THE SPATIAL STRUCTURE OF PRODUCTION WITH A LEONTIEF TECHNOLOGY II: SUBSTITUTE TECHNIQUES

by

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THE SPATIAL STRUCTURE OF PRODUCTION WITH A LEONTIEF TECHNOLOGY II: SUBSTITUTE TECHNIQUES

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ABSTRACT

The structure of the optimal spatial pattern of production is studied when there are dependencies among production units which can be described by a Leontief technology with substitute techniques, and when there is a single marketplace of final demand, the CBD. Transportation cost is proportional to distance. The various goods are produced in rings. There are a finite number of patterns in which these rings are arranged, and they can be obtained by a finite algorithm. The particular pattern depends on the final demand. Hence there is no 'Non-Substitution' theorem. 'Reswitching' of techniques can occur, that is, in an optimal pattern a technique may be operated at large and small distanced from the CBD, but not at intermediate distances; this contradicts prevailing beliefs about optimal capital/land profiles.

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1. INTRODUCTION

In this paper we study the spatial structure of production in a model described by the following elements.

a) Space. This is a featureless plane at the center of which is the center business district (CBD) of fixed radius $u_0 \geq 0$. Land outside the CBD, at distances $u \geq u_0$, is denoted exclusively to production. No production occurs within the CBD.

b) Commodities. There are n goods indexed $j = 1...n$ whose production is considered. There are two other goods. First, there is land. Land is not produced but it is essential for production of the other goods. The amount of land available for production at distance u is $\theta(u) > 0$. Secondly, there is a composite commodity called corn which is available everywhere. Corn serves as a standard and as an input for producing transportation. Land has an opportunity cost of r_A units of corn per unit area.

c) Transportation. To transport each unit of good j over one unit distance requires $\bar{t}_j$ units of corn, $\bar{t}_j > 0$, $j = 1...n$.

d) Technology. A Leontief technology with intermediate goods and with substitute techniques, but without joint production, is available for producing goods $j = 1...n$.

A vector $Q = (Q_1...Q_n)' \geqq 0$ of final demands at the CBD is specified.

There is a variety of patterns in which the production can be organized so as to meet the final demand Q at the CBD. This variety stems from two sources. Firstly, the substitute possibilities imply that the set of production techniques can be operated in different combinations of levels to produce the same demand Q. Secondly, the same combination of levels of techniques can be located in different arrangements over space

inducing different transportation flow between production units and between them and the CBD. Each pattern incurs costs of transportation and the opportunity cost of land devoted to production. The optimal pattern is the one which minimizes this cost. In the paper properties of the optimal pattern are related to the production technology and transport costs.

Earlier, Schweizer-Varaiya [1976] made a complete study of the optimal pattern for the special case where there are no substitute techniques. (There are then n techniques with technique j being the only one which produces good j. Therefore, the first source of variation in production patterns mentioned above is absent). Those of their results which are relevant to this paper may be summarized in this manner. To begin with, note that, since transportation occurs radially, different goods will be produced in different sets of concentric rings. Then, in an optimal pattern, which is unique,

(A) there are exactly n different rings (in some 'degenerate' cases there may be fewer than n rings.)

(B) if the different goods are ordered according to the rule "i precedes j if net production of good i occurs closer to the CBD than production of j," then this ordering is determined by the technology and transport costs and is <u>independent</u> of Q. (Of course, the sizes of the rings vary with Q.)

(C) all transportation moves towards the CBD, i.e., no outwards shipments of intermediate or final goods can occur.

Properties (A), (B) show that the order in which goods are optimally produced, and hence the order in which different techniques are employed,

is a unique "invariant" of the system and depends only on production technology and transport costs.

When substitute techniques are available there is no such single optimal order. Instead,

(A') there is a finite set of optimal orderings such that in every optimal pattern the techniques must be employed according to one of these.

Thus the invariant is now this finite set of optimal orderings. Once again this invariant set is determined solely by production conditions; however, the particular order which prevails in an optimal pattern depends upon the demand Q. (Thus different Q can lead to very different land use patterns.)

In some of these optimal orderings a remarkable phenomena can be observed. It can happen that a particular technique is employed in a set of disjoint rings, i.e., the technique may be efficient at small and large distances while being inefficient at intermediate distances. Thus if one identifies the use to which a piece of land is put with the technique of production adopted on it, then there is no simple relation between land use and input. In particular, if capital is defined in terms of the production technique used, such well-known results as "capital/land and output/land ratios decline with distance" need not hold when both substitute techniques and intermediate goods are present. Similarly, real estate terms like "highest and best use of land" can be ambiguous.

Property (C) listed above continues to hold for the model used here. Also it is the case that optimal production patterns can be sustained as competititve equilibria with land rents and f.o.b. prices for goods at every distance.

The paper is organized in the following manner. In the next section, some notation is introduced together with some reformulations of the fundamental theorems of Leontief systems. Section 3 consists of the optimality theorem. In section 4 it is shown that the set of optimal pattern coincides with a finite set of patterns, called the efficient patterns, when the shape of the city is ignored. In sections 5 and 6 the finite set of quasi-efficient patterns is studied. This set is interesting since it contains the efficient patterns and since it can be generated by a finite algorithm. Section 7 deals with the case where there are no substitute techniques. The three examples in section 8 illustrate the most important points raised in the paper. Some proofs are collected in the Appendix.

2. NOTATION AND PRELIMINARY RESULTS

$J = \{1 \ldots n\}$, index set of produced commodities.

$\bar{B}_j \subset R^n$ is the finite set of techniques available for producing j. Suppose technique $b = (b_1 \ldots b_n)'$ (regarded as a column vector) in $\bar{B}_j$ is operated at level $x > 0$. Then $b_j x > 0$ units of j are produced and as inputs, $b_i x \leq 0$ units of good i, $i \neq j$, and x units of land are needed.

$\bar{B} \triangleq \bar{B}_1 \cup \ldots \cup \bar{B}_n$.

$B \subset \bar{B}$ denotes a subset of $\bar{B}$ as well as the matrix consisting of the column vectors $b \in B$. The order in which these vectors are arranged will be clear from the context.

$O(B) \triangleq \{j \mid \exists\, b \in B \text{ with } b_j > 0\}$ is the set of goods which can be produced using techniques in B. If $B = \{b\}$ consists of a single technique, then $O(b) = O(\{b\})$.

$I(B) \triangleq J - O(B)$.

For $B \subset \bar{B}$, B_O, respectively B_I, denotes the sub-matrix of B consisting of all rows j such that $j \in O(B)$, respectively $j \in I(B)$. Thus $B = \begin{bmatrix} B_O \\ \cdots \\ B_I \end{bmatrix}$ (this last notation is symbolic since the rows of B_O need not be the leading rows of B.)

For any vector z, $z \geq 0$ means all its components are non-negative; $z > 0$ means $z \geq 0$ and $z \neq 0$; $z \gg 0$ means all its components are positive $R^n_+ \triangleq \{z \in R^n \mid z \geq 0\}$

<u>D2.1</u> B is <u>productive</u> iff $\exists x > 0$ such that $B_O x \gg 0$. (This is an extension of the usual notion.)

<u>D2.2</u> B is <u>Leontief</u> iff $O(B) = J$ and $|B| = n$ ($|B|$ = number of elements in B.)

Let $B \subset \bar{B}$ and set $C \triangleq B_0$. Let $k(\ell)$ be the number of rows (columns) of C. Then $O(C) \subset J$ has k elements which will be used to index components of (row) vectors in R^k. Similarly the ℓ columns of C, denoted by c, will be used to index component of (column) vectors in R^ℓ.

L2.1 Suppose C is Leontief. Then C is productive iff C is nonsingular and $C^{-1} \geq 0$ i.e. all its elements are non-negative.

Pf See Nikaido [1968] p. 90. ¤

D2.3 Let $p \in R^k$, $\beta \in R^\ell$. Then $\pi(p,\beta,C) \in R^k$ is the vector with components $\pi_j(p,\beta,C) = \max\{pc - \beta_c | c \in C, O(c) = j\}$, $j \in O(c)$.

The proofs of L2.2 and L2.4 below can be constructed using the results and methods of Gale [1960], Chapter 9.

L2.2 The following conditions on C are equivalent.

(a) C is productive.

(b) $\exists\ C^L \subset C$ such that C^L is productive and Leontief.

(c) $\exists\ x \in R^\ell_+$ such that $Cx \gg 0$.

(d) $\exists\ p \in R^k_+$, $\exists\ \beta \in R^\ell_+$ such that $\pi(p,\beta,C) \gg 0$. ¤

L2.3 Suppose C is productive. Then

(a) $\forall \beta \in R^\ell_+$, $\forall \pi \in R^k_+$ there exists a unique $p \in R^k_+$ such that $\pi(p,\beta,C) = \pi$.

(b) Furthermore if $\pi(p,\beta,C) > 0$ then $p > 0$ (if all the elements of C are non-zero then $p \gg 0$). ¤

L2.4 Suppose that all Leontief subsystems of C are productive. Then $\forall \beta \in R^\ell$, $\forall \pi \in R^k$ there is at most one $p \in R^k$ such that $\pi(p,\beta,C) = \pi$

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The assumptions A_1 and A_2 are imposed throughout.

A1 $\bar{B}$ is productive.

A2 All coefficients of $\bar{B}$ are non-zero.

A2 means that every product is directly needed as an input to produce every other product. This eliminates the need to consider separately in the proofs some "degenerate" situations. Of course, since the magnitudes of these coefficients can be arbitrarily small, the economic significance of the propositions is not reduced.

L2.5 Suppose $p \in R^n_+$ and $B \subset \bar{B}$ are such that $pB >> 0$. Then B is productive and $p_j > 0$ for $j \in O(B)$.

Pf From L2.2(d) it follows that B is productive. The remainder of the assertion follows from A2 and L2.3(b) ¤

L2.6 Suppose $B \subset \bar{B}$ and $d \in R^n$ with $d_j \geq 0$ for $j \in I(B)$ are such that $dB \geq 0$. Then

(a) $d \geq 0$.

(b) If either $d_j > 0$ for some $j \in I(B)$, or $dB \neq 0$, then $d_j > 0$ for $j \in O(B)$.

Pf (a) follows from L2.3(b), and (b) from L2.3(b) and A2. ¤

L2.7 Suppose $B \subset \bar{B}$ is productive and $\rho \in R_+$. Then there is a unique $t \in R^n_+$, denoted $t_B(\rho)$ such that (2.1), (2.2), (2.3) hold.

(2.1) $\forall b \in B,\ t_B(\rho)b \leq \rho$.

(2.2) $\forall j \in O(B),\ \exists\ b \in B \cap \bar{B}_j$ such that $t_B(\rho)b = \rho$.

(2.3) $\forall j \in I(B),\ [t_B(\rho)]_j = \bar{t}_j$.

Futhermore if $j \in O(B)$ then $[t_B(\rho)]_j$ is strictly increasing in ρ.

