ON POSSIBILITY-QUALIFICATION IN NATURAL LANGUAGES

by

Elie Sanchez

Memorandum No. UCB/ERL M77/28

26 April 1977

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

ON POSSIBILITY-QUALIFICATION IN NATURAL LANGUAGES*

Elie Sanchez

Laboratoire de Biomathématiques, Statistiques et Informatique Médicale
Faculté de Médecine, Marseille, France

Abstract

The theory of possibility, recently introduced by L.A. Zadeh, is explored in the particular subject of possibility-qualified propositions as a basis for approximate reasoning. The concept of ε-possibility related to fuzzy sets has a formulation that verifies some criteria of minimal uncertainty. It is shown how to derive possibility-qualified propositions from the classical translation rules in fuzzy logic.

*Research sponsored in part by National Science Foundation Grant MCS76-06693. This paper was written while the author was a Visiting Research Associate at the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720. This work was supported by a grant from the Institut de Recherche d'Informatique et Automatique.

1. INTRODUCTION

The purpose of this study is to explore possibility-qualified propositions introduced by Zadeh in [10], a concept different from that of modal logic [3].

In [11] Zadeh describes a theory of possibility distributions related to fuzzy sets in order to interpret the meaning, rather than the measure, of the information conveyed by a proposition, arguing that "the imprecision that is intrinsic in natural languages is, in the main, possibilistic rather than probabilistic in nature."

Based on these premises, in section 2, we introduce the dual notion of impossibility and we give the properties of possibility, impossibility and ε-possibility qualified propositions related to non fuzzy sets. We finally introduce a notion of ε-lower and ε-upper possibility along with their properties.

Remark that the formulation of the ε-possibility is different from that of Zadeh given in [11], in this sense that its extension to fuzzy sets is chosen in order to verify some criteria of "minimal uncertainty." This is discussed in section 3 which is also devoted to the different kinds of possibility qualified propositions related to fuzzy sets.

In section 4 we relate interval-valued fuzzy sets to possibilistic sub-propositions via the notion of linguistic intervals.

In section 5, possibilistic propositions are related to fuzzy relations: projections, compositional rule of inference, extension principle.

Finally in section 6 and as a main interest for applications possibilistic propositions are related to the rule of conditional composition, IF...THEN...ELSE..., and to the rule of compositional modus ponens.

2. POSSIBILISTIC PROPOSITIONS

Fuzzy Sets - Notation and Terminology

A fuzzy subset F of a universe of discourse U is characterized by a compatibility function $\mu_F: U \to [0,1]$, where $\mu_F(u)$ is interpreted as the degree of compatibility, or possibility, of the element u in U with the concept represented by F.

The support of F is the set of points in U at which $\mu_F(u)$ is positive. The height of F is the supremum of μ_F over U. F is said normal if its height is unity.

To simplify the representation of fuzzy sets we shall employ the following notation.

A non fuzzy finite set such as $U = \{u_1, \ldots, u_n\}$ will be expressed as $U = u_1 + \cdots + u_n$ or $U = \sum_{i=1}^{n} u_i$ with the understanding that + denotes the union rather than the arithmetic sum. A fuzzy subset of U will be expressed as

$$F = \mu_1/u_1 + \cdots + \mu_n/u_n \tag{2.1}$$

where μ_i, $i = 1, \ldots, n$ is the degree of compatibility of u_i in F.

When the support of a fuzzy set is a continuum we shall write

$$F = \int_U \mu_F(u)/u \tag{2.2}$$

with the understanding that $\mu_F(u)$ is the degree of compatibility of u in F and the integral denotes the union of the fuzzy singletons $\mu_A(u)/u$, $u \in U$.

A fuzzy set is of type n, $n = 2, 3, \ldots,$ if its compatibility function ranges over fuzzy sets of type n-1. The compatibility function of a fuzzy set of type 1 ranges over the interval [0,1].

Fuzzy sets of type 2 are also called Φ-fuzzy sets, see [5,7].

Intervals in [0,1]

Let Φ be the set of the intervals such as $[a,b] \subset [0,1]$, with $[a,a]$ identified with a. Each operation on $[0,1]$ induces a corresponding operation on Φ, provided that the result is an element of Φ. This is made possible by using the extension principle.

If f is a mapping from a universe of discourse U to a universe of discourse V and F is a fuzzy subset of U expressed as $F = \mu_1/u_1 + \cdots + \mu_n/u_n$, then the extension principle asserts that

$$f(F) = f(\mu_1/u_1 + \cdots + \mu_n/u_n) = \mu_1/f(u_1) + \cdots + \mu_n/f(u_n) \;. \qquad (2.3)$$

If the support of F is a continuum, that is $F = \int_U \mu_F(u)/u$, then the formulation for the extension principle is

$$f(F) = f\left(\int_U \mu_F(u)/u\right) = \int_V \mu_F(u)/f(u) \qquad (2.4)$$

with the understanding that $f(u)$ is a point in V and $\mu_F(u)$ is its grade of membership in $f(F)$, which is a fuzzy subset of V.

If $[a_1,b_1]$ and $[a_2,b_2]$ are elements of Φ and if $*$ is an operation on $[0,1]$, it follows that

$$[a_1,b_1]*[a_2,b_2] = [a_1*a_2, b_1*b_2] \quad \text{iff} \quad a_1*a_2 \leq b_1*b_2 \;. \qquad (2.5)$$

(2.5) holds for example when $*$ is replaced by one of the following operations

minimum: $a \wedge b$

maximum: $a \vee b$

product: ab

bounded sum: $a \oplus b = 1 \wedge (a+b)$, where $+$ is the arithmetic sum

bounded difference: $a \ominus b = 0 \vee (a-b)$, where $-$ is the arithmetic difference

In (2.5), when the condition $a_1 * a_2 \leq b_1 * b_2$ is not satisfied, one can define for example

$$[a_1,b_1]*[a_2,b_2] = [(a_1*a_2) \wedge (b_1*b_2), b_1*b_2] .$$

Such a formula is suitable for the resolution of composite Φ-fuzzy relational equations, see [4(V)] and [6].

Let us remark that a given structure on $[0,1]$ does not necessarily induce the same structure on Φ. For example, $[0,1]$ is a totally ordered set for the usual order relation on real numbers, but the induced relation in Φ is not a total one.

$$[a_1,b_1] \leq [a_2,b_2] \text{ iff } a_1 \leq a_2 \text{ and } b_1 \leq b_2 \tag{2.6}$$

Finally, the "complement" of $[a,b]$ in Φ is defined by

$$[a,b]' = [b',a'] = [1-b,1-a] \tag{2.7}$$

where the symbol prime (') stands for <u>complement</u>.

From the definitions of $\ominus$, ', $\oplus$, it follows that

$$[0,1] \ominus a = [0,1-a] = [0,a'] \tag{2.8}$$

$$[0,1] \ominus a = [a,1]' \tag{2.9}$$

$$a \oplus [0,1] = [a,1] \tag{2.10}$$

Let us give now some properties derived from the $\oplus$ and $\ominus$ operators which will be useful in the sequel. They are easy to verify.

$$a \vee b \leq a \oplus b \tag{2.11}$$

$$a_1 \leq a_2 \Rightarrow a_1 \oplus b \leq a_2 \oplus b \tag{2.12}$$

$$a \oplus b' = (a \wedge b) \oplus b' \tag{2.13}$$

where a, b, a_1, a_2 are elements of $[0,1]$.

$$a \ominus b \leq a \wedge b' \tag{2.14}$$

$$a' \ominus b' = b \ominus a \tag{2.15}$$

$$a \ominus b = a \ominus (a \wedge b) = (a \vee b) \ominus b \tag{2.16}$$

$$a_1 \leq a_2 \Rightarrow a_1 \ominus b \leq a_2 \ominus b \tag{2.17}$$

$$b_1 \leq b_2 \Rightarrow a \ominus b_2 \leq a \ominus b_1 \tag{2.18}$$

$$(a \oplus b)' = a' \ominus b = b' \ominus a \tag{2.19}$$

$$(a \ominus b)' = a' \oplus b \tag{2.20}$$

where a, b, a_1, a_2, b_1, b_2 are elements of $[0,1]$.

