INPUT-OUTPUT STABILITY OF INTERCONNECTED SYSTEMS USING DECOMPOSITIONS: AN IMPROVED FORMULATION

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INPUT-OUTPUT STABILITY OF INTERCONNECTED SYSTEMS USING DECOMPOSITIONS: AN IMPROVED FORMULATION

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ABSTRACT

We study the input-output stability of an arbitrary interconnection of multi-input, multi-output subsystems which may be either continuous-time or discrete-time. We consider throughout three types of dynamics: nonlinear time-varying, linear time-invariant distributed and linear time-invariant lumped. First, we use the strongly connected component decomposition to aggregate the subsystems into strongly-connected-subsystems (SCS's) and interconnection-subsystems (IS's) so that the overall system becomes a hierarchy of SCS's and IS's. Using this decomposition, we define column-subsystems (CS's). The basic structural result states that the overall system is stable if and only if every CS is stable. We then use the minimum-essential-set decomposition on each SCS so that it can be viewed as a feedback interconnection of aggregated subsystems where one of them is itself a hierarchy of subsystems. Based on this decomposition, we present results which leads to sufficient conditions for the stability of SCS. For linear

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time-invariant (transfer function) dynamics, we obtain a characteristic function which gives the necessary and sufficient condition for the overall system stability. We point out the computational saving due to the decompositions in calculating this characteristic function. We believe that decomposition techniques, coupled with other techniques such as model reduction, aggregation, singular and nonsingular perturbations, will play key roles in large scale system design.

I. Introduction

This paper considers the input-output stability of an arbitrary interconnection of multi-input multi-output subsystems. This problem can be viewed as a generalization of that dealing with the feedback interconnection of multi-input multi-output subsystems [1-6]. On the other hand, since an arbitrary interconnection can always, by suitable reformulation, be viewed as a single overall constant output feedbacks system (as is done in Fig. 2 below), the task this paper is to analyze the details of the interconnections using graph theoretic decomposition techniques and to bring them to bear on the stability study.

Basically, there are two types of stability: Lyapunov stability and input-output stability. For the Lyapunov stability, the system dynamics are restricted to ordinary and functional-differential equations [7]. In [8-13], sufficient conditions for Lyapunov stability of an arbitrary interconnection of subsystems are obtained as follows: assume each subsystem is stable with a given Lyapunov function, then try to construct either a vector or a weighted-sum Lyapunov function for the overall system. The input-output stability studied in this paper, allows much more general types of dynamics [1-3]. Papers [14-17] use this point of view and M-matrix technique to obtain sufficient conditions for input-output stability of arbitrary interconnections of subsystems. The interpretation of their results is that if each subsystem is stable, if the loop gain of each local feedback loop is smaller than one and if the gains of the interconnecting subsystems are small enough, then the overall system is stable.

The crucial difference between this paper and [8-17] is that we use graph theoretic decomposition techniques, originally proposed by Harary [18], to exploit the structure of interconnection. Furthermore, we need not assume that every subsystem is stable.[1] These graph theoretic decompositions have been used in [18-24]. In [18], Harary considered only the matrix inversion problem. In [19] and [20], Kevorkian considered solving systems of nonlinear and linear algebraic equations and, in [21], nonlinear time-invariant dynamical systems described by differential and algebraic equations. In [22], Özgüner and Perkins considered the existence of a state space description of the overall system formed by linear time-invariant subsystems interconnected through constant gain subsystems. In [23] Mayeda and Wax considered the exponential stability of systems of ordinary differential equations. In contrast to [8-12, 21-23], we use the general input-output description for our subsystems: thus our theory covers both linear and nonlinear, time-invariant and time-varying, lumped and distributed subsystems as well as the continuous-time and discrete-time cases [1-3]. A detailed comparison between our previous paper [24] and the present paper is relegated to Section X: Conclusions, so that we can make specific reference to results of the present paper.

We study the stability using three levels of aggregation. At the lowest level, we have the multi-input multi-output subsystems which are arbitrarily interconnected through summing nodes to form the overall system. By using strongly-connected-component (SCC) decomposition,

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[1] Some open-loop unstable systems occur in practice: rockets, electronic circuits with op amps, and chemical processes [25].

we aggregate the subsystems into strongly-connected-subsystems (SCS's) and interconnection-subsystems (IS's). The overall system which is the top level aggregation, becomes a hierarchy of these mid-level aggregated subsystems.

The content of this paper are as follows:

The reader is urged to give particular attention to some notational and linguistic conventions: (i) on "map" and "stable" in Sec. II; (ii) on the dimension of subsystems in Sec. III; (iii) on the relabelling due to the SCC decomposition in Sec. V; and (iv) on the relabelling due to the MES decomposition in Sec. VII.

II. Preliminaries

In this paper we consider an interconnection of subsystems with two types of dynamics: nonlinear time-varying dynamics where systems are described by operators between function spaces, and linear time-invariant dynamics where systems are described by their transfer functions. Throughout this paper, we shall use NTV and LTI to denote nonlinear time-varying operator dynamics, and linear time-invariant transfer function dynamics, respectively.

