theta^2_\zeta V^{\alpha}_{\zeta\otimes z}) d\zeta \geq 1 \ . \ \Box$$

Theorem 5.1. Under the conditions of the above lemma, define a measure $\mathcal{P}_0$ by

$$\frac{d\mathcal{P}_0}{d\mathcal{P}} = V_{z_0}$$

where V_z is given by (5.2). Define X_z by (5.1). Then,

(a) $\mathcal{P}_0$ is a probability measure

(b) X_z is a Wiener process under $\mathcal{P}_0$

(c) $\mathcal{P}_0 \sim \mathcal{P}$ and

(5.4) $$\frac{d\mathcal{P}}{d\mathcal{P}_0} = \exp\left\{ \int_{R_{z_0}} \theta_\zeta dX_\zeta - \frac{1}{2}\int_{R_{z_0}} \theta^2_\zeta d\zeta \right\} .$$

Proof. (a) From (5.3) we have $EV_{z_0} = 1$. Since V_{z_0} is clearly positive, $\mathcal{P}_0$ is a probability measure.

(b) To prove X_z is a $\mathcal{P}_0$-Wiener process it is enough to show that

$$E_0 \exp\left\{i\int_{R_{z_0}} u(\zeta) dX_\zeta\right\} = \exp\left\{-\frac{1}{2}\int_{R_{z_0}} u^2(\zeta) d\zeta\right\}$$

for all bounded deterministic u. Now,

$$E_0 \exp\left\{i\int_{R_{z_0}} u(\zeta)dX_\zeta\right\} = E\left[V_{z_0} \exp\left\{i\int_{R_{z_0}} u(\zeta)dX_\zeta\right\}\right]$$

$$= \exp\left\{-\frac{1}{2}\int_{R_{z_0}} u^2(\zeta)d\zeta\right\} E\left[\exp\left\{-\int_{R_{z_0}} [\theta_\zeta - iu(\zeta)]dX_\zeta - \frac{1}{2}\int_{R_{z_0}} [\theta_\zeta - iu(\zeta)]^2 d\zeta\right\}\right] .$$

Since u is bounded (by u_0 say)

$$\left|\exp\left\{-\int_{R_{z_0}} [\theta_\zeta - iu(\zeta)]dX_\zeta - \frac{1}{2}\int_{R_{z_0}} [\theta_\zeta - iu(\zeta)]^2 d\zeta\right\}\right|$$

$$= \left[V_{z_0} \left|\exp[\frac{1}{2}\int_{R_{z_0}} u^2(\zeta)d\zeta\right] \leq V_{z_0} \exp\left\{\frac{1}{2}u_0^2 \text{Area}(R_{z_0})\right\} .$$

Hence, $\exp\left\{-\int_{R_z} [\theta_\zeta - iu(\zeta)]dX_\zeta - \frac{1}{2}\int_{R_z} [\theta_\zeta - iu(\zeta)]^2 d\zeta\right\}$ is a square-integrable $\mathcal{P}$-martingale and

$$E_0 \exp\left\{i\int_{R_{z_0}} u(\zeta)dX_\zeta\right\} = \exp\left\{-\frac{1}{2}\int_{R_{z_0}} u^2(\zeta)d\zeta\right\}$$

as was to be proved.

(c) Since X is a $\mathcal{P}_0$-Wiener process and θ is bounded

$$\frac{1}{V_z} = \exp\left\{\int_{R_z} \theta_\zeta dX_\zeta - \frac{1}{2}\int_{R_z} \theta_\zeta^2 d\zeta\right\}$$

must satisfy (5.3) with E_0 replacing E and $\frac{1}{V_z}$ replacing V_z. Thus,

$$E_0\left(\frac{1}{V_{z_0}}\right) = 1$$

and part (c) is proved. □

Now, let $\mathcal{F}_{xz}$ denote the σ-subfield generated by $\{X_\zeta, \zeta \in R_z\}$ and denote

