# EIGEN FUZZY SETS AND FUZZY RELATIONS

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# EIGEN FUZZY SETS AND FUZZY RELATIONS*

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Abstract

When R is a fuzzy relation between the elements of a finite set X, the fuzzy sets A of X such that R A = A (MAX-MIN composition) are called eigen fuzzy sets. The main result of this paper is the determination of the greatest eigen fuzzy set associated with a given fuzzy relation and we give three methods illustrated by an example. We then state that the greatest eigen fuzzy set associated with $\hat{R}$, the transitive closure of R, is exactly the one associated with R. Finally we describe how to obtain all fuzzy relations keeping invariant a given fuzzy set.

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## 1. INTRODUCTION

Let R be a fuzzy relation between the elements of a finite set X and let A be a fuzzy set of X. The MAX-MIN composition of R and A gives B, a fuzzy set of X. When B equals A, we say that A is an eigen fuzzy set associated with the given relation R. Our purpose is to find the greatest eigen fuzzy set associated with R.

Stating basic theorems, we give three algorithmic ways to obtain the solution in a number of steps less or equal to the cardinality of X. The results are illustrated by an example.

We then prove that the greatest eigen fuzzy set associated with $\hat{R}$, the transitive closure of R, is exactly the one associated with R.

From a direct application of previous results we describe the dual problem which consists of obtaining all fuzzy relations keeping invariant a given fuzzy set.

For the simplicity of the presentation, we identify fuzzy sets and fuzzy relations with their membership functions.

## 2. FUZZY SETS AND FUZZY RELATIONS

Let L be the closed interval [0,1] of the real line in which $x \wedge y$ denotes the smaller and $x \vee y$ the larger of x and y.

Definition 1. If X is a nonempty set, a fuzzy set A of X is a function $A: X \to L$. The class of all the fuzzy sets of X is denoted by $\mathcal{L}(X)$ and in this paper we assume that X is a finite set.

Let us recall some properties of L:

L is a lattice in which for any given elements x and y the greatest lower bound or meet is $x \wedge y$ and the least upper bound or join is $x \vee y$.

$L$ is a complete lattice, i.e., each of its subsets $M$ has a least upper bound, denoted by $\sup M$ (or $\max M$ when $M$ is finite), or $\vee M$, and a greatest lower bound, denoted by $\inf M$ (or $\min M$ when $M$ is finite) or $\wedge M$, in $L$.

By a greatest element of a partially ordered set $M$ in $L$, we mean an element $b \in M$ such that $x \leq b$ for all $x \in M$.

In $L$ the meet operation is completely distributive on joins, so that $a \wedge (\vee x_i) = \vee(a \wedge x_i)$ for any set of $x_i$ and for any $a$.

Fuzzy sets according to Definition 1 are Zadeh's membership functions and we now recall the following definitions.

The fuzzy set $A \in L(X)$ is contained in the fuzzy set $B \in L(X)$ (written $A \subseteq B$) whenever $A(x) \leq B(x)$ for all $x \in X$.

It is clear that if $A$ and $B \in L(X)$ and if $A \subseteq B$, then $\vee A \leq \vee B$ and $\wedge A \leq \wedge B$.

The fuzzy sets $A$ and $B \in L(X)$ are equal (written $A = B$) whenever $A \subseteq B$ and $B \subseteq A$, i.e., $A(x) = B(x)$ for all $x \in X$.

<u>Definition 2</u>. A fuzzy relation between two nonempty sets $X$ and $Y$ is a fuzzy set $R$ of $X \times Y$, i.e., an element of $L(X \times Y)$.

<u>Definition 3</u>. Let $Q \in L(X \times Y)$ and $R \in L(Y \times Z)$ be two fuzzy relations. We define $T = R \circ Q$, $T \in L(X \times Z)$, the $\circ$-composite fuzzy relation of $R$ and $Q$ by:

$$(1) \qquad (R \circ Q)(x,z) = \bigvee_y [Q(x,y) \wedge R(y,z)]$$

where $y \in Y$, for all $(x,z) \in X \times Z$ .

This definition still holds when $Q$ is a fuzzy set; in this case $T$ becomes a fuzzy set. For example if $A \in L(X)$ and if $R \in L(X \times Y)$, $B = R \circ A$,

$B \in L(Y)$ is defined by:

(2) $$(R \circ A)(y) = \bigvee_x [A(x) \wedge R(x,y)] \quad \text{where} \quad x \in X, \quad \text{for all} \quad y \in Y \quad .$$

The $\circ$-composition is associative and it is easy to verify that:

(3) $$\text{if } Q_1 \text{ and } Q_2 \in L(X \times Y) \text{ and if } Q_1 \subseteq Q_2, \text{ then } R \circ Q_1 \subseteq R \circ Q_2, \text{ where } R \in L(Y \times Z) \ .$$

## 3. EIGEN FUZZY SETS

Let $R \in L(X \times X)$ be a given fuzzy relation. We define

(4) $$A = \{A \in L(X) | R \circ A = A\}$$

and we call the elements of $A$, eigen fuzzy sets associated with the given fuzzy relation $R$. They are the invariants of $R$ according to the $\circ$-composition (max-min).

If we think of $R$ as a system, $A$ is the class of the outputs equal to the inputs; $R$ produces no effect on the elements of $A$ which are invariants: $R \circ A = A$ or $R(A) = A$.

$A$ is a nonempty set because the null fuzzy set belongs to $A$: if $O \in L(X)$ is defined by $O(x) = 0$ for all $x \in X$, then $O \in A$.

$A$ can have numerous elements and our purpose is to set an algorithm in order to find the greatest element of $A$ (in the fuzzy inclusion sense).

Let us verify that

(5) $$\text{If } A_0 \in L(X) \text{ is defined by } A_0(x) = a_0 \text{ for all } x \in X, \text{ where } a_0 = \bigwedge_{x' \in X} \bigvee_{x \in X} R(x,x'), \text{ then } A_0 \in A.$$

For all $x' \in X$,

$$(R \circ A_0)(x') = \bigvee_x [A_0(x) \wedge R(x,x')]$$
$$= \bigvee_x [a_0 \wedge R(x,x')]$$
$$= a_0 \wedge \bigvee_x R(x,x')$$
$$= a_0 = A_0(x') .$$

Defining now $A_1 \in L(X)$ by:

(6) $$\forall x' \in X, \quad A_1(x') = \bigvee_{x \in X} R(x,x') ,$$

we can note that:

(7) $$\forall x \in X, \quad A_0(x) = a_0 = \bigwedge_{x'} A_1(x') , \quad \text{then} \quad A_0 \subseteq A_1 .$$

In Appendix I, we prove the following necessary and sufficient condition for $A_1$ to be the greatest element in $A$.

