ON RANDOM SETS AND BELIEF FUNCTIONS

by

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I. Introduction

The mathematical theory of evidence, as developed by G. Shafer ([1],[2]), is based, in the main, upon the notion of lower-probability measures in the work of A. Dempster on statistical inference (e.g.[3]). Such set-functions have been employed in many different fields such as theory of capacities (G. Choquet,[4]), stochastic geometry (G. Kendall[5], G. Matheron[6]), random fields (F. Spitzer[7]), and set-valued Markov processes (T.E. Harris[8]).

This paper deals with a closer relationship between Dempster's scheme of multi-valued mappings and Shafer's belief functions. The basic probability assignment is regarded as the probability distribution of a random set, the notion of condensability is expressed in terms of a multi-valued mapping and is related to a general notion of regularity of probability measures. These points of view are useful for applying the notion of belief to fuzzy analysis where multivalued mappings are replaced by fuzzy mappings, and propositions are of the form "X is A" where A is the label of some fuzzy set [9] of a universe of discourse, possibly a continuum.

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2. Measurability of multi-valued mappings

Let $(X, \mathcal{A})$, $(S, \mathcal{B})$, $(\mathcal{P}(S), \mathcal{B})$ be three measurable spaces, where $\mathcal{P}(s)$ denotes the collection of all subsets of the set S.

Consider a multi-valued mapping:

$$\Gamma : X \to \mathcal{P}(S).$$

We shall formulate two notions of measurability for Γ: the first one is needed for defining lower (and upper) probability measure the second one for considering random sets. Note that these notions of measurability have been investigated, for example, by G. Debreu [10] in a topological setting.

First, consider two inverses of Γ

a) Lower-inverse:

$$\Gamma_* : \mathcal{P}(S) \to \mathcal{P}(X)$$

$$T \in \mathcal{P}(S),\ \Gamma_*(T) = T_* = \{x \in X : \Gamma_x \neq \phi,\ \Gamma_x \subset T\}$$

b) Upper-inverse

$$\Gamma^* : \mathcal{P}(S) \to \mathcal{P}(X)$$

$$T \in \mathcal{P}(S),\ \Gamma^*(T) = T^* = \{x \in X : \Gamma_x \cap T \neq \phi\}.$$

<u>Remark</u>: The names of these inverses of Γ are given in the way that is related to lower and upper probability measures. The lower-inverse [resp. upper-inverse] is called upper-inverse [resp. lower-inverse] by Cl. Berge [11], and strong inverse [resp. weak inverse] by G. Debreu [10].

<u>Definition 2.1</u>

The multi-valued mapping Γ is said to be strongly measurable, with respect to $\mathcal{A}$ and $\mathcal{B}$, iff:

$\forall B \in \mathcal{B}$, $\Gamma^*(B) \in \mathcal{A}$.

Example:

Let X be a topological space, $\mathcal{A}$ its Borel σ-field; S a finite set with its discrete topology. If Γ is lower-semi continuous on X (i.e. for each $x_0 \in X$, for any V open in S such that $V \cap \Gamma_{x_0} \neq \phi$, there exists a neighborhood U of X_0 such that: $x \in U \Rightarrow V \cap \Gamma_x \neq \phi$), then Γ is strongly measurable, with respect to $\mathcal{A}$ and $\mathcal{P}(S)$, since $\forall A \subset S$ $\Gamma^*(A)$ is open in X.

Now consider Γ as a point-to-point mapping, from X to $\mathcal{P}(S)$, where "points" in $\mathcal{P}(S)$ are in fact subsets of S. The collection of all subsets of $\mathcal{P}(S)$ is denoted by $\mathcal{P}\mathcal{P}(S)$.

Let Γ^{-1} the inverse mapping of Γ, i.e.

$\Gamma^{-1} : \mathcal{P}\mathcal{P}(S) \to \mathcal{P}(X)$

$\hat{T} \in \mathcal{P}\mathcal{P}(S)$, $\Gamma^{-1}(\hat{T}) = \{x \in X : \Gamma_x \in \hat{T}\}$

If $\hat{\mathcal{B}}$ is a σ-field on $\mathcal{P}(S)$, then as usual, Γ is said to be measurable, with respect to $\mathcal{A}$ and $\hat{\mathcal{B}}$, iff:

$\forall \hat{T} \in \hat{\mathcal{B}}$, $\Gamma^{-1}(\hat{T}) \in \mathcal{A}$.