(5.5) $$L_z = E_0\left(\frac{d\mathcal{P}}{d\mathcal{P}_0}\bigg|\mathcal{F}_{xz}\right).$$

Let

(5.6) $$\Lambda_z = \frac{1}{V_z} = \exp\left\{\int_{R_z} \theta_\zeta dX_\zeta - \frac{1}{2}\int_{R_z} \theta_\zeta^2 d\zeta\right\}$$

which is of the form (3.8) with $\rho \equiv 0$. Hence, (3.9) and (3.4) yield

(5.7) $$\Lambda_z = 1 + \int_{R_z} \Lambda_\zeta \theta_\zeta dX_\zeta + \int_{R_z \times R_z} \Lambda_{\zeta'\otimes\zeta} \theta_\zeta \theta_{\zeta'} dX_\zeta dX_{\zeta'}$$

which was also derived in [4]. Since

$$L_z = E_0[\Lambda_z | \mathcal{F}_{xz}]$$

we can follow the arguments of [4] and get

(5.8) $$L_z = 1 + \int_{R_z} L_\zeta E(\theta_\zeta | \mathcal{F}_{x\zeta}) dX_\zeta + \int_{R_z \times R_z} L_{\zeta'\otimes\zeta} E(\theta_\zeta \theta_{\zeta'} | \mathcal{F}_{x\zeta'\otimes\zeta}) dX_\zeta dX_{\zeta'}$$

Now, denote

(5.9) $$\hat{\theta}(\zeta|z) = E(\theta_\zeta | \mathcal{F}_{xz})$$

and

(5.10) $$R(\zeta,\zeta'|z) = E[(\theta_\zeta - \hat{\theta}(\zeta|z))(\theta_{\zeta'} - \hat{\theta}(\zeta'|z))] .$$

Then (5.8) can be rewritten as

$$(5.11)\qquad L_z = 1 + \int_{R_z} L_\zeta \hat{\theta}(\zeta|\zeta)dX_\zeta + \iint_{R_z\times R_z} L_{\zeta'\otimes\zeta}[\hat{\theta}(\zeta|\zeta'\otimes\zeta)\hat{\theta}(\zeta'|\zeta'\otimes\zeta) + R(\zeta,\zeta'|\zeta'\otimes\zeta)]dX_\zeta dX_{\zeta'}\ .$$

Comparing (5.11) with (3.9) and using (3.8), we get

$$(5.12)\qquad L_z = \exp\left\{\int_{R_z} \hat{\theta}(\zeta|\zeta)dX_\zeta - \frac{1}{2}\int_{R_z} \hat{\theta}^2(\zeta|\zeta)d\zeta - \frac{1}{2}\int_{R_z\times R_z} R^2(\zeta,\zeta'|\zeta'\otimes\zeta)d\zeta d\zeta' + \int_{R_z\times R_z} R(\zeta,\zeta'|\zeta'\otimes\zeta)[dX_\zeta - \hat{\theta}(\zeta|\zeta'\otimes\zeta)d\zeta][dX_{\zeta'} - \hat{\theta}(\zeta'|\zeta'\otimes\zeta)d\zeta']\right\}$$

which gives an explicit representation of the likelihood ratio L_z in terms of the moments $\hat{\theta}$ and R. Such a formula was sought without success in [4]. In light of the amount of additional machinery which has been necessary to derive (5.12), the failure is hardly surprising.