<u>Theorem 1</u>. $A_1 \in A$ iff $\forall x' \in X$, $A_1(x') \leq \bigvee_{x \in F(x')} A_1(x)$, where $F(x') = \{x \in X | R(x,x') = A_1(x')\}$; moreover when $A_1 \in A$, then it is the greatest element in $A$.

Defining now the sequence $(A_n)_n$ of fuzzy sets

(8)
$$A_2 = R \circ A_1 = R^1 \circ A_1$$
$$A_3 = R \circ A_2 = R^2 \circ A_1$$
$$\cdots\cdots\cdots$$
$$A_{n+1} = R \circ A_n = R^n \circ A_1 \quad \text{for all integers} \quad n > 0$$

let us prove that

(9) $$A_0 \subseteq A_{n+1} \subseteq A_n \subseteq A_1 \quad \text{for all integers} \quad n > 0 .$$

For all $x' \in X$, $(R\circ A_1)(x') = \bigvee_x [A_1(x) \wedge R(x,x')] \leq \bigvee_x R(x,x')$ but $A_1(x') = \bigvee_x R(x,x')$. Hence $A_2 = R\circ A_1 \subseteq A_1$. From $A_0 \subseteq A_1$ we deduce $R\circ A_0 \subseteq R\circ A_1 \subseteq A_1$ or $A_0 \subseteq A_2 \subseteq A_1$, so that (9) holds with $n = 1$.

Let us assume that (9) holds with $n$ and let us prove that (9) holds with $n+1$; by induction, (9) will hold with all integers $n > 0$.

$$A_0 \subseteq A_{n+1} \subseteq A_n \subseteq A_1$$

implies $$R\circ A_0 \subseteq R\circ A_{n+1} \subseteq R\circ A_n \subseteq R\circ A_1 \;,$$

i.e., $$A_0 \subseteq A_{n+2} \subseteq A_{n+1} \subseteq A_2 \;.$$

But $A_2 \subseteq A_1$. Hence $$A_0 \subseteq A_{(n+1)+1} \subseteq A_{n+1} \subseteq A_1 \;.$$

The results (10) and (11) which we shall now state will be of great use in the algorithms hereafter described.

(10) If $\exists k$, $k$ integer $> 0$, and $\exists x' \in X$ such that $A_k(x') = a_0$, hence for all integers $n \geq k$ we have $A_n(x') = a_0$.

Let $k$ be an integer $> 0$ and $x' \in X$ such that $A_k(x') = a_0$; from (9) for all integers $n \geq k$ we deduce $A_0 \subseteq \cdots \subseteq A_n \subseteq \cdots \subseteq A_k \subseteq \cdots \subseteq A_1$ and $A_0(x') \leq \cdots \leq A_n(x') \leq \cdots \leq A_k(x')$, i.e., $a_0 \leq \cdots \leq A_n(x') \leq \cdots \leq a_0$, hence $A_n(x') = a_0$.

(11) If $\exists n$, $n$ integer $> 0$, such that $A_n \in \mathcal{A}$ then $A_n$ is the greatest element in $\mathcal{A}$; moreover $A_{n+k} = A_n$ for all integers $k \geq 0$.

Let $n$ be an integer $> 0$ such that $A_n \in \mathcal{A}$ and let $A$ be an element of $\mathcal{A}$, i.e., $R\circ A = A$. It is evident that for all integers $k > 0$, $R^k \circ A = A$.

For all $x' \in X$, $A(x') = (R\circ A)(x') = \bigvee_x [A(x) \wedge R(x,x')] \leq \bigvee_x R(x,x')$, hence $A \subseteq A_1$.

$A \subseteq A_1$ implies $R^{n-1}\circ A \subseteq R^{n-1}\circ A_1$, i.e., $A \subseteq A_n$.

Moreover $A_{n+1} = R\circ A_n = A_n$ implies $A_{n+k} = R^k\circ A_n = A_n$ for all integers $k \geq 0$.

Let us define according to (7):

(12) $$X_1 = \{x \in X \mid A_1(x) = a_0\}$$

(13) $$Y_1 = X - X_1 = \{x \in X \mid A_1(x) > a_0\} .$$

Let $R' \in L(Y_1 \times Y_1)$ denote the restriction of $R$ to $Y_1 \times Y_1$ and, for all integers $n > 0$, $A'_n \in L(Y_1)$ denote the restriction of $A_n$ to $Y_1$. We define $S_1 \in L(Y_1 \times Y_1)$ by:

(14) $$\forall (x,x') \in Y_1 \times Y_1, \quad S_1(x,x') = R'(x,x') \vee a_0 .$$

In Appendix I, we prove the following theorem helpful to reduce $R$.

<u>Theorem 2</u>. For all integers $n > 0$ and for all $x' \in Y_1$ we have:

(15) $$A_{n+1}(x') = (R\circ A_n)(x') = (S_1\circ A'_n)(x') .$$

Defining:

(16) $$a_{01} = \bigwedge_{x' \in Y_1} \bigvee_{x \in Y_1} S_1(x,x')$$

(17) $$a_1 = \bigwedge_{x' \in Y_1} A_2(x')$$

let us prove that

(18) $$a_{01} = a_1 .$$

From (15) with $n = 1$, for all $x' \in Y_1$:

$$A_2(x') = (S_1 \circ A_1')(x') = \bigvee_{x \in Y_1} [A_1'(x) \wedge S_1(x,x')] \leq \bigvee_{x \in Y_1} S_1(x,x') ,$$

hence

$$\bigwedge_{x' \in Y_1} A_2(x') \leq \bigwedge_{x' \in Y_1} \bigvee_{x \in Y_1} S_1(x,x') ,$$

i.e., $a_1 \leq a_{01}$. Moreover, for all $x' \in Y_1$:

$$\bigvee_{x \in Y_1} S_1(x,x') = a_0 \quad [\bigvee_{x \in Y_1} R'(x,x')] \quad \text{from (14)}$$

$$\leq a_0 \quad [\bigvee_{x \in X} R(x,x')] = \bigvee_{x \in X} R(x,x') = A_1(x') \quad \text{from (6) and (7).}$$

But $x' \in Y_1$ hence $A_1(x') = A_1'(x')$ and for all $x' \in Y_1$, $\bigvee_{x \in Y_1} S_1(x,x') \leq A_1'(x')$.

$$a_{01} = \bigwedge_{x' \in Y_1} \bigvee_{x \in Y_1} S_1(x,x') \leq \bigwedge_{x' \in Y_1} A_1'(x') .$$

Hence: For all $x \in Y_1$, $A_1'(x) \wedge S_1(x,x') \geq [\bigwedge_{x' \in Y_1} A_1'(x')] \wedge S_1(x,x')$.

$$A_1'(x) \wedge S_1(x,x') \geq [\bigwedge_{x' \in Y_1} \bigvee_{x \in Y_1} S_1(x,x')] \wedge S_1(x,x') = a_{01} \wedge S_1(x,x') .$$

For all $x' \in Y_1$ and from (15):

$$A_2(x') = \bigvee_{x \in Y_1} [A_1'(x) \wedge S_1(x,x')] \geq \bigvee_{x \in Y_1} [a_{01} \wedge S_1(x,x')]$$

$$= a_{01} \wedge \bigvee_{x \in Y_1} S_1(x,x') = a_{01}$$

because $a_{01} \leq \bigvee_{x \in Y_1} S_1(x,x')$ for all $x' \in Y_1$, from (16).

$$a_1 = \bigwedge_{x' \in Y_1} A_2(x') \geq a_{01} \ .$$

Hence $a_1 = a_{01}$.