For a NTV system, we adopt the following standard description [3, Sec. III.1], namely, let $\mathcal{T}$ be the time set of observation (typically $\mathcal{T} = \mathbb{R}_+$ for continuous-time case, $\mathbb{Z}_+$ for discrete-time case), $\mathcal{V}$ be a normed space with norm $|\cdot|$ (typically $\mathcal{V} = \mathbb{R}^n$ or $\mathbb{C}^n$), and $\mathcal{F}$ be the set of all the functions mapping $\mathcal{T}$ into $\mathcal{V}$. The function space $\mathcal{F}$ is a linear space over $\mathbb{R}$ (or $\mathbb{C}$) under pointwise addition and pointwise multiplication by scalars. Introducing a norm $\|\cdot\|$ on $\mathcal{F}$, we obtain a normed linear subspace $\mathcal{L}$ of the linear space $\mathcal{F}$, given by

$$\mathcal{L} \triangleq \{f : \mathcal{T} \to \mathcal{V} \,|\, \|f\| < \infty\}$$

For any $T \in \mathcal{T}$, we define $f_T(t) = f(t)$ if $t \leq T$, and zero for $t > T$. We say that f_T is obtained by truncating f at T. Associated with the normed space $\mathcal{L}$ is the extended space $\mathcal{L}_e$ defined by

$$\mathcal{L}_e \triangleq \{f : \mathcal{T} \to \mathcal{V} \,|\, \forall T \in \mathcal{T},\ \|f_T\| < \infty\}$$

We shall often write $\|f\|_T$ instead of $\|f_T\|$. From now on we take $\mathcal{V} = \mathbb{R}$. A NTV system with n_i inputs and n_o outputs is described by an input-output operator $H : \mathcal{L}_e^{n_i} \to \mathcal{L}_e^{n_o}$. An operator H is said to be causal iff for all inputs $u \in \mathcal{L}_e^{n_i}$, for all $T \in \mathcal{T}$, the corresponding output Hu satisfies

$(Hu_T)_T = (Hu)_T$. An operator H is said to be $\mathcal{L}$-stable iff there exists constants β, γ in $\mathbb{R}_+$ such that $\forall u \in \mathcal{L}_e^{n_i}$, $\forall T \in \mathcal{T}$,

$$\|Hu\|_T \leq \beta + \gamma \|u\|_T \tag{1}$$

It is well-known that when H is causal, then H is $\mathcal{L}$-stable if and only if there exists constants β, γ in $\mathbb{R}_+$ such that $\forall u \in \mathcal{L}^{n_i}$

$$\|Hu\| \leq \beta + \gamma \|u\| \tag{2}$$

Throughout this paper, we consider only causal operators (see Assumption 1 in Sec. III).The smallest γ for which (2) holds is called the gain of H, and is denoted by $\gamma[H]$, i.e.,

$$\gamma[H] \triangleq \inf\{\gamma \mid \exists \beta \in \mathbb{R}_+ \ni \forall u \in \mathcal{L}^{n_i},\ \|Hu\| \leq \beta + \gamma\|u\|\} \tag{3}$$

The incremental gain of H denoted by $\tilde{\gamma}[H]$ is defined as follows:

$$\tilde{\gamma}[H] \triangleq \inf\{\tilde{\gamma} \mid \forall u_1,u_2 \in \mathcal{L}^{n_i},\ \|Hu_1-Hu_2\| \leq \tilde{\gamma}\|u_1-u_2\|\} \tag{4}$$

Remark 1: (i) Note that the bias β in (1) and (2) which is restricted to be zero in the stability definition of [1,2,14-17] is allowed to be nonzero in our definition. This increase in generality not only allow us to consider biased operators, but also simplifies the stability analysis of the overall system (e.g. see Proof of Theorem III).
(ii) The $\mathcal{L}$-stability of H not only requires that H takes an input in $\mathcal{L}$-space into an output in $\mathcal{L}$-space, but also requires that $\gamma[H]$ be finite. (iii) H is $\mathcal{L}$-stable, or equivalently $\gamma[H] < \infty$, does not necessarily imply that $\tilde{\gamma}[H] < \infty$. (iv) It is easy to show that if there exist $\bar{u} \in \mathcal{L}^{n_i}$ such that $H\bar{u} \in \mathcal{L}^{n_o}$, then $\gamma[H] \leq \tilde{\gamma}[H]$. (v) If H is linear, then $\gamma[H] = \tilde{\gamma}[H]$. ◻

It is well-known that a very large class of linear time-invariant operators can be represented as convolution operators and if the convolution kernels are Laplace transformable, the operators can be described by transfer functions [26]. In this paper, the LTI dynamics will be described by transfer functions. Only the continuous-time case will be considered. All the results on LTI continuous-time case presented in this paper also hold for LTI discrete-time case by making corresponding changes as described in [3, Sec. IV.6].