Remark:

Let $\mathcal{J}$ be the class of all finite subsets of the set S. For $I \in \mathcal{J}$, let π_I be the projection from $\mathcal{P}(S)$ to $\mathcal{P}(I)$, i.e.

$A \in \mathcal{P}(S)$, $\pi_I(A) = A \cap I$.

A finite dimensional cylinder set in $\mathcal{P}(S)$ is a subset $\hat{A}$ of $\mathcal{P}(S)$ of the form:

$\hat{A} = \pi_I^{-1}(A)$, where $I \in \mathcal{J}$, and $A \subset \mathcal{P}(I)$.

In particular, if $A = \{I_1\}$, $I_1 \subset I$, then:

$$\hat{A} = \{B \subset S : B \supset I_1, B' \supset I - I_1\}$$

Note that if $I_1, I_2 \in \mathcal{J}$ and $I_1 \cap I_2 = \phi$, then:

$$\pi^{-1}_{I_1 \cup I_2}(I_1) = \{B \subset S : B \supset I_1, B' \supset I_2\}.$$

Let $\mathcal{C}$ denote the class of all finite dimensional cylinder sets in $\mathcal{P}(S)$, and $\mathcal{F} = \sigma(\mathcal{C})$, the σ-field of $\mathcal{P}(S)$, generated by $\mathcal{C}$. It is clear that if Γ is strongly measurable (with respect to $\mathcal{A}$ and $\mathcal{B}$) and if $\mathcal{J} \subset \mathcal{B}$, then Γ is measurable (with respect to $\mathcal{A}$ and $\mathcal{F}$).

3. Lower-probability measure and belief functions.

3.1 A source is a probability space $(X, \mathcal{A}, \underline{P})$ and a multi-valued mapping $\Gamma : X \to \mathcal{P}(S)$. For simplicity, we assume that $S^* \in \mathcal{A}$ and $\underline{P}(S^*) = 1$. Let $\mathcal{B}$ be a σ-field on S, we assume that Γ is strongly measurable (with respect to $\mathcal{A}$ and $\mathcal{B}$), and in addition:

If $T \in \mathcal{B}$, then $\tilde{T} = \{x \in X : \Gamma x \supset T\} \in \mathcal{A}$.

The lower, and upper probability measures $\underline{P}_*$, $\underline{P}^*$ are defined on respectively by:

$$\underline{P}_*(B) = \underline{P}(B_*)$$

$$\underline{P}^*(B) = \underline{P}(B^*).$$

Note that $\underline{P}^*(B) = 1 - \underline{P}_*(B')$.

A. Dempster considered also the set-function:

$$Q(B) = \underline{P}(\tilde{B}).$$

Remark: In the study of random fields [7] and set-valued Markov processes [8], the set-functions Q and $\underline{P}^*$, in the case where Γ is regarded as a random set, are called correlation function and incidence function respectively.

Let f be a set-function: $\mathcal{B} \to \mathbb{R}$. Two types of successive differences of f(B), $B \in \mathcal{B}$, with respect to parameters $B_i \in \mathcal{B}$, $i = 1,\ldots,n+1$, are defined as follows:

(i) $\nabla_1(B;B_1)_f = f(B) - f(B \cup B_1)$

$$\nabla_{n+1}(B;B_1,\ldots,B_{n+1})_f = \nabla_n(B:B_1,\ldots,B_n)_f - \nabla_n(B \cup B_{n+1};\ B_1,\ldots,B_n)_f$$

(ii) $\Delta_1(B:B_1)_f = f(B) - f(B \cap B_1)$

$$\Delta_{n+1}(B;B_1,\ldots,B_{n+1})_f = \Delta_n(B;B_1,\ldots,B_n)_f - \Delta_n(B \cap B_{n+1};\ B_1,\ldots,B_n)_f$$

Following Choquet [4], we say that

a) f is alternating of infinite order of if $\nabla_n \leq 0$ for all n

b) f is monotone of infinite order if $\Delta_n \geq 0$ for all n.

Properties of $\underline{P}_*$ and $\underline{P}^*$ can be summarized as follows:

<u>Proposition 3.1.1</u>

(i) $\underline{P}_*(\phi) = 0$, $\underline{P}_*(S) = 1$

(ii) $\underline{P}_*$ is monotone of infinte order.

