AN EXACT PENALTY FUNCTION ALGORITHM FOR OPTIMAL CONTROL PROBLEMS WITH CONTROL AND TERMINAL EQUALITY CONSTRAINTS

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ABSTRACT

The presence of control constraints, because they are non-differentiable in the space of control functions, makes it difficult to cope with terminal equality constraints in optimal control problems. Gradient projection algorithms, for example, cannot be easily employed. These difficulties are overcome in this paper by employing an exact penalty function to handle the cost and terminal equality constraints and using the control constraints to define the space of permissible search directions in the search direction sub-algorithm. The search direction sub-algorithm is, therefore, more complex than the usual linear program employed in feasible directions algorithms; the sub-algorithm (approximately) solves a convex optimal control problem to determine the search direction, the accuracy of the approximation, in the implementable version of the algorithm being automatically increased to ensure convergence.

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I. INTRODUCTION

This paper deals with optimal control problems with control and terminal equality constraints. The combination of these two types of constraints makes it difficult to obtain efficient algorithms. Algorithms of the standard penalty function type are, of course, easily developed, but are usually computationally expensive. Of course, in the absence of control constraints, the obvious analogues of finite dimensional algorithms may be employed, example of these being the gradient projection algorithms and the multiplier methods, both the versions which require, at each iteration, exact minimization of the extended Lagrangian and these which do not. Because of the non-differentiability of the control constraints (in the space of control functions) it is difficult to extend these algorithms to cope with the problem considered in this paper.

To illustrate the approach taken in the paper consider the problem P with one equality constraint, i.e., the problem of minimizing $g^0(u)$ subject to the terminal equality constraint $g^1(u) = 0$ and to the control constraint $u \in L_\infty^r[0,1]$ and $u(t) \in \Omega$ for all $t \in [0,1]$. $g^0(u) = h^0(x^u(1))$ and $g^1(u) = h^1(x^u(1))$, x^u being the solution of the system differential equation due to control u (and specified initial condition). h^0 and h^1 are continuously differentiable, Ω is convex and compact and the system differential equation satisfies standard assumptions.

We deal with the terminal equality constraint by considering the alternative problem P_c of minimizing $\tilde{\gamma}_c(u) \triangleq \max\{g^0(u)/c + g^1(u), g^0(u)/c - g^1(u)\}$ subject to the control constraint. The alternative problem P_c is equivalent to the original problem P in the sense that there exists a finite c such that a solution for P_c is also a solution for P.

Ignoring for the moment the problem of choosing c, consider problem

P_c. The nature of the cost function (the maximum of a set of continuous functions) requires special attention. If the current control is u, a search direction $\tilde{s}_c(u)$ must be ascertained which is both a descent direction of the cost and is such that $u + \tilde{s}_c(u)$ satisfies the control constraint. (Because of the convexity of Ω, $u + \lambda\tilde{s}_c(u)$ then satisfies the control constraint for all $\lambda \in [0,1]$. A suitable candidate for the search direction is the s which solves

$$\tilde{\theta}_c(u) = \min_s \max \begin{Bmatrix} g^1(u) - \gamma(u) + \langle (1/c)\nabla g^0(u) + \nabla g^1(u), s \rangle, \\ -g^1(u) - \gamma(u) + \langle (1/c)\nabla g^0(u) - \nabla g^1(u), s \rangle \end{Bmatrix}$$

subject to the constraint that u+s satisfies the control constraint, where $\gamma(u) \triangleq \max\{g^1(u), -g^1(u)\}$. Thus the control constraint is handled in a natural fashion by incorporating it into the search direction sub-problem. This sub-problem is, admittedly, more complex than usual, but is, nevertheless, a standard convex optimal control problem for which several algorithms (with proven convergence) exist. The version of the algorithm which solves this sub-problem exactly is called 'conceptual'. In the 'implementable' version of the algorithm the above problem is solved approximately (in a finite number of iterations), the degree of approximation being automatically increased as the algorithm converges.

We turn now to the choice of the parameter c, a topic which has been somewhat neglected (and even misunderstood) in the literature. It is, of course, not sufficient to establish the existence of a finite c such that P_c and P are equivalent; a means must be provided to increase c to a suitable value. Incrementing c at each iteration is not satisfactory, as c will then become excessively large causing the same computation difficulties that occur in penalty function methods. Hence c must be chosen to satisfy some test depending on the current control u; the test can be defined by $t_c(u) \le 0$. Various heuristic choices of the test function t_c for algorithms incorporating a parameter c have appeared in the literature. However, there do exist conditions that t_c should satisfy which ensure convergence of the algorithm, and these are restated in Theorem 1. Roughly speaking, these are: (i) for each c, the function t_c is continuous in u; (ii) if u is desirable (i.e., satisfies a necessary condition of optimality) for P_c and the test $t_c(u) \le 0$ is satisfied, then u is desirable for P and (iii) for each permissible control u^* there exists a neighborhood N^* of u^* and a finite c^* such that the test $t_c(u) \le 0$ is satisfied for all u in the neighborhood N^* and all $c \ge c^*$. The second condition is an obvious requirement of the test function; the first and third conditions ensure that the algorithm does not jam up at undesirable points and are the conditions always ignored in the heuristic literature.

In many algorithms the test function t_c is related fairly directly to properties of the problem, e.g., where local convexity is required (as in multiplier methods) to the positive definiteness of a Hessian matrix or, in exact penalty function algorithms, to the (approximate) multipliers, see [3]. However, obtaining a test for the problem considered in this paper was relatively difficult. The considerations involved can be appreciated by referring to Fig. 1 which shows the (reachable) set W of values attained by $(g^0(u), g^1(u))$ as u ranges over the constraint set. Clearly $\hat{y} = (g^0(\hat{u}), g^1(\hat{u}))^T$ is the optimal point in the reachable set and so $\hat{u}$ is the required solution of P. Two sets of constant cost contours of $\tilde{\gamma}_c$ are shown for $c = c_1$ and $c = c_2$ Clearly the solution of P_{c_2} $(\min\{\tilde{\gamma}_{c_2} = y^2/c_2 + y^1 | y \in W\})$ is also the solution of P. On the other hand, $\min\{\tilde{\gamma}_{c_1} = y^2/c_1 + y^1 | y \in W\}$ occurs at $\bar{y}$, corresponding to a control $\bar{u}$ which does not satisfy the equality constraint (i.e., $g^1(\bar{u}) = \bar{y}^1 \neq 0$). Clearly, the minimum value of c that is satisfactory is that defining the slope of the supporting hyperplane to W passing through $\hat{y}$. However, it is not obvious how to employ this fact to obtain a test function satisfying the conditions given above. Also, determining the slope of a supporting hyperplane to a reachable set, indirectly defined by the control constraint, is computationally expensive. Instead we propose a test function $t_c(u) \triangleq \tilde{\theta}_c(u) + \gamma(u)/c$ which is easily calculated and which can be proved (with some difficulty) to satisfy the required conditions. If the equality constraint is not satisfied, then $\gamma \neq 0$, and the test requires that c be large enough so that $\tilde{\theta}_c(u) \leq -\gamma(u)/c < 0$. $\tilde{\theta}_c(u) = 0$ is a necessary condition of optimality for problem P_c.

The complete algorithm is specified in Algorithms 1 (conceptual) and 2 (implementable) in §4 and is based on the Algorithm Model described in §2. The map A_j, which specifies an algorithm (sub-algorithm 1) for solving P_{c_j} is presented in §3, and the test function t_c in §2. The map A_j requires, in its conceptual version, exact solution of the search direction sub-problem (determination of $\tilde{\theta}_c$) defined above; hence an implementable version of A_j, requiring only approximate solutions of P_{c_j} (sub-algorithm 2) is also presented in §3. It is shown in §4 that both version of the complete algorithm have the property that all accumulation points (in the L_∞ sense) are desirable (satisfy necessary condition, of optimality). Since such accumulation points need not exist, it is shown in §5 that the complete algorithm produces sequences which always have accumulation points in the sense of control measures, and that such accumulation points satisfy (relaxed) necessary conditions of optimality. A useful corollary is that the growth of the parameter c_j must be bounded. The complete algorithm is, to the authors' knowledge, the only algorithm, with established convergence, for solving the problem considered.[†]

II. THE PROBLEM AND BUILDING BLOCKS FOR AN ALGORITHM

In this section, we define the control problem P, an equivalent problem P_c without terminal equality constraints but involving a parameter c, and present an Algorithm Model which solves P. This Algorithm Model incorporates a sub-algorithm for solving P_c, a test function t_c and a procedure for increasing c until the test $t_c(u) \leq 0$ is satisfied. Finally we propose a concrete test function t_c and show that it has the required properties. We consider optimal control problems of the following type

$$\min\{g^0(u) \mid g^j(u) = 0,\ j = 1,2,\dots,m,\quad u \in G\} \tag{2.1}$$

where

$$G \triangleq \{u \in L_\infty^r [0,1] \mid u(t) \in \Omega \text{ for all } t \in [0,1]\} \tag{2.2}$$

$$g^j(u) \triangleq h^j(x^u(1)),\ j = 0,1,2,\dots,m \tag{2.3}$$

and $x^u : [0,1] \to \mathbb{R}^n$ is the solution of

$$\dot{x}(t) = f(x(t), u(t),t) \qquad \text{a.e. on } [0,1] \tag{2.4}$$

$$x(0) = \xi \tag{2.5}$$

We begin by making three standard assumptions which will later be supplemented with a kind of constraint qualification.

Assumption 1: The function $f : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^1 \to \mathbb{R}^n$ and the functions $h^j : \mathbb{R}^n \to \mathbb{R}^1$, $j = 0,1,2,\dots,m$, are continuously differentiable. ¤

Assumption 2: There exists an $M \in (0,\infty)$ such that

$$\|f(x,u,t)\| \leq M(1 + \|x\|) \tag{2.6}$$

for all $(x,u,t) \in \mathbb{R}^n \times \Omega \times [0,1]$. ¤

[†] With the exception, of course, of classical penalty function methods which drive the penalty to infinity and which have poor computational properties.

These two assumptions ensure that all the derivatives we shall need exist, that the solutions to the differential equations exist and, if Ω is compact, that they for equicontinuous families of functions in $L_\infty^n [0,1]$.

Assumption 3: The set Ω is compact and convex. ¤

We propose to solve problem (2.1) by making use of the exact penalty function, $c \max_{j-1-m} \{|g^j(u)|\}$[†], for the equality constraints for which a suitable, finite value of $c>0$ will be constructed by the algorithm. This leads to a family of problems P_c,

$$P_c : \min\{\tilde{\gamma}_c(u) | u \in G\} \tag{2.7}$$

where

$$\tilde{\gamma}_c(u) \triangleq g^0(u)/c + \max_{j=1-m} |g^j(u)| \tag{2.8}$$

If we now define g^j, for $j = m+1, m+2, \ldots, 2m$ by

$$g^{j+m}(u) = -g^j(u) \quad j = 1,2,\ldots,m \tag{2.9}$$

then

$$\gamma(u) \triangleq \max_{j=1-m} |g^j(u)|$$

$$= \max_{j=1-2m} g^j(u) \tag{2.10}$$

and

$$\tilde{\gamma}_c(u) = \max_{j=1-2m} \tilde{g}_c^j(u) \tag{2.11}$$

where

$$\tilde{g}_c^j(u) \triangleq g^0(u)/c + g^j(u), \quad j = 1,2,\ldots,m \tag{2.12a}$$

and $$\tilde{g}_c^{j+m}(u) \triangleq -\tilde{g}_c^j(u) \quad j = 1,2,\ldots,m \tag{2.12b}$$

We shall make various uses of the following two functions from G into $\mathbb{R}^1$:

[†]We write $j = 1 - m$ for $j \in \{1,2,\ldots,m\}$, etc.

$$\theta(u) \triangleq \min_{s\in S(u)} \max_{j=1-2m} \{g^j(u) - \gamma(u) + \langle \nabla g^j(u), s \rangle_2\} \tag{2.13}$$

and

$$\begin{aligned}\tilde{\theta}_c(u) &\triangleq \min_{s\in S(u)} \max_{j=1-2m} \{\tilde{g}_c^j(u) - \tilde{\gamma}_c(u) + \langle \nabla\tilde{g}_c^j(u), s \rangle_2\} \\ &= \min_{s\in S(u)} \max_{j=1-2m} \{g^j(u) - \gamma(u) + \langle \nabla g^0(u)/c \\ &\qquad\qquad + \nabla g^j(u), s \rangle_2\}\end{aligned} \tag{2.14}$$

where for any $u \in G$,

$$S(u) \triangleq \{s | s + u \in G\} \tag{2.15}$$

is the set of permissible search directions.