We shall be concerned with two classes of convolution kernels. First, we define the algebra $\hat{\mathcal{A}}$ [3]: $f : \mathbb{C}_+ \to \mathbb{C}$ is said to be in $\hat{\mathcal{A}}$ iff $f(s) = f_a(s) + \sum_{i=0}^{\infty} f_i e^{-st_i}$ where f_a is the Laplace transform of a function in $L_1(\mathbb{R}_+)$, $f_i \in \mathbb{R}$ for all i, $\sum_{i=0}^{\infty} |f_i| < \infty$, $0 = t_0$, $0 < t_i$ for $i \geq 1$. $H : \mathbb{C}_+ \to \mathbb{C}^{n_o x n_i}$ is said to be in $\hat{\mathcal{A}}^{n_o x n_i}$ iff all its elements are in $\hat{\mathcal{A}}$. We note that (i) $f \in \hat{\mathcal{A}}$ has an inverse in $\hat{\mathcal{A}}$ if and only if $\inf_{s \in \mathbb{C}_+} |f(s)| > 0$, [27, p. 150], [3, p. 249], (ii) $H \in \hat{\mathcal{A}}^{nxn}$ has an inverse in $\hat{\mathcal{A}}^{nxn}$ if and only if $\inf_{s \in \mathbb{C}_+} |\det H(s)| > 0$ [3], and (iii) $\hat{\mathcal{A}}$ ($\hat{\mathcal{A}}^{nxn}$) is a commutative (noncommutative, resp.) algebra over the field $\mathbb{R}$ [27,3]. A LTI distributed system described by its transfer function $H : \mathbb{C}_+ \to \mathbb{C}^{n_o x n_i}$ is said to $\mathcal{A}$-stable iff $H \in \hat{\mathcal{A}}^{n_o x n_i}$. It is well-known that if a system is $\mathcal{A}$-stable then (i) for any $p \in [1,\infty]$, it takes an L_p-input into an L_p-output with a finite gain, i.e., it is $\mathcal{L}$-stable for $\mathcal{L} = L_p$, and (ii) it takes continuous and bounded inputs (periodic inputs, almost periodic inputs, resp.) into outputs belonging to the same classes resp. [3, 28].

Let $\mathbb{R}(s)$ denote the field of rational functions with real coefficients. Let $\mathbb{R}(s)^{n_o x n_i}$ denote the ring of $n_o \times n_i$ matrices whose elements are in $\mathbb{R}(s)$. By definition, A LTI lumped system is described by a transfer function in $\mathbb{R}(s)^{n_o x n_i}$. A LTI lumped system described by its transfer function $H(s) \in \mathbb{R}(s)^{n_o x n_i}$ is said to be exponentially stable (abbreviated exp-stable) iff (i) H(s) is proper (i.e. bounded at infinity) and (ii) H(s) has all its poles in the open left-half plane (i.e. H has no $\mathbb{C}_+$-pole). It is easy to see that $\mathbb{R}_e(s)$, the class of all scalar exp-stable transfer functions is an algebra over $\mathbb{R}$, in fact a subalgebra of $\hat{\mathcal{A}}$.

In either the lumped or distributed case, if the transfer function H has a domain of convergence which includes some right-half plane and if, for large Re s, it is bounded by some polynomial in s, then it is causal. (The second requirement is indispensable: viz. e^s), [3, Thm. B.3.4].

Convention on "map" and "stable"

Throughout this paper, (i) by a map H, we mean an operator H for the NTV dynamics case and a transfer function H for the LTI dynamics case and (ii) when we say that a system described by the map H is stable, we mean the operator H is $\mathcal{L}$-stable in NTV dynamics, the transfer function H is $\mathcal{A}$-stable in LTI distributed dynamics and the transfer function H is exp-stable in LTI lumped dynamics.[2] □

[2]We do not distinguish a transfer function from an operator by the usual notation of ^ because some of our results hold for all three types of dynamics with corresponding definitions of stability.

Note that when a system is described by the map H, the specification of H prescribes the inputs and the outputs of the system.[3]

Using (1), the definition of $\mathcal{L}$-stability, one can easily prove that the composition and addition of $\mathcal{L}$-stable operators are again $\mathcal{L}$-stable. Since $\hat{\mathcal{A}}$, $\mathbb{R}_e(s)$ are algebras, they are closed under multiplication and addition. Thus we have

Lemma 1 (NTV, LTI)[4]

Every series-parallel connection of stable subsystems is stable. $\square$

III. System Description and Assumptions

In this paper we consider an overall system S consisting of an arbitrary interconnection of subsystems. The subsystems are specified by an input-output map: they may be MIMO (multi-input multi-output) or SISO (single-input single-output), unstable or stable, nonlinear or linear, time-varying or time-invariant, and continuous-time or discrete-time. The interconnections are realized through m summing nodes as indicated by Fig. 1. The subsystem from node j to node i is described by the map G_{ij}. Each summing node j is fed by an external input u_j and by the outputs of the subsystems $G_{j1},\ldots,G_{jm}$. The output e_j of the summing node j is the input to the subsystems $G_{1j},\ldots,G_{mj}$. In practice, a significant portion of the subsystems G_{ij}'s are absent, hence are represented by zero maps.

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[3] As will be seen below (Remark 2), it makes a lot of difference whether H is taken to be $\tilde{G}(I-G)^{-1}$ or $G(I-G)^{-1}$.

[4] We use NTV, LTI following each lemma, theorem and corollaries to indicate the type of dynamics for which the statement holds.