Defining

(19) $$Z_2 = \{x \in Y_1 \,|\, A_2(x) = a_1\}$$

(20) $$Z_2' = \{x \in Y_1 \,|\, (S_1 \circ A_1')(x) = a_{01}\}$$
$$= \{x \in Y_1 \,|\, A_2(x) = a_{01}\} \text{ from (15) with } n = 1 \ ;$$

from (18) we deduce:

(21) $$Z_2 = Z_2' \ .$$

The following basic theorem, in which $|X|$ denotes the cardinality or number of elements (finite case) of the set $X$, proves the existence of a greatest element in $A$. The proof is in Appendix I.

Theorem 3. There exists $n \in \{1,2,3,\ldots,|X|\}$ such that $A_n$ is the greatest element in $A$; moreover $A_0 \subseteq A_n \subseteq A_1$.

In Appendix II we illustrate Theorems 2 and 3 by an example and we derive algorithms for the determination of the greatest eigen fuzzy set associated with a given fuzzy relation.

We can obtain the same result by application of the following theorem. It is not very easy to handle but it is a source of further development related to the transitive closure relation of $R$. An application is also described by an example in Appendix II.

<u>Theorem 4</u>. For all integers $n > 0$ and $\forall x' \in X$,

$$\bigvee_{x \in X} R^n(x,x') = (R^{n-1} \circ A_1)(x') = A_n(x') \quad (22)$$

<u>Proof</u>. For all integers $n > 0$, $\forall x' \in X$,

$$\begin{aligned}(R^{n-1} \circ A_1)(x') &= \bigvee_{y \in X} A_1(y) \wedge R^{n-1}(y,x') \\ &= \bigvee_y \{[\bigvee_x R(x,y)] \wedge R^{n-1}(y,x')\} \quad \text{from (6)} \\ &= \bigvee_y \bigvee_x [R(x,y) \wedge R^{n-1}(y,x')] \\ &= \bigvee_x \bigvee_y [R(x,y) \wedge R^{n-1}(y,x')] \\ &= \bigvee_x (R^{n-1} \circ R)(x,x') \\ &= \bigvee_x R^n(x,x') .\end{aligned}$$

<u>Remark</u>. In general $\mathcal{A}$ is an upper semi-lattice, but not a lattice. Let us assume that $A$ and $B \in \mathcal{A}$, $R \circ A = A$ and $R \circ B = B$.

$$R \circ (A \cup B) = (R \circ A) \cup (R \circ B) = A \cup B$$

Hence $A \cup B \in \mathcal{A}$.

On the other hand, we have $A \cap B \subseteq A$ and $A \cap B \subseteq B$, hence $R \circ (A \cap B) \subseteq R \circ A$ or $R \circ (A \cap B) \subseteq A$, and $R \circ (A \cap B) \subseteq B$, which implies $R \circ (A \cap B) \subseteq A \cap B$.

The following counterexample shows that the inclusion can be a strict one.

Let $X = \{x_1, x_2, x_3\}$ and $R$, $A$, $B$ defined by their matrix representations:

|  | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $x_1$ | .6 | .2 | .4 |
| $R = x_2$ | .3 | 1. | .5 |
| $x_3$ | .3 | .3 | .2 |

|  | $x_1$ | $x_2$ | $x_3$ |  |  |  |  |
|---|---|---|---|---|---|---|---|
| $A =$ | .6 | .3 | .4 | .6 | .3 | .4 | $= R\circ A = A$ |
| $B =$ | .3 | .7 | .5 | .3 | .7 | .5 | $= R\circ B = B$ |
| $A\cap B =$ | .3 | .3 | .4 | .3 | .3 | .3 | $= R\circ(A\cap B) \subset A\cap B$ |

We can note that $A_1$ is the greatest eigen fuzzy set associated with $R$, in our example.

|  | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $A_1 =$ | .6 | 1. | .5 |

## 4. EIGEN FUZZY SETS AND TRANSITIVE CLOSURES

Given a fuzzy relation $R$, from Theorem 3 we know that there exists an integer $n$, $n \in \{1,2,\ldots,|X|\}$, such that $A_n$ is the greatest element in $A$. Moreover, from (11), $A_{n+k} = A_n$ for all integers $k \geq 0$. In the sequel we call $n$ the smallest integer such that $A_n$ is the greatest element in $A$.

Let $\hat{R}$ be the transitive closure of $R$. We recall, see [3], that

$$\hat{R} = R \cup R^2 \cup \cdots \cup R^{|X|} \quad . \tag{23}$$

If $A \in \mathcal{A}$, then for all integers $k \geq 1$, $R^k \circ A = A$.

$$\hat{R} \circ A = (R \cup R^2 \cup \cdots \cup R^{|X|}) \circ A = (R \circ A) \cup (R^2 \circ A) \cup \cdots \cup (R^{|X|} \circ A)$$
$$= A \cup A \cup \cdots \cup A = A .$$

Hence

(24) $$\text{If } R \circ A = A, \text{ then } \hat{R} \circ A = A .$$

Furthermore from (8), we associated the sequence $(A_n)_n$ with $R$. In a similar way, we associate the sequence $(A_n^{(k)})_n$ to $R^k$ for all integers $k \geq 1$.