Also,

$$\langle x,y \rangle_2 \triangleq \int_0^1 \langle x(t), y(t) \rangle dt \tag{2.16}$$

denotes the L_2 scalar product, and $\nabla g^j(u)$, $j = 0,..,m$ is defined by:

$$\nabla g^j(u)(t) = \frac{\partial f}{\partial u} (x^u(t), u(t), t)^T \lambda_j^u(t) \tag{2.17}$$

with λ_j^u being the solution of the adjoint equation

$$\dot{\lambda}(t) = - \frac{\partial f}{\partial x} (x^u(t), u(t), t)^T \lambda^u(t) \tag{2.18}$$

$$\lambda(1) = \nabla h^j(x^u(1)) \tag{2.19}$$

$\nabla\tilde{g}_c^j(u)$, $j = 1,2,...,2m$, is similarly defined. We recall (see e.g., [1]) that $\langle \nabla g^j(u), s\rangle_2$ defines the L_∞^r [0,1] Frechet differential of g^j, $j = 0,1,...m$; similarly $\langle \nabla\tilde{g}_c^j(u),s \rangle_2$ defines the L_∞^r [0,1] Frechet differential of $\tilde{g}_c^j$, $j = 1,2,...,2m$.

We now impose a constraint qualification for the constraints specified by g^j, $j = 1,2,...,m$ and G; the qualification is analogous to the well known constraint qualification of linear independence of the equality and active inequality constraints, but is more indirect because of the nature of the constraint set G. Let $I_1, I_2, \ldots, I_{2^m}$ denote the following sets:

$$I_1 \triangleq \{1,2,3,\ldots,m\}$$

$$I_2 \triangleq \{m+1,2,3,\ldots,m\}$$

$$I_3 \triangleq \{1, m+2,3,\ldots,m\}$$

$$I_4 \triangleq \{m+1, m+2,3,\ldots,m\}$$

$$I_5 \triangleq \{1, 2, m+3,\ldots,m\}$$

$$\vdots$$

$$I_{2^m} \triangleq \{m+1, m+2, m+3,\ldots,2m\} \tag{2.20}$$

let $\mathcal{J}^*$ (the class of all such sets) be defined by:

$$\mathcal{J}^* \triangleq \{I_i \mid i = 1,2,\ldots,2^m\} \tag{2.21}$$

Hence, if $I = \{i_1,i_2,\ldots,i_{2^m}\} \in \mathcal{J}^*$ the set of constraints $\{g^{i_1},g^{i_2},\ldots g^{i_{2^m}}\}$ cannot include both g^j and $-g^j = g^{j+m}$ for any $j \in \{1,2,\ldots,m\}$. If $m = 2$, $\mathcal{J} = \{\{1,2\}, \{3,2\}, \{1,4\}, \{3,4\}\}$ corresponding to the following sets of constraints: $\{\{g^1,g^2\}, \{-g^1,g^2\}, \{g^1,-g^2\}, \{-g^1,-g^2\}\}$. Note that neither $\{g^1,-g^1\}$ nor $\{g^2,-g^2\}$ belong to the set $\mathcal{J}^*$. Finally, for each $I \in \mathcal{J}^*$, for each $c > 0$, we define the functions $\phi^I : G \to \mathbb{R}$, $\tilde{\phi}_c^I : G \to \mathbb{R}$ by:

$$\phi^I(u) \triangleq \min_{s\in S(u)} \max_{j\in I} \{g^j(u) - \gamma(u) + \langle \nabla g^j(u),s \rangle_2\} \tag{2.22}$$

$$\tilde{\phi}_c^I(u) \triangleq \min_{s\in S(u)} \max_{j\in I} \{g^j(u) - \gamma(u) + \langle \nabla g^0(u)/c + \nabla g^j(u),s \rangle_2\} \tag{2.23}$$

For any $u\in G$, let $I(u)\in\mathcal{J}^*$ be such that $g^j(u) \geq 0$ for all $j\in I(u)$ and $g^j(u) \leq 0$ for all $j\notin I(u)$, $j = 1,2,\ldots,2m$, and let $\mathcal{J}(u) \triangleq \{I(u)\}$. Note that if $\gamma(u) = 0$, then $\mathcal{J}(u) = \mathcal{J}^*$. Our constraint qualification can now be stated.

Assumption 4: For all $u \in G$, for all $I \in \mathcal{J}(u)$, $\phi^I(u) < 0$. ¤

The nature of the qualification can be appreciated from a few simple examples. If $m = 1$, Assumption 4 states that at all $u \in G$, there exists a permissible search direction reducing the maximum of g^1 and $g^2 = -g^1$. ($\mathcal{J}^* = \{\{1\}, \{2\}\}$, $\mathcal{J}(u) = \{1\}$ if $g^1(u) >$ $\mathcal{J}(u) = \{2\}$ if $g^1(u) < 0$). If $m = 2$, Assumption 4 states there exist permissible search directions which reduce, for example, $\max\{g^1(u), g^3(u)\}$ if $g^1(u) > 0$ and $g^3(u) = -g^2(u) > 0$, so that $\mathcal{J}(u) = \{1,3\}$.

Proposition 1: Let assumptions 1-4 be satisfied. For all $u \in G$ such that $\gamma(u) > 0$, $\theta(u) < 0$.

Proof:

It follows from the definitions of θ and ϕ^I that for any $u \in G$ and any $I \in \mathcal{J}(u)$,

$$\theta(u) = \min_{s \in S(u)} \max\left\{\max_{j \in I} g^j(u) - \gamma(u) + \langle \nabla g^j(u), s \rangle_2 ; \max_{j \in I^c} g^j(u) - \gamma(u) + \langle \nabla g^j(u), s \rangle_2 \right\} \tag{2.24}$$

where I^c denotes the complement of I with respect to the set $\{1,2,...,2m\}$. Let $s^I(u)$ denote the minimizing s in the definition of $\phi^I(u)$ and let $\sigma^I(u) \triangleq \max\{g^j(u) | j \in I^c\}$. Clearly $\sigma^I(u) \leq 0$ for all $I \in \mathcal{J}(u)$. Setting $s = \alpha s^I(u)$ is the right hand side of equ. (2.24) yield:

$$\theta(u) \leq \min_{\alpha \in [0,1]} \max\left\{\alpha\phi^I(u), \sigma^I(u) - \gamma(u) + \alpha \max_{j \in I^c} \langle \nabla g^j(u), s^I(u) \rangle_2 \right\} \tag{2.25}$$

Since $\phi^I(u) < 0$ and $\sigma^I(u) - \gamma(u) \leq -\gamma(u) < 0$ for all $I \in \mathcal{J}(u)$, it follows that $\theta(u) < 0$, which completes our proof. ¤

Proposition 2: Let assumptions 1-4 be satisfied. If $\hat{u} \in G$ is optimal then there exist multipliers $\psi^1, \psi^2, \ldots, \psi^m \in \mathbb{R}$ such that

$$\langle \nabla g^0(\hat{u}) + \sum_{j=1}^{m} \psi^j \nabla g^j(\hat{u}), s \rangle_2 \geq 0 \qquad (2.26)$$

for all $s \in S(\hat{u})$.

Proof:

It follows from Theorem 2.3.12 of [2] that the ray $R = \{y \in \mathbb{R}^{m+1} | y = \beta(-1,0,\ldots,0)^T, \beta > 0\}$ in $\mathbb{R}^{m+1}$ is separated from the set $W = \{y \in \mathbb{R}^{m+1} | y^{j+1} = \langle \nabla g^j(\hat{u}), s \rangle_2, j = 0,1,\ldots,m, s \in s(u)\}$ i.e., that there exist multipliers $\psi^0, \psi^1, \ldots, \psi^m \in \mathbb{R}$, not all zero, with $\psi^0 \geq 0$ such that:

$$\langle \psi^0 \nabla g^0(\hat{u}) + \sum_{j=1}^{m} \psi^j \nabla g^j(\hat{u}), s \rangle_2 \geq 0 \qquad (2.27)$$

for all $s \in S(\hat{u})$. If $\psi^0 = 0$, then not all of the multipliers $\psi^1, \psi^2, \ldots, \psi^m$ are zero and:

$$\sum_{j=1}^{m} \psi^j \langle \nabla g^j(\hat{u}), s \rangle_2 = \sum_{j=1}^{m} |\psi^j| \langle (\operatorname{sgn} \psi^j) \nabla g^j(\hat{u}), s \rangle_2 \geq 0 \qquad (2.28)$$

for all $s \in S(\hat{u})$, where sgn is defined by:

$$\operatorname{sgn}(x) = -1 \quad \text{if} \quad x < 0$$
$$= 1 \quad \text{if} \quad x \geq 0.$$

Now, since $\gamma(\hat{u}) = 0$, we have $g^j(\hat{u}) = 0$, $j = 1,2,\ldots,2m$ so that $\mathcal{J}(\hat{u}) = \mathcal{J}^*$. From assumption 4:

$$\phi^I(\hat{u}) = \max_{j \in I} \langle \nabla g^j(\hat{u}), s^I(\hat{u}) \rangle_2 < 0$$

for all $I \in \mathcal{J}^*$, where, as before, $s^I(u)$ denotes the minimizing s in eqn. (2.22). Recalling that $-g^j(\hat{u}) = g^{j+m}(\hat{u})$, $j = 1,2,\ldots,m$, we see that there exist an $I \in \mathcal{J}^*$ such that:

$$\max_{j=1-m} \langle (\operatorname{sgn} \psi^j) \nabla g^j(\hat{u}), s^I(\hat{u}) \rangle < 0 \qquad (2.29)$$

This contradicts eqn. (2.28). Hence $\psi^0 > 0$ and may be normalized to unity. ¤

Corollary 1: If assumptions 1-4 hold and $\hat{u} \in G$ is optimal, then:

$$\min\{\langle \nabla g^0(\hat{u}), s \rangle_2 \,|\, \langle \nabla g^j(\hat{u}), s \rangle_2 = 0,\ j = 1,\ldots,m,\ s \in S(\hat{u})\} = 0 \quad (2.30)$$ ¤

Corollary 2: If assumptions 1-4 hold and $\hat{u} \in G$ is optimal, then there exist no multipliers $\psi^1, \psi^2, \ldots, \psi^m \in \mathbb{R}$, not all zero, such that:

$$\sum_{j=1}^{m} \psi^j \langle \nabla g^j(\hat{u}), s \rangle_2 = 0 \quad (2.31)$$

for all $s \in S(\hat{u})$ ¤

Equation (2.30) is a weak form of the minimum principle for the problem considered.

Our Algorithm will find controls $\hat{u} \in G$ satisfying $\gamma(\hat{u}) = 0$ and the conclusion of Proposition 2, i.e., controls in the desirable set Δ defined as follows:

$$\Delta \triangleq \{\hat{u} \in G \,|\, \gamma(\hat{u}) = 0 \text{ and (2.26) is satisfied}\} \quad (2.32)$$

For the family of problem P_c in (2.7), we define the corresponding desirable sets

$$\Delta_c \triangleq \{\hat{u} \in G \,|\, \hat{\theta}_c(\hat{u}) = 0\} \quad (2.33)$$

It is easy to see by a straightforward generalization of Theorem (1.2.8) in [6] that if $\hat{u}$ is optimal for (2.7) then $\hat{u} \in \Delta_c$. We state this fact as

Proposition 3: Suppose that $\hat{u}$ is optimal for problem P_c (2.7) then $\hat{u} \in \Delta_c$. ¤

We also have the following result.

Proposition 4: Suppose $\hat{u} \in \Delta$, then there exists a $\hat{c} \geq 0$ such that $\tilde{\theta}_c(\hat{u}) = 0$ for all $c \geq \hat{c}$.

Proof:

For any $u \in G$, we define $R(u) \subset \mathbb{R}^2$ by

$$R(u) \triangleq \{y \in \mathbb{R}^2 \mid y^1 = \max_{j=1-2m} \{g^j(u) - \gamma(u) + \langle \nabla g^j(u), s \rangle_2\}$$

$$y^2 = \langle \nabla g^0(u), s \rangle_2 ; \; s \in S(u)\} \tag{2.34}$$

Then we see that for any $c > 0$,

$$\theta_c(u) = \min_{y \in R(u)} \left(\frac{1}{c} y^2 + y^1\right) \tag{2.35}$$

Let $s \in S(\hat{u})$ be arbitrary and $y(s)$ the corresponding element in $R(\hat{u})$, i.e., $y(s) = (y^1(s), y^2(s))^T$ where $y^1(s) \; \max_{j=1-2m} \; g^j(u) - \gamma(\hat{u}) + \langle \nabla g^j(u), s \rangle_2$, $y^2(s) = \langle \nabla g^0(\hat{u}), s \rangle_2$. Then, since $\gamma(\hat{u}) = 0$ ($g^j(\hat{u}) = 0$, $j = 1, \ldots, 2m$):

$$y^1(s) = \max_{j=1-2m} \langle \nabla g^j(\hat{u}), s \rangle_2 \tag{2.36}$$

Now, since $s \in L_\infty^r[0,1]$, $s \in L_2^r[0,1]$ and we can express s in the form

$$s = \sum_{j=1}^{m} a^j(s) \, \nabla g^j(\hat{u}) + \sigma(s) \tag{2.37}$$

where $\langle \nabla g^j(\hat{u}), \sigma(s) \rangle_2 = 0$ for $j = 1-m$, (and hence also for $j = 1-2m$).