Now, (3.4) takes on the form

$$(5.12\text{-}1)\qquad \hat{\theta}(\zeta'|\zeta'\otimes z) = \hat{\theta}(\zeta'|\zeta') + \int_{R_z} I(\zeta\curlywedge\zeta')R(\zeta,\zeta'|\zeta'\otimes\zeta)[dX_\zeta - \hat{\theta}(\zeta|\zeta'\otimes\zeta)d\zeta]$$

$$(5.12\text{-}2)\qquad \hat{\theta}(\zeta|z\otimes\zeta) = \hat{\theta}(\zeta|\zeta) + \int_{R_z} I(\zeta\curlywedge\zeta')R(\zeta,\zeta'|\zeta'\otimes\zeta)[dX_{\zeta'} - \hat{\theta}(\zeta'|\zeta'\otimes\zeta)d\zeta']\ .$$

It follows that $\{L_z,\ z\in R_{z_0}\}$ is completely specified by $\{\hat{\theta}(z|z),\ z\in R_{z_0}\}$ and $\{R(z,z'|z\vee z'),\ z,z'\in R_{z_0}\}$. Furthermore, if θ and W are jointly Gaussian under $\mathcal{P}$, then $R(\zeta,\zeta'|\zeta\vee\zeta')$ is a deterministic function, and $\{L_z,\ z\in R_{z_0}\}$ is completely determined by $\hat{\theta}(z|z),\ z\in R_{z_0}$. This implies, for example, that a detector for testing between the hypotheses:

$$H\colon X_z = \int_{R_z} \phi_\zeta d\zeta + W_z \quad \text{and} \quad W \text{ is a Wiener process}$$

$$H_0\colon X_z \quad \text{is a Wiener process}$$

can be implemented by a filtering operation which yields $\hat{\theta}(z|z),\ z\in R_{z_0}$.

Equation (5.12) represents a constraint on the various conditional moments. The existence of such a constraint is surprising and could hardly have been predicted a priori. As such, (5.12) has considerable interest in its own right.

Finally, we observe that a natural concommitant of the likelihood ratio formulas is the behavior of martingales under such transformation of measures. Theorems of the Girsanov type [2], representation theorems for martingales and weak martingales are all to be expected. Much of this body of results is already in hand and will be reported in a subsequent paper.

Appendix A. Stochastic Integrals for 2-Parameter Wiener Processes

As in section 1, define a Wiener process $\{W_z, \mathcal{F}_z, z \in R_{z_0}\}$ as a strong martingale such that W is also a Gaussian process with $EW_z = 0$ and

$$EW_z W_{z'} = \text{Area}(R_{z \vee z'}) \quad . \tag{A.1}$$

Provided that a separable version is chosen, a Wiener process is sample continuous, and for rectangles A and B

$$EW(A)W(B) = \text{Area}(A \cap B) \quad . \tag{A.2}$$

Let $\{\phi_z, z \in R_{z_0}\}$ be a process satisfying the following conditions:

(A.3) (a) ϕ is a bimeasurable function of (ω, z)

and

(b) $\int_{R_{z_0}} E\phi_z^2 dz < \infty$

or

(b') $\mathcal{P}\left(\left\{\omega: \sup_z |\phi(\omega,z)| < \infty\right\}\right) = 1$

and for each z

either (c_0) ϕ_z is $\mathcal{F}_z$-measurable

or (c_i) ϕ_z is $\mathcal{F}_z^i$-measurable, $i = 1,2$.

Then the stochastic integral

$$(\phi \circ W)_z = \int_{R_z} \phi_\zeta \, dW_\zeta \tag{A.4}$$

is well-defined for each $z \in R_{z_0}$. The process $(\phi \circ W)$ is a square-integrable strong martingale with respect to $\{\mathcal{F}_z\}$ if (a), (b) and (c_0) are satisfied, a square-integrable i-martingale if (a), (b) and (c_i) are satisfied. In each case a sample continuous version can be chosen, and

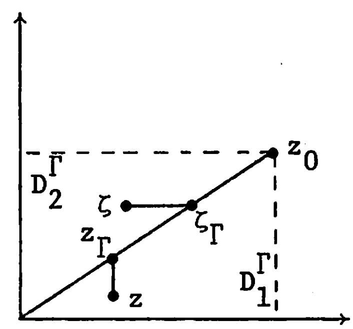
$$E(\phi \circ W)_z^2 = \int_{R_z} E\phi_\zeta^2 d\zeta \quad . \tag{A.5}$$