(25)
$$\begin{aligned} A_1^{(k)}(x') &= \bigvee_x R^k(x,x') \qquad \forall x' \in X \\ A_2^{(k)} &= R^k \circ A_1^{(k)} \\ A_3^{(k)} &= R^k \circ A_2^{(k)} \\ &\cdots\cdots\cdots \\ A_{n+1}^{(k)} &= R^k \circ A_n^{(k)} \end{aligned}$$

$$\begin{aligned} A_1^{(k)}(x') &= \bigvee_x R^k(x,x') \qquad \forall x' \in X \\ &= A_k(x') \qquad \text{from (22)} \end{aligned}$$

Hence $$A_1^{(k)} = A_k .$$

$$A_2^{(k)} = R^k \circ A_1^{(k)} = R^k \circ A_k = A_{k+k} = A_{2k}$$
$$A_3^{(k)} = R^k \circ A_2^{(k)} = R^k \circ A_{2k} = A_{2k+k} = A_{3k}$$

By induction we have

(26) $$A_i^{(k)} = A_{i \times k} \quad \text{for all integers } k \geq 1 \text{ and } i \geq 1 .$$

Let $n_k$ be the smallest integer such that $A_{n_k}^{(k)}$ is the greatest eigen fuzzy set associated with $R^k$. In particular $n_1 = n$.

From Theorem 3 we have

(27) $$R^k \circ A_{n_k}^{(k)} = A_{n_k}^{(k)} .$$

<u>Theorem 5</u>.

(28) For all integers $k \geq 1$, $A_{n_k}^{(k)} = A_n$ .

In other words, for all integers $k \geq 1$, every fuzzy relation $R^k$ has the same greatest eigen fuzzy set; it is the one associated with $R$.

If $k = 1$, $A_{n_1}^{(1)} = A_{n_1 \times 1} = A_{n_1} = A_n$.

Let us assume that there exists $k > 1$ such that $A_{n_k}^{(k)} \neq A_n$. We know that for all integers $i$, $i \geq n_k$, $A_i^{(k)} = A_{n_k}^{(k)}$, hence

$$\exists k > 1 \text{ such that } \forall i \geq n_k,\ A_i^{(k)} \neq A_n, \text{ or } A_{i\times k} \neq A_n$$

from (26).

Let $j = \mathrm{MAX}(n_k, n)$. We have

$$\exists k > 1 \text{ such that } A_{j\times k} \neq A_n, \text{ since } j \geq n_k .$$

But $j \times k > n$ since $j \geq n$ and $k > 1$. $j \times k > n$ implies $A_{j\times k} = A_n$, which gives the desired contradiction.

<u>Theorem 6</u>.

(29) For all integers $k \geq 1$, $n_k$ is the smallest integer $i$ such that $k \times i \geq n$. In particular for all $k \geq n$, $n_k = 1$.

From (26) and (28) we have:

$$\text{for all integers } k \geq 1, \quad A_{k\times n_k} = A_n \ .$$

From the definition of $n$ we deduce that $k\times n_k \geq n$. If we define $a_k$ as the smallest integer $i$ such that $k\times i \geq n$, we have $a_k \leq n_k$. $k\times a_k \geq n$, hence $k\times a_k = n+j$, $j$ integer $\geq 0$, and from (11) we deduce that $A_{k\times a_k} = A_n$, or $A^{(k)}_{a_k} = A_n$. From (28), $A^{(k)}_{n_k} = A_n$, hence $A^{(k)}_{a_k} = A^{(k)}_{n_k}$, which implies $a_k \geq n_k$ from the definition of $n_k$. Finally we have $a_k = n_k$.

As a consequence of Theorem 6, we can show that

(30) $\qquad n_k$ is a decreasing function of $k$

(31) $\qquad$ In particular, for all integers $k \geq 1$, $\quad n_k \leq n$ .

Let us prove that $p \geq q$ implies $n_p \leq n_q$. From (29), we have $q\times n_q \geq n$, and from $p \geq q$ we deduce that $p\times n_q \geq q\times n_q \geq n$. From (29) and from $p\times n_q \geq n$ we deduce $n_p \leq n_q$. In particular from $n_1 = n$ we deduce $n_k \leq n$ for all integers $k \geq 1$.

Theorem 7. The greatest eigen fuzzy set associated with $\hat{R}$, the transitive closure of $R$, is exactly the one associated with $R$.

In other words, $A_n$ is the greatest eigen fuzzy set associated with $\hat{R}$.

As a consequence of (24) we have $\hat{R}\circ A_n = A_n$. From (8), we associated the sequence $(A_k)_k$ with $R$. In a similar way, we can associate a sequence $(B_k)_k$ with $\hat{R}$. By induction, one easily shows that

$$\forall k, \quad 1 \leq k \leq |X|, \quad B_k = A_k \ .$$

Applying now Theorem 3 to $\hat{R}$ and $(B_k)_k$, let m be the smallest k such that $B_k$ is the greatest eigen fuzzy set associated with $\hat{R}$ and let us assume that $B_m \neq A_n$.

Let $j = MAX(m,n)$.

$$j \geq m \Rightarrow B_j = B_m \text{ , hence } B_j \neq A_n$$
$$j \geq n \Rightarrow A_j = A_n \text{ .}$$

But we have proved that $B_j = A_j$ and from this contradiction we derive $B_m = A_n$.

## 5. FUZZY RELATIONS KEEPING INVARIANT A GIVEN FUZZY SET

Let $A \in L(X)$ be a given fuzzy set; we define:

(32) $$\mathcal{R} = \{R \in L(X \times X) \mid R \circ A = A\} \text{ .}$$

$\mathcal{R}$ is the set of all the fuzzy relations keeping invariant, according to the max-min or $\circ$-composition a given fuzzy set. In terms of system theory, we are describing now all the systems which produce no effect on a given input.

$\mathcal{R}$ is a non empty set because if we define $R \in L(X \times X)$ by:

$$\forall (x,x') \in X \times X, \quad R(x,x') = R(x') \text{ ,}$$

then $R \in \mathcal{R}$ as one can verify.

$$\forall x' \in X, \quad (R \circ A)(x') = \bigvee_x A(x) \wedge R(x,x') = \bigvee_x A(x) \wedge A(x')$$
$$= A(x') \wedge \bigvee_x A(x) = A(x') \text{ ,}$$

hence $R \circ A = A$ and $R \in \mathcal{R}$.

Let us note that in [5] and in [6] when L, defining fuzzy sets, is a complete Brouwerian lattice, X being not necessarily a finite set, we find the resolution of $\{R|R\circ Q = T\}$ and $\{Q|R\circ Q = T\}$ in which R, Q and T are fuzzy relations, giving when it exists the greatest element of solutions.

The resolution of (32) is a direct application of results from [7].

Let us recall the definition of the $\alpha$ and $\sigma$ operators and the definition of the ⓐ and ⓢ compositions.

If a and b are elements of $L = [0,1]$, we define $a\,\alpha\,b$ and $a\,\sigma\,b$, elements of L, by:

(33) $$a\,\alpha\,b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

(34) $$a\,\sigma\,b = \begin{cases} 0 & \text{if } a < b \\ b & \text{if } a \geq b \end{cases}$$

If $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ are two fuzzy sets, we define $A \circledalpha B$ and $A \circledsigma B$, elements of $\mathcal{L}(X \times Y)$, by:

(35) $$\forall (x,y) \in X \times Y, \quad (A \circledalpha B)(x,y) = A(x)\,\alpha\,B(y)$$

(36) $$\forall (x,y) \in X \times Y, \quad (A \circledsigma B)(x,y) = A(x)\,\sigma\,B(y)\ .$$

Let us note that the definitions (35) and (36) are enlarged when A and B are fuzzy relations.