Possibility-Qualified Propositions

In the sequel, the examples chosen to illustrate the propositions will be based on:

$$\text{the universe of discourse } U = \mathbb{R}, \text{ the real line} \tag{2.21}$$

and

$$\text{the subset } F = [a,b] \text{ of } U \tag{2.22}$$

Let U be a universe of discourse, X, a variable that takes values in U and F a subset of U, the proposition

$$X \text{ is in } F \longrightarrow R(X) = F \tag{2.23}$$

where F is a subset of U which is assigned to the restriction R(X) on the values that may be assigned to X [9].

For example,

$$\begin{aligned} \mu_F(u) &= 1 \quad \text{for} \quad a \le u \le b \\ &= 0 \quad \text{elsewhere} \end{aligned} \tag{2.24}$$

$$X \text{ is in } F \text{ is } \underline{\text{possible}} \longrightarrow R(X) = F^+ \tag{2.25}$$

where F^+ is a fuzzy subset of U interval-valued, i.e., of type 2, which is assigned to R(X) and defined by, see [10,11]

$$F^+ = F \oplus \Pi \tag{2.26}$$

where

$$\Pi = \int_U [0,1]/u \tag{2.27}$$

In other words,

$$\begin{aligned} \mu_{F^+}(u) &= \mu_F(u) \oplus \mu_\Pi(u) = \mu_F(u) \oplus [0,1] \\ &= [\mu_F(u), 1] \ . \end{aligned} \tag{2.28}$$

A more general definition should require

$$\Pi = \int_U L/u \tag{2.29}$$

and

$$\mu_{F^+}(u) = \mu_F(u) \oplus L \ , \tag{2.30}$$

where L is the range of the compatibility functions under study. $L = \{0,1\}$ in this case, so that $\mu_{F+}(u) = \{\mu_F(u), 1\}$, $L = [0,1]$ in the case of fuzzy sets of type 1, etc. But here we are merely interested in fuzzy possibilistic propositions related to at most fuzzy sets of type 1 and (2.28) will serve, in the same form, for the generalization when F is a fuzzy subset of U.

For example,

$$\mu_{F^+}(u) = 1 \quad \text{for } a \le u \le b \qquad (2.31)$$
$$= [0,1] \quad \text{elsewhere}$$

Intuitively, (2.31) means that, whereas "X is in F" implies that the degree of possibility that X is outside F is zero, "X is in F is possible" implies that there is total uncertainty outside F, i.e., the degree of possibility that X is outside F is unknown, which gives the total interval [0,1] as a representation.

Impossibility-Qualified Propositions

We define impossibility-qualified propositions by the following translation rule.

$$X \text{ is in } F \text{ is } \underline{\text{impossible}} \longrightarrow R(X) = F^- \qquad (2.32)$$

where F^- is a fuzzy subset of U, interval-valued, which is assigned to R(X). F^- is defined by

$$F^- = \Pi \ominus F \qquad (2.33)$$

where Π is defined in (2.27). In other words,

$$\mu_{F^-}(u) = [0,1] \ominus \mu_F(u) = [0,\mu_{F'}(u)] \ . \qquad (2.34)$$

For example,

$$\mu_{F^-}(u) = 0 \quad \text{for } a \le u \le b \qquad (2.35)$$
$$= [0,1] \quad \text{elsewhere}$$

Let us now indicate some properties related to possibility and impossibility qualified propositions.

1) $$F \subset F^{+} \tag{2.36}$$

whereas, in general, $F^{-} \not\subset F$.

2) An equivalent definition of F^{-} is the following one:

$$F^{-} = F^{+} \ominus F \tag{2.37}$$

3) One can define F^{+} from F^{-} by

$$F^{+} = F \oplus F^{-} \tag{2.38}$$

4) $$F = F^{+} \ominus F^{-} \tag{2.39}$$

5) From (2.7) one deduces

$$F^{-} = (F^{+})' \tag{2.40}$$

and

$$F^{+} = (F^{-})' \tag{2.41}$$

6) If F and G are subsets of U:

$$F \subset G \text{ iff } F^{+} \subset G^{+} \tag{2.42}$$

$$F \subset G \text{ iff } G^{-} \subset F^{-} \tag{2.43}$$

7) For all subsets F and G of U,

$$F^{-} \subset G^{+} \tag{2.44}$$

$$(F \cup G)^{+} = F^{+} \cup G^{+} = F^{+} \cup G = F \cup G^{+} \tag{2.45}$$

$$(F \cap G)^{+} = F^{+} \cap G^{+} \tag{2.46}$$

$$(F \cup G)^- = F^- \cap G^- = F^- \cap G' = F' \cap G^- \tag{2.47}$$

$$(F \cap G)^- = F^- \cup G^- \tag{2.48}$$

$$(FG)^+ = F^+ G^+ \tag{2.49}$$

$$(FG)^- = F^- \hat{+} G^- \tag{2.50}$$

$$F^- G^- = (F \hat{+} G)^- \; , \tag{2.51}$$

where in (2.50) and (2.51), the symbol $\hat{+}$ stands for the algebraic sum: $a \hat{+} b = a + b - ab$ for a and b in $[0,1]$, [4(I)].

$$(F \oplus G)^+ = F^+ \oplus G^+ = F^+ \oplus G = F \oplus G^+ \tag{2.52}$$

$$(F \oplus G)^- = F^- \ominus G = G^- \ominus F \tag{2.53}$$

$$F^- \oplus G^- = (F \ominus G')^- = (G \ominus F')^- \tag{2.54}$$

$$(F \ominus G)^- = F^- \oplus (G')^- \tag{2.55}$$

$$F^- \ominus G^- = (F \oplus G')^- = F^- \ominus G' \tag{2.56}$$

<u>Extremal Cases</u>

Let θ denote the empty set

$$\theta = \int_U 0/u \tag{2.57}$$

$$X \text{ is in } \theta \longrightarrow R(X) = \theta \tag{2.58}$$

$$U = \int_U 1/u \tag{2.59}$$

$$X \text{ is in } U \longrightarrow R(X) = U$$

$$X \text{ is in } \theta \text{ is } \underline{\text{possible}} \longrightarrow R(X) = \theta^{+} \tag{2.60}$$

$$\theta^{+} = \theta \oplus \Pi = \Pi$$

θ^{+} and Π correspond to the truth-value interpreted as unknown and denoted by ?, see [9(II)].

$$\theta^{+} = \Pi = ? \tag{2.61}$$

$$X \text{ is in } U \text{ is } \underline{\text{possible}} \longrightarrow R(X) = U^{+} \tag{2.62}$$

$$U^{+} = U \oplus \Pi = U \tag{2.63}$$

$$X \text{ is in } \theta \text{ is } \underline{\text{impossible}} \longrightarrow R(X) = \theta^{-} \tag{2.64}$$

$$\theta^{-} = \Pi \ominus \theta = \Pi \tag{2.65}$$

$$\theta^{-} = \theta^{+} = \Pi = ? \tag{2.66}$$

$$X \text{ is in } U \text{ is } \underline{\text{impossible}} \longrightarrow R(X) = U^{-} \tag{2.67}$$

$$U^{-} = \Pi \ominus U = \theta \tag{2.68}$$

ε-Possibility Qualified Propositions

Let ε be a point in $[0,1]$. We define ε-possibility qualified propositions by the following translation rule.

$$X \text{ is in } F \text{ is } \varepsilon\text{-}\underline{\text{possible}} \longrightarrow R(X) = F(\varepsilon) \tag{2.69}$$

where $F(\varepsilon)$ is a fuzzy subset of U, interval-valued, which is assigned to $R(X)$. $F(\varepsilon)$ is defined by

$$\mu_{F(\varepsilon)}(u) = [\varepsilon \wedge \mu_F(u), \varepsilon \vee \mu_{F'}(u)] , \quad u \in U \tag{2.70}$$

The concept of ε-possibility is due to Zadeh who gives the following expression corresponding to the right-hand member of (2.70).

$$[\varepsilon \wedge \mu_F(u), \varepsilon \oplus \mu_{F'}(u)] \;, \quad u \in U \tag{2.71}$$

Recalling that we deal with non fuzzy subsets F of U, when $\mu_F(u) = 0$ or when $\mu_F(u) = 1$, the expressions given by (2.70) and (2.71) coincide for when $\mu_F(u) \in \{0,1\}$, $\varepsilon \vee \mu_{F'}(u) = \varepsilon \oplus \mu_{F'}(u)$ for all ε in $[0,1]$.