Consequently,

$$y^1(s) = \max_{j=1-2m} \sum_{k=1}^{m} a^k(s) \langle \nabla g^j(\hat{u}), \nabla g^k(\hat{u}) \rangle_2 \tag{2.38}$$

Let Γ be a m×m matrix with jkth element $\langle \nabla g^j(\hat{u}), \nabla g^k(\hat{u}) \rangle_2$ and let $a = (a^1, a^2, \ldots, a^m)^T$. Then, setting $d(s) = \Gamma a(s)$, we see that

$$y^1(s) = \max_{j=1-m} \{d^j(s), -d^j(s)\}$$

$$= \|d(s)\|_\infty = \|\Gamma a(s)\|_\infty \geq 0 \tag{2.39}$$

Now assumption 5 implies that the functions $\nabla g^j(\hat{u})$, $j = 1,2,\ldots,m$ are linearly independent and hence the matrix Γ must be nonsingular. Consequently,

$$a(s) = \Gamma^{-1}\,\Gamma a(s) \tag{2.40}$$

and hence

$$\begin{aligned} \|a(s)\|_\infty &\le \|\Gamma^{-1}\|_\infty \, \|\Gamma a(s)\|_\infty \\ &= \|\Gamma^{-1}\|_\infty \, y^1(s) \end{aligned} \tag{2.41}$$

Next, making use of (2.26) and (2.37) we get,

$$\begin{aligned} y^2(s) &= \langle \nabla g^0(\hat{u}),s \rangle_2 \ge \sum_{j=1}^{m} -\psi^j \langle \nabla g^j(\hat{u}),s \rangle_2 \\ &= \langle -\Gamma\psi, a(s) \rangle \\ &\ge -m\|\Gamma\psi\|_\infty \, \|a(s)\|_\infty \\ &\ge -m\|\Gamma\psi\|_\infty \, \|\Gamma^{-1}\|_\infty \, y^1(s) \end{aligned} \tag{2.42}$$

Setting $\hat{c} \triangleq m\|\Gamma\psi\|_\infty \, \|\Gamma^{-1}\|_\infty$, we see that

$$\frac{1}{c}\, y^2(s) + y^1(s) \ge 0 \tag{2.43}$$

for all $c \ge \hat{c}$, for all $s \in S(\hat{u})$, i.e. (c.f. (2.35)) $\theta_c(\hat{u}) = 0$ for all $c \ge \hat{c}$ (since the zero functions is in $S(\hat{u})$). ⌑

We propose to construct our algorithm in conformity with an Algorithm Model presented in [3] and stated below for ease of reference. The model requires a strictly monotonically increasing sequence $\{c_j\}_{j=0}^{\infty}$ (e.g., the sequence generated by $c_{j+1} = c_j+\omega$, $\omega>0$ or by $c_{j+1} = \omega c_j$, $\omega>1$) together with corresponding sequences of costs $\{\tilde{\gamma}_j\}_{j=0}^{\infty}$, $\tilde{\gamma}_j \triangleq \tilde{\gamma}_{c_j}$ and desirable sets $\{\Delta_j\}_{j=0}^{\infty}$, $\Delta_j \triangleq \Delta_{c_j}$; Δ_j is the set of desirable points for the problem

P_{c_j}: $\min\{\tilde{\gamma}_j(u) \mid u \in G\}$. The model also requires a sequence of test functions $\{t_j\}_{j=0}^{\infty}$ and a sequence of iteration functions $\{A_j\}_{j=0}^{\infty}$ mapping G into G. A_j defines an algorithm for finding desirable points for P_{c_j}: all accumulation points of any sequence $\{u_i\}_{i=0}^{\infty}$ satisfying $u_{i+1} \in A_j(u_i)$, $i = 0,1,2,\ldots$ lie in Δ_j.

Given $u_i \in G$, the Alogirthm Model utilizes the iteration function A_j if $t_j(u_i) \leq 0$ to generate a new control $u_{i+1} \in A_j(u_i)$ and increases j (and therefore c_j) if $t_j(u_i) > 0$. The Model employs two counters, i and j: $\{u_i\}$ is the sequence of controls generated by the model and $\{\bar{u}_j\}$ is a sequence made up of elements of $\{u_i\}$ which contains that subset of $\{u_i\}$ corresponding to increases in c_j, so that $t_j(\bar{u}_j) > 0$. A particular u_i may appear many times in $\{\bar{u}_j\}$. An example of a sequence generated by the model is:

$(c = c_0)$	$t_0(u_0) \leq 0$,	determine $u_1 \in A_0(u_0)$
	$t_0(u_1) \leq 0$,	determine $u_2 \in A_0(u_1)$
	$t_0(u_2) \leq 0$,	determine $u_3 \in A_0(u_2)$
	$t_0(u_3) > 0$,	set $\bar{u}_0 = u_3$, $c = c_1$
$(c = c_1)$	$t_1(u_3) > 0$,	set $\bar{u}_1 = u_3$, $c = c_2$
$(c = c_2)$	$t_2(u_3) \leq 0$,	determine $u_4 \in A_2(u_3)$
	$t_2(u_4) \leq 0$,	determine $u_5 \in A_2(u_4)$
	$t_2(u_5) > 0$,	set $\bar{u}_2 = u_5$, $c = c_3$
$(c = c_3)$	$t_3(u_5) \leq 0$,	determine $u_6 \in A_3(u_5)$

etc. Theorem 1 gives conditions under which accumulation points of $\{u_i\}$ are desirable for the original problem.

<u>Algorithm Model</u>

<u>Data</u>: $u_0 \in G$.

<u>Step 0</u>: Set $i = 0$, $j = 0$.

Step 1: If $t_j(u_i) > 0$ go to Step 2; else go to Step 3.

Step 2: Set $\bar{u}_j = u_i$, set $j = j+1$ and go Step 1.

Step 3: Compute a $u \in A_j(u_i)$

Step 4: If $\tilde{\gamma}_j(u) < \tilde{\gamma}_j(u_i)$ set $u_{i+1} = u$, $i = i+1$ and go to Step 1; else stop. ⌑

Note that c is increased to a satisfactory value in the loop comprising Steps 1 and 2.

The following convergence result, given in [3], is required.

Theorem 1: Suppose: (i) For each j, A_j is such that any accumulation point $\hat{u}$ of an infinite sequence $\{u_i\}_{i=0}^{\infty}$ constructed according to $u_{i+1} \in A_j(u_i)$ and satisfying, for all i, $\tilde{\gamma}_j(u_{i+1}) < \tilde{\gamma}_j(u_i)$ lies in Δ_j (i.e. $\hat{u} \in \Delta_j$), and that $\tilde{\gamma}_j(u') \geq \tilde{\gamma}_j(u)$ for any $u' \in A_j(u)$ implies that $u \in \Delta_j$. (ii) The test functions $t_j(\cdot)$, $j = 0,1,2,\ldots$, are continuous. (iii) For $j = 0,1,2,\ldots$, $\{u \in \Delta_j | t_j(u) \leq 0\} \subset \Delta$. (iv) For every $u^* \in G$ there exists an integer j^* such that if $u_k \to u^*$ (in the L_∞ sense) then, for some k_0, $t_j(u_k) \leq 0$ for all $t \geq k_0$, for all $j \geq j^*$. Under these assumptions, (i) if the algorithm model constructs a finite sequence $\{u_i\}_{i=0}^{k}$ (so that $\{\bar{u}_j\}$ is also finite) then the last element, u_k, is in Δ. (ii) If the Algorithm Model constructs a finite sequence $\{\bar{u}_j\}$ and $\{u_i\}$ is infinite, then every L_∞ accumulation point u^* of $\{u_i\}$ is in Δ. (iii) If the sequence $\{\bar{u}_j\}$ is infinite, then it has no L_∞ accumulation points. ⌑

In the next section we shall propose an algorithm (A_j in the Algorithm Model) for solving the problems P_{c_j}. Here we define a set of test functions $\{t_j\}_{j=0}^{\infty}$ and show that they satisfy the properties required in Theorem 1, under (ii, (iii) and (iv).

For any $c > 0$ we define the test function t_c: $G \to \mathbb{R}^1$ by

$$t_c(u) \triangleq \tilde{\theta}_c(u) + \gamma(u)/c \tag{2.44}$$

Since γ is continuous by inspection, to prove that t_c is continuous, we must show that $\tilde{\theta}_c$ is continuous. At the same time it is convenient to establish the continuity of θ, ϕ^I and $\tilde{\phi}_c^I$.

Lemma 1: The functions θ, $\tilde{\theta}_c$, ϕ^I, $\tilde{\phi}_c^I$ are continuous on G, for all $c > 0$, for all $I \in \mathcal{I}^*$.

Proof: Because of Assumptions 1 and 2 the functions g^j, $\tilde{g}_c^j$, ∇g^j, $\nabla\tilde{g}_c^j$, $j = 0,1,\ldots,m$, $c \geq 0$, are all continuous. Let u^* be any control in G and $\{u_i\}_{i=0}^{\infty}$ any infinite sequence in G such that $u_i \to u^*$ (in the L_∞ sense).

We first prove that θ is upper-semi-continuous (u.s.c). There exists a $\bar{u}^* \in G$ such that:

$$\theta(u^*) = -\gamma(u^*) + \max_{j=1-2m} \{g^j(u^*) + \langle \nabla g^j(u^*), \bar{u}^* - u^* \rangle_2\} \tag{2.45}$$

(Clearly $\bar{u}^* - u^*$ is the minimizing $s \in S(u^*)$ in (2.13).) From the definition of θ ((2.13)) it follows that:

$$\theta(u_i) \leq -\gamma(u_i) + \max_{j=1-2m} \{g^j(u_i) + \langle \nabla g^j(u_i), \bar{u}^* - u_i \rangle_2\}$$

and, hence that

$$\overline{\lim}\, \theta(u_i) \leq \overline{\lim} \left[-\gamma(u_i) + \max_{j=1-2m} \{g^j(u_i) + \langle \nabla g^j(u_i), \bar{u}^* - u_i \rangle_2\}\right]$$

$$= \theta(u^*) \tag{2.46}$$

where the last equality follows from the continuity of γ, g^j, ∇g^j, $j=1-2m$. Hence θ is u.s.c. at u^*.

We next establish that θ is lower-semi-continuous (l.s.c.). For each element u_i of the sequence $\{u_i\}$, introduced above, there exists a $\bar{u}_i \in G$ such that:

$$\theta(u_i) = -\gamma(u_i) + \max_{j=1-2m} \{g^j(u_i) + \langle \nabla g^j(u_i), \bar{u}_i - u_i \rangle_2\} \tag{2.47}$$

Suppose, contrary to what is to be proven, that:

$$\underline{\lim}\, \theta(u_i) = \theta(u^*) - \varepsilon$$

for some $\varepsilon>0$. Then there exists a subsequence, indexed by $K \subset \{1,2,3,\ldots\}$, such that:

$$\theta(u_i) \overset{K}{\to} \theta(u^*) - \varepsilon$$

and there exists a $K' \subset K$ such that

$$\gamma(u_i) \overset{K'}{\to} \gamma(u^*)$$

$$g^j(u_i) \overset{K'}{\to} g^j(u^*), \quad j = 1,2,\ldots,m$$

$$\langle \nabla g^j(u_i), \bar{u}_i - u_i \rangle_2 \overset{K'}{\to} \alpha^j, \quad j = 1,2,\ldots,m$$

It follows that:

$$-\gamma(u^*) + \max_{j=1-2m} g^j(u^*) + \alpha^j = \theta(u^*) - \varepsilon$$

Hence, there exists an integer i_0 such that

$$-\gamma(u^*) + \max_{j=1-2m} \{g^j(u^*) + \langle \nabla g^j(u\ ), \bar{u}_i - u_i\rangle_2\}$$

$$= -\gamma(u^*) + \max_{j=1-2m} \{g^j(u^*) + \langle \nabla g^j(u^*), \bar{u}_i - u_i\rangle_2$$

$$+ \langle \nabla g^j(u_i) - \nabla g^j(u^*), \bar{u}_i - u_i\rangle_2\} \leq \theta(u^*) - \varepsilon/2 \qquad (2.48)$$

for all $i \geq i_0$, $i \in K'$. Since $\nabla g^j(u_i) \to \nabla g^j(u^*)$ as $i \to \infty$, $i \in K'$; it follows from the second line of (2.48) and the compactness of Ω that there exists an integer $i_1 > i_0$ such that

$$-\gamma(u^*) + \max_{j=1-2m} \{g^j(u^*) + \langle \nabla g^j(u^*), \bar{u}_i - u_i\rangle_2\} \leq \theta(u^*) - \varepsilon/2$$

for all $i \geq i_1$, $i \in K'$. But this contradicts the fact that:

$$\theta(u^*) \leq -\gamma(u^*) + \max_{j=1-2m} \langle \nabla g^j(u^*), \bar{u}_i - u_i\rangle_2$$

for all i. Hence

$$\underline{\lim}\ \theta(u_i) = \theta(u^*), \qquad (2.49)$$

i.e. θ is l.s.c. at u^*. Hence θ is continuous.

The continuity of $\tilde{\theta}_c$, ϕ^I, $\tilde{\phi}^I_c$ for all $c \geq 0$, for all $I \in \mathcal{J}^*$, can be similarly established. ¤

The following is obvious.