The following results hold:

-- $A \circledalpha A$ and $A \circledsigma A$ are elements of $\mathcal{R}$

-- $\check{R} = A \circledalpha A$ is the greatest element of $\mathcal{R}$

-- The minimal elements $R_i$ of $\mathcal{R}$ are defined by: for all $x' \in X$, the only possible element $R_i(x,x') \neq 0$ is just $R_i(x_i,x') = A(x')$ for some $x_i \in X$ such that $A(x_i) \geq A(x')$.

-- The fuzzy union of all minimal elements of $\mathcal{R}$ is equal to $A \circledcirc A$.

-- For all $R \in L(X \times X)$ such that $A \circledcirc A \subseteq R \subseteq A \circledalpha A$, we have $R \in \mathcal{R}$.

<u>Example</u>. Let $X = \{x_1, x_2, x_3, x_4\}$ and $A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ .1 & .4 & .1 & .5 \end{bmatrix}$, $A \in L(X)$.

$\check{R} = A \circledalpha A$:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | 1. | 1. | 1. | 1. |
| $x_2$ | .1 | 1. | 1. | 1. |
| $x_3$ | .1 | .4 | 1. | .5 |
| $x_4$ | .1 | .4 | 1. | 1. |

$A \circledcirc A$:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | .1 | .0 | .0 | .0 |
| $x_2$ | .1 | .4 | .0 | .0 |
| $x_3$ | .1 | .4 | 1. | .5 |
| $x_4$ | .1 | .4 | .0 | .5 |

Example of a minimal element (it is sufficient to keep here a non zero element in each column of $A \circledcirc A$):

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $x_1$ | .0 | .0 | .0 | .0 |
| $x_2$ | .1 | .4 | .0 | .0 |
| $x_3$ | .0 | .0 | 1. | .5 |
| $x_4$ | .0 | .0 | .0 | .0 |

## APPENDIX I

Proof of Theorem 1. For all $x' \in X$, $F(x') = \{x \in X | R(x,x') = A_1(x')\}$, hence from (6), for all $x \notin F(x')$, $R(x,x') < A_1(x')$, i.e., $\bigvee_{x \notin F(x')} R(x,x') < A_1(x')$ and we deduce:

(I.1) $$\bigvee_{x \notin F(x')} A_1(x) \wedge R(x,x') \leq \bigvee_{x \notin F(x')} R(x,x') < A_1(x') .$$

Let us assume that $A_1 \in A$, i.e., $R \circ A_1 = A_1$.

$$\forall x' \in X, \quad A_1(x') = (R \circ A_1)(x') = \bigvee_{x \in X} A_1(x) \wedge R(x,x')$$

$$= [\bigvee_{x \in F(x')} A_1(x) \wedge R(x,x')] \vee [\bigvee_{x \notin F(x')} A_1(x) \wedge R(x,x')]$$

$$= \bigvee_{x \in F(x')} A_1(x) \wedge R(x,x') \quad \text{from (I.1)}$$

$$= \bigvee_{x \in F(x')} A_1(x) \wedge A_1(x') = A_1(x') \wedge [\bigvee_{x \in F(x')} A_1(x)]$$

Hence, $\forall x' \in X$, $A_1(x') \leq \bigvee_{x \in F(x')} A_1(x)$.

Let us now assume that $\forall x' \in X$, $A_1(x') \leq \bigvee_{x \in F(x')} A_1(x)$.

$$\forall x' \in X, \quad (R \circ A_1)(x') = \bigvee_{x \in X} A_1(x) \wedge R(x,x')$$

$$= [A_1(x') \wedge \bigvee_{x \in F(x')} A_1(x)] \vee [\bigvee_{x \notin F(x')} A_1(x) \wedge R(x,x')]$$

$$= A_1(x') \vee [\bigvee_{x \notin F(x')} A_1(x) \wedge R(x,x')]$$

$$= A_1(x') \quad \text{from (I.1)}$$

Hence $R \circ A_1 = A_1$, i.e., $A_1 \in A$.

$\forall A \in A, \quad R \circ A = A$

$$\forall x' \in X, \quad A(x') = (R \circ A)(x') = \bigvee_{x \in X} A_1(x) \wedge R(x,x') \leq \bigvee_{x \in X} R(x,x') ;$$

hence from (6), $A(x') \leq A_1(x')$, i.e., $A \subseteq A_1$ and when $A_1 \in A$, then it is the greatest element in $A$.

Proof of Theorem 2. From (10) and (12) we deduce:

(I.2) For all integers $n > 0$, $\forall x \in X_1$, $A_n(x) = a_0$ .

From (9) we deduce:

(I.3) For all integers $n > 0$, $\forall x \in X$, $A_n(x) \geq a_0$ .

Let us define: For all integers $n > 0$, $G_n = R' \circ A'_n$, $G_n \in L(Y_1)$, i.e.,

(I.4) $\forall x' \in Y_1$, $G_n(x') = \bigvee_{x \in Y_1} A'_n(x) \wedge R'(x,x') = \bigvee_{x \in Y_1} A_n(x) \wedge R(x,x')$ .

Let $n$ be an integer $> 0$; $\forall x' \in Y_1$ we have:

$$\begin{aligned} A_{n+1}(x') &= \bigvee_{x \in X} A_n(x) \wedge R(x,x') \\ &= [\bigvee_{x \in X_1} A_n(x) \wedge R(x,x')] \vee G_n(x') \quad \text{from (I.4)} \\ &= [\bigvee_{x \in X_1} a_0 \wedge R(x,x')] \vee G_n(x') \quad \text{from (I.2)} \\ &= [a_0 \wedge \bigvee_{x \in X_1} R(x,x')] \vee G_n(x') \ . \end{aligned}$$

On the other hand, $\forall x' \in Y_1$ we have:

$$\begin{aligned} (S_1\ A'_n)(x') &= \bigvee_{x \in Y_1} A'_n(x) \wedge S_1(x,x') \\ &= \bigvee_{x \in Y_1} [A'_n(x) \wedge (R'(x,x') \vee a_0)] \quad \text{from (14)} \\ &= \bigvee_{x \in Y_1} [(A'_n(x) \wedge R'(x,x')) \vee (A'_n(x) \wedge a_0)] \end{aligned}$$

$$= G_n(x') \vee [\bigvee_{x \in Y_1} (A'_n(x) \wedge a_0)] \quad \text{from (I.4)}$$

$$= G_n(x') \vee [a_0 \wedge \bigvee_{x \in Y_1} A'_n(x)] \quad .$$

From (I.3) we deduce:

$$\forall x \in Y_1, \quad A'_n(x) = A_n(x) \geq a_0, \quad \text{hence} \quad \bigvee_{x \in Y_1} A'_n(x) \geq a_0 \ .$$