When $\mu_F(u) \in [0,1]$, $\varepsilon \vee \mu_{F'}(u) \leq \varepsilon \oplus \mu_{F'}(u)$, see (2.11). Hence for all u in U, (2.71) is included (set inclusion or inclusion in the sense of (2.6)) in the right-hand member of (2.70). The choice of (2.70) is discussed in the next section. From (2.70),

$$F(0) = F^- \tag{2.72}$$

$$F(1) = F^+ \tag{2.73}$$

$$F^- \subset F(\varepsilon) \subset F^+ \;, \quad \varepsilon \in [0,1] \tag{2.74}$$

$$\begin{aligned} \mu_{F(\varepsilon)}(u) &= \varepsilon && \text{for } u \in F \\ &= [0,1] && \text{for } u \notin F \end{aligned} \tag{2.75}$$

For example,

$$\begin{aligned} \mu_{F(\varepsilon)}(u) &= \varepsilon && \text{for } a \leq u \leq b \\ &= [0,1] && \text{elsewhere} \end{aligned} \tag{2.76}$$

When ε increases from 0 to 1, the ε-possibility increases from the impossibility, formulated in (2.32), to the possibility, formulated in (2.25).

Intuitively, from (2.75), we can say that the subset F of U is evenly spread to a degree ε and there is total uncertainty outside it.

For all ε in $[0,1]$, we define the fuzzy subset ε of U by

$$\varepsilon = \int_U \varepsilon/u \;, \tag{2.77}$$

that is, we identify a "constant fuzzy set" with the unique value taken by its compatibility function. In particular $\theta = 0$ and $U = 1$.

From (2.70) it is easy to verify that

$$F(\varepsilon) = (\varepsilon \cap F^+) \cup F^- \tag{2.78}$$

or

$$F(\varepsilon) = (\varepsilon \cup F^-) \cap F^+ \ , \tag{2.79}$$

When F is non fuzzy, one can easily find many equivalent formulations for $F(\varepsilon)$, for example

$$F(\varepsilon) = (\varepsilon \cap F) \oplus F^- \tag{2.80}$$

$$F(\varepsilon) = F^+ \ominus (\varepsilon' \cap F) \ , \tag{2.81}$$

and many other expressions, replacing the intersections by products, the bounded sums by unions, etc. They are all equivalent formulations for $\mu_F(u) \in \{0,1\}$ for all u in U, but when F is a fuzzy set this result is no longer true.

Remark that the expression (2.80) is the same as (2.71), given by Zadeh, according to (2.13).

From (2.32), (2.40) and (2.72) we can write

X is in F is <u>impossible</u> ,
X is in F is <u>not</u> <u>possible</u> and (2.82)
X is in F is <u>0-possible</u>

have the same translation rule given by

$$R(X) = F^- = (F^+)' = F(0) \ .$$

For all ε in $[0,1]$,

$$\theta(\varepsilon) = \Pi \tag{2.83}$$

and

$$U(\varepsilon) = \varepsilon \tag{2.84}$$

where θ and U are considered as functions of ε, and ε in the right-hand member of (2.84) stands for a constant fuzzy set, see (2.77).

ε-Lower-Possibility Qualified Propositions

Let ε be a point in $[0,1]$. We define ε-lower-possibility qualified propositions by the following translation rule

$$X \text{ is in } F \text{ is } \underline{\varepsilon\text{-lower-possible}} \longrightarrow R(X) = F(\varepsilon)_* \tag{2.85}$$

where $F(\varepsilon)_*$ is a fuzzy subset of U, interval-valued, which is assigned to $R(X)$. $F(\varepsilon)_*$ is defined by

$$\begin{aligned} \mu_{F(\varepsilon)_*}(u) &= \mu_F(u) \oplus [0,\varepsilon] \\ &= [\mu_F(u), \mu_F(u) \oplus \varepsilon] \end{aligned} \tag{2.86}$$

where $[0,\varepsilon]$ can be considered as a "lower subset" of Π, see (2.26) and (2.27).

When ε increases from 0 to 1, the ε-lower-possibility increases from the certainty in F, formulated in (2.23), to the possibility, formulated in (2.25)

$$F(0)_* = F \tag{2.87}$$

and

$$F(1)_* = F^+ \tag{2.88}$$

$$F \subset F(\varepsilon)_* \subset F^+ , \quad \varepsilon \in [0,1] \tag{2.89}$$

$$\mu_{F(\varepsilon)_*}(u) = 1 \quad \text{for} \quad u \in F$$
$$= [0,\varepsilon] \quad \text{for} \quad u \notin F \tag{2.90}$$

For example,

$$\mu_{F(\varepsilon)_*}(u) = 1 \quad \text{for} \quad a \leq u \leq b$$
$$= [0,\varepsilon] \quad \text{elsewhere} \tag{2.91}$$

Intuitively, from (2.90), we can say that, whereas "X is in F" implies that the degree of possibility that X is outside F is zero, "X is in F is <u>ε-lower-possible</u>" implies that there is a "lower uncertainty" bounded by ε, outside F.

<u>ε-Upper-Possibility Qualified Propositions</u>

This notion is the dual of the previous one.

$$X \text{ is in } F \text{ is } \underline{\varepsilon\text{-upper-possible}} \longrightarrow R(X) = F(\varepsilon)^* \tag{2.92}$$

where ε is a point in $[0,1]$ and

$$\mu_{F(\varepsilon)^*}(u) = [\varepsilon,1] \ominus \mu_F(u)$$
$$= [\varepsilon \ominus \mu_F(u), \mu_{F'}(u)] \tag{2.93}$$

$[\varepsilon,1]$ can be considered as an "upper subset" of Π, see (2.33) and (2.27).

When ε increases from 0 to 1, the ε-upper-possibility increases from the impossibility, formulated in (2.32) to the certainty in not F.

$$F(0)^* = F^- \tag{2.94}$$

and

$$F(1)^* = F' \tag{2.95}$$

$$F^- \subset F(\varepsilon)^* \subset F' , \quad \varepsilon \in [0,1] \tag{2.96}$$

$$\mu_{F(\varepsilon)^*}(u) = 0 \quad \text{for} \quad u \in F \qquad (2.97)$$
$$= [\varepsilon,1] \quad \text{for} \quad u \notin F$$

For example,

$$\mu_{F(\varepsilon)^*}(u) = 0 \quad \text{for} \quad a \le u \le b \qquad (2.98)$$
$$= [\varepsilon,1] \quad \text{elsewhere}$$

Intuitively, from (2.97), we can say that, whereas "X is in not F" implies that the degree of possibility that X is outside F is one, "X is in F is ε-upper-possible" implies that there is an "upper uncertainty" bounded by ε, outside F.

One can easily verify that

$$\theta(1)_* = \theta(0)^* = \Pi \qquad (2.99)$$

$$F(\varepsilon)_* = F \oplus \theta(\varepsilon)_* \qquad (2.100)$$

$$F(\varepsilon)^* = \theta(\varepsilon)^* \ominus F \qquad (2.101)$$

$$\left(F(\varepsilon)_*\right)' = F(\varepsilon')^* \qquad (2.102)$$

and

$$\left(F(\varepsilon)^*\right)' = F(\varepsilon')_* \qquad (2.103)$$

3. POSSIBILISTIC PROPOSITIONS RELATED TO FUZZY SETS

In the sequel, the examples chosen to illustrate the propositions are based on:

-- the universe of discourse U equal to the set of the non negative real numbers,

-- the fuzzy subset of U, F = small defined by $0 < a < b$ and

$$\mu_F(u) = 1 - S(u;a,\tfrac{a+b}{2},b) \qquad (3.1)$$

where S is the S-function [10] whose parameters may be adjusted to fit a specified compatibility function related to the concept small. In other terms,

$$\begin{aligned} \mu_F(u) &= 1 && \text{for } u \leq a \\ &= 1 - 2\left(\frac{u-a}{b-a}\right)^2 && \text{for } a \leq u \leq \frac{a+b}{2} \\ &= 2\left(\frac{u-b}{b-a}\right)^2 && \text{for } \frac{a+b}{2} \leq u \leq b \\ &= 0 && \text{for } u \geq b \end{aligned} \tag{3.2}$$

The shape of μ_F is illustrated in Figure 1.