Convention on dimension of subsystems

To alleviate burdensome notations which are peripheral to the main ideas of the paper, we denote each subsystem as if it were SISO, (i.e. for NTV dynamics, $G_{ij} : \mathcal{L}_e \to \mathcal{L}_e$; for LTI dynamics, G_{ij} is a scalar transfer function) the results presented in this paper still hold for MIMO subsystems, (i.e. for NTV dynamics, $G_{ij} : \mathcal{L}_e^{n_{jo}} \to \mathcal{L}_e^{n_{io}}$; for LTI dynamics, G_{ij} is a matrix transfer function), by modifying the "dimension" of the product spaces accordingly.[5] ¤

Assumption 1: Causality

Throughout this paper, we assume that all the subsystems G_{ij} are causal. ¤

The summing-node equations read

$$e_i = u_i + \sum_{j=1}^{m} G_{ij} e_j \quad \text{for } i = 1,\dots,m. \tag{10}$$

In matrix notation, we have

$$\begin{bmatrix} I-G_{11} & -G_{12} & \cdots & -G_{1m} \\ -G_{21} & I-G_{22} & & -G_{2m} \\ \vdots & & \ddots & \vdots \\ -G_{m1} & \cdots & \cdots & I-G_{mm} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \tag{11}$$

[5] This simplified description of the subsystems does not affect the validity of the formulas below. The simplified description avoids the messy bookkeeping of three levels of aggregation of subsystems.

Define $G \triangleq \begin{bmatrix} G_{11} & \cdots & G_{1m} \\ \vdots & & \vdots \\ G_{m1} & \cdots & G_{mm} \end{bmatrix}$, $e \triangleq \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}$, $u \triangleq \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$.

With these definitions, the system equations (10), or equivalently (11) become

$$(I-G)e = u \tag{12}$$

Assumption 2: Unique Solvability

Throughout this paper, we assume that

(a) for NTV dynamics, $(I-G)^{-1} : u \longmapsto e$ is a map from $\mathcal{L}_e^m$ into $\mathcal{L}_e^m$, and is a causal map.

(b) for LTI distributed dynamics, $\forall$ sequences $(s_i)_{i=1}^{\infty} \subset \mathbb{C}_+$, where $|s_i| \to \infty$, $\liminf_{i \to \infty} |\det(I-G)(s_i)| > 0$.

(c) for LTI lumped dynamics, $\det(I-G)(\infty) \neq 0$. ¤

Conditions under which Assumption 2(a) is satisfied can be found in [2, Ch.2], [3, Sec. III.5].

Definition of the overall system stability

The overall system S is said to be stable iff $\forall i, j = 1,\ldots,m$, the maps $u \longmapsto G_{ij}[(I-G)^{-1}u]_j$ are stable.

Remark 2: (i) At first sight, one might want to choose as definition of overall system stability: the map $(I-G)^{-1}$: $u \longmapsto e$ be stable. Note that without requiring G to be linear,

$$I + G(I-G)^{-1} = (I-G)(I-G)^{-1} + G(I-G)^{-1} = (I-G)^{-1} \tag{13}$$

Hence the map $(I-G)^{-1}$ is stable if and only if the map $G(I-G)^{-1}$ is stable, i.e. for $i = 1,\ldots,m$ the maps $u \longmapsto \sum_{j=1}^{m} G_{ij}e_j$ are stable. By definition the overall system S is stable iff $\forall i,j = 1,\ldots,m$, $u \longmapsto G_{ij}e_j$ are stable. Hence by Lemma 1, if the overall system S is stable, then the map $(I-G)^{-1}$ is stable.

(ii) The converse of the last statement is not true because two unstable terms, say $G_{i\ell}e_\ell$ and $G_{ik}e_k$ may cancel each other and give a stable sum $\sum_{j=1}^{m} G_{ij}e_j$. An example to illustrate the point: Let $m = 2$ and all subsystems be SISO LTI lumped. It is easy to check that if $G_{11}(s) = (2s+3)/(s+1)$, $G_{12}(s) = 0\ \forall s$, $G_{21}(s) = -(s-4)/(2s-1)$ and $G_{22}(s) = (s-4)/(2s-1)$, then a) $(I-G)^{-1}$ and $G(I-G)^{-1}$ are exponentially stable, b) the maps $u_1 \longmapsto G_{21}e_1$ and $u_1 \longmapsto G_{22}e_2$ are both unstable (pole at $s = 0.5$); but their sum is of course stable since it is the (2,1) element of $G(I-G)^{-1}$. ◻

In order to formulate the definition of overall system stability, let $G_{.j}$ denotes the jth column of G. Let

$$\tilde{G} \triangleq \mathrm{diag}(G_{.1},\ldots,G_{.m}) \tag{14}$$

Let I be the mxm identity matrix and let $\tilde{K} \in \mathbb{R}^{m \times m^2}$ be given by

$$\tilde{K} \triangleq [I \,\vdots\, \ldots \,\vdots\, I] \tag{15}$$

$$\text{Let} \quad \tilde{y} \triangleq \begin{bmatrix} G_{11}e_1 \\ G_{21}e_1 \\ \vdots \\ G_{m1}e_1 \\ \vdots \\ G_{mm}e_m \end{bmatrix} \triangleq (\tilde{y}_{ij})_{i=1,\ldots,m,j=1,\ldots,m} \tag{16}$$

Note that from (14) and (15)

$$G = \tilde{K}\tilde{G} \tag{17}$$

Then the overall system S can be viewed as a constant output feedback system as shown in Fig. 2. Thus the overall system S is stable if and only if the map $\tilde{G}(I-\tilde{K}\tilde{G})^{-1} = \tilde{G}(I-G)^{-1} : u \longmapsto \tilde{y}$ is stable.