If condition (b) is replaced by (b') then there exists a sequence $\{\phi_n\}$ satisfying (b) such that $(\phi_n \circ W)$ converges uniformly with probability 1, and $\phi_n \to \phi$ almost surely. Hence, $\phi \circ W$ can be defined as the uniform limit of a sequence of continuous strong martingales (resp. i-martingales), if ϕ satisfies conditions (a), (b) and (c_0) $((c_i))$. Convergence being uniform, $\phi \circ W$ is sample continuous. We shall call $(\phi \circ W)$ under these conditions a local martingale (or local i-martingale).

The integral $\phi \circ W$ can be generalized still further. Let Γ be an increasing path connecting the origin to z_0. For each $z \in R_{z_0}$ let z_Γ denote the smallest point on Γ greater than z (with respect to the ordering). The path Γ divides R_{z_0} into two parts, say D_i^Γ, i = 1,2, where D_1^Γ is the area below Γ and D_2^Γ is the area to the left of Γ, i.e.,

$$D_1^\Gamma = \{\zeta \in R_{z_0} : \zeta \otimes \zeta_\Gamma = \zeta_\Gamma\}$$

$$D_2^\Gamma = \{\zeta \in R_{z_0} : \zeta_\Gamma \otimes \zeta = \zeta_\Gamma\}$$



Now, suppose that instead of (A.3c), ϕ satisfies

(A.3c') For each $z \in R_{z_0}$, ϕ_z is $\mathcal{F}_{z_\Gamma}$-measurable.

Define for i = 1,2

(A.6) $$\phi_{iz} = \phi_z \quad \text{if} \quad z \in D_i^\Gamma$$
$$= 0 \quad \text{otherwise} .$$

Then ϕ_{iz} is $\mathcal{F}_z^i$-measurable, $i = 1,2$, and $\phi_z = \phi_{1z} + \phi_{2z}$ for almost all z. Hence, we can define

(A.7) $$\phi \circ W = (\phi_1 \circ W) + (\phi_2 \circ W)$$

and so defined $\phi \circ W$ is a sample-continuous martingale (or local martingale) on Γ. It is also a weak martingale if ϕ_z is $\mathcal{F}_z$-adapted.

Consider a process X_z, $z \in R_{z_0}$, defined by

(A.8) $$X_z = \int_{R_z} f(z,\zeta) dW_\zeta .$$

In general, because the integrand depends on the endpoint, X is not a martingale of any kind. However, suppose that f satisfies the conditions:

(A.9) For each $z \in R_{z_0}$ and each $\zeta \in R_z$

(a_i) $$f(z,\zeta) = f(\zeta \otimes z, \zeta) \quad \text{for} \quad i = 1$$
$$= f(z \otimes \zeta, \zeta) \quad \text{for} \quad i = 2$$

and

(b_i) $$f(z,\zeta) \text{ is } \mathcal{F}_{\zeta \otimes z}\text{-measurable}, \ i = 1$$
$$f(z,\zeta) \text{ is } \mathcal{F}_{z \otimes \zeta}\text{-measurable}, \ i = 2 .$$

Then, X is an adapted i-martingale (local i-martingale). The intuitive reason for this is clear. For $i = 1$ let z and $z' \succ z$ be two points on the same horizontal line, then $\zeta \otimes z = \zeta \otimes z'$ so that

$$X_{z'} - X_z = \int_{R_{z'} - R_z} f(\zeta \otimes z, \zeta) dW_\zeta$$

and

$$E[X_{z'}-X_z|\mathcal{F}_z^1] = 0 \ .$$

Similarly, for $i = 2$, we have $E(X_{z'}-X_z|\mathcal{F}_z^2) = 0$ whenever z, z' lie on the same vertical line.