(iii) If $B_n \in \mathcal{B}$ is a decreasing sequence, then:

$$\underline{P}_*(B_n) \downarrow \underline{P}_*(\bigcap_n A_n).$$

In a dual way:

<u>Proposition 3.1.2</u>

(i) $\underline{P}^*(\phi) = 0$, $\underline{P}^*(S) = 1$

(ii) $\underline{P}^*$ is alternating of infinte order.

(iii) If $B_n \in \mathcal{B}$ is an increasing sequences, then:

$$\underline{P}^*(B_n) \uparrow \underline{P}^*(\bigcup_n B_n).$$

These facts can be seen from the definition of $\underline{P}_*$ and $\underline{P}^*$ in terms of $\underline{P}$, and the fact that:

$$\Gamma_*(\bigcap_i B_i) = \bigcap_i \Gamma_*(B_i)$$

$$\Gamma^*(\bigcup_i B_i) = \bigcup_i \Gamma^*(B_i).$$

Remarks:

a) We have only $\Gamma_*(\bigcup_i B_i) \supseteq \bigcup_i \Gamma_*(B_i)$.

b) In particular, the lower-probability measure $\underline{P}_*$ [Resp. $\underline{P}^*$] is strongly super-additive [resp. strongly sub-additive].

c) For the time being, no topological notions are considered. For further application to fuzzy analysis, where $S = [0,1]$ or some compact set of the real line, the topology will play an important role. Let us point out a result in [6] (Choquet's theorem) concerning a functional associated with a random closed set: [this functional plays the role of probability distribution function of a real random variable]: If S is a locally compact space, the space F of closed subsets of S is topologized in some suitable way, σ_F denotes its Borel σ-field, and T is a set-function defined on the space $\mathcal{K}$ of compact sets of S, then the following are equivalent:

(i) T is an alternating Choquet Capacity of infinite order such that T takes values in $[0,1]$ and $T(\phi) = 0$.

(ii) There exists a unique probability measure $\hat{\underline{P}}$ on σ_F such that

$$T(K) = \hat{\underline{P}}[\{A \in F : A \cap K \neq \phi\}], \ \forall K \in \mathcal{K}.$$

3.2. We recall here the notion of belief function on a finite set S.

A belief function Bel on S is a set-function from $\mathcal{P}(S)$ to $[0,1]$ such that:

(i) Bel $(\phi) = 0$

(ii) Bel $(S) = 1$

(iii) For any k, $\text{Bel}\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\substack{I \neq \phi \\ I \subset \{1,\ldots,k\}}} (-1)^{|I|+1} \text{Bel}\left(\bigcap_{i \in I} A_i\right)$

where $|I|$ denotes the number of elements in I.

Note that a belief function Bel is increasing and there exists a set-function:

$m : \mathcal{P}(S) \to [0,1]$ such that:

a) $m(\phi) = 0$

b) $\sum_{A \in \mathcal{P}(S)} (A) = 1$

c) $\text{Bel}(A) = \sum_{B \subseteq A} m(B)$.

m is called the basic probability assignment [2], and

$$m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} \text{Bel}(B).$$

Note also that (iii) is equivalent to the non-negativity of m.

Remark: The representation problem of belief functions in terms of measure algebra and allocation of probability has been fully discussed in [1].

4. Random sets and belief functions

Consider a source $(X, \mathcal{A}, \underline{P})$, $\Gamma : X \to \mathcal{P}(S)$.

Let $\mathcal{B}$ be a σ-field on S. We assume that Γ is strongly measurable. (with respect to $\mathcal{A}$ and $\mathcal{B}$).

Proposition 4.1 The lower-probability measure $\underline{P}_*$ on $\mathcal{B}$ is deduced from the probability distribution of Γ considered as a random set.