Pf The matrix B is partitioned as $B = \begin{bmatrix} B_O \\ \cdots \\ B_I \end{bmatrix}$, and each column b of B as $b = \begin{bmatrix} b_O \\ \cdots \\ b_I \end{bmatrix}$. Partition the transportation cost vector in a

similar manner as $\bar{t} = [\bar{t}_0 \vdots \bar{t}_I]$. Set $-\beta = \bar{t}_I B_I$. Then $\beta \geq 0$ since the coefficients of B_I are non-positive. By L2.3 there is a unique vector $t_0 \geq 0$ such that $\pi_j(t_0, \beta, B_0) = \rho$ for each $j \in O(B_0) = O(B)$. Setting $t = [t_0 \vdots \bar{t}_I]$ proves the first part of the assertion. To prove the second part let $\rho_1 < \rho_2$ and set $d = t_B(\rho_2) - t_B(\rho_1)$. Then $d_j = \bar{t}_j - \bar{t}_j = 0$ for $j \in I(B)$ and so, by L2.3(b) and A2, $d_j > 0$ for $j \in O(B)$. ¤

3. OPTIMAL ALLOCATIONS

An allocation is a specification of production plans $x(u)$ for each distance u, together with a specification of transportation flows of the various goods, such that the demands for intermediate inputs at each u, and the final demand Q at the CBD, are met. A production plan at u is merely the set of activity levels at which each activity must be operated; thus $x(u) = \{x_b(u) \mid b \in \bar{B}\}$. The transporation flow can consist of flow towards the CBD as well as flow towards the periphery since <u>a priori</u> one cannot assert that the latter flow will be absent in an optimal allocation. A precise definition follows.

D3.1 An <u>allocation</u> (with final output Q) is a 4-tuple $\omega = (\bar{u}, x(\cdot), f(\cdot), \phi(\cdot))$ where (i) $\bar{u} \geq u_0$ is the maximum distance at which production occurs, (ii) $x(u) = \{x_b(u) \mid b \in \bar{B}\}$, $x(u) \geq 0$ is the production plan at u, $u_0 \leq u \leq \bar{u}$, (iii) $f(u) \in R^n$, respectively $\phi(u) \in R^n$, is the amount of net local production at u which is shipped towards the CBD, respectively away from the CBD; and such that these feasibility conditions are satisfied:

a) $0 \leq \sum_{b \in \bar{B}} x_b(u) \leq \theta(u), \quad u_0 \leq u \leq \bar{u},$

b) $y(u) \triangleq \bar{B}\, x(u) = \sum_{b \in \bar{B}} x_b(u) b = f(u) + \phi(u), \quad u_0 \leq u \leq \bar{u},$

c) if $s(u)$ is obtained from the differential equation

$$\dot{s}(u) \triangleq \frac{ds}{du}(u) = -f(u), \; s(\bar{u}) = 0, \tag{3.1}$$

then $s(u) \geq 0$, $u_0 \leq u \leq \bar{u}$ and $s(u_0) = Q$,

d) if $\sigma(u)$ is obtained from the differential equation

$$\dot{\sigma}(u) = \phi(u), \quad \sigma(u_0) = 0, \tag{3.2}$$

then $\sigma(u) \geq 0$, $u_0 \leq u \leq \bar{u}$.

Condition a) states that land devoted to production at u cannot exceed the available land $\theta(u)$; b) states that commodity flows originating at u equal the net production y(u) at u. The material balance conditions c) and d) can be deduced from Fig. 3.1.

The cost incurred by an allocation ω is the sum of the transport cost and the opportunity cost of land,

$$C(\omega) \triangleq \int_{u_0}^{\bar{u}} \bar{t}[s(u) + \sigma(u)]du + \int_{u_0}^{\bar{u}} r_A \underset{\sim}{1}\, x(u)du^{1}. \tag{3.3}$$

<u>D3.2</u> An allocation $\omega^* = (\bar{u}^*, x^*(\cdot), f^*(\cdot), \phi^*(\cdot))$ is <u>optimal</u> if $C(\omega^*) \leq C(\omega)$ for all allocations ω (with final output Q).

<u>T3.1</u> (Optimality conditions) ω^* is optimal if and only if there exists an absolutely continuous price system $p^*(u) \geq 0, u_0 \leq u \leq \bar{u}^*$, such that the following conditions are satisfied:

a) For $u_0 \leq u \leq \bar{u}^*$

$$[p^*(u)\bar{B}-r_A\underset{\sim}{1}]x^*(u) = \mathrm{Max}\{[p^*(u)\bar{B}-r_A\underset{\sim}{1}]x \,|\, x\geq 0,\ \sum_{b\in\bar{B}} x_b \leq \theta(u)\} \tag{3.3}$$

b) If $s^*(u)$, $\sigma^*(u)$ are the solutions of (3.1), (3.2) corresponding respectively to $f^*(u)$, $\phi^*(u)$, then for $u_0 \leq u \leq \bar{u}^*$ and $j = 1...n$,

$$s_j^*(u) > 0 \text{ implies } \sigma_j^*(u) = 0 \tag{3.4}$$

$$\dot{p}_j^*(u) = \begin{cases} -\bar{t}_j & \text{if} \quad s_j^*(u) > 0 \\ +\bar{t}_j & \text{if} \quad s_j^*(u) < 0 \\ \in[-\bar{t}_j, +\bar{t}_j] & \text{if} \quad s_j^*(u) = \sigma_j^*(u) = 0 \end{cases} \tag{3.5}$$

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[1] Here $\underset{\sim}{1}$ denotes the vector all of whose components equal 1 so that $\underset{\sim}{1}\, x(u) = \Sigma\, x_b(u)$ which is the amount of land at u necessary for production at level x(u).

c) At the edge of the city

(3.6) $$[p^*(\bar{u}^*)\bar{B}-r_A\underset{\sim}{1}]x^*(\bar{u}^*) = 0,$$

(3.7) $$\sigma(\bar{u}^*) = 0$$

Pf See Appendix 1. ¤

The optimality conditions have a straightforward economic interpretation. (3.3) asserts that $x^*(u)$ is the maximum profit activity vector when $p^*(u)$ is the vector of f.o.b. prices, provided that the profit/land ratio exceeds r_A in which case all available land, $\theta(u)$, is used for production. If this ratio is less than r_A, then $x^*(u) = 0$. According to (3.6), at the edge of the city the ratio equals r_A. It will be shown later that the profit/land ratio declines with distance (a consequence of these conditions which is not at all obvious), so that, in fact, all available land between u_0 and $\bar{u}^*$ is indeed used for production as might be expected intuitively. (3.5) is a version of the condition first observed by Samuelson [1952]. It asserts that if it is optimal at u to ship the jth good inwards (outwards), then the price of j must decrease (increase) at the rate of the transport cost; on the other hand, to prevent arbitrage the prices cannot change faster than the transport cost. (3.4) merely asserts that it cannot be optimal to simultaneously ship a good inwards and outwards. From these conditions it is not evident that it is not optimal to have outward shipment of goods, although this cannot occur at $\bar{u}^*$ due to (3.7).

These optimality conditions also give some information about the pattern of the optimal production. This will be presented later. For the moment the economic interpretation given above immediately

leads to two corollaries of interest.

C3.1 The system of f.o.b. prices $p^*(u)$, $u_0 \leq u \leq \bar{u}^*$, and land rents $r^*(u)$ given by

$$r^*(u) = [\theta(u)]^{-1} p^*(u)\bar{B}x^*(u) \tag{3.8}$$

sustains the optimal allocation ω^* as a competitive allocation.

Pf With these rents and prices $x^*(u)$ is a maximum profit activity vector and the maximum profit (after payment of land rents) equals zero. The result follows. ¤

C3.2 Let $p^*(u)$, $r^*(u)$ be a price and rent system which sustain ω^* as in T 3.1. Let ω be any other optimal allocation. Then p^*, r^* also sustain ω.

Pf See Appendix 2. ¤

4. EFFICIENT PATTERNS

The optimality conditions cannot be solved to obtain an optimal allocation. This is due to the fact that (3.5) is a differential correspondence and not a differential equation, hence the conditions do not give a classical boundary value problem as normally arises in calculus of variations or optimal control theory. A more refined analysis is necessary in which a <u>finite</u> set of patterns (called 'efficient patterns' below) is introduced <u>a priori</u> and the optimality conditions are then used to show that optimal allocations must belong to this set.

To motivate the consideration of efficient patterns make the intuitive hypothesis that for $u < \bar{u}^*$, $[p^*(u)\bar{B}-r_A\underset{\sim}{1}]x^*(u) > 0$, that is. only at the city's margin is the net profit zero (cf. (3.6)) whereas all intra-marginal locations yield positive profits. By (3.3), it will then be the case that

(4.1) $$p^*(u)\bar{B}x^*(u) = \mathrm{Max}\{p^*(u)\bar{B}x \mid x \geq 0,\ \underset{\sim}{1}x \leq \theta(u)\},\ \underset{\sim}{1}\ x^*(u) = \theta(u),$$

i.e. $x^*(u)$ contains only those techniques operated at a positive level which show the most profit, and all available land is devoted to production. In terms of

(4.2) $$r^*(u) \triangleq \mathrm{Max}\{p^*(u)\ b \mid b \in \bar{B}\}\quad ,\quad B^*(u) \triangleq \{b \in \bar{B} \mid p^*(u)\ b = r^*(u)\},$$

the first condition in (4.1) says that $x_b^*(u) > 0$ only if $b \in B^*(u)$.

$B^*(u)$ is the subset of activities of $\bar{B}$ which are most profitable at u. Since $p^*(u)$ changes continuously with u, and since $\bar{B}$ is finite, therefore as u decreases from $\bar{u}^*$ to u_0 $B^*(u)$ will change discretely i.e. it will be a piecewise constant function of u. At this point make the second hypothesis that $B^*(u)$ changes only a finite number

of times for $u \in [u_0, \bar{u}^*]$. That is, there is a set of distances $u_0 < u_1 < \ldots < u_N \triangleq \bar{u}^*$ and subsets $B^1, B^2, \ldots, B^N$ of $\bar{B}$ such that

$$B^*(u) = B^k, \quad u_{k-1} < u < u_k, \quad k = 1, \ldots, N \tag{4.3}$$

Production will then be arranged in N concentric rings with techniques from B^k being employed in the kth ring (u_{k-1}, u_k). At the boundary of a ring, say u_k, it must be that

$$B^*(u_k) = B^k \cup B^{k+1}, \quad k = 1, \ldots, N-1. \tag{4.4}$$

This follows from the definition (4.2) of $B^*(u)$ and the continuity of $p^*(u)$. Note that the number of rings N, and hence the number of production patterns $B^1, \ldots, B^N$, cannot yet be fixed.