Now, let $\psi_{z,z'}$, $(z,z') \in R_{z_0} \times R_{z_0}$, be a random function satisfying the following conditions

(A.10) (a) ψ is a measurable function of (ω,z,z')

(b) $\displaystyle\int_{R_{z_0} \times R_{z_0}} I(z \curlywedge z')E\psi^2_{z,z'}dzdz' < \infty$

(alternatively, (b') $\mathcal{P}\left(\sup_{z,z'} |\psi_{z,z'}| < \infty\right) = 1$)

(c) For each (z,z') $\psi_{z,z'}$ is $\mathcal{F}_{z \vee z'}$-measurable.

For such a ψ we define the integral

$$\int_{R_z \times R_z} \psi_{\zeta,\zeta'} d\mu(\zeta) d\mu'(\zeta')$$

where μ and μ' can each be either W or the Lebesgue measure, as follows:

Let $I(\zeta \curlywedge \zeta')$ be equal to 1 or 0 according as $\zeta \curlywedge \zeta'$ or not. Define

$$\psi_\mu(z,\zeta') = \int_{R_z} I(\zeta \curlywedge \zeta')\psi_{\zeta,\zeta'} d\mu(\zeta)$$

$(d\mu(\zeta) = dW_\zeta$ or $d\zeta)$ and

$$\tilde{\psi}_\mu(z,\zeta) = \int_{R_z} I(\zeta \curlywedge \zeta')\psi_{\zeta,\zeta'} d\mu(\zeta') \ .$$

Observe that $\psi_\mu(z,\zeta)$ satisfies the condition (A.9b$_1$) and $\tilde{\psi}_\mu(z,\zeta)$ (A.9b$_2$). It can be shown that

$$\int_{R_z} \psi_\mu(z,\zeta')d\tilde{\mu}(\zeta') = \int_{R_z} \tilde{\psi}_{\tilde{\mu}}(z,\zeta)d\mu(\zeta) \qquad (\mu,\tilde{\mu} = W \text{ or Lebesgue})$$

so that

$$(\psi\circ\mu\tilde{\mu})_z = \int_{R_z \times R_z} \psi_{\zeta,\zeta'} d\mu(\zeta) d\tilde{\mu}(\zeta')$$

is well-defined. $(\psi\circ\mu W)$ is a 1-martingale, $(\psi\circ W\mu)$ is a 2-martingale, and $\psi\circ WW$, being both a 1-martingale and a 2-martingale, is a martingale.

We should note that for any ψ

$$\int_{R_{z_0} \times R_{z_0}} \psi_{\zeta,\zeta'} I(\zeta \curlywedge \zeta') d\mu(\zeta) d\tilde{\mu}(\zeta') = \int_{R_{z_0} \times R_{z_0}} \psi_{\zeta,\zeta'} d\mu(\zeta) d\mu(\zeta')$$

so that only values of $\psi_{\zeta,\zeta'}$ for $\zeta \curlywedge \zeta'$ affect the integral $(\psi\circ\mu\tilde{\mu})$. We also note that the definition for $\psi\circ WW$ given here is slightly different from the symmetrized definition given in [5], but is the same as the one introduced in [1,6] and used in all our papers on the subject since that time.

Appendix B. Differentiation Formulas

The simplest differentiation formulas for two-dimensional stochastic integrals are those associated with horizontal and vertical paths. Let X_{kz} be processes defined by

$$X_{kz} = X_{k0} + \int_{R_z} u_k(z,\zeta)dW_\zeta + \int_{R_z} v_k(z,\zeta)d\zeta \ , \quad k = 1,2,\ldots,n \tag{B.1}$$

where u_k and v_k satisfy conditions (A.9-a_i,b_i). We shall call X an i-semimartingale. For i = 1, X_k is a continuous semi-martingale on horizontal paths, and for i = 2, X is a continuous semi-martingale on vertical paths. If we consider the differentiation formula for continuous one-parameter martingales on horizontal paths in the case of i = 1, and on vertical paths in the case of i = 2, we get the formulas [7]