By Assumptions 1 and 2, $\tilde{G}(I-\tilde{K}\tilde{G})^{-1} : u \longmapsto \tilde{y}$ is a <u>well-defined causal</u> map.

For theoretical development, Fig. 2 is convenient. However it does not take advantage of the particular structure of the interconnection, namely that a number of G_{ij}'s are zero maps. Our objective is to take the structure into consideration using graph theoretic decomposition techniques.

IV. Overall System Stability Without Using Decomposition

In this section we consider the overall system stability <u>without</u> using decomposition. Theorem I gives a sufficient condition for the overall system stability; it is quite general since it holds for NTV dynamics with $\mathcal{L}$-stability, LTI distributed dynamics with $\mathcal{A}$-stability, and LTI lumped dynamics with exp-stability.

<u>Theorem I</u> (NTV, LTI)

Consider the overall system S described by (11) and satisfying Assumptions 1 and 2. If (a) G_{ij} is stable $\forall i \neq j$, $i,j = 1,\ldots,m$ (b) for every <u>unstable</u> G_{ii}, $(I-G_{ii})^{-1}$ is stable and (c) $(I-G)^{-1}$ is stable then the overall system S is stable. ◻

Remark 3: (i) By Theorem I, under its assumptions (a) and (b), the stability of $(I-G)^{-1}$ is equivalent to the overall system stability. (ii) If every G_{ij}, $i,j = 1,\dots,m$ is stable, then Assumptions (a) and (b) are satisfied. ◻

For the remaining part of this section we study the $\mathcal{A}$-stability (exp-stability, resp.) of the overall system S in LTI distributed (lumped, resp.) dynamics. Due to the symmetry between the distributed and lumped cases, the results are presented in pairs: we use D(L, resp.) to denote distributed (lumped, resp.) case.

Let $\mathbb{R}[S]$ denote the commutative ring of polynomials with real coefficients. Let $H(s) \in \mathbb{R}(s)^{n_o \times n_i}$, $N_r(s) \in \mathbb{R}[s]^{n_o \times n}$, $D(s) \in \mathbb{R}[s]^{n \times n}$, $N_\ell(s) \in \mathbb{R}[s]^{n \times n_i}$, then (N_r,D,N_ℓ) is said to be a right left coprime factorization (r.l.c.f) of H iff (i) $H = N_r D^{-1} N_\ell$, (ii) there exist $U_r(s) \in \mathbb{R}[s]^{n \times n_o}$, $U_\ell(s) \in \mathbb{R}[s]^{n_i \times n}$, $V_r(s)$, $V_\ell(s) \in \mathbb{R}[s]^{n \times n}$ such that $\det[U_r N_r + V_r D](s) \neq 0, \forall s \in \mathbb{C}$ and $\det[N_\ell U_\ell + DV_\ell](s) \neq 0$, $\forall s \in \mathbb{C}$. Let $H(s) \in \mathbb{R}(s)^{n_o \times n_i}$, $N_r(s) \in \mathbb{R}[s]^{n_o \times n_i}$, $D_r(s) \in \mathbb{R}[s]^{n_i \times n_i}$, then (N_r,D_r) is said to be a right coprime factorization (r.c.f.) of H iff (N_r,D_r,I) is a r.l.c.f. of H.A left coprime factorization (l.c.f.) is similarly defined.

Let $H : \mathbb{C}_+ \to \mathbb{C}^{n_o \times n_i}$, $N_r \in \hat{\mathcal{A}}^{n_o \times n}$, $D \in \hat{\mathcal{A}}^{n \times n}$, $N_\ell \in \hat{\mathcal{A}}^{n \times n_i}$, then (N_r,D,N_ℓ) is said to be a pseudo right left coprime factorization (p.r.l.c.f.) of H iff (i) $H = N_r D^{-1} N_\ell$, (ii) there exist $U_r \in \hat{\mathcal{A}}^{n \times n_o}$, $U_\ell \in \hat{\mathcal{A}}^{n_i \times n}$, V_r, $V_\ell \in \hat{\mathcal{A}}^{n \times n}$ such that $\det[U_r N_r + V_r D](s) \neq 0$, $\forall s \in \mathbb{C}_+$ and $\det[N_\ell U_\ell + DV_\ell](s) \neq 0$, $\forall s \in \overline{\mathbb{C}_+}$, (iii) $\forall$ sequences $(s_i)_{i=1}^{\infty} \subset \mathbb{C}_+$

where $|s_i| \to \infty$, $\liminf_{i \to \infty} |\det D(s_i)| > 0$. Let $H : \mathbb{C}_+ \to \mathbb{C}^{n_o \times n_i}$, $N_\ell \in \hat{\mathcal{A}}^{n_o \times n_i}$, $D_\ell \in \hat{\mathcal{A}}^{n_o \times n_o}$, then (N_ℓ, D_ℓ) is said to be a <u>pseudo left coprime factorization</u> (p.l.c.f.) of H iff (I, D_ℓ, N_ℓ) is a p.r.l.c.f. of H. A <u>pseudo right coprime factorization</u> (p.r.c.f.) is similarly defined.