Possibility-Qualified Propositions Related to Fuzzy Sets

Let U be a universe of discourse, X a variable that takes values in U and F a fuzzy subset of U, the proposition

$$X \text{ is } F \longrightarrow R(A(X)) = F \tag{3.3}$$

where A is an implied attribute of X, $A(X)$ a fuzzy variable which takes values in U, $R(A(X))$ a fuzzy restriction on the values that may be assigned to $A(X)$ and F is a unary fuzzy relation which is assigned to $R(A(X))$. For example,

$$\text{John is } \underline{\text{small}} \longrightarrow R(\text{Height(John)}) = \underline{\text{small}} \tag{3.4}$$

Extending now (2.25) with F being a fuzzy subset of U,

$$X \text{ is } F \text{ is } \underline{\text{possible}} \longrightarrow R(A(X)) = F^+ \tag{3.5}$$

where, see [10,11],

$$F^+ = F \oplus \Pi \tag{3.6}$$

$$\mu_{F^+}(u) = [\mu_F(u),1] = \mu_F(u) \oplus [0,1] \quad . \tag{3.7}$$

For example, see Figure 1,

$$\mu_{F^+}(u) = [1 - S(u;a,\tfrac{a+b}{2},b),1] \tag{3.8}$$

or

$$\mu_{\underline{small}^+}(u) = [\mu_{\underline{small}}(u),1] \tag{3.9}$$

Impossibility-Qualified Propositions Related to Fuzzy Sets

Extending (2.32) with F being a fuzzy subset of U,

$$X \text{ is } F \text{ is } \underline{\text{impossible}} \longrightarrow R(A(X)) = F^- \;, \tag{3.10}$$

where

$$F^- = \Pi \ominus F \tag{3.11}$$

$$\mu_{F^-}(u) = [0,\mu_{F'}(u)] = [0,1] - \mu_F(u) \tag{3.12}$$

For example, see Figure 2,

$$\mu_{F^-}(u) = [0,S(u;a,\tfrac{a+b}{2},b)] \tag{3.13}$$

or

$$\mu_{\underline{small}^-}(u) = [0,\mu_{\underline{not}\ \underline{small}}(u)] \tag{3.14}$$

One can note that properties from (2.36) to (2.68), that is the properties related to F, F^+ and F^-, still hold when F is a fuzzy set.

ε-Possibility-Qualified Propositions Related to Fuzzy Sets

When ε increases from 0 to 1, one may expect the ε-possibility to express all the variations of possibility from the impossibility, formulated in (3.10), to the possibility, formulated in (3.5).

As we already noted it, in the case of non fuzzy sets, one can find many different, but equivalent, expressions for the ε-possibility.

Unfortunately, the extensions of all expressions to fuzzy sets don't give equivalent formulations. Considering, for example, (2.80) and (2.81) with $\varepsilon = 0.7$ and $\mu_F(u) = 0.4$, we derive

from (2.80): $(0.7 \wedge 0.4) \oplus [0,0.6] = [0.4,1]$

and

from (2.81): $[0.4,1] \ominus (0.3 \wedge 0.4) = [0.1,0.7]$.

Let us denote by $F_1(\varepsilon)$ the left-hand member of (2.81) and by $F_2(\varepsilon)$ the left-hand member of (2.80), where $F_2(\varepsilon)$ corresponds to the expression given by Zadeh for the concept of ε-possibility.

$$F_1(\varepsilon) = F^+ \ominus (\varepsilon' \cap F) \tag{3.15}$$

$$F_2(\varepsilon) = (\varepsilon \cap F) \oplus F^- \tag{3.16}$$

The two following properties holds for all ε in $[0,1]$

$$F_1(\varepsilon) \subset F_2(\varepsilon) \tag{3.17}$$

and

$$F_1(\varepsilon') = (F_2(\varepsilon))' \tag{3.18}$$

The illustrations for $F_1(\varepsilon)$ and $F_2(\varepsilon)$ based on F = <u>small</u> are given in Figure 3.

In order to study $F(\varepsilon)$ when F is a fuzzy set, let u be an element of U and let us denote by f the quantity $\mu_F(u)$. Then

$$\mu_{F(\varepsilon)}(u) = [a(\varepsilon,f),b(\varepsilon,f)] \tag{3.19}$$

As a first consideration for the extension of $F(\varepsilon)$, we require the following extremal conditions:

C.1 $\quad a(0,f) = 0$

and

C.2 $$b(0,f) = f' ,$$

that is, when $\varepsilon = 0$, (3.19) should stand for the impossibility, see (3.12).

C.3 $$a(1,f) = f$$

and

C.4 $$b(1,f) = 1 ,$$

that is, when $\varepsilon = 1$, (3.19) should stand for the possibility, see (3.7).

C.5 $$a(\varepsilon,0) = 0$$

C.6 $$b(\varepsilon,0) = 1$$

and

C.7 $$a(\varepsilon,1) = \varepsilon$$

C.8 $$b(\varepsilon,1) = \varepsilon ,$$

that is, when F is non fuzzy, (3.19) should coincide with the ε-possibility, see (2.69) and (2.75), which was described in the previous section.

In order to reduce fuzziness in the possibility, one should require for $F(\varepsilon)$ the minimal uncertainty, in the sense that for all u in U, $b(\varepsilon,f) - a(\varepsilon,f)$ should be minimal for the expressions of a and b verifying C.1 to C.8. In other words, for all u in U, $\mu_{F(\varepsilon)}(u)$ should be contained (in the set inclusion sense) in the expressions for the ε-possibilities verifying the extremal conditions.

For a given f, when ε increases from 0 to 1, that is when we pass from the impossibility to the possibility, it is natural to require from C.1, C.3 and C.2, C.4

$$0 \leq a(\varepsilon,f) \leq f \tag{3.20}$$

and

$$f' \leq b(\varepsilon,f) \leq 1 \tag{3.21}$$

Now, for a given ε and a given u, for the f's closer to 0 or 1, that is for the less fuzzy f's, one should require less uncertainty, i.e., a should increase and b should decrease, so that C.5, C.7 and C.6, C.8 imply

$$0 \leq a(\varepsilon,f) \leq \varepsilon \tag{3.22}$$

and

$$\varepsilon \leq b(\varepsilon,f) \leq 1 \tag{3.23}$$

Requiring now a "minimal uncertainty" for $F(\varepsilon)$, from (3.20) and (3.22) we derive

$$a(\varepsilon,f) \leq \text{greatest lower bound of } f \text{ and } \varepsilon \ ,$$

that is,

$$a(\varepsilon,f) \leq \varepsilon \wedge f \tag{3.24}$$

and from (3.21) and (3.23) we derive

$$b(\varepsilon,f) \geq \text{least upper bound of } \varepsilon \text{ and } f' \ ,$$

that is,

$$b(\varepsilon,f) \geq \varepsilon \vee f' \quad . \tag{3.25}$$

So that under the above considerations, we choose

$$\mu_{F(\varepsilon)}(u) = [\varepsilon \wedge \mu_F(u), \varepsilon \vee \mu_{F'}(u)] \ , \quad u \in U \tag{3.26}$$

which may be interpreted in our example by

$$F(\varepsilon) = [\varepsilon \cap \underline{\text{small}}, \varepsilon \cup \underline{\text{not}}\ \underline{\text{small}}] \ , \tag{3.27}$$

see Figure 4 and section 4.

(3.27) is an extension of (2.70) with F being a fuzzy set. We define, for ε in [0,1],

$$X \text{ is } F \text{ is } \varepsilon\text{-possible} \longrightarrow R(A(X)) = F(\varepsilon) \tag{3.28}$$

where $F(\varepsilon)$ is defined in (3.27).