$$\begin{aligned} F(X_z) &= F(X_0) + \int_{R_z} F_k(X_{\zeta\otimes z})[u_k(z,\zeta)dW_\zeta + v_k(z,\zeta)d_\zeta] \\ &\quad + \frac{1}{2}\int_{R_z} F_{k\ell}(X_{\zeta\otimes z})u_k(z,\zeta)u_\ell(z,\zeta)d\zeta \end{aligned} \tag{B.2-1}$$

$$\begin{aligned} F(X_z) &= F(X_0) + \int_{R_z} F_k(X_{z\otimes\zeta})[u_k(z,\zeta)dW_\zeta + v_k(z,\zeta)d\zeta] \\ &\quad + \frac{1}{2}\int_{R_z} F_{k\ell}(X_{z\otimes\zeta})u_k(z,\zeta)u_\ell(z,\zeta)d\zeta \end{aligned} \tag{B.2-2}$$

where F(x), $x \in R^n$, has continuous partials $F_k(x) = \frac{\partial}{\partial x_k}F(x)$ and $F_{k\ell}(x) = \frac{\partial^2}{\partial x_k \partial x_\ell}F(x)$ and every repeated index implies summation from 1 to n.

Now suppose that X_{kz} satisfy

$$\begin{aligned} X_{kz} &= X_{k0} + \int_{R_z} \phi_{k\zeta}dW_\zeta + \int_{R_z} \theta_{k\zeta}d\zeta \\ &\quad + \int_{R_z\times R_z} \psi_{k,\zeta,\zeta'}dW_\zeta dW_{\zeta'} + \int_{R_z\times R_z} f_{k,\zeta,\zeta'}d\zeta dW_{\zeta'} \\ &\quad + \int_{R_z\times R_z} g_{k,\zeta,\zeta'}dW_\zeta d\zeta' \ . \end{aligned} \tag{B.3}$$

Then, it can be rerepresented in the form (B.1) in two ways: as a 1-semimartingale and a 2-semimartingale. Defining

$$\text{(B.4-1)} \qquad u_k(z,\zeta') = \phi_{k\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta')[\psi_{k,\zeta,\zeta'} dW_\zeta + f_{k,\zeta,\zeta'} d\zeta]$$

$$v_k(z,\zeta') = \theta_{k\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta') g_{k,\zeta,\zeta'} dW_\zeta$$

$$\text{(B.4-2)} \qquad \tilde{u}_k(z,\zeta) = \phi_{k\zeta} + \int_{R_z} I(\zeta \curlywedge \zeta')[\psi_{k,\zeta,\zeta'} dW_{\zeta'} + g_{k,\zeta,\zeta'} d\zeta']$$

$$\tilde{v}_k(z,\zeta) = \theta_{k\zeta} + \int_{R_z} I(\zeta \curlywedge \zeta') f_{k,\zeta,\zeta'} dW_{\zeta'}$$

we can write

$$\text{(B.5-1)} \qquad X_{kz} = X_{k0} + \int_{R_z} u_k(z,\zeta') dW_{\zeta'} + \int_{R_z} v_k(z,\zeta') d\zeta'$$

$$\text{(B.5-2)} \qquad X_{kz} = X_{k0} + \int_{R_z} \tilde{u}_k(z,\zeta) dW_\zeta + \int_{R_z} \tilde{v}_k(z,\zeta) d\zeta$$

which in turn yield

$$\text{(B.6-1)} \quad X_{k,\zeta'\otimes z} = X_{k,\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta')[\tilde{u}_k(\zeta'\otimes\zeta,\zeta) dW_\zeta + \tilde{v}_k(\zeta'\otimes\zeta,\zeta) d\zeta]$$

$$\text{(B.6-2)} \quad X_{k,z\otimes\zeta} = X_{k,\zeta} + \int_{R_z} I(\zeta \curlywedge \zeta')[u_k(\zeta'\otimes\zeta,\zeta') dW_{\zeta'} + v_k(\zeta'\otimes\zeta,\zeta') d\zeta'].$$

The differentiation formula (B.2-i) as applied to (B.5-i), can be taken together with (B.4-i) and (B.6-i) to yield a two-dimensional differentiation formula which represents $F(X_z)$ where X_z is the sum of five integrals as in (B.3)) as the sum of integrals of these five types once again. This formula, given in [7], involves mixed partial derivatives of F through the fourth order. It is not necessary for our purpose here.