The next step is to study (3.5) to see how $p^*(u)$ can vary with u. This will, through (4.2), also help to restrict the possible production patterns. Start with the hypothesis that within each ring the rates of change of $p^*(u)$ is constant, and let

$$t^k \triangleq -\dot{p}^*(u), \quad u_{k-1} < u < u_k, \quad k = 1, \ldots N. \tag{4.5}$$

Because of (3.5) it must be that $-\bar{t} \leq t^k \leq \bar{t}$. Let

$$p^N \triangleq p^*(\bar{u}^*) \text{ and } p^k \triangleq p^N + \sum_{\ell=k+1}^{N} (u_\ell - u_{\ell-1}) t^\ell, \quad k = 0, \ldots, N-1 \tag{4.6}$$

so that, because of (4.5), $p^*(u_k) = p^k$. Since $\dot{p}^*(u)$ is constant inside each ring therefore, from (3.5) and (3.8), $\dot{r}^*(u)$ is also constant within each ring. Let

$$\rho^k \triangleq \dot{r}^*(u), \quad u_{k-1} < u < u_k, \quad k = 1, \ldots, N, \tag{4.7}$$

$$r^N \triangleq r_A \text{ and } r^k \triangleq r^N + \sum_{\ell=k+1}^{N} (u_\ell - u_{\ell-1}) \rho^\ell, \quad k = 0, \ldots, N-1, \tag{4.8}$$

so that $r^*(u_k) = r^k$. From (4.2) it follows that

(4.9) $$r^k = \mathrm{Max}\{p^k\, b \mid b \in \bar{B}\}\ , \qquad k = 0,\ldots,N.$$

The final step is to relate the production pattern $B^1,\ldots,B^N$ to the prices and rents $p^0,\ldots,p^N$ and $r^0,\ldots,r^N$. First note from (4.4) that $B^k \subset B^*(u_k)$ and $B^k \subset B^*(u_{k-1})$ that is the techniques in B^k must be most profitable at both u_k and u_{k-1}. Hence

(4.10) $$r^k = p^k b,\ r^{k-1} = p^{k-1} b \quad \text{for } b \in B^k\ , \quad k = 1,\ldots,N.$$

Secondly, in each ring k all the inputs needed for production using techniques in B^k, that is the commodities $I(B^k)$, must be imported from the 'outer' rings $k+1,\ldots,N$ or the 'inner' rings $1,\ldots,k-1$, and hence these inputs must be produced there. Let $\hat{S}^k \subset J = \{1,\ldots,n\}$ be the commodities imported from the outer rings, and $\hat{\Sigma}^k \subset J$ the ones imported from the inner rings. Then

(4.11) $$I(B^k) \subset \hat{S}^k \cup \hat{\Sigma}^k\ , \quad k = 1,\ldots,N.$$

Now, if $j \in \hat{S}^k$ then this commodity must have been produced in some outer ring $\ell > k$ i.e., $j \in O(B^\ell)$, and it must have been shipped across the intervening rings $k,\ldots,\ell$. Let S^ℓ (Σ^ℓ) denote the set of commodities which are shipped inwards (outwards). Then it must be that

(4.12) $$\hat{S}^k = \bigcup_{\ell=k+1}^{N} [O(B^\ell) \cap S^k \cap \ldots \cap S^\ell]\ , \quad k = 1,\ldots,N$$

and a similar argument shows that

(4.13) $$\hat{\Sigma}^k = \bigcup_{\ell=1}^{k-1} [O(B^\ell) \cap \Sigma^\ell \cap \ldots \cap \Sigma^k]\ , \quad k = 1,\ldots,N.$$

Next, in each ring k there must be at least one output $j \in O(B^k)$ which is produced at a positive level and this output must be shipped either inwards or outwards, hence

(4.14) $$0(B^k) \cap [(S^k \cap S^{k-1}) \cup (\Sigma^k \cap \Sigma^{k+1})] \neq \phi \ , \quad k = 1,\ldots,N$$

where evidently $S^0 = J$, $\Sigma^{N+1} = \phi$. Finally, by (3.5) there is a relation between the t^k and S^k, Σ^k, namely

(4.15) $$S^k = \{j \mid t_j^k = \bar{t}_j\} \ , \ \Sigma^k = \{j \mid t_j^k = -\bar{t}_j\} \ , \quad k = 1,\ldots,N.$$

An efficient pattern $\{B^k, t^k, \rho^k\}$, $k = 1,\ldots,N$ is any sequence which satisfies the conditions derived above.

<u>D4.1</u> A sequence $\{B^k, t^k, \rho^k\}$, $k = 1,\ldots,N$ is an <u>efficient pattern</u> if for each k, $B^k \subset \bar{B}$, $-\bar{t} \leq t^k \leq \bar{t}$, $\rho^k \in R$, and if there are distances $u_0 < u_1 < \ldots < u_N$, and $p^N \geq 0$, such that the following conditions are satisfied:

Let

$$p^k = p^N + \sum_{\ell=k+1}^{N} (u_\ell - u_{\ell-1}) t^\ell, \ r^k = r^N + \sum_{\ell=k+1}^{N} (u_\ell - u_{\ell-1}) \rho^\ell, \ k = 0,\ldots,N-1$$

with $r^N = r_A$, and let

$$S^k = \{j \mid t_j^k = \bar{t}_j\} \ , \quad \Sigma^k = \{j \mid t_j^k = -\bar{t}_j\} \ , \quad k = 1,\ldots,N,$$

$$\hat{S}^k = \bigcup_{\ell=k+1}^{N} [0(B^\ell) \cap S^k \cap \ldots \cap S^\ell], \ \hat{\Sigma}^k = \bigcup_{\ell=1}^{k-1} [0(B^\ell) \cap \Sigma^\ell \cap \ldots$$

$$\cap \Sigma^k] \ , \quad k = 1,\ldots,N;$$

then, $p^k \geq 0$ for each k and

(4.16) $$b \in B^k \Leftrightarrow p^k b = r^k \text{ and } p^{k-1} b = r^{k-1} \ , \quad k=1,\ldots,N$$

(4.17) $$b \notin B^k \Rightarrow p^k b \leq r^k \text{ and } p^{k-1} b \leq r^{k-1} \ , \quad k=1,\ldots,N$$

(4.18) $$I(B^k) \subset \hat{S}^k \cup \hat{\Sigma}^k \ , \quad k=1,\ldots,N$$

(4.19) $$0(B^k) \cap [(S^k \cap S^{k-1}) \cup (\Sigma^k \cap \Sigma^{k+1})] \neq \phi \ , \quad k=1,\ldots,N.$$

L4.1 Suppose $\{B^k, t^k, \rho^k\}$, $k=1,\ldots,N$, is efficient. Then for each k

(4.20) $$j \in \hat{S}^k \Rightarrow t_j^k = \bar{t}_j, \quad j \in \hat{\Sigma}^k \Rightarrow t_j^k = -\bar{t}_j$$

(4.21) $$b \in B^k \Rightarrow t^k b = \rho^k$$

(4.22) $$b \in B^k,\ b \notin B^{k-1} \Rightarrow t^{k-1} b < \rho^{k-1}$$

(4.23) $$b \in B^{k-1},\ b \notin B^k \Rightarrow t^k b > \rho^k$$

Pf (4.20) follows from the definition of S^k, Σ^k and the fact that these are contained in $\hat{S}^k$, $\hat{\Sigma}^k$ respectively. If $b \in B^k$ then, by (4.16), $(p^{k-1}-p^k)b = r^{k-1}-r^k$. But $p^{k-1}-p^k = (u_k-u_{k-1})t^k$, $r^{k-1}-r^k = (u_k-u_{k-1})\rho^k$, so that substitution yields (4.21). If $b \in B^k$ and $b \notin B^{k-1}$ then, by (4.16) and (4.17), $(p^{k-1}-p^k)b < r^{k-1}-r^k$ and so $t^{k-1}b < \rho^{k-1}$ proving (4.22). The last assertion is proved in a similar manner. ¤

T4.1 Suppose $\{B^k, t^k, \rho^k\}$, $k=1,\ldots,N$, is efficient. Then

(4.24) $\rho^N > 0$, $\rho^k \geq \rho^{k+1}$ and equality holds only if $B^k = B^{k+1}$, $t^k = t^{k+1}$

(4.25) B^k is productive for each k

(4.26) $\hat{\Sigma}^k = \phi$ for each k

(4.27) $N \leq \bar{N}$ where $\bar{N}$ is a number depending only on $\bar{B}$

Pf First (4.24) and (4.25) will be proved by backward induction on k. Consider the case $k = N$. By (4.16) $p^N b = r^N = r_A \geq 0$ for $b \in B^N$ and so by L2.2 B^N is productive.[1] Next partition B^N as $\begin{bmatrix} B_0^N \\ \cdots \\ B_I^N \end{bmatrix}$ (using the notation of §2) and similarly partition t^N as $[t_0^N \vdots t_I^N]$ so that

$$t^N B^N = t_0^N B_0^N + t_I^N B_I^N$$

[1] L2.2 applies when $r_A > 0$. If $r_A = 0$ the argument needs to be modified as in the proof of C4.2.

By (4.21) $t^N B^N = \rho^N \underset{\sim}{1}$ so that

$$t_0^N B_0^N = \rho^N \underset{\sim}{1} - t_I^N B_I^N . \tag{4.28}$$

Now by (4.18) $I(B^N) \subset \hat{S}^N \cup \hat{\Sigma}^N$, but $\hat{S}^N = \phi$ so that $I(B^N) \subset \hat{\Sigma}^N$; hence, by (4.20) $t_I^N = -\bar{t}_I$ and since $B_I^N \leq 0$ therefore $-t_I^N B_I^N \leq 0$. Suppose $\rho^N \leq 0$. Then from (4.28), $t_0^N B_0^N \leq 0$ and so, by L2.2, $t_0^N \leq 0$. But by (4.19), since $\Sigma^{N+1} = \phi$ therefore $O(B^N) \cap S^N \neq \phi$. Hence there exists $j \in O(B^N)$ such that $t_j^N > 0$ which contradicts $t_0^N \leq 0$. Hence $\rho^N > 0$.

Next assume that (4.24), (4.25) hold for k+1,...,N and consider the case k. Let $\{p^k, r^k\}$ be the sequence of prices and rents in D4.1. Since $r^k = r^N + \sum_{\ell=k+1}^{N} (u_\ell - u_{\ell-1})\rho^\ell$ and since $\rho^\ell > 0$ by the induction hypothesis, therefore, $r^k > 0$. By (4.16) $p^k b = r^k$ for $b \in B^k$ and so by L2.2 B^k is productive, proving (4.25). To prove (4.24) assume in contradiction that

$$\rho^k < \rho^{k+1} \tag{4.29}$$

and set $d = t^k - t^{k+1}$. Then, by (4.29), (4.21), (4.23),

$$dB^k << 0. \tag{4.30}$$

Now, by (4.18), $I(B^k) \subset \hat{S}^k \cup \hat{\Sigma}^k$. If $j \in I(B^k) \cap \hat{S}^k$ then $j \in S^k \cap S^{k+1}$ and so $t_j^k = t_j^{k+1} = \bar{t}_j$ and so $d_j = 0$; whereas if $j \in \hat{\Sigma}^k$ then $j \in \Sigma^k$ and so $t_j^k = -\bar{t}_j$ and so $d_j = t_j^k - t_j^{k+1} = -\bar{t}_j - t_j^{k+1} \leq 0$. Thus

$$d_j \leq 0 \qquad \text{for} \quad j \in I(B^k). \tag{4.31}$$

From (4.30), (4.31) and L2.6 it follows that

$$d_j < 0 \qquad \text{for} \quad j \in O(B^k). \tag{4.32}$$

On the other hand, by (4.19), there exists $j \in O(B^k)$ such that either

$j \in S^k$ or $j \in \Sigma^k \cap \Sigma^{k+1}$. In the first case $t_j^k = \bar{t}_j$ and so $d_j \geq 0$ contradicting (4.32), and in the second case $t_j^k = t_j^{k+1} = \bar{t}_j$ so that $d_j = 0$ again contradicting (4.32). Hence (4.29) must be false thus proving the first part of (4.24). To prove the second part suppose $\rho^k = \rho^{k+1}$. Then, instead of (4.30), one has $dB^k = 0$ which implies $d = 0$ since B^k is productive and so $t^k = t^{k+1}$. But then from (4.16), (4.17) $B^k = B^{k+1}$. Thus (4.24), (4.25) are proved.