One can verify that

$$F(\varepsilon) = (\varepsilon \cap F^+) \cup F^- \tag{3.29}$$

and

$$F(\varepsilon) = (\varepsilon \cup F^-) \cap F^+ \tag{3.30}$$

$$F(\varepsilon') = (F(\varepsilon))' \tag{3.31}$$

Considering F as a function of ε, when ε increases from 0 to 1, F increases from F^- to F^+ in the fuzzy inclusion sense.

$$F^- \subset F(\varepsilon) \subset F^+ , \quad \varepsilon \in [0,1] \tag{3.32}$$

With $F_1(\varepsilon)$ and $F_2(\varepsilon)$ defined in (3.15) and (3.16), respectively,

-- in the set inclusion sense:

$$\mu_{F(\varepsilon)}(u) \subseteq \mu_{F_1(\varepsilon)}(u) , \quad u \in U \tag{3.33}$$

and

$$\mu_{F(\varepsilon)}(u) \subseteq \mu_{F_2(\varepsilon)}(u) , \quad u \in U \tag{3.34}$$

-- in fact, in the set intersection sense:

$$\mu_{F(\varepsilon)}(u) = \mu_{F_1(\varepsilon)}(u) \cap \mu_{F_2(\varepsilon)}(u) \tag{3.35}$$

ε-Lower-Possibility Qualified Propositions Related to Fuzzy Sets

Let ε be a point in [0,1], extending (2.85) and (2.86) with F being a fuzzy subset of U. We define

$$X \text{ is } F \text{ is } \underline{\varepsilon\text{-lower-possible}} \longrightarrow R(A(X)) = F(\varepsilon)_* \qquad (3.36)$$

where

$$\mu_{F(\varepsilon)_*}(u) = \mu_F(u) \oplus [0,\varepsilon] \quad , \qquad (3.37)$$

see Figure 5(a) for an illustration with F being the fuzzy set labelled small.

When ε increases from 0 to 1, the ε-lower-possibility increases from the certainty in F, $F(0)_* = F$, to the possibility, $F(1)_* = F^+$.

ε-Upper-Possibility Qualified Propositions Related to Fuzzy Sets

Let ε be a point in [0,1], extending (2.92) and (2.93) with F being a fuzzy subset of U. We define

$$X \text{ is } F \text{ is } \underline{\varepsilon\text{-upper-possible}} \longrightarrow R(A(X)) = F(\varepsilon)^* \qquad (3.38)$$

where

$$\mu_{F(\varepsilon)^*}(u) = [\varepsilon,1] \ominus \mu_F(u) \ , \qquad (3.39)$$

see Figure 5(b) for an illustration.

When ε increases from 0 to 1, the ε-upper-possibility increases from the impossibility, $F(0)^* = F^-$, to the certainty in not F, $F(1)^* = F'$.

The properties expressed from (2.99) to (2.103) still hold when the ε-lower (or upper) -possibilistic propositions are related to fuzzy sets.

4. INTERVAL-VALUED FUZZY SETS AND POSSIBILISTIC SUB-PROPOSITIONS

Let U be a universe of discourse and C a fuzzy subset of U interval-valued. Let us denote by

$$\mu_C(u) = [a(u),b(u)] \ , \quad u \in U \ , \tag{4.1}$$

where a(u) and b(u) are elements of [0,1] and $a(u) \leq b(u)$. Let us define the fuzzy subsets A and B of U by

$$A = \int_U (\inf \mu_C(u))/u = \int_U a(u)/u \tag{4.2}$$

and

$$B = \int_U (\sup \mu_C(u))/u = \int_U b(u)/u \ . \tag{4.3}$$

Then,

$$\mu_C(u) = [\mu_A(u),\mu_B(u)] \ , \tag{4.4}$$

which we denote symbolically by

$$C = [A,B] \tag{4.5}$$

C is interpreted as a linguistic interval whose bounds are fuzzy sets. For example, see (3.9),

$$\underline{\text{small}}^+ = [\underline{\text{small}},U] \ , \tag{4.6}$$

which denotes all the fuzzy subsets of U containing, in the fuzzy sense the fuzzy subset of U labelled small, or, see (3.14),

$$\underline{\text{small}}^- = [\theta,\underline{\text{not}}\ \underline{\text{small}}] \ , \tag{4.7}$$

which denotes all the fuzzy subsets of U included, in the fuzzy sense, into the fuzzy subset of U labelled not small.

One easily shows that

$$C = [A,B] \quad \text{iff} \quad C = A^{+} \ominus B^{-} \tag{4.8}$$

$$\text{"X is C, } C = [A,B]\text{"} \tag{4.9}$$

$$\equiv \text{"X is A is \underline{possible}"} \ominus \text{"X is B is \underline{impossible}"}$$

(4.8) relates interval-valued fuzzy sets to the possibilistic sub-propositions A^{+} and B^{-}. For example, see (3.27),

$$\underline{\text{small}}(\varepsilon) = [\varepsilon \cap \underline{\text{small}}, \ \varepsilon \cup \underline{\text{not}} \ \underline{\text{small}}] \ . \tag{4.10}$$

From (4.8):

$$\underline{\text{small}}(\varepsilon) = (\varepsilon \cap \underline{\text{small}})^{+} \ominus (\varepsilon \cup \underline{\text{not}} \ \underline{\text{small}})^{-} \tag{4.11}$$

$$\begin{aligned}
\underline{\text{small}}(0) &= \theta^{+} \ominus (\underline{\text{not}} \ \underline{\text{small}})^{-} \\
&= [\theta, U] \ominus [\theta, \ \underline{\text{not}} \ \underline{\text{not}} \ \underline{\text{small}}] \\
&= [\theta, \ \underline{\text{not}} \ \underline{\text{small}}] \\
&= \underline{\text{small}}^{-}
\end{aligned}$$

$$\begin{aligned}
\underline{\text{small}}(1) &= \underline{\text{small}}^{+} \ominus U^{-} \\
&= \underline{\text{small}}^{+} \ominus \theta \\
&= \underline{\text{small}}^{+}
\end{aligned}$$

5. POSSIBILISTIC PROPOSITIONS RELATED TO FUZZY RELATIONS

The basic translation rules in fuzzy logic of this section and the next one are referred to in [1].

In section 3, we were concerned by translation rules of Type I, that is translation rules for operations that involve attribute modification. They apply to fuzzy propositions of the form

$$p \triangleq X \text{ is } mF \tag{5.1}$$

where F is a fuzzy subset of a universe of discourse U, m is a modifier and either X or A(X), where A is an implied attribute of X, is a fuzzy variable that takes values in U.

The modifier rule asserts that the translation of a fuzzy proposition of the form (5.1) is expressed by

$$X \text{ is } mF \longrightarrow R(A(X)) = mF \tag{5.2}$$

where m is interpreted as an operator which transforms the fuzzy set F into the fuzzy set mF.

In section 3, replacing F by mF, one has possibilistic propositions related to translation rules of Type I. For example,

$$X \text{ is } \underline{\text{very}}\ \underline{\text{small}} \text{ is } \underline{\text{possible}} \longrightarrow R(\text{Height}(X)) = (\underline{\text{very}}\ \underline{\text{small}})^{+} = (\underline{\text{small}}^{2})^{+} \ . \tag{5.3}$$

With small defined in (3.1), very small = mF = F^2

$$\mu_{F^2}(u) = \mu_{\underline{\text{very}}\ \underline{\text{small}}}(u) = \left(1 - S(u;a,\tfrac{a+b}{2},b)\right)^2 \tag{5.4}$$

and

$$\mu_{(\underline{\text{very}}\ \underline{\text{small}})^{+}}(u) = [\mu_{\underline{\text{very}}\ \underline{\text{small}}}(u),\ 1] \tag{5.5}$$

or

$$(\underline{\text{very}}\ \underline{\text{small}})^{+} = [\underline{\text{very}}\ \underline{\text{small}},\ U] \ . \tag{5.6}$$

An n-ary fuzzy relation R is a fuzzy subset of a cartesian product of n universes of discourse; hence one can apply the results of section 3 and 4. Our purpose is now to develop some results which are specific to fuzzy relations. For the simplicity of the presentation we shall deal with 2-ary fuzzy relations.

Translation rules of Type II apply to composite fuzzy propositions which are generated from fuzzy propositions of the form "X is F" through the use of various kinds of binary connectives such as the conjunction and, the disjunction or, the conditional if...then..., etc.

Possibilistic Propositions and Projections of Fuzzy Relations

Let U and V be two possibly different universes of discourse, X a variable that takes values in U, Y a variable that takes values in V and S a fuzzy relation from U to V.

$$X \text{ and } Y \text{ are } S \longrightarrow R(A(X),B(Y)) = S \tag{5.7}$$

where A and B are implied attributes of X and Y, respectively.