Corollary: For any $c > 0$, the function t_c: $L^m_\infty [0,1] \to \mathbb{R}$, defined by (2.34), is continuous. ¤

Lemma 2: For any $c > 0$, the set $\{u \in \Delta_c | t_c(u) \leq 0\}$ is contained in the set Δ.

Proof: Suppose $\bar{u} \in \Delta_c$ is such that $t_c(\bar{u}) \leq 0$. Then, by (2.33) $\tilde{\theta}_c(\bar{u}) = 0$ and, hence, (2.44) implies that $\gamma(\bar{u}) = 0$, i.e. that $\bar{u}$ is feasible. Now, by (2.14)

$$\tilde{\theta}_c(\bar{u}) = \min_{s \in S(\bar{u})} \max_{j=1-2m} \{ g^j(\bar{u}) - \gamma(\bar{u}) + \frac{1}{c} \langle \nabla g^0(\bar{u}), s \rangle_2 + \langle \nabla g^j(\bar{u}), s \rangle_2 \}$$

$$= \min_{s \in S(\bar{u})} \{ \frac{1}{c} \langle \nabla g^0(\bar{u}), s \rangle_2 + \max_{j=1-2m} \langle \nabla g^j(\bar{u}), s \rangle_2 \} = 0 \qquad (2.50)$$

since $\gamma(\bar{u}) = 0$, $g^j(\bar{u}) = 0$ also for $j = 1,2,\ldots,2m$). Suppose $\bar{u} \notin \Delta$. Then there exists an $\bar{s} \in S(\bar{u})$, such that $\langle \nabla g^0(\bar{u}), \bar{s} \rangle_2 < 0$ and $\langle \nabla g^j(\bar{u}), \bar{s} \rangle_2 = 0$ for $j = 1,2,\ldots,2m$. Substituting this $\bar{s}$ into (2.39) implies that $\tilde{\theta}_c(\bar{u}) < 0$, which is a contradiction. Hence $\bar{u} \in \Delta$. ¤

Lemma 3: For any $\bar{u} \in G$ there exists a $\bar{c} > 0$ such that for any infinite sequence $\{u_i\}$ in G converging to $\bar{u}$ (in the $L^r_\infty [0,1]$ sense) there exists and $i_1 \geq 0$ such that $t_c(u_i) \leq 0$ for all $i \geq i_1$, for all $c \geq \bar{c}$.

Proof: Let $\bar{u}$ and $\{u_i\}$ be as stated above. From assumption 4, $\phi^I(\bar{u}) \leq -\delta$, for some $\delta > 0$, for all $I \in \mathcal{J}(\bar{u})$. Because of the continuity of ϕ^I (Lemma 1), there exists an $i_0 \geq 0$ such that

$$\phi^I(u_i) \leq -\delta/2 \qquad (2.51)$$

for all $I \in \mathcal{I}(\bar{u})$ for all $i \geq i_0$.

As before, let $\sigma^I(u) \triangleq \max\{g^j(u) | j \in I^c\}$ (I^c being the complement of I in $\{1,2,\ldots,2m\}$). Clearly $\sigma^I(u) \leq 0$ for all $I \in \mathcal{I}(u)$. Let $s^I(u)$ denote the minimizing s in the definition of $\phi^I(u)$ (2.22) and $\tilde{s}_c^I(u)$ the minimizing s in the definition of $\tilde{\phi}_c^I(u)$ (2.23). From the definition of ϕ^I, $\tilde{\phi}_c^I$ it follows, for all $u \in G$, all $I \in \mathcal{I}^*$, all $c \geq 0$, that:

$$\tilde{\phi}_c^I(u) \leq \max_{j \in I}\{ g^j(u) - \gamma(u) + \langle (1/c)\nabla g^0(u) + \nabla g^j(u), s^I(u) \rangle_2\}$$

$$= \phi^I(u) + \langle (1/c)\nabla g^0(u), s^I(u) \rangle_2 \tag{2.52}$$

Hence, since Ω is compact, there exists a $M \in (0,\infty)$ such that

$$\tilde{\phi}_c^I(u) \leq \phi^I(u) + M/c$$

for all $u \in G$, all $I \in \mathcal{I}^*$, all $c > 0$. Let $c_1 = \max\{4M/\delta, 1\}$, so that $M/c \leq \delta/4$ for all $c \geq c_1$. Clearly,

$$\tilde{\phi}_c^I (u_i) \leq \phi^I (u_i) + M/c \leq - \delta/2 + \delta/4 = - \delta/4 \tag{2.53}$$

for all $c \geq c_1$, all $i \geq i_0$, all $I \in \mathcal{I}(\bar{u})$.

(a) Suppose $\gamma(\bar{u}) > 0$. Then there exists an $i_1 \geq i_0$ such that for all $i \geq i_1$, for all $j \in I^c$, for all $I \in \mathcal{I}(\bar{u})$.

$$g^j(u_i) - \gamma(u_i) \leq -\gamma(\bar{u})/2 \tag{2.54}$$

and

$$\gamma(u_i) \leq 3/2\ \gamma(\bar{u}) \tag{2.55}$$

(The first inequality follows from the fact that if $I \in \mathcal{I}(\bar{u})$, $g^j(\bar{u}) \leq 0$ for all $j \in I^c$, for all $u \in G$, and the fact that

$\gamma(u_i) \to \gamma(\bar{u})$ as $i \to \infty$). Now, for all $I \in \mathcal{J}^*$

$$\tilde{\theta}_c(u_i) = \min_{s \in S(u)} \max \left\{ \max_{j \in I} \{ g^j(u_i) - \gamma(u_i) + \langle (1/c)\nabla g^0(u_i) + \langle \nabla g^j(u_i), s \rangle_2 \}, \right.$$

$$\max_{j \in I^c} \{ g^j(u_i) - \gamma(u_i) + \langle (1/c)\nabla g^0(u_i)$$

$$\left. + \langle \nabla g^j(u_i), s \rangle_2 \} \right\} \qquad (2.56)$$

Substituting $\alpha \tilde{s}_c^I(u)$ for s yields, for all $I \in \mathcal{J}^*$:

$$\tilde{\theta}_c(u_i) \leq \min_{\alpha \in [0,1]} \max \left\{ \alpha \tilde{\phi}_c^I(u_i), \ \max_{j \in I^c} \{ g^j(u_i) - \gamma(u_i) + \alpha b \} \right\} \qquad (2.57)$$

where $b \triangleq \sup \left\{ (\|\nabla g^0(u)\|_2 / c + \|\nabla g^j(u)\|_2) \|s\|_2 \,\middle|\, s \in S(u),\ u \in G,\ c \geq c_1,\ j = 1,2,\ldots,m \right\} < \infty$,

and $\|y\|_2 \triangleq [\langle y,y \rangle_2]^{1/2}$. Hence, choosing any $I \in \mathcal{J}(\bar{u})$ and adding $\gamma(u_i)/c$ to both terms in (2.57), we obtain

$$t_c(u_i) = \tilde{\theta}_c(u_i) + \gamma(u_i)/c$$

$$\leq \min_{\alpha \in [0,1]} \max \left\{ -\alpha\delta/4 + \gamma(u_i)/c, \ \max_{j \in I^c} \{ g^j(u_i) - \gamma(u_i) + \alpha b \right.$$

$$\left. + \gamma(u_i)/c \} \right\}$$

$$\leq \min_{\alpha \in [0,1]} \max \left\{ 3\gamma(\bar{u})/2c - \alpha\delta/4, \ -\gamma(\bar{u})/2 + \alpha b + 3\gamma(\bar{u})/2c \right\} \qquad (2.58)$$

for all $i \geq i_1$, $c \geq c_1$. The first term on the right hand side of (2.58) is negative if:

$$\alpha > \alpha_1 \triangleq 6\gamma(\bar{u})/\delta c \qquad (2.59)$$

and the second is negative if

$$0 \leq \alpha < \alpha_2 \triangleq \gamma(\bar{u})(1-3/c)/2b \qquad (2.60)$$

Obviously, there exists a $c_2 \geq c_1$, such that $\alpha_1 < 1$, and $0 < \alpha_1 < \alpha_2$ for all $c \geq c_2$. Hence, letting $\alpha = \alpha_1$, we obtain $t_c(u_i) \leq 0$ for all $i \geq i_1$, all $c \geq c_2$.

(b) Suppose now $\gamma(\bar{u}) = 0$. Hence, $g^j(\bar{u}) = 0$ for $j = 1,2,\ldots,2m$, $\mathcal{J}(\bar{u}) = \mathcal{J}$ and $\mathcal{J}(u_i) \subset \mathcal{J}(\bar{u})$ for all i. Since $I \in \mathcal{J}(u_i)$ implies that $g^j(u_i) \leq 0$ for all $j \in I^c$ we see from eqn. (2.57) that:

$$\tilde{\theta}_c(u_i) \leq \min_{\alpha\in[0,1]} \max\{\alpha\tilde{\phi}_c^I(u_i), -\gamma(u_i) + \alpha b\} \tag{2.61}$$

for all $I \in \mathcal{J}(u_i)$. Since $\mathcal{J}(u_i) \subset \mathcal{J}^*$ for all i, we can make use of (2.53) to obtain

$$\tilde{\theta}_c(u_i) \leq \min_{\alpha\in[0,1]} \max\{-\alpha\delta/4, -\gamma(u_i) + \alpha b\} \tag{2.62}$$

for all $i \geq i_1$, $c \geq c_1$.

Hence:

$$t_c(u_i) \leq \min_{\alpha\in[0,1]} \max\{\gamma(u_i)/c - \alpha\delta/4, -\gamma(u_i) + \gamma(u_i)/c + \alpha b\} \tag{2.63}$$

The first term on the right hand side of (2.63) is negative of

$$\alpha > \alpha_1(u_i) \triangleq 4\,\gamma(u_i)/c\delta \tag{2.64}$$

and the second is negative if

$$0 \leq \alpha < \alpha_2(u_i) \triangleq \gamma(u_i)(1-1/c)/b \tag{2.65}$$

Hence, if $\gamma(u_i) \neq 0$, there exists a $\bar{c} \geq c_2$ such that $\alpha_1(u_i) < 1$ and $\alpha_1(u_i) \leq \alpha_2(u_i)$ for all i. Hence for all $i \geq i_1$ such that $\gamma(u_i) \neq 0$, for all $c \geq c_3$, $t_c(u_i) \leq 0$.

If $\gamma(u_i) = 0$ then $t_c(u_i) = \tilde{\theta}_c(u_i) + \gamma(u_i)/c \leq 0$ for all $c > 0$.

Combining (a) and (b) we obtain $t_c(u_i) \leq 0$ for all $i \geq i_1$ all $c \geq \bar{c}$. ¤

We can summarize our findings as follows:

Theorem 2: Given any monotonically increasing sequence $\{c_j\}_{j=0}^{\infty}$, $c_j > 0$, with $c_j \to \infty$, the test functions $t_j \triangleq t_{c_j} : G \to R$ (defined by (2.44)) satisfy the assumption of Theorem 1. $\square$

We can now proceed with the construction of the maps A_j which define subalgorithms to solve problems P_{c_j}.

III. SUBALGORITHMS FOR $\min\{\tilde{\gamma}_c(u) \mid u \in G\}$

In this section we present a 'conceptual' and an 'implementable' algorithm for solving problem P_c and establish convergence. The following subalgorithm has a structural relationship with the Topkis-Veinott method of feasible directions [6] which is analogous to the relationship between the Demjanov method for min max problems [4] and the Zoutendijk method of feasible directions[5]. Its main advantage over Demjanov's method is that it requires only one evaluation of the optimality function $\tilde{\theta}_c(u)$ per iteration, versus several in the Demjanov version. The evaluation of the optimality functions in an optimal control application is much costlier than in a nonlinear programming problem. Also, we have here a built-in bounded set G, defining the S(u), rather than the L_∞ hypercube used in nonlinear programming. Hence the reasons for preferring a Zoutendijk type algorithm to one of Topkis-Veinott form, do not transfer from the nonlinear programming situation to optimal control.

For the purpose of simplifying exposition we state the subalgorithm defining the maps $A_c(A_j)$ for solving problem $P_c(P_{c_j})$ first in a relatively simple, but conceptual, form, then in a somewhat more complex, but implementable form. In subalgorithm 1 below the map A_c (A_j for $c = c_j$) is defined in steps 0 to 2.

Subalgorithm 1.

Data: $u_0 \in G$, $\beta \in (0,1)$, $c > 0$.

Step 0: Set $i = 0$.

Step 1: Compute $\tilde{\theta}_c(u_i)$ and a corresponding $s_i \in S(u_i)$ (where $\tilde{\theta}_c$ was defined in (2.14)).

Step 2: If $\tilde{\theta}_c(u_i) = 0$, stop; else compute the smallest integer $k_i \geq 0$ such that

$$\tilde{\gamma}_c(u_i + \beta^{k_i} s_i) - \gamma_c(u_i) \leq \beta^{k_i}\tilde{\theta}_c(u_i)/4 \qquad (3.1)$$

Step 3: Set $u_{i+1} = u_i + \beta^{k_i} s_i$, set $i = i + 1$ and go to step 1. ¤

The fact that the subalgorithm is well defined, i.e. (3.1) is always satisfied with a finite k_i, can be deduced from the proof the Theorem 3 below, which states the convergence properties of the subalgorithm.