Let us recall that our purpose is to prove that for all integers $n > 0$ and for all $x' \in Y_1$, $A_{n+1}(x') = (S_1 \circ A'_n)(x')$; but our present result is:

(I.5) $$\forall x' \in Y_1, \quad A_{n+1}(x') = [a_0 \wedge \bigvee_{x \in X_1} R(x,x')] \vee G_n(x')$$

$$(S_1 \circ A'_n)(x') = a_0 \vee G_n(x') \ .$$

Let us now define:

(I.6) $$\forall x' \in Y_1, \quad H(x') = \{x \in Y_1 \mid R'(x,x') \geq a_0\} \ .$$

If there exists $x' \in Y_1$ such that $H(x') = \emptyset$, then $\forall x \in Y_1$, $R'(x,x') < a_0$. $a_0 = \bigwedge_{x' \in X} \bigvee_{x \in X} R(x,x')$, hence $\bigvee_{x \in X} R(x,x') \geq a_0$ so that we deduce: there exists $x_i \in X$ such that $R(x_i,x') \geq a_0$. From $H(x') = \emptyset$ we can now deduce: there exists $x_i \in X_1$ such that $R(x_i,x') \geq a_0$ and therefore $\bigvee_{x \in X_1} R(x,x') \geq a_0$, which implies:

$$A_{n+1}(x') = a_0 \vee G_n(x') = (S_1 \circ A'_n)(x') \ .$$

Let us now assume that $\forall x' \in Y_1$, $H(x') \neq \emptyset$, i.e., there exists $x_j \in Y_1$ such that $R'(x_j,x') \geq a_0$.

$$G_n(x') = \bigvee_{x \in Y_1} A'_n(x) \wedge R'(x,x') \geq \bigvee_{x \in H(x')} A'_n(x) \wedge R'(x,x')$$

$$\geq A'_n(x_j) \wedge R'(x_j,x') \geq A'_n(x_j) \wedge a_0 = a_0 \quad \text{from (I.3).}$$

Moreover, $a_0 \wedge \bigvee_{x \in X_1} R(x,x') \leq a_0 \leq G_n(x')$, hence:

$$A_{n+1}(x') = G_n(x') = S_1 \circ A'_n(x') .$$

<u>Proof of Theorem 3</u>. Let us point out that if $A_1 = A_0$, then from (5) and (11) with $n = 1$, $A_1$ is the greatest element in $A$.

Moreover, if $A_1 \in A$ (see Theorem 1), from (11) with $n = 1$, $A_1$ is the greatest element in $A$.

The interesting result of this theorem is when $A_1 \notin A$.

Let us assume that $A_1 \notin A$. It is sufficient to prove that there exists $n \in \{2,3,\ldots,|X|\}$ such that $A_n \in A$; from (11) $A_n$ will be the greatest element in $A$; $A_0 \subseteq A_n \subseteq A_1$ is given by (9).

From (12) and (I.2) we have: for all integers $n > 0$, $\forall x \in X$ such that $A_1(x) = a_0$, we deduce $A_n(x) = a_0$.

From $A_1 \notin A$ we deduce $Y_1 \neq \emptyset$ or $|Y_1| < |X|$.

For all integers $n > 0$, we know the values $A_n(x)$ when $x \in X_1$. Our purpose is to evaluate $A_n(x)$ on $Y_1$ or on a subset of $Y_1$.

From Theorem 2, on $Y_1$, $S_1$ gives the same results that $R$, and $a_{01}$ defined in (16) will enjoy the same properties with $S_1$ that $a_0$ with $R$; hence the analogous property of (I.2) is:

> For all integers $n > 1$, $\forall x \in X_2$, $A_n(x) = a_{01}$, where
> $X_2 = \{x \in Y_1 | A_2(x) = a_{01}\}$.

This property holds because $X_2 = Z_2 = Z'_2$ (see (21)).

If $Y_2 = Y_1 - X_2 \neq \emptyset$ or $|Y_2| < |Y_1|$ and if $S'_1 \in L(Y_2 \times Y_2)$ denotes the restriction of $S_1$ to $Y_2 \times Y_2$, for all integers $n > 1$, $A''_n \in L(Y_2)$ denotes the restriction of $A'_n$ to $Y_2$, we define $S_2 \in L(Y_2 \times Y_2)$ by:

$$\forall (x,x') \in Y_2 \times Y_2, \quad S_2(x,x') = S'_1(x,x') \vee a_{01} .$$

Then we have the analogous property of (15):

for all integers $n > 1$ and for all $x' \in Y_2$,

$A''_{n+1}(x') = S_2 \circ A''_n(x')$ .

We then define $a_{02} = \bigwedge_{x' \in Y_2} \bigvee_{x \in Y_2} S_2(x,x')$ and $a_2 = \bigwedge_{x' \in Y_2} A_3(x')$; we have $\{x \in Y_2 | A_2(x) = a_2\} = X_3 = \{x \in Y_2 | (S_2\ A'_2)(x) = a_{02}\}$ and for all integers $n > 2$, $\forall x \in X_3$, $A_n(x) = a_{02}$.

If $Y_3 = Y_2 - X_3 \neq \emptyset$ or $|Y_3| < |Y_2|$, we continue, but $X$ being a finite set, there exists $n \in \{2,3,\ldots,|X|\}$ such that $Y_n = Y_{n-1} - X_n = \emptyset$, i.e., $X_n = Y_{n-1}$ and $|Y_{n-1}| < |Y_{n-2}| < \cdots < |Y_2| < |Y_1|$.

$$X = \bigcup_{i=1}^{n} X_i \quad \text{and} \quad X_i \cap X_j = \emptyset \quad \text{if} \quad i \neq j \ ;$$

$\forall x \in X$, $\exists i \in \{1,2,\ldots,n\}$ such that $x \in X_i$ and for all integers $k \geq i$, $\forall x \in X_i$,

$$A_k(x) = a_{0i-1}$$

$$A_n(x) = \begin{cases} a_0 & \text{if } x \in X_1 \\ a_{01} & \text{if } x \in X_2 \\ a_{02} & \text{if } x \in X_3 \\ \cdots & \cdots \\ a_{0n-1} & \text{if } x \in X_n \end{cases}$$

$$A_{n+1}(x) = A_n(x) \ ,$$

i.e., $(R \circ A_n)(x) = A_n(x)$ and $A_n \in A$.