We recall that the projection (shadow) of S on U is a 1-ary fuzzy relation (i.e., a fuzzy set) S_1 on U which is defined by

$$\begin{aligned} S_1 &\triangleq \text{Proj } S \text{ on } U \triangleq P_1 S \\ &\triangleq \int_U \sup_{v \in V} \mu_S(u,v)/u \end{aligned} \tag{5.8}$$

Similarly, the projection of S on V is a fuzzy subset S_2 of V which is defined by

$$\begin{aligned} S_2 &\triangleq \text{Proj } S \text{ on } V \triangleq P_2 S \\ &\triangleq \int_V \sup_{u \in U} \mu_S(u,v)/v \end{aligned} \tag{5.9}$$

One can show that projections commute with possibility,

$$(S^+)_1 = (S_1)^+ \tag{5.10}$$

and

$$(S^+)_2 = (S_2)^+ \tag{5.11}$$

Possibilistic Propositions and the Compositional Rule of Inference

Let F be a fuzzy subset of U and the proposition

$$X \text{ is } F \longrightarrow R(C(X)) = F \,, \qquad (5.12)$$

where C is an implied attribute of X.

The compositional rule of inference asserts that the solution of the two above relational equations, say (5.7) and (5.12), is given by

$$Y \text{ is } G \longrightarrow R(D(Y)) = G \,, \qquad (5.13)$$

where D is an implied attribute of Y and $G = F \circ S$.[1]

Considering now possibilistic propositions, one can show that

$$(F \circ S)^+ = F^+ \circ S^+ \,, \qquad (5.14)$$

$$F \circ (S^+) = (F \circ S)^+ \quad \text{iff} \quad F \text{ is normal} \,, \qquad (5.15)$$

i.e., iff $\sup_{u \in U} \mu_F(u) = 1$,

$$F^+ \circ S = (F \circ S)^+ \quad \text{iff} \quad S_2 = V \,, \qquad (5.16)$$

where S_2 is the projection of S on V, see (5.9).

As a simple illustration of these results, (5.16) for example, let us assume that (see Notation for fuzzy sets in section 2)

$$U = V = 1 + 2 + 3 + 4 \qquad (5.17)$$

---

[1] We recall that if R is a fuzzy relation from U to V (or, equivalently, a fuzzy subset of $U \times V$) and if S is a fuzzy relation from V to W, then the composition of R and S is a fuzzy relation from U to W denoted by $R \circ S$ and defined by

$$R \circ S = \int_{U \times V} \sup_{v \in V} \left(\mu_R(u,v) \wedge \mu_S(v,w)\right)/(u,w) \quad .$$

Remark that the notation $S \circ R$ replaces in some papers the notation $R \circ S$.

$$F = \underline{small} = 0.9/1 + 0.6/2 + 0.2/3 \tag{5.18}$$

$$S = \underline{approximately}\ \underline{equal} \tag{5.19}$$

$$S = 1/[(1,1)+(2,2)+(3,3)+(4,4)]$$
$$+ 0.5/[(1,2)+(2,1)+(2,3)+(3,2)+(3,4)+(4,3)]$$

S verifies $\sup_u \mu_S(u,v) = 1$ for all v in V, that is $S_2 = V$. Hence, as one can verify

$$(F \circ S)^+ = F^+ \circ S = [0.9,1]/1 + [0.6,1]/2 + [0.5,1]/3 + [0.2,1]/4 \tag{5.20}$$

Finally,

$$\begin{array}{c} X \text{ is } \underline{small} \text{ is } \underline{possible} \\ X \text{ and } Y \text{ are } \underline{approximately}\ \underline{equal} \\ \hline Y \text{ is } \underline{more}\ \underline{or}\ \underline{less}\ \underline{small} \text{ is } \underline{possible} \end{array} \tag{5.21}$$

where more or less small is possible is a linguistic term that may approximate $(F \circ S)^+$, see [1] in which a linguistic approximation of

$$(\underline{small}) \circ (\underline{approximately}\ \underline{equal})$$

is given by (5.22)

$$\underline{more}\ \underline{or}\ \underline{less}\ \underline{small}\ .$$

Possibilistic Propositions and the Extension Principle

Let U and V be two possibly different universes of discourse, F a fuzzy subset of U and f a mapping from U to V, then the extension principle, see (2.4), asserts that

$$f(F) = \int_V \mu_F(u)/f(u) \tag{5.23}$$

or, equivalently,

$$\mu_{f(F)}(v) = \sup_{u \in f^{-1}(v)} \mu_F(u) , \quad v \in V . \tag{5.24}$$

Let X be a variable that takes values in U and $f(X)$ a variable that takes values in V. One can show that

if X is F is possible, then f(X) is f(F) is possible . (5.25)

In other terms,

$$f(F^+) = (f(F))^+ , \tag{5.26}$$

where $f(F^+)$ is defined by the extension principle.

More generally, let $*$ be a binary operation defined on $U \times V$ with values in W. Thus, if $u \in U$ and $v \in V$, then

$$w = u * v , \quad w \in W . \tag{5.27}$$

Suppose now that F and G are fuzzy subsets of U and V, respectively. Then, by using the extension principle, one may define

$$\begin{aligned} F * G &= \int_W (\mu_F(u) \wedge \mu_G(v))/u*v \\ &= \int_W \mu_{F\times G}(u,v)/u*v \end{aligned} \tag{5.28}$$

and it is easily shown that

$$(A * B)^+ = A^+ * B^+ . \tag{5.29}$$

Let us note that we already used the fact that the notion of possibility is universal, in the sense that we can compare, in terms of possibility, fuzzy subsets of different universes of discourse.

6. POSSIBILISTIC PROPOSITIONS RELATED TO THE CONDITIONAL COMPOSITION AND THE COMPOSITIONAL MODUS PONENS

Possibilistic Propositions and the Rule of Conditional Composition

Let U and V be two possibly different universes of discourse, X and Y be variables that take values in U and V, respectively, and let F and G be fuzzy subsets of U and V, respectively.

Conditional fuzzy propositions of Type II of the form "If X is F then Y is G" have a translation rule, referred to as the rule of conditional composition, which may be expressed as[2]

$$\text{If X is F, then Y is G} \longrightarrow R(A(X),B(Y)) = (\bar{F})' \oplus \bar{G} \ , \qquad (6.1)$$

where $\oplus$ denotes the bounded sum, and $\bar{F}$ and $\bar{G}$ denote the cylindrical extensions of F and G, respectively.

$$\bar{F} = \int_{U\times V} \mu_F(u)/(u,v) \qquad (6.2)$$

and

$$\bar{G} = \int_{U\times V} \mu_G(v)/(u,v) \ . \qquad (6.3)$$

That is, F is the projection of $\bar{F}$ on U and G is the projection of $\bar{G}$ on V.

One can show that the operator of cylindrical extension commutes with the operator of possibility. For example, in the case of 2-ary fuzzy relations for simplicity, and with the notations of "Possibilistic propositions and projections of fuzzy relations" in section 5,

$$(\bar{S}_1)^+ = \overline{(S_1{}^+)} \ , \qquad (6.4)$$

---

[2]It is tacitly understood that this rule is non interactive in nature. In the form defined by (6.1), it is consistent with the definition of implication in L_{aleph_1} logic, see [8].

where $\bar{S}_1$ is the cylindrical extension of S_1, with S_1 constituting the base of $\bar{S}_1$.

Moreover,

$$(\bar{F})' \oplus \overline{G^+} = \left((\bar{F})' \oplus \bar{G}\right)^+ \tag{6.5}$$

which can be interpreted as

$$\begin{aligned} &\text{If (X is F) then (Y is G is } \underline{\text{possible}}) \\ &\equiv \left(\text{If (X is F) then (Y is G)}\right) \text{ is } \underline{\text{possible}} \end{aligned} \tag{6.6}$$

Let us consider now the conditional fuzzy proposition

$$\text{If X is F then Y is G else Y is H} \tag{6.7}$$

which is interpreted as the conjunction of the propositions

$$\text{If X is F then Y is G} \tag{6.8}$$

and

$$\text{If X is } \underline{\text{not}} \text{ F then Y is H} \tag{6.9}$$

has the following translation rule:

$$R(A(X),B(Y)) = (\bar{F}' \oplus \bar{G}) \cap (\bar{F} \oplus \bar{H}) \ . \tag{6.10}$$

From (6.5) and (2.46) we derive

$$(\bar{F}' \oplus \overline{G^+}) \cap (\bar{F} \oplus \overline{H^+}) = \left((\bar{F}' \oplus \bar{G}) \cap (\bar{F} \oplus \bar{H})\right)^+ \tag{6.11}$$

which is interpreted as

$$\begin{aligned} &\text{If (X is F) then (Y is G is } \underline{\text{possible}}) \text{ else (Y is H is } \underline{\text{possible}}) \\ &\equiv \left(\text{If (X is F) then (Y is G) else (Y is H)}\right) \text{ is } \underline{\text{possible}} \end{aligned} \tag{6.12}$$