Theorem 3: Suppose that Subalgorithm 1 construct an infinite sequence of controls $\{u_i\}_{i=0}^{\infty}$ in G_0. Then every $L_\infty^m[0,1]$ accumulation point $\hat{u}$ of this sequence is in Δ_c, i.e. $\tilde{\theta}_c(\hat{u}) = 0$.

Proof: We shall prove this theorem using the methods employed in proving theorem (1.3.9) in [6]. Thus, suppose that $\hat{u} \in G$ is an $L_\infty^r[0,1]$ accumulation point of $\{u_i\}_{i=0}^{\infty}$ and that $\hat{u} \notin \Delta_c$, i.e. $\tilde{\theta}_c(\hat{u}) < 0$. Then, there exists an infinite subsequence $\{u_i\}_{i\in K}$ such that $u_i \xrightarrow{K} \hat{u}$ (in L_∞^r) and, because $\tilde{\theta}_c$ is continuous, there is an integer i_o such that

$$\tilde{\theta}_c(u_i) \leq \tilde{\theta}_c(\hat{u})/2 \quad \text{for all } i \geq i_o,\ i \in K \qquad (3.2)$$

Now, by definition

$$\tilde{\theta}_c(u_i) = \min_{s \in S(u_i)} \max_{j=1-2m} \{ \tilde{g}_c^{\,i}(u_i) - \tilde{\gamma}_c(u_i) + \langle \nabla\tilde{g}_c^{\,i}(u_i), s\rangle_2 \} \quad (3.3)$$

Because the set Ω is compact, the functions $s \in S(u_i)$ are uniformly bounded. For any $j \in \{1,2,\ldots,2m\}$, $i \geq i_o$, $i \in K$, one of the following two must hold: either

$$\hat{\gamma}_c(u_i) - \tilde{g}_c^{i}(u_i) > -\tilde{\theta}_c(u_i)/2 \qquad (3.4)$$

or

$$\tilde{\gamma}_c(u_i) - \tilde{g}_c^{i}(u_i) \leq -\tilde{\theta}_c(u_i)/2 \qquad (3.5)$$

Suppose (3.4) holds. Then because of the continuity of $\tilde{g}_c^{\,i}$, γ_c and the uniform boundedness of the s_i, there exists an integer $k' \geq 0$ such that for all $i \in K$, $i \geq i_o$, $k \geq k'$

$$\begin{aligned} \tilde{g}_c^{\,i}(u_i + \beta^k s_i) - \tilde{\gamma}_c(u_i) &\leq \tilde{\theta}_c(u_i)/4 \\ &\leq \beta^k \tilde{\theta}_c(u_i)/4 \\ &\leq \beta^k \tilde{\theta}_c(\hat{u})/8 \end{aligned} \qquad (3.6)$$

Next, suppose that (3.5) holds for some $j \in \{1,2,\ldots,2m\}$. Then, from (2.14), we must have

$$\langle \nabla \tilde{g}_c^{\,i}(u_i), s_i \rangle_2 \leq \tilde{\theta}_c(u_i)/2 \qquad (3.7)$$

Hence, applying a first order Taylor expansion with remainder we get

$$\begin{aligned} &\tilde{g}_c^{\,j}(u_i + \lambda s_i) - \tilde{g}_c^{\,i}(u_i) \\ &= \lambda \langle \nabla \tilde{g}_c^{\,j}(u_i), s_i \rangle_2 + \int_0^1 \langle \nabla g^j(u_i + \xi \lambda s_i) - \nabla g^j(u_i), s_i \rangle_2 \, d\xi \\ &\leq \lambda \tilde{\theta}_c(u_i)/2 + \sup_{\xi \in [0,1]} \| \nabla g^u(u_i + \xi \lambda s_i) - \nabla g^i(u_i) \|_\infty \| s_i \|_\infty \end{aligned} \qquad (3.8)$$

Since it follows from assumptions 1 and 2 that $\tilde{g}_c^{\,j}$ and $\nabla \tilde{g}_c^{\,j}$, $j=0,1,\ldots m$, are continuous in $L_m^\infty[0,1]$, there exists an integer $k'' \geq k'$, finite, such that for all $k \geq k''$

$$\tilde{g}_c^{\,j}(u_i + \beta^k s_i) - \tilde{\gamma}_c(u_i) \leq \tilde{g}_c^{\,j}(u_i + \beta^k s_i) - \tilde{g}_c^{\,j}(u_i) \leq \beta^k \,\tilde{\theta}_c(u_i)/4$$

$$\leq \beta^k \,\tilde{\theta}_c(\hat{u})/8 \qquad (3.9)$$

Hence, combining (3.9) with (3.6) we conclude that for all $i \geq i_o$, $i \in K$

$$\tilde{\gamma}_c(u_i + \beta^k s_i) - \tilde{\gamma}_c(u_i) \leq \beta^k \,\tilde{\theta}_c(u_i)/4$$

$$\leq \beta^k \,\tilde{\theta}_c(\hat{u})/8 \qquad (3.10)$$

and hence that $k_i \leq k''$ for all $i \geq i_o$, $i \in K$. Consequently, for all $i \in K$, $i \geq i_o$,

$$\tilde{\gamma}_c(u_{i+1}) - \tilde{\gamma}_c(u_i) \leq \beta^{k_i} \,\tilde{\theta}_c(\hat{u})/8 \leq \beta^{k''} \,\tilde{\theta}_c(\hat{u})/8 \triangleq -\delta < 0 \qquad (3.11)$$

Note that (3.10) also shows that the subalgorithm is well-defined. Now, $\{\tilde{\gamma}_c(u_i)\}$ is a bounded, monotonically decreasing sequence which must converge to $\tilde{\gamma}_c(\hat{u})$ since $\tilde{\gamma}_c(u_i) \xrightarrow{K} \tilde{\gamma}_c(\hat{u})$. But

$$\tilde{\gamma}_c(\hat{u}) - \tilde{\gamma}_c(u_{i_o}) = \sum_{\substack{i \in K \\ i \geq i_o}} \tilde{\gamma}_c(u_{i+1}) - \tilde{\gamma}_c(u_i)$$

$$+ \sum_{\substack{i \notin K \\ i \geq i_o}} \tilde{\gamma}_c(u_{i+1}) - \tilde{\gamma}_c(u_i) \qquad (3.12)$$

It now follows from (3.11) that $\tilde{\gamma}_c(\hat{u}) = -\infty$, which is clearly a contradiction of the boundedness of $\tilde{\gamma}_c(\cdot)$ on G and hence we must have $\hat{u} \in \Delta_c$, i.e. $\tilde{\theta}_c(\hat{u}) = 0$. ¤

Subalgorithm 1 suffers from the drawback that it is only a conceptual algorithm because the evaluation of $\tilde{\theta}_c(u_i)$, which can be carried out by means of the Meyer-Polak algorithm [6] requires an infinite number of interactions. We now state an implementable version. This version computes approximations $\tilde{\theta}'_{c,\varepsilon}(u_i)$ and $\tilde{\theta}''_{c,\varepsilon}(u_i)$ to $\tilde{\theta}_c(u_i)$ and direction vectors s_i satisfying, for a given $\varepsilon > 0$,

$$\tilde{\theta}'_{c,\varepsilon}(u_i) \le \tilde{\theta}_c(u_i) \le \tilde{\theta}''_{c,\varepsilon}(u_i) \tag{3.13}$$

$$\tilde{\theta}''_{c,\varepsilon}(u_i) - \tilde{\theta}_c(u_i) \le \varepsilon \tag{3.14}$$

$$\tilde{\theta}''_{c,\varepsilon}(u_i) = \max_{j=1-2m}\{\tilde{g}_c^j(u_i) - \tilde{\gamma}_c(u_i) + \langle \nabla \tilde{g}_c^j(u_i), s_i \rangle_2\} \tag{3.15}$$

The details of the computation of $\tilde{\theta}_c'$, $\tilde{\theta}_c''$ and s_i are given in the Appendix.

Subalgorithm 2:

Data: $u_0 \in G$, $\beta \in (0,1)$, $c > 0$, $\varepsilon_o > 0$.

Step 0: Set $i = o_\tau$ $\varepsilon = \varepsilon_o$.

Step 1: Compute $\tilde{\theta}''_{c,\varepsilon}(u_i)$ and a corresponding vector s_i by means of subalgorithm 3 (see Appendix).

Step 2: If $\tilde{\theta}''_{c,\varepsilon}(u_i) \le -\varepsilon$, go to step 3; else set $\varepsilon = \varepsilon/2$ and go to step 1.

Step 3: Compute the smallest integer k_i such that

$$\tilde{\gamma}_c(u_i + \beta^{k_i} s_i) - \tilde{\gamma}_c(u_i) \le \beta^{k_i}\, \tilde{\theta}_{c,\varepsilon}(u_i)/4 \tag{3.16}$$

Step 4: Set $u_{i+1} = u_i + \beta^{k_i} s_i$, set $i = i + 1$ and go to step 1. ¤

The adjustment of ε in subalgorithm 2 is based on the algorithm model (1.3.26) in [6] which can also be used to prove the theorem below, in a reasonably straightforward manner.

Theorem 4: (i) Suppose that subalgorithm 2 constructs an infinite sequence of controls in G. Then every $L_\infty^r[0,1]$ accumulation point $\hat{u}$ of this sequence is in Δ_c, i.e. $\tilde{\theta}_c(\hat{u}) = 0$. (ii) If subalgorithm 2 jams up at a u_i, cycling in steps 1 and 2, then $u_i \in \Delta_c$. ¤

Theorems 3 and 4 are relatively weak results because the set G is not $L_\infty^m[0,1]$ compact and hence there is no guarantee that a sequence $[u_i]_{=0}^{\infty}$ constructed by subalgorithms 1 or 2 has $L_\infty^r[0,1]$ accumulation points. However, assumptions 1 and 2 and the compactness of Ω quarantee that the corresponding sequence of trajectories $\{x^{u_i}\}_{i=0}^{\infty}$, constructed according to (2.4), (2.5), always has accumulation points x^* in $L_\infty^r[0,1]$. The trajectories $x^*(\cdot)$ are absolutely continuous and can be realized by a relaxed control. In section V we shall show that all the limit trajectories $x^*(\cdot)$ together with the generating relaxed controls satisfy an optimality condition for the relaxed optimal control problem corresponding to the optimal control problem P_c (2.7). This is a stronger result than theorems 3 and 4.

IV. THE COMPLETE ALGORITHM

Since an implementable version requires modification of the test functions t_c defined in §II as well as some further control of the precision parameter ε in Subalgorithm 2, resulting in a considerable increase in complexity, we shall first state our algorithm in a conceptual form and then in an expanded implementable form. The convergence of the conceptual form follows from theorem 1. We shall omit a proof of convergence of the implementable version since it is a straightforward but tedious exercise. In both cases, we make use of a function $\zeta: \mathbb{R} \to \mathbb{R}$ to generate the sequence $\{c_j\}$ according to $c_{j+1} = \xi(c_j)$, e.g. $e_{j+1} = c_j + 1$. We use the abbreviated notation $\tilde{\gamma}_j$, $\tilde{\theta}_j$, t_j instead of $\tilde{\gamma}_{c_j}$, $\tilde{\theta}_{c_j}$, t_{c_j}.

Algorithm 1 (Conceptual Version)

Data: $u_0 \in G$, $\beta \in (0,1)$, $\tau > 0$, $c_0 > 0$, $\zeta : \mathbb{R} \to \mathbb{R}$.

Step 0: Set $i = 0$, $j = 0$.

Step 1: Compute x^{u_i}, λ^{u_i}, $g^j(u_i)$, $\nabla g^i(u_i)$, $j = 0,1,2,\ldots,m$, $\gamma(u_i)$, $\tilde{\theta}_j(u_i)$,

$s_i \in S(u_i)$, $t_j(u_i)$ according to (2.4),(2.17)-(2.19),(2.3),(2.10),(2.14),(2.44).

Step 2: If $t_j(u_i) \leq 0$, go to step 3; else go to step 5.

Step 3: If $\tilde{\theta}_j(u_i) = 0$, stop; else compute the smallest integer $k_i \geq 0$ such that

$$\tilde{\gamma}_j(u_i + \beta^{k_i} s_i) - \tilde{\gamma}_j(u_i) \leq \beta^{k_i} \tilde{\theta}_j(u_i)/4 \qquad (4.1)$$

Comment: The computation of k_i requires that the differential equation (2.4) be integrated once for each value of $k = 0,1,2,...$ tried until (4.1) is satisfied.

Step 4: Set $u_{i+1} = u_i + \beta^{k_i} s_i$, set $i = i+1$ and go to step 1.

Step 5: set $c_{j+1} = \zeta(c_j)$, set $j = j+1$ and go to step 1.0 † ⌑

Since we have already shown that all the blocks of algorithm 1 satisfy the assumptions of theorem 1, we can summarize its convergence properties, as follows.