Next (4.26) will be proved by forward induction on k. For k=1 certainly $\hat{\Sigma}^1 = \phi$. Now assume that $\hat{\Sigma}^k = \phi$ and consider the case for k+1. Then

$$\hat{\Sigma}^{k+1} = O(B^k) \cap \Sigma^k \cap \Sigma^{k+1}. \tag{4.33}$$

Furthermore, by (4.18) $I(B^k) \subset \hat{S}^k$ and so, by (4.20), $t_j^k = \bar{t}_j > 0$ for $j \in I(B^k)$. Now, by (4.21), $t^k b = \rho^k > 0$. Hence by L2.6 $t_j^k > 0$ for $j \in O(B^k)$ which implies that $\Sigma^k \cap O(B^k) = \phi$. By (4.33) this implies $\hat{\Sigma}^{k+1} = \phi$. Thus (4.26) is proved.

It remains to prove (4.27). By L2.4 and L2.7 for each triple (B^k, S^k, Σ^k) there are at most two pairs (t, ρ) fulfilling (4.21), and the conditions $t_j = \bar{t}_j$, $j \in S^k$ and $t_j = -\bar{t}_j$, $j \in \Sigma^k$ and (4.14). Since there are only a finite number of distinct (B^k, S^k, Σ^k) and since repetition is eliminated by (4.24) therefore (4.27) follows. ¤

<u>R4.1</u> Note that in the proof of (4.24), (4.25) the condition $\hat{\Sigma}^0 = \phi$ was not used. This means that even if 'imports' from the CBD were permitted, (4.24), (4.25) continue to remain valid. In particular, rents decline with distance from the CBD even if imports are permitted.

<u>C4.1</u> Suppose $\{B^k, t^k, \rho^k\}$, $k = 1,\ldots,N$ is efficient. Then $t^k \gg 0$ for each k.

Pf Since $\hat{\Sigma}^k = \phi$ by (4.26) therefore, by (4.18), $I(B^k) \subset \hat{S}^k$ and so $t_j^k = \bar{t}_j > 0$ for $j \in I(B^k)$. By (4.21) $t^k B^k = \rho^1 \underset{\sim}{1} >> 0$ so that by L2.6 $t_j^k > 0$ for $j \in O(B^k)$. ¤

C4.2 Suppose $\{B^k, t^k, \rho^k\}$, $k = 1,\ldots,N$ is efficient. Then $\{B^N, t^N, \rho^N\}$ is uniquely determined by $\bar{B}$, $\bar{t}$, and the boundary price p^N is the unique multiple of t^N, $p^N = \lambda t^N$ such that $\lambda\rho^N = r_A$. Furthermore for $b \notin B^N$ $t^N b < \rho^N$.

Pf In the outermost ring both $\hat{\Sigma}^N = \phi$ and $\hat{S}^N = \phi$ so that $I(B^N) = \phi$ and $O(B^N) = J$. Therefore B^N, p^N satisfy

$$b \in B^N \Rightarrow p^N b = r_A$$

$$b \in \bar{B} \Rightarrow p^N b \leq r_A$$

$$\forall_j \in J,\ B^N \cap \bar{B}_j \neq \phi$$

By L2.3 these conditions determine p^N uniquely. Let $\hat{B} = \{b \in \bar{B} | p^N b = r_A\}$. By L4.1 B^N, $\hat{B}$, t^N and ρ^N satisfy

$$b \in B^N \Rightarrow t^N b = \rho^N \tag{4.34}$$

$$b \in \hat{B},\ b \notin B^N \Rightarrow t^N b < \rho^N \tag{4.35}$$

$$\forall_j \in O(B^N) = O(\hat{B}) = J,\ \exists\ b \in B^N \cap \bar{B}_j \text{ such that } t^N b = \rho^N \tag{4.36}$$

By L2.7 these conditions uniquely determine t^N as a function of ρ^N, $t^N = t^N(\rho^N)$ and $[t^N(\rho^N)]_j$ is strictly increasing in ρ^N. By (4.19), there is $j \in O(B^N)$ for which $[t^N(\rho^N)]_j = \bar{t}_j$. Hence by the strict monotonicity ρ^N is uniquely determined by

$$\rho^N = \max\{\rho | t^N(\rho) \leq \bar{t}\}$$

Next, from (4.34), (4.35) and the uniqueness of p^N it follows that $p^N = \frac{r_A}{\rho^N} t^N$. Finally, if $p^N >> 0$ then $B^N = \hat{B}$ whereas if $p^N = 0$ then $\hat{B} = \bar{B}$. In either case therefore $t^N b < \rho^N$ for $b \notin B^N$. ¤

From the discussion at the beginning of this section and from T4.1 it follows that every optimal allocation yields an efficient pattern.

<u>T4.2</u> Let $x^*(u)$, $u_0 \leq u \leq \bar{u}^*$, be an optimal allocation and let $p^*(u)$, $r^*(u)$ be the corresponding price and rent functions. Then there is an efficient pattern $\{B^k, t^k, \rho^k\}$, $k = 1,\ldots,N$, and distances $u_0 < u_1 <\ldots< u_N = \bar{u}^*$ such that for each k and $u_{k-1} < u < u_k$,

$$x_b^*(u) > 0 \Leftrightarrow b \in B^k$$

$$\dot{p}^*(u) = -t^k << 0, \quad p(\bar{u}^*) = p^N$$

$$\dot{r}^*(u) = -\rho^k < 0, \quad r(\bar{u}^*) = r_A$$

In particular, all transportation flows towards the CBD; prices decline with u; and the rent is a declining, convex function of u.

<u>Pf</u> Only the last statement needs a proof which follows from the fact that $\rho^1 > \rho^2 >\ldots> \rho^N$ according to (4.24) ¤

Thus every optimal allocation gives rise to an efficient pattern. To study the converse proposition consider an efficient pattern $\{B^k, t^k, \rho^K\}$, $k = 1,\ldots,N$, and the associated distances $u_0 < u_1 <\ldots< u_n \triangleq \hat{\bar{u}}$. Construct the price function $\hat{p}(u)$ with $\hat{p}(\hat{\bar{u}}) = p^N$, and

$$\dot{\hat{p}}(u) = -t^k, \quad u_{k-1} < u < u_k, \quad k = 1,\ldots,N \tag{4.37}$$

One can also attempt to construct production plans $\hat{x}(u)$, $u_0 \leq u \leq \hat{\bar{u}}$ such that in the kth ring $u_{k-1} < u < u_k$, $\hat{x}_b(u) > 0$ only if $b \in B^k$ and $\underset{\sim}{1}\,\hat{x}(u) = \theta(u)$. However, these plans will not in general yield an allocation since the material balance conditions (3.1), (3.2) may not be able to be satisfied. But it seems reasonable to expect that

if the amount of land available for production, $\hat{\theta}(u)$, were adjustable then $\hat{x}(u)$ would give an allocation and such an allocation would be optimal for the new 'geography' $\hat{\theta}(u)$.

<u>T4.3</u> Let $\{B^k, t^k, \rho^k\}$, $k = 1,\ldots,N$, be an efficient pattern and let $u_0 < u_1 <\ldots< u_N = \hat{\bar{u}}$, be associated distances. Then there exists

(a) a geography of the form

$$\hat{\theta}(u) \equiv \theta^k > 0 , \qquad u_{k-1} < u < u_k$$

(b) an allocation $\hat{\omega} = (\hat{\bar{u}}, \hat{x}(\cdot), \hat{f}(\cdot), \hat{\phi}(\cdot))$ which is optimal for this geography and such that the production plans $x(\cdot)$ have the form

$$\hat{x}(u) \equiv x^k$$

with

$$x_b^k > 0 \Rightarrow b \in B^k,$$

and such that commodities are not transported towards the periphery,

$$\hat{\phi}(u) \equiv 0$$

<u>Pf</u> (The idea of the proof is to assume $\hat{\theta}$, $\hat{x}$ have the specified form and to obtain relations among the $\{\theta^k\}$, $\{x^k\}$ such that $\hat{\omega}$ satisfies (3.1), (3.2) with $\hat{\phi} \equiv 0$.) Let S^k, etc. be as in D4.1, and define

$$K(k,j) = \{\ell \mid \ell \geq k \text{ and } j \in O(B^\ell) \cap S^k \cap\ldots\cap S^\ell\} \tag{4.38}$$

Let $\hat{Q} \geq 0$ be any final demand vector such that

$$\hat{Q}_j > 0 \quad \Leftrightarrow \quad K(N,j) \neq \phi \tag{4.39}$$

Next a sequence $\{Q^k, x^k\}$, $k = 1,\ldots,N$, will be constructed inductively with $Q^1 = \hat{Q}$ and with these properties:

(i) $x_b^k = 0$ if $b \notin B^k$, $Q^{k+1} = Q^k - \bar{B}\, x^k \geq 0$,

(ii) $\exists\, j \in O(B^k)$ such that $Q_j^k > 0$

(iii) if $Q_j^k > 0$, then $K(k,j) \neq \phi$

(iv) $x^k \geq 0$ is chosen depending on Q^k such that

$$[\bar{B}\, x^k]_j = \begin{cases} Q_j^k & \text{if } K(k,j) = \{k\} \\ \frac{1}{2} Q_j^k & \text{if } K(k,j) \supset \{k,k+1\} \text{ and } Q_j^k > 0 \\ < 0 \text{ with } x_b^k > 0 \text{ for } b \in B^k \cap \bar{B}_j & \text{if } K(k,j) \supset \{k,k+1\} \text{ and } Q_j^k = 0 \\ 0 & \text{if } j \in O(B^k) \text{ and } j \notin S^k \end{cases}$$

To prove the existence of such a sequence begin by noting that since B^k is productive (by T4.1) therefore if $Q^k > 0$ then there always exists $x^k > 0$ such that (iv) holds. Such x^k must have the property

(4.40). $$[\bar{B}\, x^k]_j < 0 \qquad \text{if} \quad j \in I(B^k)$$

The verification of (i)-(iii) proceeds by induction on k. For k=1, $Q^1 = \hat{Q}$ and so (iii) follows from (4.39). Since $\hat{\Sigma}^1 = \phi$ by T4.1, therefore $O(B^1) \cap S^1 \neq \phi$ and so there exists $j \in O(B^1)$ such that $K(1, j) \neq \phi$ which proves (ii), finally (i) follows from (iv).