Equations (B.2) are basically one-parameter differentiation formulas for horizontal and vertical paths, and can be generalized to any increasing path. Let Γ be an increasing path and let X_{kz}, $z \in \Gamma$, be processes of the form

$$X_{kz} = X_{k0} + \int_{R_z} u_k(\Gamma,\zeta)dW_\zeta + \int_{R_z} v_k(\Gamma,\zeta)d\zeta \ , \qquad k = 1,2,\ldots,m \tag{B.7}$$

where $u_k(\Gamma,z)$ and $v_k(\Gamma,z)$ are $\mathcal{F}_{z_\Gamma}$-measurable for each $z \in R_{z_0}$. Then X_k are semimartingales on Γ and (B.2) now takes the form

$$\begin{aligned} F(X_z) &= F(X_0) + \int_{R_z} F_k(X_{\zeta_\Gamma})[u_k(\Gamma,\zeta)dW_\zeta + v_k(\Gamma,\zeta)d\zeta] \\ &\quad + \frac{1}{2}\int_{R_z} F_{k\ell}(X_{\zeta_\Gamma})u_k(\Gamma,\zeta)u_\ell(\Gamma,\zeta)d\zeta \ . \end{aligned} \tag{B.8}$$

If X_k are defined by (B.3) then they can be put into the form of (B.7) by suitably identifying u and v as follows. Consider an integral of the form

$$Y_z = \int_{R_z \times R_z} \psi_{\zeta,\zeta'} d\mu(\zeta)d\tilde{\mu}(\zeta') \tag{B.9}$$

as introduced in appendix A where μ and $\tilde{\mu}$ can each be a Wiener process on the Lebesgue measure, and $\psi_{\zeta,\zeta'}$ is $\mathcal{F}_{\zeta\vee\zeta'}$-measurable. This can be reexpressed as

$$\begin{aligned} Y_z &= \int_{\zeta'\otimes\zeta \in D_1^\Gamma} \psi_{\zeta,\zeta'} I(\zeta \curlywedge \zeta')d\mu(\zeta)d\tilde{\mu}(\zeta') \\ &\quad + \int_{\zeta'\otimes\zeta \in D_2^\Gamma} \psi_{\zeta,\zeta'} I(\zeta \curlywedge \zeta')d\mu(\zeta)d\tilde{\mu}(\zeta') \\ &= \int_{R_z} \int_{\zeta'\otimes\zeta \in D_2^\Gamma} \psi_{\zeta,\zeta'} I(\zeta \curlywedge \zeta')d\mu(\zeta) \ d\tilde{\mu}(\zeta) \\ &\quad + \int_{R_z} \int_{\zeta'\otimes\zeta \in D_2^\Gamma} \psi_{\zeta,\zeta'} I(\zeta \curlywedge \zeta')d\tilde{\mu}(\zeta') \ d\mu(\zeta) \end{aligned} \tag{B.10}$$

which is of the form

$$\text{(B.11)} \qquad Y_z = \int_{R_z} \alpha(\Gamma,\zeta)\,d\tilde{\mu}(\zeta) + \int_{R_z} \beta(\Gamma,\zeta)\,d\mu(\zeta)$$

where for each ζ, $\alpha(\Gamma,\zeta)$ and $\beta(\Gamma,\zeta)$ are $\mathcal{F}_{\zeta_\Gamma}$-measurable. Reexpressing each of the double integrals in (B.3) in this way puts it into the form of (B.7).

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