Possibilistic Propositions and the Rule of Compositional Modus Ponens

From (6.1) and (5.13), we obtain the rule of compositional modus ponens, which reads

$$\frac{\begin{array}{c}X \text{ is } F \\ \text{If } X \text{ is } G \text{ then } Y \text{ is } H\end{array}}{Y \text{ is } F \circ (\bar{G}' \oplus H)} \tag{6.13}$$

Combining now (5.14), (5.15), (5.16) and (6.5), one can show that

$$F^{+} \circ (\bar{G}' \oplus \overline{H^{+}}) = (F \circ (\bar{G}' \oplus \bar{H}))^{+} \tag{6.14}$$

$$F \circ (\bar{G}' \oplus \overline{H^{+}}) = (F \circ (\bar{G}' \oplus \bar{H}))^{+} \quad \text{iff} \quad F \text{ is normal} \tag{6.15}$$

$$F^{+} \circ (\bar{G}' \oplus \bar{H}) = (F \circ (\bar{G}' \oplus \bar{H}))^{+} \tag{6.16}$$

iff the height of H' is less or equal than the height of G' ,

i.e.,

$$\text{iff} \quad \sup_{u \in U} \mu_{G'}(u) \geq \sup_{v \in V} \mu_{H'}(v) \ . \tag{6.17}$$

Let us note that if $U = V$, the condition $G \subset H$ implies (6.17). Moreover, if H' and G' are normal (U possibly different from V), then (6.17) is verified.

ACKNOWLEDGMENT

Our work was stimulated by discussions with Professor L.A. Zadeh and by the lectures he gave in Berkeley and Stanford.

REFERENCES

[1] R.E. Bellman and L.A. Zadeh, Local and fuzzy logics, ERL Memo. M-584, University of California, Berkeley (1976). To appear in Modern Uses of Multiple-Valued Logics, D. Epstein, ed., D. Reidel, Dordrecht (1976).

[2] B.R. Gaines and L.J. Kohout, Possible automata, Proc. Int. Symp. on Multiple-Valued Logics, University of Indiana, Bloomington, Ind., 183-196 (1975).

[3] G.E. Hughes and M.J. Cresswell, An Introduction to Modal Logic, Methuen, London (1968).

[4] A. Kaufmann, Introduction à la Théorie des Sous-ensembles flous, Masson, Paris. Tome I: Eléments Théoriques de Base (1973); Tome II: Langages, sémantique et logique (1975); Tome III: Classification et reconnaissance des formes, automates (1975); Tome IV: Compléments et nouvelles applications (1977); Tome V (E. Sanchez, coll.): Nouveaux compléments (to appear). Also English trans. of Vol. I: Theory of Fuzzy Subsets, Academic Press, New York (1975).

[5] R. Sambuc, Fonctions Φ-floues. Application à l'aide au diagnostic en pathologie thyroidienne. Thèse de Doctorat en Médecine, Marseille (1975).

[6] E. Sanchez, Resolution of composite fuzzy relation equations, Inf. and Control 30, 38-48 (1976).

[7] E. Sanchez and R. Sambuc, Relations floues. Fonctions Φ-floues. Application à l'aide au diagnostic en pathologie thyroidienne, Medical Data Processing Symposium, IRIA, Toulouse (1976). Published by Taylor and Francis, Ltd., London.

[8] L.A. Zadeh, Fuzzy logic and approximate reasoning (in memory of Grigore Moisil), Synthese 30, 407-428 (1975).

[9] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Inf. Sci., Part I, 8, 199-249 (1975); Part II, 8, 301-357 (1975); Part III, 9, 43-80 (1975).

[10] L.A. Zadeh, Theory of fuzzy sets, UCB/ERL Memo M77/1, University of California, Berkeley (1977). To appear in Encyclopedia of Computer Science and Technology, J. Belzer, A. Holzman and A. Kent (eds.), Marcel Dekker, New York (1977).

[11] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, UCB/ERL Memo M77/12, University of California, Berkeley (1977). To appear in the Int. J. of Fuzzy Sets and Systems.

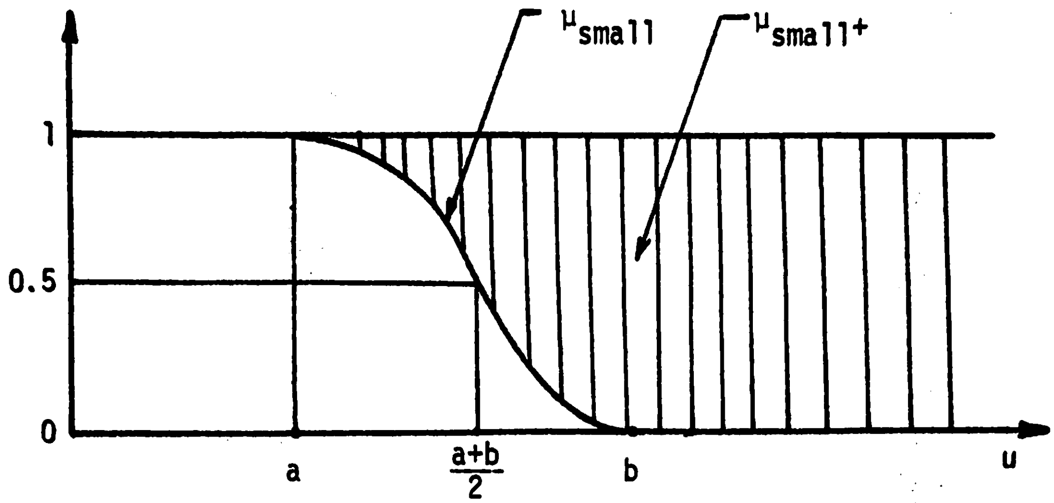


Figure 1. "X is small is possible"

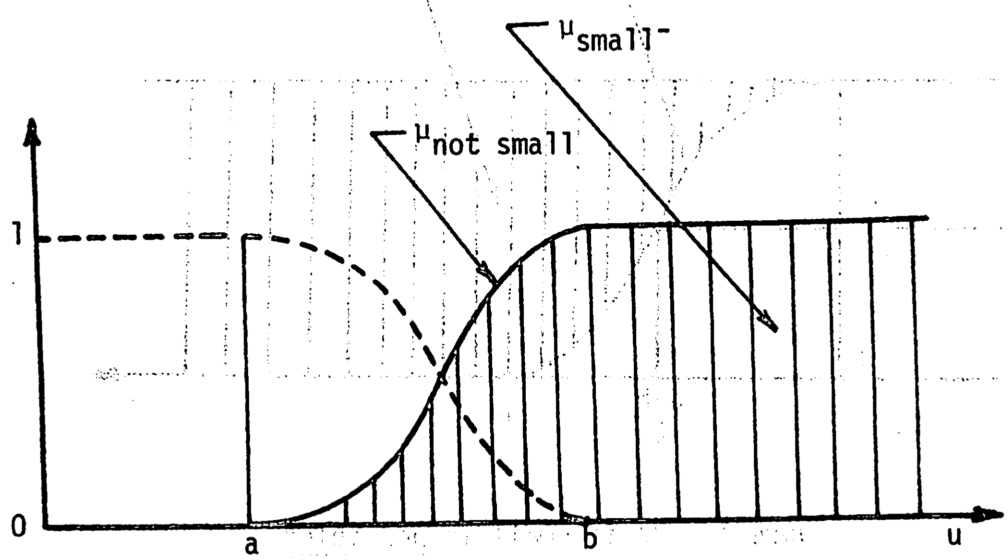
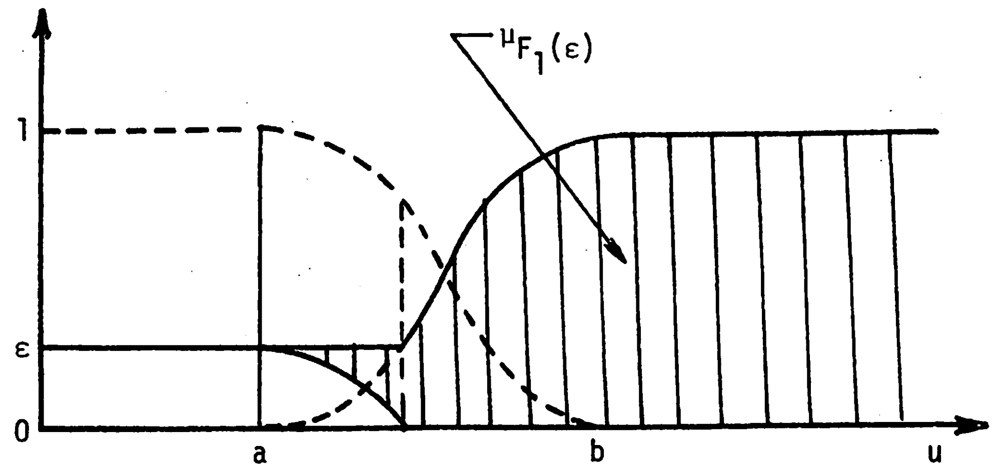
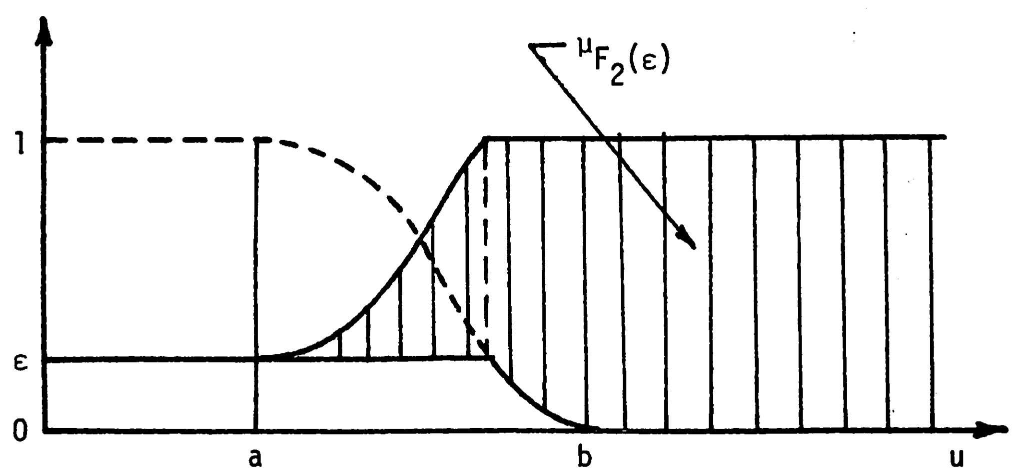