# APPENDIX II

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, $|X| = 5$ and let us consider the given fuzzy relation $R \in L(X \times X)$:

$R =$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $x_1$ | .1 | .7 | .2 | (.8) | (.7) |
| $x_2$ | .0 | .6 | .4 | .3 | .5 |
| $x_3$ | (.3) | (1.) | .0 | .1 | .4 |
| $x_4$ | (.3) | .3 | (.8) | .1 | .0 |
| $x_5$ | .0 | .0 | .7 | .5 | .0 |

The encircled elements are the greatest ones in each column. They allow us to define $A_1$ and $A_0$. From Theorem 3 we know that there exists $n \in \{1,2,3,4,5\}$ such that $A_n$ is the greatest fuzzy set $A$ verifying $R\ A = A$; moreover $A_0 \subseteq A_n \subseteq A_1$.

## First Determination of $A_n$

$$a_0 = .3 \wedge 1. \wedge .8 \wedge .8 \wedge .7 = .3$$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $A_1 =$ | $\underline{.3}$ | 1. | .8 | .8 | .7 |
| $A_0 =$ | .3 | .3 | .3 | .3 | .3 |

$A_0 \in A$ (it is easy to verify it) and $A_0 \subseteq A_1$.

$A_1 \notin A$; we can verify that $R \circ A_1 \neq A_1$ or apply Theorem 1 to $x' = x_4$ for example:

$$A_1(x') = A_1(x_4) = .8$$
$$F(x') = F(x_4) = \{x \in X | R(x, x_4) = .8\} = \{x_1\}$$

$\bigvee_{x \in F(x_4)} A_1(x) = A_1(x_1) = .3$ and we don't have $A_1(x_4) \leq \bigvee_{x \in F(x_4)} A_1(x)$, hence $A_1 \notin A$.

From (12), (13) and (I.2) we have: $X_1 = \{x_1\}$, $Y_1 = \{x_2, x_3, x_4, x_5\}$ and $\forall n \geq 1$, $A_n(x_1) = .3$. So we underline .3 in position $x_1$ in $A_1$; it is the first invariant element.

$$A_2 = R \circ A_1 = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ [ & \underline{.3} & .8 & .8 & \underline{.5} & .5 & ] \end{matrix} \subseteq A_1$$

$$a_{01} = a_1 = \bigwedge_{x' \in Y_1} A_2(x') = A_2(x_2) \wedge A_2(x_3) \wedge A_2(x_4) \wedge A_2(x_5) = .5$$

$$X_2 = \{x \in Y_1 | A_2(x) = a_{01}\} = \{x_4, x_5\}, \qquad Y_2 = \{x_2, x_3\} ,$$

$\forall n \geq 2$, $\forall x \in X_2$, $A_n(x) = .5$. So we underline .5 in positions $x_4$ and $x_5$ in $A_2$; they are invariant in the following compositions:

$$A_3 = R \circ A_2 = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ [ & \underline{.3} & .8 & \underline{.5} & \underline{.5} & \underline{.5} & ] \end{matrix} \subseteq A_2$$

$$a_{02} = a_2 = \bigwedge_{x' \in Y_2} A_3(x') = A_3(x_2) \wedge A_3(x_3) = .5$$

$$X_3 = \{x \in Y_2 | A_3(x) = a_{02}\} = \{x_3\}, \qquad Y_3 = \{x_2\} ,$$

$\forall n \geq 3$, $\forall x \in X_3$, $A_n(x) = .5$. So we underline .5 in position $x_3$ in $A_3$.

$$A_4 = R \circ A_3 = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ [ & \underline{.3} & \underline{.6} & \underline{.5} & \underline{.5} & \underline{.5} & ] \end{matrix} \subseteq A_3$$

$$a_{03} = a_3 = .6 , \qquad X_4 = \{x_2\} , \qquad Y_4 = \emptyset ;$$

$\forall n \geq 4$, $\forall x \in X_4$, $A_n(x) = .6$ and we underline .6 in position $x_2$ in $A_4$.

$A_n$, with $n = 4$, is now completely determined. $Y_4 = \emptyset$ implies that $A_4 \in A$, as one can easily verify $R \circ A_4 = A_4$.

Summary of the First Method

-- Encircle the greatest elements in each column of R and set them in $A_1$ in their corresponding positions.

-- Underline in $A_1$ the smallest of its elements (it is not necessarily in a unique position). It (or they) will be invariant after the next compositions.

-- Compute $R \circ A_1 = A_2$ and underline in $A_2$ the smallest of the non underlined elements.

-- Compute $R \circ A_2 = A_3$, $R \circ A_3 = A_4$, etc., underlining at each step the smallest of the non underlined elements. When all elements are underlined, we get $A_n$ the greatest eigen fuzzy set. We know that $n \leq |X|$, so that convergence is fairly fast.

Second Determination of $A_n$

In the following method we need not evaluate MAX-MIN or $\circ$-composition to obtain $A_n$. The idea is to get the underlined elements of the first method replacing R of order $|X|$ by $S_1$ of order $|Y_1| < |X|$, etc. The results are derived from direct application of (14), (15), (18) and (21). Let us now describe the different steps, R being given in a tabulated form.

-- Encircle the greatest elements in each column of R. The smallest of these elements is equal to $a_0$. It is .3 in the $x_1$'th column in our example.

-- Delete from R the columns containing the smallest of the greatest elements, and the same rows, say the column $x_1$ and the row $x_1$. We have got R', the first reduction of R.

It is important to remark that we don't delete the rows passing through the positions of the value $a_0 = .3$, say the row $x_3$ and the row $x_4$.

-- Set in $A_n$ (n is not known yet) the value of $a_0$, in the position of the deleted columns, say $x_1$.

-- Return to the first step with R' instead of R, but with the following restriction: $a_{01}$ denoting the smallest of the greatest elements in each column of R, set in the appropriate position of $A_n$:

$$a_{01} \text{ if } a_{01} \geq a_0$$
$$a_0 \text{ if } a_{01} < a_0 \text{ , according to (14) and (15)}$$

From R' we derive .5 in positions $x_4$ and $x_5$ and .5 is greater than .3. So we insert .5 in the positions $x_4$ and $x_5$ of $A_n$.

For the k'th reduction, this restriction becomes: Set in $A_n$,

$$a_{0k} \text{ if } a_{0k} \geq a_{0,k-1}$$
$$a_{0,k-1} \text{ if } a_{0k} < a_{0,k-1}$$

From R" we derive .4 which is smaller than .5; thus we set .5, instead of .4 in the position $x_3$ of $A_n$.

-- Return to the first step until all reductions are exhausted.