Theorem 5: Consider a sequence of controls $\{u_i\}$ constructed by algorithm 1. (i) If the sequence $\{u_i\}_{i=0}^{k}$ is finite, then its last element $u_k(\cdot)$ satisfies (2.21), i.e. $u_k \in \Delta$. (ii) If the sequence $\{u_i\}_{i=0}^{\infty}$ is infinite and the indices j remain bounded by some $j^* < \infty$, i.e. $j \leq j^*$ for all j, then every $L_\infty^r[0,1]$ accumulation point $\hat{u}$ of $\{u_i\}_{i=0}^{\infty}$ is in Δ. (iii) If the sequence $\{u_i\}$ is infinite and $j \to \infty$, then $\{u_i\}$ has at least one subsequence which has no $L_\infty^r[0,1]$ accumulation points. ⌑

In the next section, we shall see that in the topology of relaxed controls case (iii) of theorem 5 cannot arise and hence only cases (i) and (ii) are of importance.

Before leaving this section, we present an implementable version of

† Note there is no need to store the subsequence $\{\bar{u}_j\}$ which was defined in the Algorithm Model for the sake of proofs only.

Algorithm 1, based on Subalgorithm 2. Since we shall now be using approximations $\tilde{\theta}''_{c,\varepsilon}$ to $\tilde{\theta}_c$, the test functions t_c also must be modified, which we propose to do as follows: For any $\varepsilon > 0$, $c > 0$,

$$t_{c,\varepsilon}(u) \triangleq \theta''_{c,\varepsilon}(u) + \frac{1}{c}\min\{\gamma(u),\tau\} - \varepsilon \tag{4.2}$$

where $\tau > 0$.

This leads to the following implementable version.

Algorithm 2. (Implementable version.)

Data: $u_0 \in G$, $\beta \in (0,1)$, $\tau > 0$, $c_0 > 0$, $\varepsilon_0 > 0$, $\zeta\colon \mathbb{R}^1 \to \mathbb{R}^1$.

Step 0: Set $i = 0$, $j = 0$, $\varepsilon = \varepsilon_0$.

Step 1: Compute x^{u_i}, λ^{u_i}; $g^j(u_i)$, $\nabla g^j(u_i)$, $j = 0, 1, \cdots, m$, $\gamma(u_i)$, $\tilde{\theta}''_{j,\varepsilon}(u_i)$, $s_i \in S(u_i)$, $t_{j,\varepsilon}(u_i)$, according to (2.4),(2.17)-(2.19),(2.3),(2.10), (3.13)-(3.15), (4.2).

Step 2: If $t_{j,\varepsilon}(u_i) \leq 0$, go to step 3; else go to step 6.

Step 3: If $\tilde{\theta}''_{j,\varepsilon}(u_i) \leq -\varepsilon$, go to step 3; else set $\varepsilon = \varepsilon/2$ and go to step 1.

Step 4: Compute the smallest integer $k_i \geq 0$ such that

$$\tilde{\gamma}_j(u_i + \beta^{k_i} s_i) - \tilde{\gamma}_j(u_i) \leq \beta^{k_i}\tilde{\theta}''_{j,\varepsilon}(u_i)/4 \tag{4.3}$$

Step 5: Set $u_{i+1} = u_i + \beta^{k_i} s_i$, set $i = i + 1$ and go to step 1.

Step 6: Set $c_{j+1} = \zeta(c_j)$, set $j = j+1$ and go to step 1. ◻

By making use of Algorithm Model (1.3.26) in [6], Theorem 1 and the various results established for the conceptual version, we can deduce, with some effort, the following not altogether surprising result.

Theorem 6: (i) If Algorithm 2 jams up at a control u_i, cycling in steps 1 to 3, then $u_i \in \Delta$.

(ii) If Algorithm 2 constructs an infinite sequence of controls $\{u_i\}_{i=0}^{\infty}$ and the index j remains bounded, i.e. $j \leq j^*$ for all j, then every $L_\infty^r[0,1]$ accumulation point $\hat{u}$ of $\{u_i\}$ is in Δ.

(iii) If Algorithm 2 constructs an infinite sequence of controls $\{u_i\}_{i=0}^{\infty}$ and the index $j \to \infty$ as $i \to \infty$, then $\{u_i\}_{i=0}^{\infty}$ contains at least one subsequence which has no L_∞^r [0,1] accumulation points. ⌑

Again, as in the case of the conceptual algorithm 1, we shall see in the next section that case (iii) above, cannot arise in the control measures topology and hence only cases (i) and (ii) of Theorem 6 are of importance.

V. CONVERGENCE IN THE SENSE OF CONTROL MEASURES

We now present an analysis of our algorithm in a more abstract, but also more satisfying setting, namely, in the sense of convergence of control measures. In doing so, we follow the methodology proposed in Williamson and Polak [7]. The reason for turning to relaxed controls is that, unlike in $L_\infty^r[0,1]$, a sequence of bounded relaxed controls always has accumulation points. Hence, case (iii) in Theorem 6 can be ruled out, i.e. we can show that the index j remains bounded. Furthermore, as we shall see, the accumulation points generated by our algorithm satisfy an appropriate optimality condition for the relaxed optimal control problem, which we are about to define. We begin with a few definitions and results which are standard in the relaxed control literature (see e.g. [8], [9]).

Let $\underset{\sim}{V}$ be the set of probability measures on $\Omega(\underset{\sim}{v} \in \underset{\sim}{V} \Rightarrow \int_\Omega d\underset{\sim}{v}(u) = 1)$.

For any continuous function ϕ: $\mathbb{R}^n \times \Omega \times [0,1] \rightarrow \mathbb{R}^p$, the corresponding <u>relaxed function</u> ϕ_r: $\mathbb{R}^n \times \underset{\sim}{V} \times [0,1]$ is defined by[†]

$$\phi_r(x,\underset{\sim}{v},t) \triangleq \int_\Omega \phi(x,u,t)d\underset{\sim}{v}(u) \tag{5.1}$$

A <u>relaxed control</u> is any function $\underset{\sim}{v}$: $[0,1] \rightarrow \underset{\sim}{V}$. A relaxed control is said to be <u>measurable</u> if for any polynomial $p(u)$ in (the components of) u, the function θ: $[0,1] \rightarrow \mathbb{R}$ is measurable, with $\theta(t) \triangleq p_r(\underset{\sim}{v}(t)) = \int_\Omega p(u) d\underset{\sim}{v}(t)(u)$. Let $\underset{\sim}{G}$ be the set of measurable relaxed controls, then the relaxed optimal control problem (corresponding to (2.6)) is

$$\min\{\hat{g}^0(\underset{\sim}{v}) | \hat{g}^i(\underset{\sim}{v}) = 0,\ j = 1,2,\cdot\cdot,m,\ \underset{\sim}{v} \in \underset{\sim}{G}\} \tag{5.2}$$

where

$$\hat{g}^j(\underset{\sim}{v}) \triangleq h^j(x^{\underset{\sim}{v}}(1)),\ j = 0,\ 1,2,..,m \tag{5.3}$$

and $x^{\underset{\sim}{v}}$: $[0,1] \rightarrow \mathbb{R}^n$ is the solution of

$$\mathring{x}(t) = f_r(x(t),\underset{\sim}{v}(t),t) \qquad \text{a.e. on } [0,1] \tag{5.4}$$

$$x(0) = \xi. \tag{5.5}$$

In addition to (5.4), (5.5), we define also a set of adjoint systems defining the functions $\lambda_j^{\underset{\sim}{v}}$: $[0,1] \rightarrow \mathbb{R}^n$, $j = 0,1,2,..,m$, as solutions of

$$\mathring{\lambda}(t) = -\left(\frac{\partial f}{\partial x}\right)_r (x^{\underset{\sim}{v}}(t),\underset{\sim}{v}(t),t)^T\lambda(t) \tag{5.6}$$

$$\lambda(1) = \nabla h^j(x^{\underset{\sim}{v}}(1)). \tag{5.7}$$

We now collect some relevant results and definitions.

[†]Note that we use here the symbols θ, ϕ, Δ for different objects than in the preceding section. This should cause no difficulty because of the entirely different context.

Proposition 5: If $\underset{\sim}{v} \in \underset{\sim}{G}$ and ϕ: $\Omega \times [0,1] \to \mathbb{R}^p$ is continuous, then the function θ: $[0,1] \to \mathbb{R}^p$ definitely $\theta(t) \triangleq \phi_r(\underset{\sim}{v}(t),t) = \int_\Omega \phi(u,t)dv(t)(u)$ is measurable. ⌑

For a proof, see Young [8] p. 290.

Proposition 6: Under Assumptions 1-3, there exist absolutely continuous functions $x^{\tilde{v}}$, $\lambda_j^{\tilde{v}}$, $j = 0,1,2,..,m$, which are the unique solutions of the system (5.4)-(5.7). ⌑

For a proof, see Young [9] p. 291-292, 298.

Proposition 7: Under Assumptions 1-3, there exists a $\delta \in (0,\infty)$ such that for all $t \in [0,1]$,

$$x^{\tilde{v}}(t) \in X \tag{5.8}$$

$$\lambda_j^{\tilde{v}}(t) \in X, \; j = 0,1,2,..,m, \tag{5.9}$$

where

$$X \triangleq \{\xi \in \mathbb{R}^n \,|\, \|\xi\| \leq \delta\} \tag{5.10}$$

This proposition can be deduced from a similar result in [11].

We shall say (see [8], [9]) that an infinite sequence $\{\underset{\sim}{v}_i\}$ in $\underset{\sim}{G}$ converges to $\underset{\sim}{v}^* \in \underset{\sim}{G}$ in the sense of control measures (i.s.c.m.) if for every continuous function ϕ: $\Omega \times [0,1] \to \mathbb{R}$ and every subinterval Δ of $[0,1]$,

$$\int_\Delta \phi_r(\underset{\sim}{v}_i(t),t)dt \to \int_\Delta \phi_r(\underset{\sim}{v}^*(t),t)dt \tag{5.11}$$

as $i \to \infty$.

The following result follows directly from Lemma 1 in Williamson and

Polak [7].

Proposition 8: Let ℓ: $X \times \Omega \times [0,1] \to W$, with W a bounded subset of $\mathbb{R}^p$, be continuous in $X \times \Omega$ and measurable in [0,1]. If x_i: $[0,1] \to X$ $i = 1,2,3,..$, converge uniformly to x^* on [0,1] and $\underset{\sim}{u}_i \in \underset{\sim}{G}$, $i = 1,2,3,...$, converge to $\underset{\sim}{u}^* \in \underset{\sim}{G}$ i.s.c.m., then for each subinterval Δ of [0,1]

$$\int_\Delta \ell_r(x_i(t), \underset{\sim}{u}_i(t),t)dt \to \int_\Delta \ell_r(x^*(t),\underset{\sim}{u}^*(t),t)dt \tag{5.12}$$

It is now straightforward to show, using propositions 2, 3 and 4 that

Corollary: Under Assumptions 1-3, if $\{\underset{\sim}{v}_i\}$ is an infinite sequence in $\underset{\sim}{G}$ converging to $\underset{\sim}{v}^*$, i.s.c.m., then the corresponding sequences of trajectories $\{x^{\underset{\sim}{v}_i}\}$, $\{\lambda_j^{\underset{\sim}{v}_i}\}$ 1 = 0,1,2,..,m, converge uniformly on [0,1] to x^{v^*}, $\lambda_j^{v^*}$, j = 0,1,..,m respectively. ¤

Now, before proceeding further, we note that for any $u \in G$ and $s \in S(u)$, $j = 0,1,2,...,2m$

$$\langle \nabla g^j(u),s \rangle_2 = \langle \nabla h^j(x^u(1)),z^{u,u+s}(1) \rangle \tag{5.13}$$

where $z^{u,u+s}$: $[0,1] \to \mathbb{R}^n$ is the unique solution of

$$\mathring{z}(t) = \frac{\partial f}{\partial x}(x^u(t),u(t),t)z(t) + \frac{\partial f}{\partial u}(x^u(t),u(t),t)s(t) \tag{5.14}$$

$$z(0) = 0. \tag{5.15}$$

Hence, if we define

$$R(u) = \{z^{u,w}(1) \mid (w-u) \in S(u)\} \tag{5.16}$$

then we see that the functions θ and $\tilde{\theta}_c$ can be rewritten as

$$\theta(u) = \min_{z\in R(u)} \max_{j=1-2m} h^j(x^u(1)) - \gamma(u) + \langle \nabla h^j(x^u(1), z \rangle \tag{5.17}$$

$$\tilde{\theta}_c(u) = \min_{z\in R(u)} \max_{j=1-2m} h^j(x^u(1)) - \gamma(u) + \langle (1/c)\nabla h^0(x^u(1)) + \nabla h^j(x^u(1), z \rangle \tag{5.18}$$