Now assume that (i)-(iv) are verified for k and let $Q^{k+1} = Q^k - \bar{B}\, x^k$. Then $Q^{k+1} \geq 0$ and so (iv) holds and this implies (i). To check (iii) suppose $K(k+1, j) = \phi$. Then, from (4.38), either $K(k,j) = \phi$ or $K(k, j) = \{k\}$. In the first case $Q_j^k = 0$ by (iii) and $[\bar{B}\, x^k]_j = 0$ by (iv) and so $Q_j^{k+1} - [\bar{B}\, x^k]_j = 0$ also; in the second case, by (iv) $[\bar{B}\, x^k]_j = Q_j^k$ and so again $Q_j^{k+1} = 0$. Thus (iii) is verified. Finally to verify (ii) suppose in contradiction that $Q_j^{k+1} = 0$ for every $j \in O(B^{k+1})$. By (iv) this is possible only if for each $j \in O(B^{k+1})$, $j \notin S^{k+1}$, so that $O(B^{k+1}) \cap S^{k+1} = \phi$. But since $\hat{\Sigma}^{k+1} = \phi$ by T4.1 therefore by (4.19) $O(B^{k+1}) \cap S^{k+1} = \phi$. Thus (ii) is verified.

Having obtained the sequence $\{Q^k, x^k\}$, k = 1,...,N, define

$$\theta^k = \frac{\underset{\sim}{1}\, x^k}{u_k - u_{k-1}}$$

Now consider the allocation $\hat{\omega} = (\hat{\bar{u}}, \hat{x}(\cdot), \hat{f}(\cdot), \hat{\phi}(\cdot))$ where for $u_{k-1} < u < u_k$ $\hat{x}(u) = x^k$, $\hat{f}(u) = \bar{B}\, x^k$, $\hat{\phi}(u) = 0$. It is easy to verify that the transportation flow $\hat{s}(u)$ defined by (3.1),

$$\dot{\hat{s}}(u) = -\hat{f}(u), \quad \hat{s}(\hat{\bar{u}}) = 0$$

satisfies $\hat{s}(u) \geq 0$. Furthermore $\hat{s}(u_k) = Q_{k-1}$, and so $\hat{s}(u_0) = \hat{Q}$. Hence $\hat{\omega}$ is feasible for the geography $\hat{\theta}$. Furthermore if $\hat{p}(u)$ is defined by (4.36), then it is immediate that it satisfies the optimality condition of T3.1. If follows that $\hat{\omega}$ is optimum. ¤

5. THE CANONICAL PATTERN

In the previous section it was shown that the set of efficient patterns is an "invariant" of the technology $\bar{B}$ and transport coefficients $\bar{t}$. It is a minimal invariant in the sense that up to an arbitrary geography this set characterizes all optimal allocations (T4.2,T4.3). Furthermore this set is finite (see (4.27)). Unfortunately the definition D4.1 does not appear to yield a finite algorithm to determine this set. This is because the conditions in D4.1 on the variables $u_1,\ldots,u_N$ seem not to be verifiable by a finite algorithm.

In the next section a method is proposed to enumerate a finite set of sequences which contains all the efficient ones. Some preliminary results are collected here.

<u>L5.1</u> Suppose $\{B^k,t^k,\rho^k\}$, k=1...N, is efficient. Fix k and let $B = B^k \cup B^{k-1}$, $M = 0(B^{k-1})$, $\rho = \rho^{k-1}$, $t = t^{k-1}$. Then $\{M,t,\rho\}$ is a solution (not necessarily unique) to the relations (5.1)-(5.3).

(5.1) $\quad b \in B \Rightarrow tb \leq \rho$

(5.2) $\quad j \in M \Leftrightarrow j \in 0(B)$ and $\max\{tb \mid b \in B \cap \bar{B}_j\} = \rho$

(5.3) $\quad 0 \leq t \leq \bar{t}$, $t_j = \bar{t}_j$ for $j \notin M$ and $\exists j \in M$ with $t_j = \bar{t}_j$.

<u>Pf</u> (5.1) and (5.2) follow from (4.21), (4.22). Since $t^{k-1} >> 0$ by C4.1 therefore $0 \leq t \leq \bar{t}$. If $j \notin M$ then $j \in I(B^{k-1})$ and so by (4.26), (4.18) $j \in S^{k-1}$ so that $t_j^{k-1} = \bar{t}_j$. By (4.19), (4.26) there exists $j \in S^{k-1}$ and so $t_j^{k-1} = \bar{t}_j$. ¤

Recall the definition $t_B(\rho)$ introduced in L2.7. By L5.1 the set B in L5.1 is a member of $\mathcal{B}$ introduced below.

D5.1 $\mathcal{B}$ is the set of all subsets $B \subset \bar{B}$ such that B is productive and $t_B(0) \leq \bar{t}$.

D5.2 For $B \in \mathcal{B}$, $t(B) \geq 0$, $\rho(B) \geq 0$ are defined by the conditions: $t(B) = t_B(\rho(B))$, $\rho(B) = \max\{\rho \mid 0 \leq t_B(\rho) \leq \bar{t}\}$. (Note that $[t(B)]_j = \bar{t}_j$ for at least one $j \in 0(B)$.)

L5.2 Suppose $\{B^k, t^k, \rho^k\}$, $k = 1 \ldots N$, is efficient. Then $t^k = t(B^k)$, $\rho^k = \rho(B^k)$.

Pf By T4.1 $\rho^k > 0$, $t^k > 0$, $t^k > 0$ and $t_j^k = \bar{t}_j$ for $j \in I(B^k)$. Hence, by L2.7 $t^k = t_{B^k}(\rho^k)$. By T4.1 and (4.19) there is $j \in 0(B^k)$ such that $t_j^k = \bar{t}_j$. Since $[t_{B^k}(\rho^k)]_j$ increases strictly monotonically with ρ^k, therefore $\rho^k = \max\{\rho \mid t_{B^k}(\rho) \leq \bar{t}\}$. ¤

D5.3 A sequence $\{B^k\}$, $k = 1 \ldots N$, is called <u>efficient</u> if $\{B^k, t(B^k), \rho(B^k)\}$, $k = 1 \ldots N$, is an efficient pattern.

Because of L5.1, given $B \in \mathcal{B}$, it is natural to study the solutions $\{M, t, \rho\}$ of (5.1)-(5.3).

T5.1 Let $B \in \mathcal{B}$. There are exactly m solutions $\{M^\mu, t^\mu, \rho^\mu\}$, $\mu = 1 \ldots m$ of (5.1)-(5.3) where m depends on B. These can be arranged so that

(5.4) $$M^1 \subsetneq M^2 \subsetneq \ldots \subsetneq M^m$$

(5.5) $$t^1 > t^2 > \ldots > t^m \gg 0; \quad \rho^1 > \rho^2 > \ldots > \rho^m > 0$$

(5.6) $$M^\mu = \{j \in M^{\mu+1} \mid t_j^{\mu+1} < \bar{t}_j\}$$

(5.7) $$t_j^\mu = t_j^{\mu+1} \Leftrightarrow t_j^{\mu+1} = \bar{t}_j$$

(5.8) $$M^m = 0(B), \quad t^1 = \bar{t}.$$

In particular $m \leq |0(B)| \leq n$ ($|0(B)|$ = number of elements in $0(B)$)

Pf The last statement is immediate from (5.4). The proof of the remainder is conducted in a series of steps.

Step 1 It is claimed that given $M \subset 0(B)$ there is at most one pair (t,ρ) satisfying (5.1)-(5.3).

To prove this, note that by L2.7, for each M, ρ, there is at most one vector t, denoted $t(M,\rho)$ satisfying (5.1)-(5.3). Furthermore, (5.3) and L2.7 uniquely specify ρ as $\rho(M)$ where

$$\rho(M) = \max\{\rho \mid t(M,\rho) \leq \bar{t}\}$$

Thus $t = t(M,\rho(M))$, $\rho = \rho(M)$ give the unique pair. Note that if M^μ is determined, then $t^\mu = t(M^\mu,\rho(M^\mu))$

Step 2 It is claimed that if $M \subset 0(B)$, $\tilde{M} \subset 0(B)$ give pairs (t,ρ) and $(\tilde{t},\tilde{\rho})$ respectively, then

(5.9) $\rho \geq \tilde{\rho} \Rightarrow t \geq \tilde{t}$

(5.10) $\rho = \tilde{\rho} \Rightarrow M = \tilde{M}$

(5.11) $\rho > \tilde{\rho} \Rightarrow t_j > \tilde{t}_j$ for $j \in M$

(5.12) $\rho > \tilde{\rho} \Rightarrow M \subset \tilde{M}$.

To prove this set $B_M \triangleq \{b \in B \mid tb = \rho\}$, $d \triangleq t - \tilde{t}$. By (5.3) if $j \notin M$, $d_j = \bar{t}_j - \tilde{t}_j \geq 0$, and if $\rho \geq \tilde{\rho}$ then $dB_M \geq 0$. By L2.6(a), $d \geq 0$ proving (5.9). By (5.9) if $\rho = \tilde{\rho}$, then $t = \tilde{t}$ and so $M = \tilde{M}$ because of (5.2), thus proving (5.10). The proof of (5.11) is similar to that of (5.9) using L2.6(b). Finally let $\rho > \tilde{\rho}$ and suppose there is a $j \in M$, $j \notin \tilde{M}$. Then, by (5.3) $\tilde{t}_j = \bar{t}_j \geq t_j$. But by (5.11) $t_j > \tilde{t}_j$ which is a contradiction. Hence $M \subset \tilde{M}$.

Steps 1, 2 together prove (5.4), (5.5) and, once (5.6) is proved they imply (5.7). Furthermore by (5.11), (5.12)

(5.13) $M^{\mu-1} \subset M^\mu - \{j \mid t_j^\mu = \bar{t}_j\}$

Step 3 It is claimed that (5.6) holds

Let $\hat{M}$ denote the right-hand side of (5.13). Set $B^{\mu} = \{b \in B | 0(b) \in M^{\mu}\}$. Then $\{M^{\mu}, t^{\mu}, \rho^{\mu}\}$ is a solution of (5.1)-(5.3) (with B replaced by B^{μ}.) Set $\hat{B} = \{b \in B^{\mu} | 0(b) \in \hat{M}\}$. Then by L2.7 $t_{\hat{B}}(\rho^{\mu}) = t^{\mu} \leq \bar{t}$. Let $\hat{\rho} = \max\{\rho | t_{\hat{B}}(\rho) \leq \bar{t}\}$. Then $\hat{\rho} > \rho^{\mu}$ and $\hat{t} = t_{\hat{B}}(\hat{\rho}) > t^{\mu}$. Hence $\{\hat{M}, \hat{t}, \hat{\rho}\}$ is another solution of (5.1)-(5.3) (with $B = B^{\mu}$). Now if $b \in B - B^{\mu}$, then $\hat{t}b = t_{\hat{B}}(\hat{\rho})b \leq t_{\hat{B}}(\rho^{\mu})b = t^{\mu}b \leq \rho^{\mu} < \hat{\rho}$. Therefore $\{\hat{M}, \hat{t}, \hat{\rho}\}$ is a solution of (5.1)-(5.3). But this is possible only if $\hat{M} = M^{\mu-1}$, $\hat{t} = t^{\mu-1}$, $\hat{\rho} = \hat{\rho}^{\mu-1}$, thus proving (5.13).