Proof: Let $\hat{\mathcal{B}}$ be the σ-field on $\mathcal{P}(S)$ defined by:

$$\hat{T} \in \hat{\mathcal{B}} \Leftrightarrow \Gamma^{-1}(\hat{T}) \in \mathcal{A}.$$

Thus, with respect to $\mathcal{A}$ and $\hat{\mathcal{B}}$, Γ is a measurable mapping. We say that Γ is a rondom set by specifying its probability distribution $\hat{\underline{P}}$ on $\mathcal{B}$:

$$\hat{T} \in \mathcal{B}, \ \hat{\underline{P}}(\hat{T}) = \underline{P}[\Gamma^{-1}(\hat{T})]$$

If $A \in \mathcal{P}(S)$, denote by $I(A)$ the principal ideal generated by A, i.e., $I(A) = \{B \subset S : B \subset A\}$, then $\forall B \in \mathcal{B}$, $I(B) \in \mathcal{B}$. Indeed: $\Gamma^{-1}(I(B)) = B_* \in \mathcal{A}$ by strong measurability of Γ. It follows that: $\hat{\underline{P}}[I(B)] = \underline{P}_*(B)$, $\forall B \in \mathcal{B}$.

Proposition 4.2. In the finite case, the probability distribution of the random set Γ is precisely the basic probability assignment.

Proof: Since S is finite, and we assume that $\tilde{A} \in \mathcal{A}$ for all $A \subseteq S$, it is clear that $\hat{\mathcal{B}} = \mathcal{P}\mathcal{P}(S)$.

On the other hand, since:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \underline{P}_*(B)$$

$$\Rightarrow P_*(B) = \sum_{B \subset A} m(B) = \sum_{B \in I(A)} m(B) = \hat{m}[I(A)]$$

where $\hat{m}$ is the probability measure on $\mathcal{P}\mathcal{P}(S)$ with density m. But

$$\hat{\underline{P}}[I(A)] = \underline{P}_*(A) \Rightarrow \hat{\underline{P}}(\{A\}) = \sum_{B \subset A} (-1)^{|A|-|B|} \underline{P}_*(B)$$

$$= \hat{m}(\{A\}).$$

Remarks:

(i) For $A \subset S$, let $F(A)$ be the principal filter generated by A, then: $\forall A \subset S$ (or more generally, $A \in \mathcal{B}$, in the infinite case)

$\tilde{A} \in \mathcal{A} \Leftrightarrow F(A) \in \mathcal{B}$.

(ii) Let X be a topological space, $\mathcal{A}$ its Borel σ-field. S be a be a Hausdorff, locally compact space, $\mathcal{B}$ its Borel σ-field.

F, $\mathcal{G}$, $\mathcal{K}$ denote respectively the collection of all closed, open, compact subsets of S. As a topological space [where the topology is generated by $\{F^K,\ K \in \mathcal{K}\}$ and $\{F_G,\ G \in \mathcal{G}\}$, with:

$$F^K = \{A \in F : A \cap K = \phi\}$$

$$F_G = \{A \in F : A \cap G \neq \phi\}$$

the space F is a Hausdorff, compact space. If $\Gamma : X \to \mathcal{F}$ is continuous, then:

$$\forall B \in \mathcal{B},\ \{x \in X : \Gamma_x = B\} \in \mathcal{A}.$$

Note that if A_* and $\tilde{A} \in \mathcal{A}$ then

$$A_* \cap \tilde{A} = \{n : \Gamma_x = A\} \in \mathcal{A}.$$

(iii) In this finite case, the existence of the biunivocal correspondence between belief functions on S and probability distributions of random sets is established by using the fact that to construct $\underline{\hat{P}}$, it is sufficient to construct its density on $\mathcal{P}(S)$, on one hand; and on the other hand, given a set function v (belief function) on $\mathcal{P}(S)$, we define $\underline{\hat{P}}[I(A)] = v(A)$, and we are in conditions of application of the Mobius inversion theorem [12] to obtain $\underline{\hat{P}}(\{A\})$ via the Mobius function:

$$\mu(A,B) = (-1)^{|A|-|B|},\ A \subset B.$$

(iv) If $\mathcal{R}(\Gamma)$ denotes the range of Γ, it is sufficient to consider $I(A) = \{B \in \mathcal{R}(\Gamma) : B \subset A\}$.

<u>Example</u> Let $E = \{A_t,\ t \in [0,1]\}$ be a family of subsets of S such that:

a) $A_0 = S$

b) $A_1 = \phi$

c) $s \leq t \Leftrightarrow A_s \supseteq A_t$.

Let $\mathcal{E}$ be the σ-field on E defined as follows:

$\hat{T} \in \mathcal{E} \Leftrightarrow \hat{T} = \{A_t\}_{t \in T}$, where $T \in \mathcal{B}_1$ the Borel σ-field of the unit inverval [0,1].