Consequently, we introduce the functions $z^{\underset{\sim}{v},w} : [0,1] \to \mathbb{R}^n$, with $\underset{\sim}{v} \in \underset{\sim}{G}$ and $w \in G$, defined as the unique solutions of

$$\dot{z}(t) = (f_x)_r (x^{\underset{\sim}{v}}(t), \underset{\sim}{v}(t), t) z(t) + (f_u)_r (x^{\underset{\sim}{v}}(t), \underset{\sim}{v}(t), t) w(t) - \phi_r(x^{\underset{\sim}{v}}(t), \underset{\sim}{v}(t), t) \tag{5.19}$$

$$z(0) = 0, \tag{5.20}$$

where

$$\phi(x,u,t) \triangleq f_u(x,u,t)u \tag{5.21}$$

Defining the set value map $\hat{R}$: $\underset{\sim}{G} \to 2^{\mathbb{R}^n}$

$$\hat{R}(\underset{\sim}{u}) = \{z^{\underset{\sim}{u},w}(1) \mid \underset{\sim}{u} \in \underset{\sim}{G},\ w \in G\} \tag{5.22}$$

we can define extensions of θ and $\tilde{\theta}_c$ as follows: Let $\hat{\gamma}$: $\underset{\sim}{G} \to \mathbb{R}$, $\hat{\theta}$: $\underset{\sim}{G} \to \mathbb{R}$, and, for $c > 0$, $\hat{\theta}_c$: $\underset{\sim}{G} \to \mathbb{R}$, $\hat{t}_c$: $\underset{\sim}{G} \to \mathbb{R}$, be defined by

$$\hat{\gamma}(\underset{\sim}{v}) \triangleq \max_{j=1-2m} \hat{g}^j(\underset{\sim}{v}) = \max_{j=1-2m} g^j(\underset{\sim}{v})\ h^j(x^{\underset{\sim}{v}}(1)) \tag{5.23}$$

$$\hat{\theta}(\underset{\sim}{v}) \triangleq \min_{z \in \hat{R}(\underset{\sim}{v})} \max_{j=1-2m} \{h^j(x^{\underset{\sim}{v}}(1)) - \hat{\gamma}(\underset{\sim}{v}) + \langle \nabla h^j(x^{\underset{\sim}{v}}(1), z \rangle \} \tag{5.24}$$

$$\hat{\theta}_c(\underset{\sim}{v}) \triangleq \min_{z \in \hat{R}(\underset{\sim}{v})} \max_{j=1-2m} \{h^j(x^{\underset{\sim}{v}}(1)) - \hat{\gamma} + \langle (1/c)\nabla h^0(x^{\underset{\sim}{v}}(1) + \nabla h^j(x^{\underset{\sim}{v}}(1)), z\rangle \} \tag{5.25}$$

$$\hat{t}_c(\underset{\sim}{v}) \triangleq \hat{\theta}_c(\underset{\sim}{v}) + \hat{\gamma}(\tilde{v})/c \tag{5.26}$$

It is easy to show that if $\hat{\underset{\sim}{v}}$ is optimal for the relaxed problem corresponding to P then $\hat{\theta}(\hat{\underset{\sim}{v}}) = 0$ and if $\hat{v}$ is optimal for the recaped penalized problem, (corresponding to P_c), then $\hat{\theta}_c(\hat{\underset{\sim}{v}}) = 0$. It now follows directly from Corollary 1 (which also applies to the functions $z^{\underset{\sim}{v,w}}$) that if $\{\underset{\sim}{v}_i\}_{i=0}^{\infty}$ is a sequence in $\underset{\sim}{G}$ converging i.s.c.m. to a $\underset{\sim}{v}^* \in \underset{\sim}{G}$, then the set $\hat{R}(\underset{\sim}{v}_i) \to \hat{R}(\underset{\sim}{v}^*)$ in the Hausdorff metric, and hence it follows easily that $\hat{\theta}(\underset{\sim}{v}_i) \to \hat{\theta}_c(\underset{\sim}{v}_i) \to \hat{\theta}_c(\underset{\sim}{v}^*)$ (for any c) and $\hat{t}_c(\underset{\sim}{v}_i) \to \hat{t}_c(\underset{\sim}{v}^*)$, i.e. all these functions are sequentially continuous in the topology of relaxed controls. We thus obtain the analogies of Lemmas 1 and 2. Similarly, tracing through the proof of Lemma 3, we find that we can substitute relaxed controls for controls in G to conclude that for any $\underset{\sim}{v}^* \in \underset{\sim}{G}$ there exists a $c^* > 0$ such that if $\underset{\sim}{v}_i \to \underset{\sim}{v}^*$ i.s.c.m., then there exists an $i_o \geq 0$ such that $t_c(\underset{\sim}{v}_i) \leq 0$ for all $i \geq i_o$ and $c \geq c^*$.

Now, with each (ordinary) control $u_i \in G$ generated by our algorithms we can associate the relaxed control $\bar{\underset{\sim}{u}}_i \in \underset{\sim}{G}$ which is wholly concentrated at $u_i(t)$, i.e.,

$$\int_{\{u_i(t)\}} d\bar{u}_i(t)(u) = 1 \tag{5.27}$$

Since as we have just seen, all the functions which we have used and which were $L_\infty^r[0,1]$ continuous are also sequentially continuous i.s.c.m., it is quite straightforward to establish the following results, by essentially retracing the steps in the proofs of Theorems 3, 4, 5 and 6.

Theorem 7: Suppose that $\{u_i\}_{i=0}^{\infty}$ is an infinite sequence of (ordinary) controls constructed by Subalgorithm 1 or Subalgorithm 2. Let $\{\bar{\underset{\sim}{u}}_i\}_{i=0}^{\infty}$ be the associated sequence of relaxed controls. Then every accumulation point $\hat{\underset{\sim}{u}}$ of $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$, i.s.c.m., satisfies $\hat{\theta}_c(\hat{\underset{\sim}{u}}) = 0$. ◻

Theorem 8: Suppose that $\{u_i\}_{i=0}^{\infty}$ is an infinite sequence of (ordinary) controls constructed by Algorithm 1 or Algorithm 2 and let $\{\bar{\underset{\sim}{u}}_i\}_{i=0}^{\infty}$ be the associated sequence of relaxed controls. Then there exists an index $j^* < \infty$ such that $j \leq j^*$ throughout the computation, and every accumulation point $\hat{\underset{\sim}{u}} \in G$ of $\{\underset{\sim}{u}_i\}_{i=0}^{\infty}$, i.s.c.m., satisfies for some multipliers ψ^j, $j = 1,2,\ldots,m$, the optimality condition

$$\hat{\underset{\sim}{u}} \in \hat{\Delta} \triangleq \{\underset{\sim}{u} \in \underset{\sim}{G} | \hat{\gamma}(\underset{\sim}{u}) = 0; \langle \nabla h^0(x^{\underset{\sim}{u}}(1)) + \sum_{j=1}^{m} \psi^j \nabla h^j(x^{\underset{\sim}{u}}(1)), z \rangle \geq 0, \quad \forall z \in \hat{R}(\underset{\sim}{u})\} \tag{5.28}$$

Note that case (iii) of Theorems 5 and 6 is ruled out in Theorem 8 because any infinite subsequence of $\{\bar{\underset{\sim}{u}}_i\}$ must have accumulation points i.s.c.m.. Thus, by carrying out our analysis in terms of relaxed controls we have reaped two benefits: we have eliminated the need to establish that $L_\infty^r[0,1]$ accumulation points will exist for the sequences constructed, and we have also been able to show that the growth of the penalty c_j cannot be unbounded.

VI CONCLUSION

In this paper we have shown how to solve an optimal control problem with control and terminal equality constraints. The terminal equality constraints are handled by defining an equivalent control problem with control constraints but <u>without</u> terminal equality constraints. This is done by defining an exact penalty function involving a parameter c. The two problems are equivalent if c is sufficiently large (but finite). The control constraint is handled by incorporating it as a constraint in the search direction subproblem. The complete algorithm incorporates an inner loop to increase the parameter c to a satisfactory value. It is shown that accumulation points, both in the L_∞ sense and in the sense of control measures, satisfy necessary conditions of optimality; accumulation points, in the sense of control measures, always exist for the control problem considered. This fact is used to show that the sequence of parameters $\{c_j\}$ produced by the algorithm is finite. Except for algorithms of the penalty function type, no other algorithms with proven convergences, for this class of problems are known to the authors.

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APPENDIX

The conceptual version of the algorithm requires at each iteration the determination of a search direction $s_c(u)$ which is a solution of:

$$\tilde{\theta}_c(u) = \min_{s \in S(u)} \max_{j=1-2m} g^j(u) - \gamma(u) + \langle (1/c)\nabla g^0(u) + \nabla g^j(u), s \rangle_2$$

$$= \min_{s \in S(u)} \Pi_c^u(s) \tag{A1}$$

where, for each $u \in G$, $c > 0$, Π_c^u: $S(u) \rightarrow \mathbb{R}$ is defined as follows:

$$\Pi_c^u(s) \triangleq \max_{j=1-2m} \{g^j(u) - \gamma(u) + \langle (1/c)\nabla g^0(u) + \nabla g^j(u), s \rangle_2\} \tag{A2}$$

Consider the reachable set R_c^u in $\mathbb{R}^{2m}$ defined by:

$$R_c^u \triangleq \{\xi \in \mathbb{R}^{2m} | \xi^j = g^j(u) - \gamma(u) + \langle (1/c)\nabla g^0(u) + \nabla g^j(u), s \rangle_2,$$
$$j = 1,\ldots,2m \;\; s \in S(u)\}$$
$$= \{\xi \in \mathbb{R}^{2m} | \xi^j = g^j(u) - \gamma(u) + \langle (1/c)\nabla h^0(x^u(1)) + \nabla h^j(x^u(1)),$$
$$z^{u,u+s}(1)\rangle \;\; j = 1,\ldots,2m, \;\; s \in S(u)\} \tag{A3}$$

where $z^{u,u+s}$, defined in (5.13), (5.14), is the solution of a linear time varying differential equation due to a "control" s. It is well known that R_c^u is convex and compact. Eq. (A1) can be rewritten in the form:

$$\tilde{\theta}_c(u) = \min_{\xi \in R_c^u} \max_{j=1-2m} \xi^j \tag{A4}$$

This is a standard convex control problem for which several algorithms exist. These algorithms make use of the fact that points on the boundary of R_c^u can be easily determined; more precisely, given any vector $d \in \mathbb{R}^{2m}$, the point $\hat{\xi}_c^u(d)$ in R_c^u which maximizes $\langle d, \xi \rangle$ can be easily determined, the j^{th} component of $\hat{\xi}_c^u(d)$ being given by:

$$(\hat{\xi}_c^u(d))^j = g^j(u) - \gamma(u) + \langle (1/c)\nabla h^0(x^u(1)) + \nabla h^j(x^u(1)), z^{u,u+\hat{s}_c^u(d)}(1)\rangle \tag{A5}$$

$j = 1,\ldots,2m$, where the search direction $\hat{s}_c^u(d) \in S(u)$ satisfies (by the Pontyagin Minimum Principle), for all $t \in [0,1]$:

$$\langle \hat{s}_c^u(d)(t), B^u(t)\lambda_c^u(t,d) \rangle \leq \langle w-u(t), B^u(t)\lambda_c^u(t,d) \rangle \tag{A6}$$

for all $w \in \Omega$. B_u: $[0,1] \to \mathbb{R}^{n\times r}$ is defined by:

$$B_u(t) \triangleq f_u(x^u(t), u(t), t) \tag{A7}$$

and $\lambda_c^u(\cdot,d)$: $[0,1] \to \mathbb{R}^n$ is the solution of eqn. (2.18) with boundary condition:

$$\lambda(1) = \sum_{j=1}^{2m} d^j[(1/c)\nabla h^0(x^u(1)) + \nabla h^j(x^u(1))] \tag{A8}$$

The problem defined by (A1) is convex, and we can use this fact to obtain an approximation to this problem to any required degree of accuracy in a finite number of iterations. We shall employ the Meyer-Polak algorithm, described in (5.3.14) in [6], modified so that the algorithm terminates when the desired accuracy is attained. The algorithm employs two sets, R_c^u, defined above, and the set $D_c^u(a)$ defined, for all $a \in \mathbb{R}$, by:

$$D^u_c(a) \triangleq \{\eta \in \mathbb{R}^{2m} | \eta^j \in [(b^u_c)^j, a], j = 1,\ldots 2m\} \tag{A9}$$

where:

$$(b^u_c)^j \triangleq \hat{\xi}^u_c(e_j)^j \tag{A10}$$

for $j = 1,\ldots 2m$, where e_j denotes the j^{th} unit vector in $\mathbb{R}^{2m}$. Let $b^u_c \in \mathbb{R}^{2m}$ denote the vector whose j^{th} component is $(b^u_c)^j$. The two sets R^u_c, $D^u_c(a)$ are illustrated in Fig. 2.