Step 4 It is claimed that (5.8) holds.

Since $B \in \mathcal{B}$ therefore $t_B(0) \leq \bar{t}$. Set $\rho^m = \max\{\rho | t_B(\rho) \leq \bar{t}\}$, $t^m = t_B(\rho^m)$ and $M^m = 0(B)$. Then by L2.7 $\{M^m, t^m, \rho^m\}$ is a solution of (5.1)-(5.3). Finally set $t^1 = \bar{t}$, $\rho^1 = \max\{t^1 b | \in B\}$, $M^1 = \{j | \exists b \in 0(b)$ with $0(b) = j$ and $t^1 b = \rho^1\}$. Evidently $\{M^1, t^1, \rho^1\}$ is a solution of (5.1)-(5.3).

The theorem is proved. ¤

Theorem T5.1 permits the introduction of the following notation.

D5.4 For $B \in \mathcal{B}$ let $B^{\mu} = \{b \in B | t^{\mu}b = \rho^{\mu}\}$, $\mu = 1 \ldots m$, where $\{M^{\mu}, t^{\mu}, \rho^{\mu}\}$ is the μ-th "solution" in T5.2. For convenience define $F(B) = B^m$ and $G(B) = B^{m-1}$. (Note that m depends on B). For $B = \bar{B}$, the notation $\bar{B}^{\mu}$, $\bar{M}^{\mu}$ etc. is used and the sequences $\{\bar{B}^{\mu}, \bar{t}^{\mu}, \bar{\rho}^{\mu}\}$, $\mu = 1 \ldots m$ and $\{\bar{B}^{\mu}\}$, $\mu = 1 \ldots m$ are called the <u>canonical pattern</u> and <u>canonical sequence</u> respectively.

L5.3 The operators F, G satisfy the following properties

(5.14) $0(F(B)) = 0(B)$

(5.15) $F(F(B)) = F(B)$

(5.16) If $\{B^k\}$, $k = 1...N$, is efficient, then $B^N = F(\bar{B})$, $B^{N-1} = G(B^N)$

Pf (5.14) is a consequence of (5.8) and D5.4. From Step 4 in the proof of T5.1 it is seen that for any $B \in \mathcal{B}$

$$F(B) = \{b \in B | t(B)b = \rho(B)\}$$

where $t(B)$, $\rho(B)$ are defined in D5.2. (5.15) is then immediate. From C4.2 it follows that $B^N = F(\bar{B})$. Finally suppose $b \notin B^N$. Then $t(\bar{B})b < \rho(\bar{B})$ and so from (4.16) $b \notin B^{N-1}$, hence $B^{N-1} \subset B^N$. But then by T5.1 $B^{N-1} = G(B^N)$. ¤

If $\bar{B}$ is a Leontief technology without substitute techniques and if $\{B^k\}$, $k = 1...N$, is efficient then as shown in §7 below, $B^{N-2} \subset B^{N-1}$, which turns out to be equivalent to $G(B^N) = G(\bar{B})$. Unfortunately, when substitute techniques are present this need not hold (see §8). Mathematically, this is what creates most of the difficulties. From an economic viewpoint this implies that the Non-Substitution theorem need not hold when space is introduced.

The canonical pattern is not always efficient. However it does have the following attractive properties.

T5.2 (a) If $B \in \mathcal{B}$ and $\bar{\rho}^\mu \geq \rho(B) > \bar{\rho}^{\mu+1}$ then $O(B) \subset \bar{M}^\mu$ and $t(B) \geq \bar{t}^\mu$

(b) If the canonical pattern is not degenerate i.e., if $|\bar{B}^\mu| = |O(\bar{B}^\mu)|$ for every μ, then it is a subsequence of an efficient pattern; whereas if it is degenerate then there exists $\bar{\bar{B}}^\mu \subset \bar{B}^\mu$ with $O(\bar{\bar{B}}^\mu) = O(\bar{B}^\mu)$ and such that $\{\bar{\bar{B}}^\mu\}$, $\mu = 1...m$, is a subsequence of an efficient sequence.

Pf See Appendix. ¤

D5.5 An efficient sequence $\{B^k\}$, $k = 1...N$, is called <u>maximal</u> if it is not a strict subsequence of another efficient sequence.

R5.2 From the viewpoint of qualitative theory it must be considered a central problem in connection with von Thünen models to determine all the maximal sequences generated by a technology $\bar{B}$ and transport costs $\bar{t}$. This is because these sequences yield the most complex land use patterns possible. In general, given $\bar{B}$ and $\bar{t}$, there may be more than one maximal sequence (cf. §8, Example 1). However when $\bar{B}$ contains no substitute technique there is only one maximal sequence (T7.1).

For future reference the following relation between the initial and final segments of the canonical sequence and every maximal sequence is useful.

T5.3 Let $\{B^k\}$, $k = 1...N$, be a maximal sequence and $\{\bar{B}^\mu\}$, $\mu = 1...m$, the canonical sequence. Then $B^N = \bar{B}^m$, $B^{N-1} = \bar{B}^{m-1} = G(\bar{B}^m)$; $B^1 \subset \bar{B}^1$, $O(B^1) = O(\bar{B}^1)$ and $t^1 \triangleq t(B^1) = \bar{t}$.

Pf See Appendix. ¤

6. THE QUASI-EFFICIENT SEQUENCES

The algorithm proposed here enumerates a finite set of sequences which contains all the efficient ones.

D6.1 For $B \in \mathcal{B}$ let $A(B) \triangleq \{b \notin B | t(B)b \geq \rho(B)\}$ and write, $B \to \tilde{B}$ if either $\tilde{B} = G(B)$ or $\tilde{B} = F(B \cup \{b\})$ for some $b \in A(B)$.

D6.2 Let $\mathcal{E}$ denote the smallest family of subsets $B \subset \bar{B}$ which satisfy (6.1), (6.2),

(6.1) $\quad F(\bar{B}) \in \mathcal{E}$

(6.2) $\quad B \in \mathcal{E}$ and $B \to \tilde{B} \Rightarrow \tilde{B} \in \mathcal{E}$.

(The definition makes sense since the intersection of two families satisfying (6.1), (6.2) also satisfy them.)

D6.3 A sequence $\{B^k\}$, $k = 1 \ldots N$, is quasi-efficient if it is a subsequence of a sequence $\{E^\mu\}$, $\mu = 1 \ldots m$, such that $E^\mu \in \mathcal{E}$, $E^1 = B^1$, $E^m = B^N$ and

(6.3) $\quad E^m \to E^{m-1} \to \ldots \to E^1$.

It follows from these definitions that if $B \in \mathcal{E}$ then there is a quasi-efficient sequence $\{B^k\}$, $k = 1 \ldots N$ with $B^1 = B$ and $B^N = F(\bar{B})$. Recall D5.2.

L6.1 Suppose A, B are in $\mathcal{B}$. Set $A_0 = \{b \in A | 0(b) \notin 0(B)\}$. Suppose $t(B)b \geq \rho(B)$ for $b \in A_0$. Then $\rho(A \cup B) \geq \rho(B)$.

Pf Case 1

Suppose $A_0 = \phi$. Assume $\rho(A \cup B) < \rho(B)$ and put $d = t(B) - t(A \cup B)$. Then for $b \in F(B)$, $db = t(B)b - t(A \cup B)b \geq \rho(B) - \rho(A \cup B) > 0$. By L2.6 $d_j > 0$ for $j \in 0(B) = 0(A \cup B)$. But by T5.2$[t(A \cup B)]_j = \bar{t}_j$ for at least one $j \in 0(A \cup B)$ and so the corresponding $d_j \leq 0$. which is a contradiction. Hence $\rho(A \cup B) \geq \rho(B)$.

Case 2 Suppose $A_0 = A$, and set $d = t_{A \cup B}(\rho(B)) - t_B(\rho(B)) = t_{A \cup B}(\rho(B)) - t(B)$. Consider $b \in \tilde{B} = A \cup F(B)$. If $b \in A$ then $db = \rho(B) - t(B)b \leq 0$ and if $b \in F(B)$ then $db = \rho(B) - \rho(B) = 0$. Hence $d\tilde{B} \leq 0$. Also if $j \in I(\tilde{B}) \subset I(B)$ then $d_j = \bar{t}_j - \bar{t}_j = 0$. By L2.6 therefore it follows that $d \leq 0$. Hence $t_{A \cup B}(\rho(B)) \leq t(B) \leq \bar{t}$. By definition $\rho(A \cup B) = \max\{\rho \mid t_{A \cup B}(\rho) \leq \bar{t}\}$. Hence $\rho(A \cup B) \geq \rho(B)$.

General Case

$\rho(B) \leq \rho(A_0 \cup B)$ by Case 2 and $\rho(A_0 \cup B) \subset \rho(A \cup B)$ by Case 1.

¤

L6.2(a) If $0(b) \in 0(B)$, then $\rho(B \cup \{b\}) > (=) \rho(B)$ according as $t(B)b > (\leq) \rho(B)$; furthermore $b \in F(B \cup \{b\})$ if and only if $t(B)b \geq \rho(B)$.

(b) If $0(b) \notin 0(B)$, then $\rho(B \cup \{b\}) > (=) [<] \rho(B)$ according as $t(B)b > (=) [<] \rho(B)$; furthermore $b \in F(B \cup \{b\})$.

Pf The method of proof is very similar to that of L6.1. Hence it is omitted. ¤

C6.1(a) If $B \to \tilde{B}$ then $\rho(B) \leq \rho(\tilde{B})$. $\rho(B) = \rho(\tilde{B})$ if and only if $\tilde{B} = F(B \cup \{b\})$, $b \in A(B)$ and $t(B)b = \rho(B)$ and then $t(B) = t(\tilde{B})$ $\rho(B) = \rho(\tilde{B})$, $A(\tilde{B}) \subsetneq A(B)$.

(b) If $\{B^k\}$, $k = 1 \ldots N$, is quasi-efficient and $N > 1$, then $B^1 \neq B^N$. In particular the length of all quasi-efficient sequences is uniformly bounded.