Let Γ be a random set taking values in (E, $\mathcal{E}$) with probability distribution $\hat{\underline{P}}$:

$$\hat{\underline{P}}[\Gamma \in \hat{T}] = \Lambda(T)$$

where Λ is the Lebesque measure on [0,1].

Let $I(A_t) = \{A_s : A_s \subseteq A_t\}$

Then $I(A_t) \in \mathcal{E}$ for all $t \in [0,1]$, since

$$I(A_t) = \{A_s\}_{s \in [t,1]}$$

Define a belief function v on E by:

$$v(A_t) = \hat{\underline{P}}[\Gamma \in I(A_t)] = 1-t$$

5. Regularity and Condensability.

In this paragraph, given a scheme $(X, \mathcal{A}, \underline{P})$, $\Gamma : X \to \mathcal{P}(S)$, we assume that Γ is strongly measurable with respect to $\mathcal{A}$ and $\mathcal{P}(S)$. Thus, the belief function $\underline{P}_*$ and the upper probability measure $\underline{P}^*$ are defined on $\mathcal{P}(S)$. Following Shafer [1], we say that the upper probability measure $\underline{P}^*$ is condensable iff $\underline{P}^*$ has the following approximation property:

$$\forall A \in \mathcal{P}(S),\ \underline{P}^*(A) = \sup_{B \in \mathcal{J} \cap \mathcal{P}(A)} \underline{P}^*(B) \qquad (1)$$

where $\mathcal{J}$ denotes the collection of all finite subsets of S. Recall that, if A_n is an increasing sequence in $\mathcal{P}(S)$, then:

$$\underline{P}^*(\bigcup_n A_n) = \sup_n \underline{P}^*(A_n)$$

The condensability of $\underline{P}^*$ is stronger than this sequential increasing continuity. In fact [1], $\underline{P}^*$ is condensable if and only if for any upward net A_i in $\mathcal{P}(S)$, $i \in I$, we have:

$$\underline{P}^*(\bigcup_I A_i) = \sup_I \underline{P}^*(A_i). \qquad (2)$$

The fact that (2) implies (1) can be seen as follows: let $A \in \mathcal{P}(S)$, and $T = \mathcal{J} \cap \mathcal{P}(A)$. It is obvious that T is an upward net in $\mathcal{P}(S)$, and $A = \bigcup_{I \in T} I$ thus:

$$\underline{P}^*(A) = \underline{P}^*(\bigcup_{I \in T} I) = \sup_{I \in \mathcal{J} \cap \mathcal{P}(A)} \underline{P}^* \qquad (I)$$

Recall also that the upper-inverse Γ^* of Γ maps $\mathcal{P}(S)$ into $\mathcal{A}$, since Γ is strongly measurable, and:

(i) Γ^* is increasing

(ii) $\Gamma^* (\bigcup_I A_i) = \bigcup_I \Gamma^*(A_i)$

As a consequence, if A_i is an upward net in $\mathcal{P}(S)$, then $\Gamma^*(A_i)$ is an upward net in $\mathcal{A}$.

We now proceed to give a first characterization of condensability of $\underline{P}^*$ in terms of Γ.

Let $\hat{\mathcal{A}}(\underline{P})$ be the subset of $\mathcal{P}(\mathcal{A})$ defined by:

$$\hat{A} \in \hat{\mathcal{A}}(\underline{P}) \Leftrightarrow \begin{cases} \bigcup_{A \in \hat{A}} A \in \mathcal{A} \text{ and} \\ \underline{P}[\bigcup_{A \in \hat{A}} A] = \sup_{A \in \hat{A}} \underline{P}(A) \end{cases}$$

Let $U[\mathcal{P}(S)]$ be the set of all upward nets in $\mathcal{P}(S)$. Define the mapping $\hat{\Gamma}$, from $\mathcal{P}\mathcal{P}(S)$ into $\mathcal{P}\mathcal{P}(X)$ [in fact into $\mathcal{P}\mathcal{P}(\mathcal{A})$], induced by $\hat{\Gamma}^*$, as follows:

$$\hat{\Gamma}(\hat{A}) = \{\Gamma^*(A), A \in \hat{A}\}$$

Proposition 5.1 A necessary and sufficient condition for the condensability of $\underline{P}^*$ is that $\hat{\Gamma}$ maps $U[\mathcal{P}(S)]$ into $\hat{\mathcal{A}}(\underline{P})$.