It is easy to see that the definitions of coprime factorizations given above are equivalent to those defined in the literature [e.g.3,5,6,24,29]. The reason for introducing new definitions is to achieve <u>symmetry</u> between the distributed and the lumped cases.

It is well-known that if (N_r, D, N_ℓ) is r.l.c.f. of H, and H is proper then H is exp-stable if and only if det D(s) has no $\mathbb{C}_+$-zero [29]. By similar reasoning as in [5], it is easy to show that if (N_r, D, N_ℓ) is a p.r.l.c.f. of H, then H is $\mathcal{A}$-stable if and only if $\inf_{s \in \mathbb{C}_+} |\det D(s)| > 0$.

The following lemma in spite of its simple proof has far reaching consequences: it gives us a characteristic function mapping from $\mathbb{C}_+$ into $\mathbb{C}$ such that the overall system S is $\mathcal{A}$-stable (exp-stable, resp.) if and only if the infimum over $\mathbb{C}_+$ of the absolute value of that characteristic function is positive.

<u>Lemma 2D(L)</u> (LTI)

Consider the LTI distributed (lumped, resp.) constant output feedback system shown in Fig. 3 where H is the transfer function in the forward path and K is a constant matrix. Let (N_r, D, N_ℓ) be a p.r.l.c.f. (resp. r.l.c.f.) of H. (For the lumped case assume that H is proper.) Assume that $\forall$ sequences $(s_i)_{i=1}^{\infty} \subset \mathbb{C}_+$ where $|s_i| \to \infty$, $\liminf_{i \to \infty} |\det(I-KH)(s_i)| > 0$. ($\det(I-KH)(\infty) \neq 0$, resp.) Under these conditions, $H(I-KH)^{-1} : v \longmapsto z$

is $\mathcal{a}$-stable (exp-stable, resp.) if and only if

$$\inf_{s \in \mathbb{C}_+} |\det(D-N_\ell K N_r)(s)| > 0 \tag{18}$$

$(\det(D-N_\ell K N_r)(s)$ has no $\mathbb{C}_+$ - zero, resp.) ◻

<u>Remark 4</u>: (i) Lemma 2D(L) is proved by showing that if (N_r, D, N_ℓ) is a p.r.l.c.f. (resp. r.l.c.f.) of H then $(N_r, D-N_\ell K N_r, N_\ell)$ is a p.r.l.c.f. (resp. r.l.c.f.) of $H(I-KH)^{-1}$. It is well-known that if x is the state used in a finite dimensional <u>minimal</u> SSSD (state space system description) of H, then the SSSD of the constant output feedback system $H(I-KH)^{-1}$, using the same state x, will also be <u>minimal</u>, since constant output feedback preserves both complete controllability and complete observability when the <u>same</u> state is used [30, p. 365]. Since the <u>r.l.c.f.</u> in PMSD (polynomial matrix system description) is a counterpart of the <u>minimal</u> realization in SSSD, [31,32], Lemma 2L can be viewed as a counterpart in PMSD of the above well known fact in SSSD.

(ii) Applying Lemma 2D(L) to the feedback system considered in [5,33], we can easily obtain all the characteristic functions (resp. polynomials) given in these papers.

(iii) Note that Lemma 2D(L) will still hold if K has elements in $\hat{\mathcal{a}}$ (if K is a polynomial matrix, resp.). ◻

Under Assumptions 1 and 2, applying Lemma 2D(L) to the overall system S shown in Fig. 2 and using the particular form of $\tilde{K}$, we obtain a characteristic function (polynomial, resp.) for the overall system S.

<u>Theorem II D(L)</u> (LTI)

Consider the LTI overall system S described by (11) and satisfying Assumptions 1 and 2. For j = 1,...,m, let $G_{\cdot j}$ denote the <u>jth</u> column of G, and let $(N_{.j}, D_j)$ be a p.r.c.f. (r.c.f., resp.) of $G_{.j}$. (For the lumped case, assume that G is proper.) Let

$$N \triangleq [N_{.1} \vdots \cdots \vdots N_{.m}], \quad \tilde{D} \triangleq \text{diag}(D_1, \ldots, D_m) \tag{20}$$

With these definitions, the overall system S is $\mathcal{a}$-stable (exp-stable, resp.) if and only if

$$\inf_{s \in \mathbb{C}_+} |\det(\tilde{D}-N)(s)| > 0 \tag{21}$$

($\det(\tilde{D}-N)$ has no $\mathbb{C}_+$-zero, resp.) ¤

Remark 5: (i) Note that $G = N\tilde{D}^{-1}$ but that N, $\tilde{D}$ are not necessarily p.r.c. (resp. r.c.). (ii) Clearly by using p.l.c.f. of $G_{.j}$'s, one can obtain a characteristic function for the overall system S. However since the size[7] of the output $\tilde{y}$ of S is m^2, such characteristic function will be the determinant of a $m^2 \times m^2$ matrix. The characteristic function given by Theorem II D(L) is the determinant of a $m \times m$ matrix where m is the size of the input u of S (see Fig. 2). ¤

In Sec. IX below, we will discuss how the SCC and MES decompositions to be described in Sec. V and VII, simplify the necessary and sufficient stability test given by (21).