Pf (a) If $\tilde{B} = G(B)$ then $\rho(\tilde{B}) > \rho(B)$ by T5.1. If $\tilde{B} = F(B \cup \{b\})$ for some $b \in A(B)$ then $\rho(\tilde{B}) \geq \rho(B)$ by L6.1. The remainder of the assertion follows from L6.2. (b) is an immediate consequence of (a).

¤

L6.3 Let $B \in \mathcal{B}$, $b_i \in \bar{B}$, $i = 1,2$. Set $B_i = F(B \cup \{b_i\})$ and suppose $t(B)b_i \geq \rho(B)$, $i = 1,2$ and $\rho(B_1) \leq \rho(B_2)$. Then either $t(B_1)b_2 \geq \rho(B_1)$ or $0(b_2) \in 0(B)$.

Pf Set $\hat{B} = F(B \cup \{b_1,b_2\})$ and suppose

(6.4) $$t(B_1)b_2 < \rho(B_1).$$

Then it must be shown that

(6.5) $$0(b_2) \in 0(B).$$

Several cases need be distinguished.

Case 1

Suppose $t(B_2)b_1 < \rho(B_2)$. Then, following the notation of D5.3, $B_1 = \hat{B}^{\mu}$, $B_2 = \hat{B}^{\lambda}$ for some λ, μ. By L6.2, $t(B)b_1 \geq \rho(B_1)$ implies $b_1 \in B_1$ and $t(B_2)b_1 < \rho(B_2)$ implies $b_1 \notin B_2$. Then by T5.1, $\rho(B_1) \neq \rho(B_2)$. Then, by T5.1 again, $\lambda < \mu$ and $0(B_1) = 0(B) \cup 0(b_1) \underset{\neq}{\supset} 0(B_2) = 0(B) \cup 0(b_2)$. Hence $0(b_2) \in 0(B)$ proving (6.5).

Case 2

Suppose $t(B_2)b_1 = \rho(B_2)$. In this case $B_2 \cup \{b_1\} = F(\hat{B})$ and, by a similar argument to the one above, $\rho(B_2) < \rho(B_1)$. But by assumption $\rho(B_1) \leq \rho(B_2)$, so that this case cannot occur.

Case 3

Suppose $t(B_2)b_1 > \rho(B_2)$. By L6.2 it follows that $\rho(B_2) < \rho(\hat{B})$, and as in the above $\rho(\hat{B}) \leq \rho(B_1)$. Hence $\rho(B_2) < \rho(B_1)$ and this case cannot occur either. ¤

L6.4 Let $B \in \mathcal{E}$ and let $A \subset \bar{B}$ be such that $t(B)b \geq \rho(B)$ for $b \in A$. Then the sequence B, $\hat{B}$ of length two is quasi-efficient where $\hat{B} = F(B \cup A)$.

Pf The result is proved by induction on $k = |O(\hat{B})| - |O(B)|$.

Suppose first $k = 0$. Then $O(A) \subset O(B)$. Let

(6.5) $\{b_1,\ldots,b_\ell\} = \{b \in A | t(B)b = \rho(B)\}$

and put $B_1 = F(B \cup \{b_1\})$, $B_2 = F(B_1 \cup \{b_2\}),\ldots,B_\ell = F(B_{\ell-1} \cup \{b_\ell\})$. Then $B \to B_1 \to B_2 \to \ldots \to B_\ell$ and so the sequence B_1, B_ℓ is quasi-efficient. Also $B_\ell \neq \hat{B}$ if and only if there is $b \in \hat{B}$ such that $t(B_\ell)b > \rho(B_\ell)$. Define $B_{\ell+1} = F(B_\ell \cup \{b\})$. Then $B_\ell \to B_{\ell+1}$ and by L6.2 $\rho(B_{\ell+1}) > \rho(B_\ell)$. Furthermore $O(B_{\ell+1}) = O(\hat{B})$. Define $A_1 = \{b \in \hat{B} | b \notin B_{\ell+1}, t(B_{\ell+1})b \geq \rho(B_{\ell+1})\}$. If $A_1 = \phi$ then $B_{\ell+1} = \hat{B}$. Otherwise repeat the process above starting with $B_{\ell+1}$ in place of B_ℓ. Eventually this process must stop at some stage m with $A_m = \phi$ and $B_{\ell+m} = \hat{B}$. Hence B, $\hat{B}$ is quasi-efficient.

Suppose the assertion is true for $k = |O(\hat{B})| - |O(B)|$, and consider the case for k+1. Set $A_0 = \{b \in A | O(b) \notin O(B)\}$ and let $b_1 \in A_0$ be such that $\rho(B \cup \{b_1\}) = \min\{\rho(B \cup \{b\} | b \in A_0\}$, and let $B_1 = F(B \cup \{b_1\})$. By L6.3 if $b \in \hat{B}$, $O(b) \notin O(B_1)$ then $t(B_1)b \geq \rho(B_1)$. Thus, if $A_1 = \{b \in \hat{B} | t(B_1)b \geq \rho(B_1)\}$, then $O(B_1) \cup O(A_1) = O(\hat{B})$ and by the induction hypothesis the sequence B_1, $F(B_1 \cup A_1)$ is quasi-efficient. But then B, $F(B_1 \cup A_1)$ and quasi-efficient. It only remains to show that $F(B_1 \cup A_1)$, $\hat{B}$ is quasi-efficient. But this follows readily using T5.1 and the fact that $O(F(B_1 \cup A_1)) = O(\hat{B})$.

¤

T6.1 An efficient sequence is quasi-efficient.

Pf Suppose $\{B^k, t^k, \rho^k\}$, $k = 1\ldots N$, is an efficient pattern. Let $A = \{b \in B^{k-1} | b \notin B^k\}$. By L4.1 $t^k b = t(B^k)b \geq \rho(B^k)$. Hence $B^{k-1} = F(B^k \cup A)$. By L6.4 the sequence B^k, B^{k-1} is quasi-efficient.

¤

Using T5.1 it is readily seen that

$$G(B) = F(\{b \in B | \mathcal{O}(b) = j \text{ and } t_j(b) \neq \bar{t}_j\})$$

Given $B \subset \bar{B}$ it is an easy matter to describe a finite computer algorithm to obtain $F(B)$. Using D6.1, D6.2 it is then possible to enumerate in a finite number of steps the set of all quasi-efficient sequences. By T6.1 this includes all the efficient sequences. However if there are many substitute techniques it may not be an easy matter to use (4.16), (4.17) to eliminate the sequences which are not efficient from the quasi-efficient ones.

<u>C6.2</u> The canonical sequence is quasi-efficient.

<u>Pf</u> Follows from T5.2(b) and T6.1. ¤

7. APPLICATION TO THE CASE WITHOUT SUBSTITUTE TECHNIQUES

Suppose $\bar{B}$ has no substitute techniques and let $\{\bar{B}^{\mu}\}$, $\mu = 1...m$ be the canonical sequence. Then by T5.2(a) $\bar{B}^{\mu} \subsetneq \bar{B}^{\mu+1}$ for each μ. Hence

$$G(\bar{B}) = G(F(\bar{B}))$$

It is then easy to see that the canonical sequence is the only maximal sequence. Furthermore the length of the sequence $m \leq n$ where n is the number of commodities.

<u>T7.1</u> Suppose $\bar{B}$ has no substitute techniques. Then the CBD price vector p^0 uniquely determines the land-use pattern.

<u>Pf</u> Let $\{\bar{B}^{\mu}, \bar{t}^{\mu}, \bar{\rho}^{\mu}\}$, $\mu = 1...m$, be the canonical pattern. Using the notation of D4.1, and the unique final price vector p^N given by C4.2

$$p^0 - p^N = \sum_{\ell=1}^{m} (u_\ell - u_{\ell-1})\bar{t}^{\ell} \tag{7.1}$$

But using T5.1 it is easy to see that the vectors $\bar{t}^1, \ldots, \bar{t}^m$ are linearly independent, so that (7.1) uniquely determines the size of the rings $(u_\ell - u_{\ell-1})$, and hence the land-use pattern. ¤

Of course the solutions $(u_\ell - u_{\ell-1})$ of (7.1) are non-negative only if $p^0 \in p^N + \{\sum_{\ell=1}^{m} \alpha_\ell \bar{t}^{\ell} | \alpha_\ell \geq 0\}$. If p^0 does not belong to this set then there is no corresponding equilibrium allocation which does not import commodities from the CBD.

8. EXAMPLES

These numerical examples are given to illustrate the arguments given above. Throughout $r_A = 0$. Hence the final price vector $p^N = 0$. In the first example, $n = 3$ and there are four techniques, b^1, $\hat{b}^1$, b^2, b^3 with b^1, $\hat{b}^1$ substitute techniques for producing 1, b^2 and b^3 are the only techniques for producing goods 2 and 3 respectively.

$$b^1 = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}, \quad \hat{b}^1 = \begin{bmatrix} 5 \\ -1 \\ -3 \end{bmatrix}, \quad b^2 = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}, \quad b^3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

The transport cost vector is $\bar{t} = [4,3.2,1]$. By T5.3 the final two elements of every efficient pattern are uniquely determined. These are $\{B^4,t^4,\rho^4\}$, $\{B^3,t^3,\rho^3\}$ where

$$B^4 = \{b^1,b^2,b^3\}, \quad t^4 = [1,1,1], \quad \rho^4 = 2$$

$$B^3 = \{b^1,b^2\}, \quad t^3 = [3.2,3.2,1], \quad \rho^3 = 8.6$$

B^3 cannot belong to the canonical sequence since $t^3 \cdot \hat{b}^1 = 9.8 > \rho^3$. There are exactly two subsets B^{2i}, $i = 1,2$ such that $B^2 \to B^{2i}$. These are

$$B^{21} = \{b^1\} = G(B^2) \text{ and } B^{22} = \{\hat{b}^1,b^2\} = F(B^2 \cup \{\hat{b}^1\}).$$

It is immediate that if $B^{21} \to B$ or if $B^{22} \to B$ then

$$B = B^1 = \{\hat{b}^1\}$$

Thus all quasi-efficient sequences are subsequences of the two sequences

(8.1) $\quad B^1,B^{21},B^3,B^4$ and B^1,B^{22},B^3,B^4.

The following points are worth noting:

(i) Both sequences in (8.1) are maximal and efficient.

(ii) The canonical sequence which is easily seen to be the sequence B^1,B^{22},B^4 is quasi-efficient but it is not efficient.

(iii) According to the non-substitution theorem

(Nikaido [1968], p. 189) the techniques in $B^4 = \{b^1, b^2, b^3\}$ "dominate" the other techniques. However when production requires space, there are optimal patterns in which the "dominated" technique $\hat{b}^1$ operates at a positive level.