Proof Suppose that $\underline{P}^*$ is condensable. Let A_i, $i \in I$, be an upward net in $\mathcal{P}(S)$. By strong measurability of Γ, $\Gamma(\bigcup_I A_i) \in \mathcal{A}$, thus $\bigcup_I \Gamma^*(A_i) \in \mathcal{A}$. We have:

$$\underline{P}^*(\bigcup_I A_i) = \underline{P}[(\bigcup_I A_i)^*] = \underline{P}[\bigcup_I A_i^*] = \sup_I \underline{P}^*(A_i) = \sup_I \underline{P}(A_i^*)$$

Thus $\{A_i^*, i \in I\} \in \hat{\mathcal{A}}(\underline{P})$.

The sufficiency follows immediately from the definition of $\hat{\mathcal{A}}(\underline{P})$.

There is another way to study the condensability of the upper-probability measure $\underline{P}^*$, associated with the scheme $(X, \mathcal{A}, \underline{P})$, $\Gamma : X \to \mathcal{P}(S)$, uniquely in terms of the probability space $(X, \mathcal{A}, \underline{P})$ and Γ. As before the upper-inverse Γ^* will play an important role. For this purpose, we shall first introduce a general notion of regularity for probability measures (or more generally, for measures); using this notion, we shall express the condensability of $\underline{P}^*$ in terms of Γ^* as a criterion and study some consequences.

Notion of ρ-regularity

Let $(\Omega, \mathcal{A}, \underline{P})$ be a probability space.

Let $(E, \leq)$ be a partially ordered set, and $F \subset E$. Finally, let ρ be a mapping from E to $\mathcal{A}$.

Definition 5.2

We say that the probability measure $\underline{P}$ is regular with respect to the system (E, F, ρ) [or simply ρ-regular, if E and F are fixed] iff:

$$\forall x \in E,\ \underline{P}[\rho(x)] = \sup_{A \in \hat{\rho}(x)} \underline{P}(A)$$

where

$$\hat{\rho}(x) = \{\rho(y) : y \in F,\ y \leq x\}.$$

<u>Remarks</u>:

(i) Let E, F be subclasses of $\mathcal{A}$: $F \subset E \subset \mathcal{A}$; and $\rho : E \to \mathcal{A}$ the canonical injection. Then the ρ-regularity of $\underline{P}$ is the usual one, i.e.

$$\forall A \in E,\ \underline{P}(A) = \sup_{B \in F \cap \mathcal{P}(A)} \underline{P}(B)$$

Here, $\hat{\rho}(A) = F \cap \mathcal{P}(A)$.

(ii) If $\underline{P}$ is ρ-regular, and ρ increasing, then:

$$\underline{P}[\rho(x)] = \sup_{A \in \rho(F) \cap \mathcal{P}[\rho(x)]} \underline{P}(A)$$

Consider again the scheme $(X, \mathcal{A}, \underline{P})$, $\Gamma : X \to \mathcal{P}(S)$, with Γ strongly measurable. Denote by $\mathcal{J}$ the collection of all finite subsets of S. Put $\mathcal{T}^* = \Gamma^*(\mathcal{J})$ and $\mathcal{A}^* = \Gamma^*[\mathcal{P}(S)]$.
Consider the system $(\mathcal{P}(S), \mathcal{J}, \Gamma^*)$.

We say that $\underline{P}$ is Γ^*-regular if $\underline{P}$ is regular with respect to the system $(\mathcal{P}(S), \mathcal{J}, \Gamma^*)$. Then it is straightforward that:

<u>Proposition 5.3</u>

The following are equivalent

(i) $\underline{P}^*$ is condensable

(ii) $\underline{P}$ is Γ^*-regular.

<u>Proposition 5.4</u>

If $\underline{P}$ is Γ^*-regular, then:

$$\forall A \in \mathcal{A}^*,\ \underline{P}(A) = \sup_{T \in \mathcal{T}^* \cap \mathcal{P}(A)} \underline{P}(T).$$

Proof:

Let $B \in \mathcal{P}(S)$ such that $A = \Gamma^*(B)$.