V. SCC Decomposition of the Overall System

In this section, we apply the ideas presented in [24, Sec. III] to the present formulation. In order to make this paper to some extent self-contained, we define below all the required terms.

By definition, a digraph $\mathcal{D} \triangleq (V,E)$ consists of a set of vertices V and a set of directed edges $E = \{(\nu_i, \nu_j) | \nu_i, \nu_j \in V\}$. (ν_i, ν_j) is an

[7]Recall Footnote 5.

edge directed from ν_i to ν_j and is said to be incident to both ν_i and ν_j [34,35]. A section graph of $\mathcal{G} = (V,E)$ is defined to be a digraph $\mathcal{G}(U) \triangleq (U \subseteq V, \{(\nu_i,\nu_j) \in E | \nu_i,\nu_j \in U\})$. $\mathcal{G}(U)$ is said to be connected iff disregarding the direction of the edges, every pair of vertices in U are mutually reachable by going through edges in $\mathcal{G}(U)$. $\mathcal{G}(U)$ is said to be strongly connected iff respecting the direction of the edges, every pair of vertices in U are mutually reachable by traversing along edges in $\mathcal{G}(U)$. A maximal strongly connected section graph $\mathcal{G}(U)$ is called a strongly connected component (abbr. SCC) of $\mathcal{G}$. In other words, mutually reachability between a pair of vertices is an equivalence relation (i.e. reflexive, symmetric and transitive) defined on V and the set of vertices in each SCC is an equivalence class under that equivalence relation. A connected component is similarly defined. The vertex ν_i is said to have a self-loop iff $(\nu_i,\nu_i) \in E$. A circuit of length $\ell > 1$ is defined to be an ordered set of ℓ distinct vertices $(\pi_1,\pi_2,\ldots,\pi_\ell)$ such that $(\pi_\ell,\pi_1) \in E$ and $(\pi_k,\pi_{k+1}) \in E$ for $k = 1,2,\ldots,\ell-1$. A digraph is said to be acyclic iff it does not contain any circuit. The indegree (resp. outdegree) of a vertex ν_i is defined to be the number of edges coming into (resp. out of) ν_i. The adjacency matrix of a digraph $\mathcal{G} = (V,E)$ is defined to be an nxn matrix A where n is the number of vertices in $\mathcal{G}$, such that $a_{ij} = 1$ iff $(\nu_j,\nu_i) \in E$[8] and $a_{ij} = 0$ otherwise.

[8] Most graph theorists define $a_{ij} = 1$ iff $(\nu_i,\nu_j) \in E$. Hence our adjacency matrix is the transpose of theirs.

Consider the overall system S described above. The <u>interconnection digraph</u> $\mathcal{D}_{int}$ of S is defined as follows: each summing node i of S corresponds to a vertex ν_i of $\mathcal{D}_{int}$, and $\mathcal{D}_{int}$ has a directed edge from ν_j to ν_i iff the subsystem G_{ij} is not the zero map. Since each connected component of $\mathcal{D}_{int}$ can be analyzed separately, without loss of generality, we <u>assume</u> that $\mathcal{D}_{int}$ is a <u>connected</u> digraph.

We now perform the SCC decomposition on the connected digraph $\mathcal{D}_{int}$.

Step 1: Find all the SCC's $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\mu$ of $\mathcal{D}_{int}$.

Step 2: Make a condensation of $\mathcal{D}_{int}$ with respect to these SCC's. That is, we define a new digraph called the <u>structural digraph</u> $\mathcal{D}_s$ of S as follows: each SCC $\mathcal{C}_\alpha$ of $\mathcal{D}_{int}$ corresponds to a vertex $\bar{\nu}_\alpha$ in $\mathcal{D}_s$ and there is a directed edge from $\bar{\nu}_\alpha$ to $\bar{\nu}_\beta$ iff the set of directed edges in $\mathcal{D}_{int}$ from any vertex in $\mathcal{C}_\alpha$ to any vertex in $\mathcal{C}_\beta$ is not empty. By construction, $\mathcal{D}_s$ is a <u>connected acyclic digraph</u>.

Step 3: Relabel the vertices of $\mathcal{D}_s$ so that its adjacency matrix A_s is a lower triangular matrix. Hence, with respect to the new labeling, a SCC, say $\mathcal{C}_\alpha$, can only feed its output to SCC's, say $\mathcal{C}_\beta, \mathcal{C}_\gamma, \ldots$, with a higher subscript, i.e. $\beta, \gamma > \alpha$.

Step 4: Relabel the vertices of $\mathcal{D}_{int}$ so that (a) those that belong to the same SCC are numbered consecutively and (b) those that belong to the lower numbered SCC are numbered lower than those belonging to the higher numbered SCC. ¤

Step 1, the identification of SCC's, can be done by using Tarjan's efficient algorithm STRONGCONNECT [36]. In [36] Tarjan has proved that his algorithm is correct and that its requirements for memory space and computing time are bounded by a <u>linear</u> function in the number of vertices and edges in the digraph. For somewhat less efficient but simpler algorithm, see [18, 22]. For a heuristic algorithm, see [19, 21]. Step 2 can easily be done by inspection. Step 3, labeling of a connected acyclic digraph is called <u>topological sort</u> [34, pp. 402], [37, pp. 258]. It is done in μ iterations (μ = the number of SCC's) by deleting a vertex with zero indegree and all its incident edges at each iteration and, then, by relabeling the vertices in the order they were deleted.