In the second example the numerical values are

$$b^1 = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}, \quad b^2 = \begin{bmatrix} -2 \\ 5 \\ -2 \end{bmatrix}, \quad \hat{b}^2 = \begin{bmatrix} -1 \\ 4.5 \\ -3 \end{bmatrix}, \quad b^3 = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

and $\bar{t} = [5.3, 5.2, 1]$. It turns out that there is only one maximal sequence of length five $B^1, B^2, \ldots, B^5$ where $B^5 = \{b^1, b^2, b^3\}$, $B^4 = \{b^1, b^2\}$, $B^3 = \{b^1\}$, $B^2 = \{b^1, \hat{b}^2\}$, $B^1 = \{\hat{b}^2\}$. The canonical sequence on the other hand is B^1, B^2, B^5. The point to note here is that even when there is exactly one maximal sequence, T7.1 cannot be generalized to allow for substitute techniques.

In the third and final example the data is

$$b^1 = \begin{bmatrix} 8 \\ -5 \\ -1 \end{bmatrix}, \quad \hat{b}^1 = \begin{bmatrix} 6 \\ -2 \\ -2.8 \end{bmatrix}, \quad b^2 = \begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix}, \quad b^3 = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

and $\bar{t} = [4, 2.5, 1]$. Again there is only one maximal sequence $B^1, .., B^5$ where $B^5 = \{b_1, b_2, b_3\}$, $B^4 = \{b^1, b^2\}$, $B^3 = \{\hat{b}^1, b^2\}$, $B^2 = \{\hat{b}^1\}$, $B^1 = \{b^1\}$. This sequence is also efficient. The crucial point to note is that the technique b^1 is most profitable in the first ring, and in the fourth and fifth ring but it is <u>not</u> most profitable in the inner second and third rings. This is a phenomenon somewhat akin to "double switching" (Burmeister and Dobbell [1970], p. 245). More importantly, if capital is measured in such a way that it is determined in terms of the production technique used, then a high (or low) capital intensive

technique can be most profitable at small and large distances from the CBD but not at intermediate distances. This is a surprising result in light of the usual models employed in von Thünen models in which capital/land ratios decline with distance from the CBD.

APPENDIX

1. <u>Proof of T3.1</u> The necessity of these optimality conditions follows by applying known results e.g. Hestenes [1966, p. 354], so that only sufficiency needs to be proved. Let $\omega = (\bar{u}, x(\cdot), f(\cdot), \phi(\cdot))$ be any other allocation and let $s(u) \geq 0$, $\sigma(u) \geq 0$ be the solutions of (3.1), (3.2). If $\bar{u} > \bar{u}^*$, let $p^*(u) = p^*(\bar{u}^*)$ for $u \in [\bar{u}^*, \bar{u}]$. It will be shown that

(A.1) $$C(\omega^*) \leq C(\omega).$$

From (3.5) it follows that

$$\dot{p}^*(u) + \bar{t} \geq 0 \ , \quad [\dot{p}^*(u) + \bar{t}]s^*(u) = 0,$$

$$-\dot{p}^*(u) + \bar{t} \geq 0 \ , \quad [-\dot{p}^*(u) + \bar{t}]\sigma^*(u) = 0.$$

So that, since $s(u) \geq 0$ and $\sigma(u) \geq 0$, therefore,

$$0 = [\dot{p}^*(u) + \bar{t}]s^*(u) + [-\dot{p}^*(u) + \bar{t}]\sigma^*(u)$$

$$\leq [\dot{p}^*(u) + \bar{t}]s(u) + [-\dot{p}^*(u) + \bar{t}]\sigma(u).$$

Integrating this inequality gives

$$0 = \int_{u_0}^{\bar{u}^*} p^*(u) \ [\overset{*}{s}(u) - \sigma^*(u)]du + \int_{u_0}^{\bar{u}^*} \bar{t}[\overset{*}{s}(u) + \sigma^*(u)]du$$

(A.2) $$\leq \int_{u_0}^{\bar{u}} \dot{p}^*(u) \ [s(u) - \sigma(u)]du + \int_{u_0}^{\bar{u}} \bar{t}[s(u) + \sigma(u)]du.$$

Integrating by parts, and using the boundary conditions in (3.1), (3.2), (3.7) gives

$$\int_{u_0}^{\bar{u}^*} \dot{p}^*(u) \ [s^*(u) - \sigma^*(u)]du = -p^*(u_0)Q + \int_{u_0}^{\bar{u}^*} p^*(u) \ \bar{B} \ x^*(u)du,$$

$$\int_{u_0}^{\bar{u}} \dot{p}^*(u) \ [s(u) - \sigma(u)]du = -\overset{*}{p}(u_0)Q - \overset{*}{p}(\bar{u})\sigma(\bar{u})$$

$$+ \int_{u_0}^{\bar{u}} \dot{p}^*(u) \ \bar{B} \ x \ (u)du.$$

Substituting these relations into (A.2) yields

$$\text{(A.3)} \qquad \int_{u_0}^{\bar{u}^*} \bar{t}[s^*(u) + \sigma^*(u)]du + \int_{u_0}^{\bar{u}^*} p^*(u)\ \bar{B}\ x^*(u) \leq \int_{u_0}^{\bar{u}} \bar{t}[s(u) + \sigma(u)]du$$

$$+ \int_{u_0}^{\bar{u}} p^*(u)\ \bar{B}\ x(u)du - p^*(\bar{u})\sigma(\bar{u}).$$

Now, from (3.3), for $u_0 \leq u \leq \bar{u}^*$

$$\text{(A.4)} \qquad [p^*(u)\bar{B} - r_A\underset{\sim}{1}]x^*(u) = \text{Max}\{[p^*(u)\bar{B} - r_A\underset{\sim}{1}]x \mid x \geq 0,\ \underset{\sim}{1}\ x \leq \theta(u)\} \geq 0.$$

On the other hand if $\bar{u}^* < \bar{u}$ then since $p^*(u) = p^*(\bar{u}^*)$ for $u \in [\bar{u},\bar{u}^*]$ and because of (3.6), the equality holds in (A.4) for $u \in [\bar{u},\bar{u}^*]$. It follows that whether $\bar{u}^*$ is larger or smaller than $\bar{u}$,

$$\text{(A.5)} \qquad \int_{u_0}^{\bar{u}^*} [p^*(u)\bar{B} - r_A\underset{\sim}{1}]x^*(u)du \geq \int_{u_0}^{\bar{u}} [p^*(u)\bar{B} - r_A\underset{\sim}{1}]x(u)du$$

and subtracting this inequality from (A.3) leads to

$$\int_{u_0}^{\bar{u}^*} \bar{t}[s^*(u) + \sigma^*(u)]du + \int_{u_0}^{\bar{u}^*} r_A\underset{\sim}{1}\ x^*(u)du \leq \int_{u_0}^{\bar{u}} t[s(u) + \sigma(u)]du$$

$$+ \int_{u_0}^{\bar{u}} r_A\underset{\sim}{1}\ x(u)du - p^*(\bar{u})\sigma(\bar{u}),$$

or

$$C(\omega^*) \leq C(\omega) - p^*(\bar{u})\sigma(\bar{u}).$$

Since $p^*(\bar{u}) \geq 0$ and $\sigma(\bar{u}) \geq 0$, this implies (A.1). ¤

2. <u>Proof of C2.2</u> Let $\omega = (\bar{u},x(\cdot),f(\cdot),\phi(\cdot))$ be another optimal allocation. Then $C(\omega^*) = C(\omega)$. Hence equality must hold in (A.5) and so it

must be the case that

$$[p^*(u)\bar{B} - r_A \underset{\sim}{1}]x(u) = \mathrm{Max}\{[p^*(u)\bar{B} - r_A \underset{\sim}{1}]x \mid x \geq 0,\ \underset{\sim}{1}\ x \leq \theta(u)\}$$

so that ω is sustained by p^*, r^*.

¤

3. <u>Proof of T5.2</u> The proof of part (a) is essentially the same as Step 2 in the proof of T5.1. Part (b) is proved by backward induction on μ. Assume that the subsequence $\bar{\bar{B}}^{k+1},\ldots,\bar{\bar{B}}^{m}$ has been constructed such that

$$\bar{\bar{B}}^{\ell} \subset \bar{B}^{\ell},\ 0(\bar{\bar{B}}^{\ell}) = 0(\bar{B}^{\ell}) \text{ for } \ell = \mu+1,\ldots,m$$

and such that it is a subsequence of an efficient sequence B^{ν}, $\nu = 1\ldots N$. Let $\{B^{\nu},t^{\nu},\rho^{\nu}\}$, $\nu = 1\ldots N$ be the associated efficient pattern and $u_0 < u_1 < \ldots < u_N$, $p^0,\ldots,p^N$, $r^0,\ldots,r^N$ the associated distances, prices and rents (see D4.1). Suppose that $\bar{\bar{B}}^{\mu+1} = B^{\nu}$ for some ν. It can be assumed that

$$p^{\nu}b \leq r^{\nu} \quad \text{for} \quad b \in \bar{B},$$
$$p^{\nu}b = r^{\nu} \Leftrightarrow b \in \bar{\bar{B}}^{\mu+1}$$

Set $B^{\nu,1} = G(B^{\nu}) = G(\bar{\bar{B}}^{\mu+1})$. Then by (5.6) $0(B^{\nu,1}) = 0(\bar{B}^{\mu})$. Let $t^{\nu,1} = t(B^{\nu,1})$, $\rho^{\nu,1} = \rho(B^{\nu,1})$ and suppose $b \in \bar{B}$ is such that $t^{\nu,1}b > \rho^{\nu,1}$. Then by (4.24) and part (a) above, $b \in \bar{B} - B^{\nu,1}$ and $0(b) \in 0(\bar{B}^{\mu})$. Such b exists if and only if $B^{\nu,1} \not\subset \bar{B}^{\mu}$, in which case choose the b such that

$$(p^{\nu} + \alpha t^{\nu})b = r^{\nu} + \rho^{\nu}\alpha$$

for the smallest $\alpha > 0$. Set $B^{\nu,2} = F(B^{\nu,1} \cup \{b\})$. Then again $0(B^{\nu,2}) = 0(\bar{B}^{\mu})$, and either $B^{\nu,2} \subset \bar{B}^{\mu}$ or the procedure can be repeated. Evidently at some step k it must be the case that

$0(B^{\nu,k}) = 0(\bar{B}^{\mu})$ and $B^{\nu,k} \subset \bar{B}^{\mu}$. Setting $\bar{\bar{B}}^{\mu} = B^{\nu,k}$ proves the result for μ and part (b) now follows by induction.

¤

4. <u>Proof of T5.3</u> This follows readily from the proof of T5.2.

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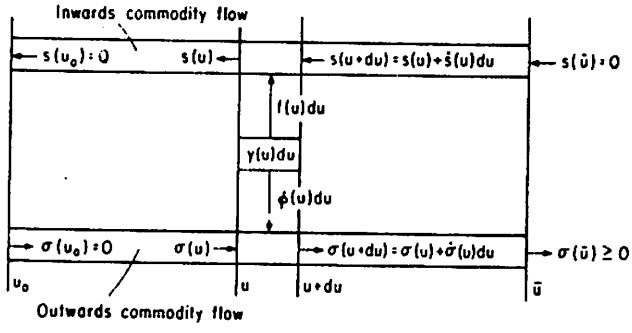

Fig. 3.1. Commodity flows in an allocation.