We have:

$$\underline{P}(A) = \underline{P}[\Gamma^*(B)] = \sup_{T \in \hat{\Gamma}^*(B)} \underline{P}(T)$$

where $\hat{\Gamma}^*(B) = \{\Gamma^*(I),\ I \in \mathcal{J},\ I \subset B\}$.

Thus:

$$\underline{P}(A) = \sup_{\begin{cases} I \in \mathcal{J} \\ I \subset B \end{cases}} \underline{P}[\Gamma^*(I)]$$

$$\leq \sup_{\begin{cases} I \in \mathcal{J} \\ \Gamma^*(I) \subset \Gamma^*(B) \end{cases}} \underline{P}[\Gamma^*(I)] \quad \text{since } \Gamma^* \text{ is increasing.}$$

We obtain, in fact, equality since $\underline{P}$ is increasing.

Remark

If $(\Omega, \mathcal{A}, \underline{P})$ is a probability space and $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$, we say that $\underline{P}$ is (inner) regular on $\mathcal{B}$ if:

$$\forall B \in \mathcal{B},\ \underline{P}(B) = \sup_{T \in \mathcal{C} \cap \mathcal{P}(B)} \underline{P}(T)$$

More generally, let ψ a mapping from $\mathcal{B}$ into $\mathcal{P}(\mathcal{C})$ such that $\psi(B) \subset \mathcal{C} \cap \mathcal{P}(B)$ for all $B \in \mathcal{B}$. We can say that $\underline{P}$ is regular with respect to $(\mathcal{C}, \mathcal{B}, \psi)$ iff: $\forall B \in \mathcal{B},\ \underline{P}(B) = \sup_{T \in \psi(B)} \underline{P}(T)$

If the upper-inverse Γ^* is injective [11], i.e.

$$A \neq B \Rightarrow \Gamma^*(A) \cap \Gamma^*(B) = \phi$$

then $\underline{P}^*$ is condensable if and only if $\underline{P}$ is regular with respect to $(\mathcal{J}^*, \mathcal{A}^*, \psi)$ where:

$A \in \mathcal{A}^*,\ A \neq \phi$

$\psi(A) = \{\Gamma^*(I),\ I \in \mathcal{J},\ I \subset B\}$ where B is the unique element of $\mathcal{P}(S)$ such that $A = \Gamma^*(B)$.

Proposition 5.5

If for each $B \in \mathcal{P}(S)$, there exists a sequence $\{I_n\}_{n \in N}$ elements of $\mathcal{J}$ such that:

$$\Gamma^*(B) = \bigcup_n \Gamma^*(I_n)$$

then $\underline{P}$ is regular on $\mathcal{A}^*$ with respect to $\mathcal{J}^*$.

Proof: Let $A \in \mathcal{A}^*$, $A = \Gamma^*(B)$ for some $B \in \mathcal{P}(S)$.

Since $\mathcal{J}$ is closed under finite union, and Γ^* preserves (arbitrary) unious, we can assume that the sequence $\{\Gamma^*(I_n)\}_{n \in N}$ is increasing.

By monotone continuity of $\underline{P}$, we have:

$$\underline{P}(A) = \underline{P}[\Gamma^*(B)] = \sup_{n.} \underline{P}[\Gamma^*(I_n)]$$

$$\leq \sup_{\substack{I \in \mathcal{J} \\ \Gamma^*(I) \subset A.}} \underline{P}(\Gamma^*(I))$$

We then get equality since $\underline{P}$ is increasing.

Proposition 5.6

If S is countable, then $\underline{P}$ is Γ^* regular.

Proof:

Each $B \in \mathcal{P}(S)$ can be written as:

$$B = \bigcup_n I_n \text{ with } I_n \in \mathcal{J}, \text{ and } I_n \text{ increasing.}$$

$$\Gamma^*(B) = \bigcup_n \Gamma^*(I_n) \text{ with } \{\Gamma^*(I_n)\}_n \text{ increasing.}$$

Thus: $\underline{P}[\Gamma^*(B)] = \sup_n \underline{P}[\Gamma^*(I_n)]$.

$$\leq \sup_{\substack{I \in \mathcal{J} \\ I \subset B}} \underline{P}[\Gamma^*(I)]$$

$$\leq \sup_{\substack{I \in \mathcal{J} \\ \Gamma^*(I) \subset \Gamma^*(B)}} \underline{P}[\Gamma^*(I)] \quad \leq \underline{P}[\Gamma^*(B)]$$

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