## Illustration of the Second Method

-- From R we first get:

$$A_n = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ [ & .3 & \times & \times & \times & \times & ] \end{matrix} \qquad a_0 = .3$$

-- We delete from R the column and the row corresponding to $x_1$ and we get R', reduction of R:

R' =

| | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|
| $x_2$ | .6 | .4 | .3 | (.5) |
| $x_3$ | (1.) | .0 | .1 | .4 |
| $x_4$ | .3 | (.8) | .1 | .0 |
| $x_5$ | .0 | .7 | (.5) | .0 |

-- We deduce $a_{01} = .5$ in positions $x_4$ and $x_5$ and $A_n$ becomes:

$$A_n = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ [ & .3 & \times & \times & .5 & .5 & ] \end{matrix}$$

because $a_{01}$ is greater than $a_0$.

-- We delete from R' the columns and the rows corresponding to $x_4$ and $x_5$ and we get R", reduction of R':

$$R'' = \begin{matrix} & x_2 & x_3 \\ x_2 & .6 & \textcircled{.4} \\ x_3 & \textcircled{1.} & .0 \end{matrix}$$

-- We deduce $a_{02} = .4$ in position $x_3$ but we insert, in position $x_3$ of $A_n$, .5 instead of .4 because $a_{02} < a_{01}$ and $A_n$ becomes:

$$A_n = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ [ & .3 & \times & .5 & .5 & .5 & ] \end{matrix}$$

-- We delete from R" the column and the row corresponding to $x_3$ and we get R"', reduction of R":

$$R''' = \begin{matrix} & x_2 \\ x_2 & .6 \end{matrix}$$

$a_{03} = .6$ in position $x_2$ and we get the final expression of $A_n$:

$$A_n = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ [ & .3 & .6 & .5 & .5 & .5 & ] \end{matrix}$$

because $a_{03}$ is greater than $a_{02}$.

<u>Third Determination of $A_n$</u>

This third method is based on a direct application of Theorems 3 and 4.

-- Determine first $A_1$ with the elements corresponding to the greatest element in each column of R.

-- Evaluate $R^2 = R \circ R$ and encircle the greatest elements in each column of $R^2$ to determine $A_2$ according to:

$$\forall x' \in X, \quad \bigvee_{x} R^2(x,x') = A_2(x') \qquad \text{(see (22))}$$

-- Compare $A_2$ to $A_1$; if they are different, evaluate $R^3 = R \circ R^2$ to get $A_3$ according to:

$$\forall x' \in X, \quad \bigvee_{x} R^3(x,x') = A_3(x')$$

-- Compare $A_3$ to $A_2$; if they are different, evaluate $R^4 = R \circ R^2$ to get $A_4$, etc. Stop when you find n such that $A_{n+1} = A_n$, i.e., $R \circ A_n = A_n$.

This method is not very easy to handle but it can be interesting if in a problem under study one has already evaluated $R^2$, $R^3$, etc.

In our example, we get:

$R^2 =$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $x_1$ | (.3) | .6 | (.8) | (.5) | (.5) |
| $x_2$ | (.3) | .6 | .5 | (.5) | (.5) |
| $x_3$ | .1 | .6 | .4 | .4 | (.5) |
| $x_4$ | (.3) | (.8) | .3 | .3 | .4 |
| $x_5$ | (.3) | .7 | .5 | .1 | .4 |

and we deduce:

$$A_2 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ .3 & .8 & .8 & .5 & .5 \end{bmatrix}, \text{ etc.}$$

## CONCLUDING REMARKS

We believe that an eigen fuzzy set theory deserves to be developed thinking of its potential applied areas such as belief systems, transportation problems, fuzzy clustering, human decision processes, pattern recognition, medical diagnosis assistance.

We are now inserting the results of this paper in already proposed applied models in medical diagnosis [5], [7], [8]. They are based on max-min composite equations [6], [7] derived from fuzzy meta-implications [3], conditioned fuzzy set [11]. In fuzzy logic it is the rule of inference stated by Zadeh [12], [1]. One can infer diagnosis or prognosis from observed symptoms by means of a specific medical knowledge.

The model of eigen fuzzy sets provides a methodology for searching invariants in therapeutic recommendations. Such assistance proposed to physicians aims to avoid non optimal treatments in health programs.

The inference rules involved in medical diagnosis problems can be viewed in terms of belief functions. Based on relevant data (or evidence) expressed by observed symptoms, one can assign degrees of belief on diagnosis or diseases. We plan to investigate the role played by fuzzy relations in such problems.

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## REFERENCES


1. BELLMAN, R.E. and ZADEH, L.A., "Local and Fuzzy Logics," Memorandum No. ERL M584, May 11, 1976, Electronics Research Laboratory, University of California, Berkeley.

2. BIRKHOFF, G.,"Lattice Theory," 3rd Ed., American Mathematical Society Colloquium Publications, Vol. XXV, Providence, R.I., 1967.

3. KAUFMANN, A., Introduction to the Theory of Fuzzy Subsets, Vol. 1, Academic Press, New York, 1975.

4. KAUFMANN, A., Introduction à la Théorie des Sous-Ensembles Flous, Tomes I, II, III, IV, Ed. Masson, Paris, 1973-1976.

5. SANCHEZ, E., "Equations de Relations Floues," Thèse Biologie Humaine, Faculté de Médecine, Marseille, 1974.

6. SANCHEZ, E., "Resolution of Composite Fuzzy Relation Equations," Information and Control, Vol. 30, pp. 38-48, 1976.

7. SANCHEZ, E., "Solutions in Composite Fuzzy Relation Equations. Application to Medical Diagnosis in Brouwerian Logic," in Fuzzy Automata and Decision Processes, American Elsevier Publ. Co., New York, to appear in 1977.

8. SANCHEZ, E. and SAMBUC, R., "Relations Floues. Fonctions Φ-Floues. Application à l'Aide au Diagnostic en Pathologie Thyroidienne," Medical Data Processing Symposium, IRIA, Toulouse, 1976, to be published by Taylor and Francis Ltd., London.

9. SANCHEZ, E., "Eigen Fuzzy Sets," Session Approximate Reasoning and Approximate Algorithms in Computer Science, NCC, New York City, June 7-10, 1976.

10. SANCHEZ, E., "Resolution of Eigen Fuzzy Sets Equations," ORSA/TIMS, Miami, November 3-6, 1976.

11. ZADEH, L.A., "Fuzzy Sets," Information and Control, Vol. 8, pp. 338-353, 1965.

12. ZADEH, L.A., "The Concept of a Linguistic Variable and its Application to Approximate Reasoning," Information Sciences, Part I, Vol. 8, pp. 199-249, 1975; Part II, Vol. 8, pp. 301-357, 1975; Part III, Vol. 9, pp. 43-80, 1975.