The Meyer-Polak algorithm determines the infimum value of a such that $R^u_c \cap D^u_c(a) \neq \phi$ and, hence, $\tilde{\theta}_c(u)$ and the corresponding search direction $\tilde{s}_c(u)$ (the minimizing s in (A1)). The modified algorithm yields, for a given $\varepsilon > 0$ (ε is the precision parameter) approximations $\tilde{\theta}'_{c,\varepsilon}(u)$, $\tilde{\theta}''_{c,\varepsilon}(u)$ to $\tilde{\theta}_c(u)$, and a search direction $\bar{s}^{\varepsilon}_c(u) \in S(u)$ satisfying:

$$\tilde{\theta}'_{c,\varepsilon}(u) = \leq \tilde{\theta}_c(u) \leq \tilde{\theta}''_{c,\varepsilon}(u) \tag{A11}$$

$$\tilde{\theta}''_{c,\varepsilon}(u) - \tilde{\theta}'_{c,\varepsilon}(u) \leq \varepsilon \tag{A12}$$

$$\tilde{\theta}''_{c,\varepsilon}(u) = \Pi^u_c(\bar{s}^{\varepsilon}_c(u)) \tag{A13}$$

Thus the algorithm yields a search direction $\bar{s}^{\varepsilon}_c(u)$ such that

$\tilde{\theta}_c(u) \in [\Pi^u_c(\bar{s}^{\varepsilon}_c(u) - \varepsilon, \Pi^u_c(\bar{s}^{\varepsilon}_c(u)]$.

In the algorithm description which follows ξ denotes a point in R^u_c and η a point outside R^u_c. At iteration i of the algorithm:

$$\tilde{\theta}'_{c,\varepsilon}(u) = \max_{j=1-2m} \eta^j_i \tag{A14}$$

$$\tilde{\theta}''_{c,\varepsilon}(u) = \max_{j=1-2m} \xi_i^j \tag{A15}$$

The algorithm terminates when $\tilde{\theta}''_{c,\varepsilon}(u) - \tilde{\theta}'_{c,\varepsilon}(u) \leq \varepsilon$.

<u>Subalgorithm 3</u>.

(Truncated version of Meyer-Polak algorithm)

<u>Data</u>: $u \in G$, $\varepsilon > 0$, $c \geq 0$.

<u>Step 0</u>: Set $i = 0$, set $s_0 = 0$.

Set $\xi_0 = (g^1(u) - \gamma(u), \ldots, g^{2m}(u) - \gamma(u))^T$,

Set $\eta_0 = b_c^u$.

<u>Step 1</u>: If $\|\eta_i - \xi_i\|_\infty \leq \varepsilon$,

set $\tilde{\theta}'_{c,\varepsilon}(u) = \|\eta_i\|_\infty$,

set $\tilde{\theta}''_{c,\varepsilon}(u) = \|\xi_i\|_\infty$,

set $s_i = \bar{s}_c^\varepsilon(u)$,

and stop.

<u>Step 2</u>: Set $\sigma_i = \eta_i - \xi_i$.

Set $\bar{\xi}_i = \hat{\xi}_c^u(\sigma_i)$.

Set $\bar{s}_i = \hat{s}_c^u(\sigma_i)$.

Compute a $\bar{\eta}_i$ which solves

$\min\{\|\eta\|_\infty \mid \langle \sigma_i, \eta \rangle = \langle \sigma_i, \bar{\xi}_i \rangle\}$.

<u>Step 3</u>: Compute $\xi_{i+1} \in [\xi_i, \bar{\xi}_i]$, $\eta_{i+1} \in [\eta_i, \bar{\eta}_i]$ such that:

$\|\xi_{i+1} - \eta_{i+1}\| = \min\{\|\xi-\eta\| \mid \xi \in [\xi_i, \bar{\xi}_i], \eta \in [\eta_i, \bar{\eta}_i]\}$

Set $s_{i+1} = s_i + (\bar{s}_i - s_i) \|\xi_i - \xi_{i+1}\| / \|\xi_i - \bar{\xi}_i\|$.

Set i = i+1.

Go to Step 1. ⌑

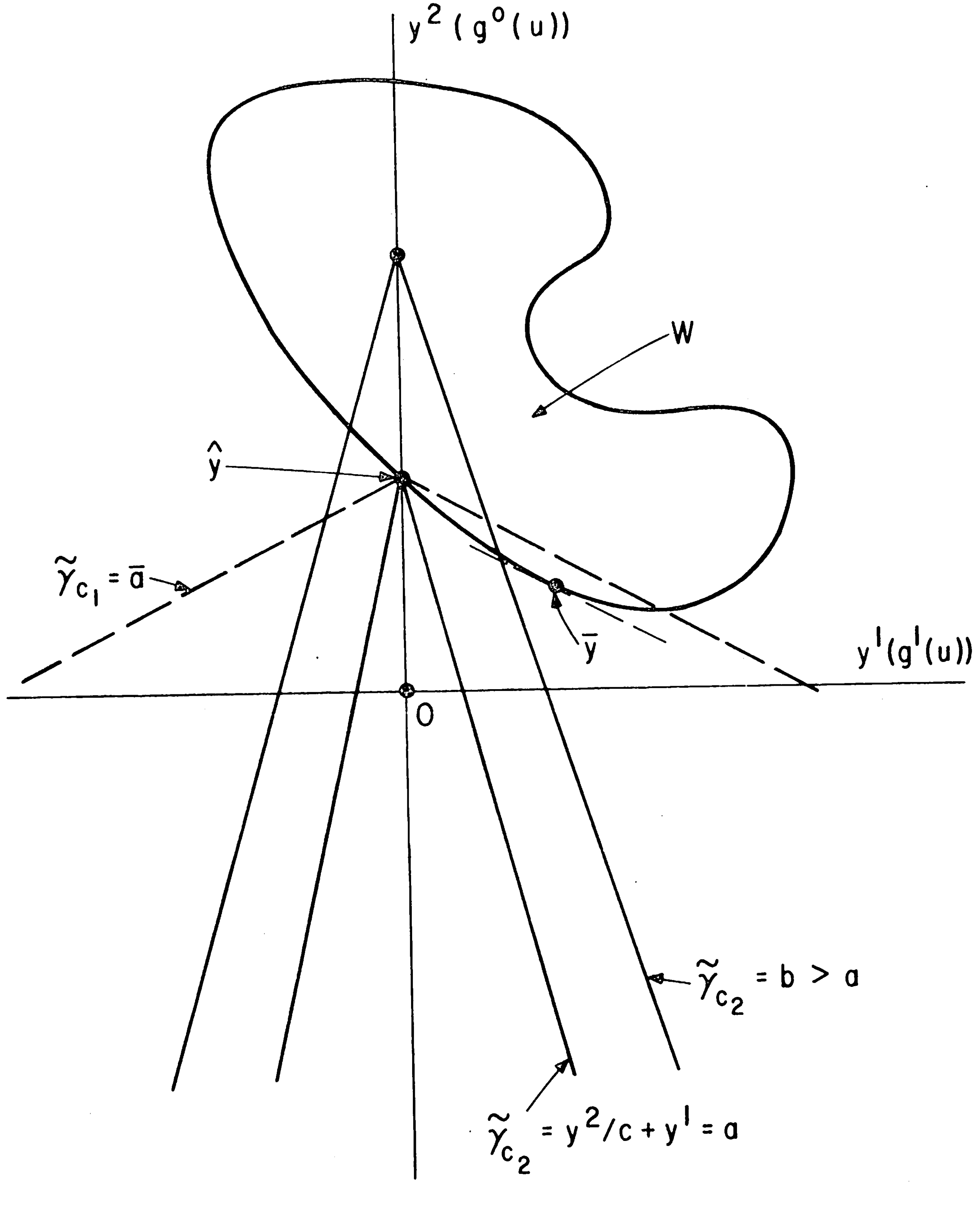
Note that s_i is the search direction corresponding to ξ_i in the sense that ξ_i is that point in R^u_c generated by s_i, i.e.

$$\xi^j_i = g^j(u) - \gamma(u) + \langle (1/c)\nabla g^0(u) + \nabla g^j(u), s_i \rangle_2 \tag{A16}$$

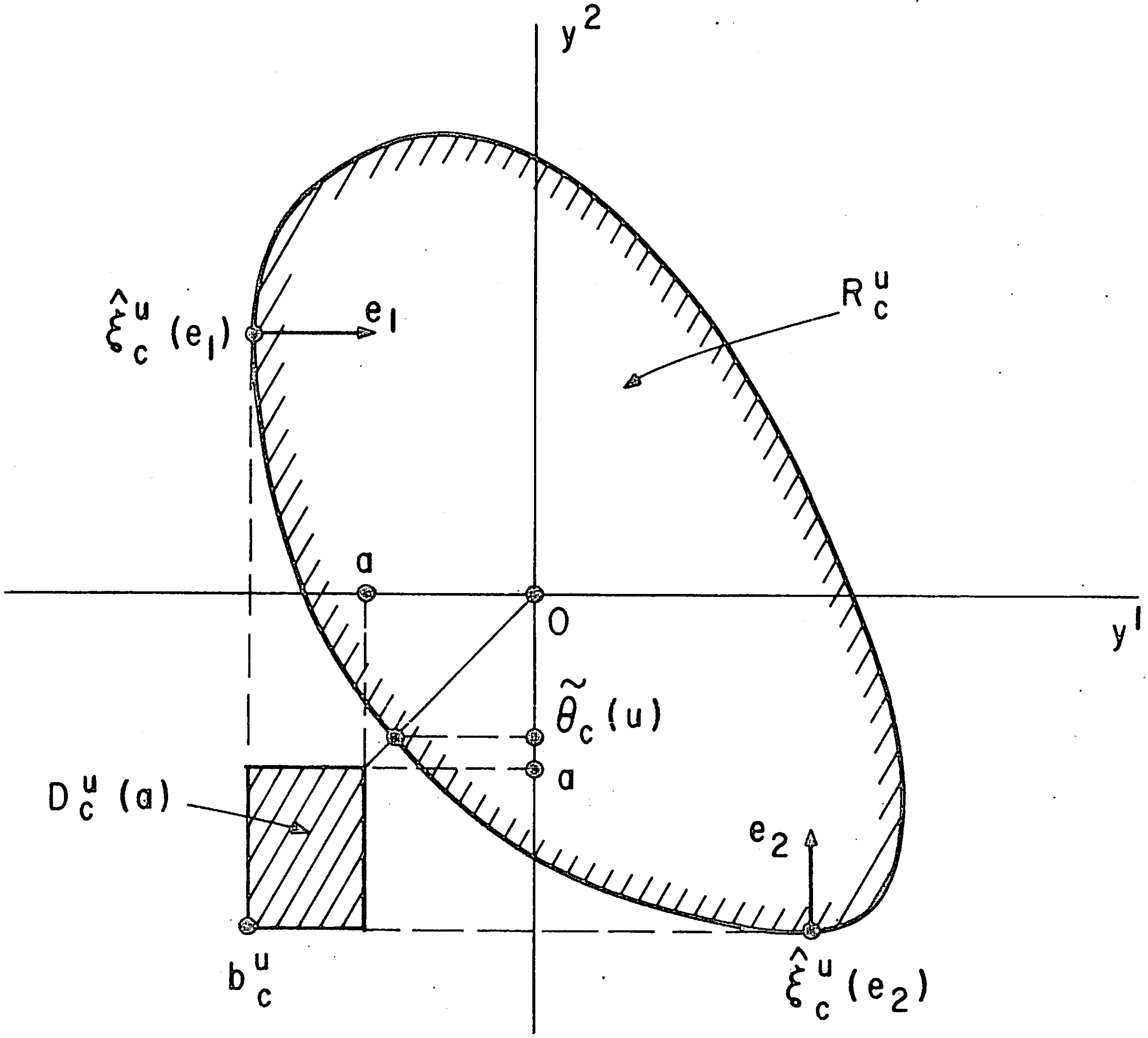
for j = 1,...,2m.

The following result follows directly from the proven convergence of algorithm (5.3.14) in [6] by identifying C with R^u_c and $\mathcal{R}(\alpha)$ with $D^u_c(a)$ and α with a. Since $\|\xi_i - \eta_i\| \to 0$ in the original algorithm it follows that $\|\xi_i - \eta_i\|_\infty \to 0$ and hence that $\tilde{\theta}''_{c,\varepsilon}(u) - \tilde{\theta}'_{c,\varepsilon}(u) \leq \varepsilon/2$ after a finite number of iterations.

<u>Proposition</u>. For any $u \in G$, any $c > 0$, any $\varepsilon > 0$ such that $\tilde{\theta}_c(u) < 0$ there exists a finite integer $k_{c,\varepsilon}(u)$ such that Subalgorithm 3 terminates in $k_{c,\varepsilon}(u)$ iterations yielding a search direction $\bar{s}^\varepsilon_c(u) \in S(u)$ and bounds $\tilde{\theta}''_{c,\varepsilon}(u)$ satisfying (A11)-(A13). ⌑



$y^2(g^0(u))$
W
$\hat{y}$
$\tilde{\gamma}_{c_1} = \bar{a}$
$\bar{y}$
$y^1(g^1(u))$
O
$\tilde{\gamma}_{c_2} = b > a$
$\tilde{\gamma}_{c_2} = y^2/c + y^1 = a$



y 2
R u c
ξ̂ u c (e1)
e1
a
O
y 1
θ̃ c (u)
a
D u c (a)
e2
b u c
ξ̂ u c (e2)