Figure 2. "X is small is impossible"



(a)



(b)

Figure 3. Illustrations from the fuzzy set small of (a) $F_1(\varepsilon)$, (b) $F_2(\varepsilon)$

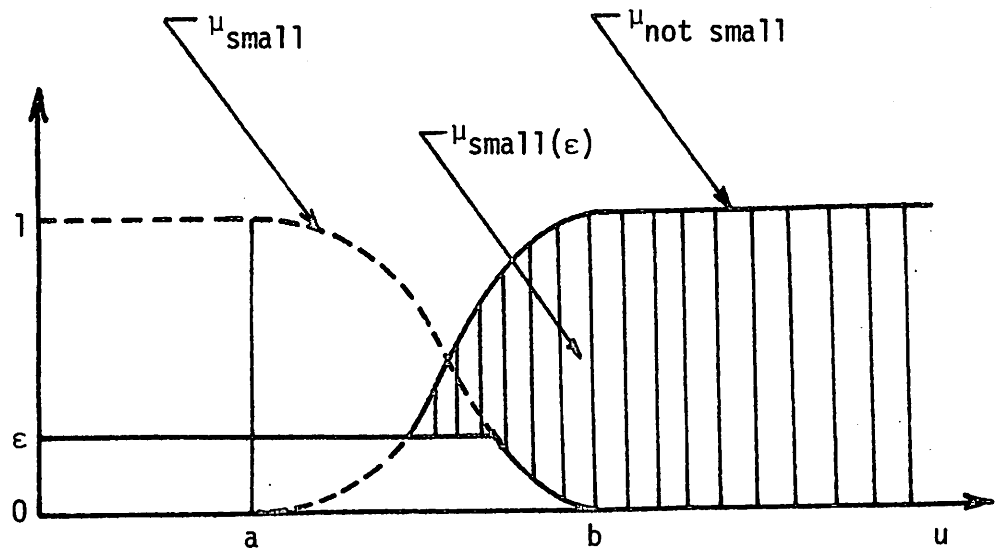


Figure 4. "X is small is ε-possible"

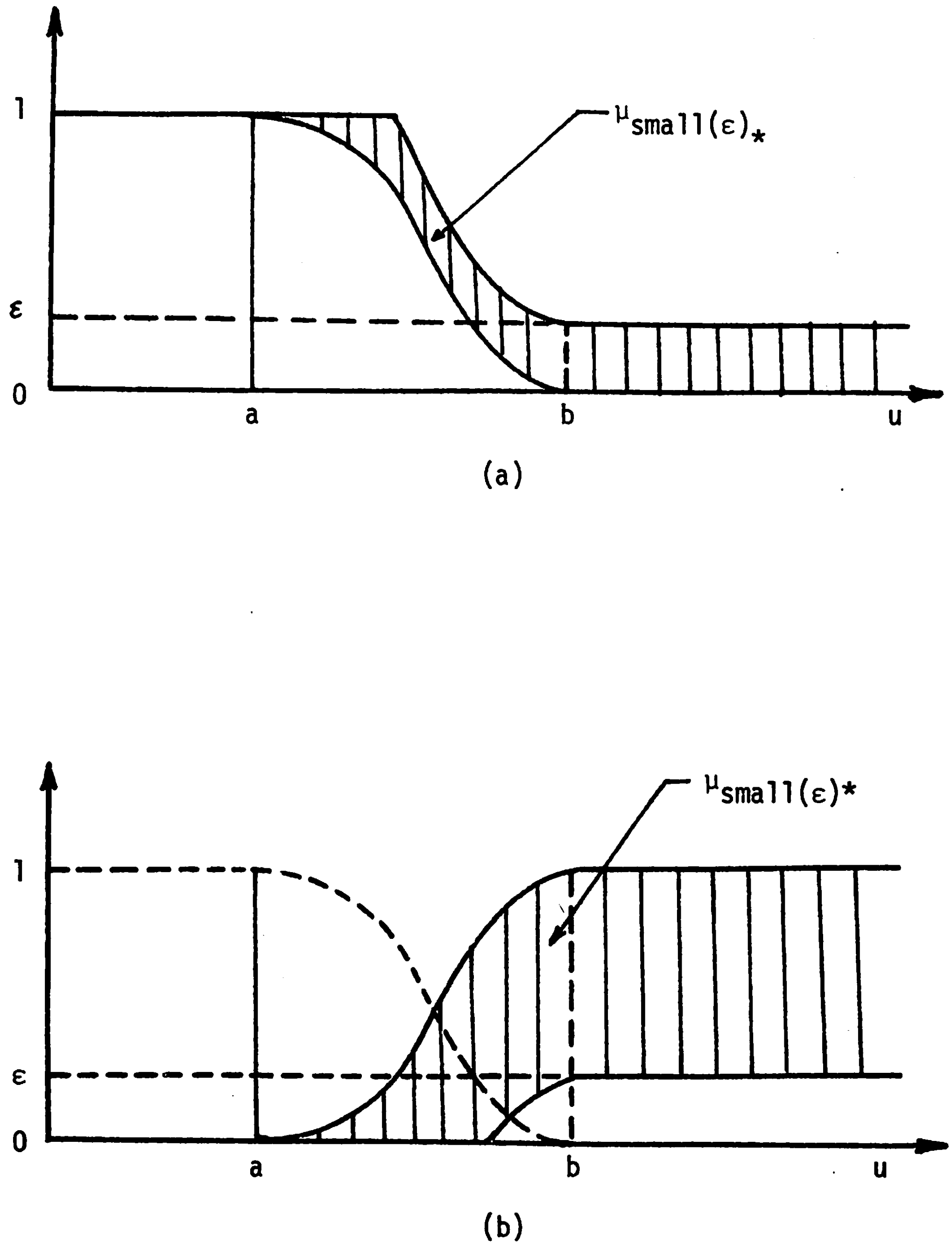


Figure 5. (a) "X is <u>small</u> is ε-<u>lower-possible</u>"
(b) "X is <u>small</u> is ε-<u>upper-possible</u>"