A little thought reveals that the adjacency matrix A_{int} of $\mathcal{D}_{int}$ after Step 4 will be in the lower block triangular form:

$$A_{int} = \begin{array}{c} \\ m_1^c \\ m_2^c \\ \vdots \\ m_\mu^c \end{array} \begin{array}{c} \begin{array}{cccc} m_1^c & m_2^c & \cdots & m_\mu^c \end{array} \\ \begin{bmatrix} A_{11}^c & 0 & \cdots & 0 \\ A_{21}^c & A_{22}^c & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_{\mu 1}^c & A_{\mu 2}^c & \cdots & A_{\mu\mu}^c \end{bmatrix} \end{array} \tag{22}$$

where (i) m_α^c is the number of vertices in $\mathcal{C}_\alpha$, (ii) each diagonal block $A_{\alpha\alpha}^c$ is the adjacency matrix of $\mathcal{C}_\alpha$ and (iii) each off diagonal block $A_{\alpha\beta}^c$, $\alpha > \beta$ is the adjacency matrix of $\mathcal{C}_{\alpha\beta}$ which is defined to be the bipartite digraph [34, pp. 168] consisting of (a) all the vertices of $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$, and (b) all edges of $\mathcal{D}_{int}$ directed from a vertex in $\mathcal{C}_\beta$ to a vertex in $\mathcal{C}_\alpha$.

Notational Convention

From now on, without loss of generality, we assume that we start out with the overall system S which has been relabeled after the SCC decomposition. ¤

For $\alpha \geq \beta = 1,\ldots,\mu$, we define

$$V_\alpha^c \triangleq \text{the set of vertices in SCC } \mathcal{C}_\alpha \tag{23}$$

$$u_\alpha^c \triangleq \text{the } m_\alpha^c - \text{vector } (u_i)_{i \in V_\alpha^c} \tag{24}$$

$$e_\alpha^c \triangleq \text{the } m_\alpha^c - \text{vector } (e_i)_{i \in V_\alpha^c} \tag{25}$$

$$G_{\alpha\beta}^c \triangleq \text{the } m_\alpha^c \times m_\beta^c \text{ matrix } [G_{ij}]_{i \in V_\alpha^c,\ j \in V_\beta^c} \tag{26}$$

$$\tilde{G}_{\alpha\beta}^c \triangleq \text{the } (m_\alpha^c \cdot m_\beta^c) \times m_\beta^c \text{ matrix: diag(columns of } G_{\alpha\beta}^c) \tag{27}$$

$$\tilde{K}_{\alpha\beta}^c \triangleq \text{the } m_\alpha^c \times (m_\alpha^c \cdot m_\beta^c) \text{ matrix: } [I \vdots \ldots \vdots I] \tag{28}$$

$$\tilde{y}_{\alpha\beta}^c \triangleq \text{the } (m_\alpha^c \cdot m_\beta^c) - \text{vector } (G_{ij}e_j)_{i \in V_\alpha^c,\ j \in V_\beta^c} \tag{29}$$

(Compare (27), (28), (29) with (14), (15), (16)). Note that

$$G_{\alpha\beta}^c = \tilde{K}_{\alpha\beta}^c \tilde{G}_{\alpha\beta}^c \tag{30}$$

and

$$\tilde{y}_{\alpha\beta}^c = \tilde{G}_{\alpha\beta}^c e_\beta^c \tag{31}$$

Equation (11) can now be written as

$$\begin{bmatrix} I-G_{11}^c & 0 & \cdots & 0 \\ -G_{21}^c & I-G_{22}^c & & \vdots \\ \vdots & & & 0 \\ -G_{\mu 1}^c & \cdot & \cdot & I-G_{\mu\mu}^c \end{bmatrix} \begin{bmatrix} e_1^c \\ e_2^c \\ \vdots \\ e_\mu^c \end{bmatrix} = \begin{bmatrix} u_1^c \\ u_2^c \\ \vdots \\ u_\mu^c \end{bmatrix} \tag{32}$$

From (32) we can write

$$e^c_\beta = (I-G^c_{\beta\beta})^{-1} \; (u^c_\beta + \sum_{j=1}^{\beta-1} G^c_{\beta j} \; e^c_j)$$

Using (30) and (31) we have

$$\tilde{y}^c_{\alpha\beta} = \tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1} \; (u^c_\beta + \sum_{j=1}^{\beta-1} \tilde{K}^c_{\beta j} \; \tilde{y}^c_{\beta j}) \tag{33}$$

Observe that due to block-lower-triangular form of G after relabeling, e^c_β, $\tilde{y}^c_{\alpha\beta}$ do not depend on u^c_j for $j > \beta$.

For $\alpha = 1,\ldots,\mu$, we denote by S^c_α, the <u>strongly-connected-subsystem</u> (SCS) associated with the SCC $\mathcal{C}_\alpha$ of $\mathcal{D}_{int}$: it is obtained from the overall system S by removing all the summing nodes and subsystems which do not appear in $\mathcal{C}_\alpha$. Hence its input is u^c_α, its output is $\tilde{y}^c_{\alpha\alpha}$ and it is described by the map $\tilde{G}^c_{\alpha\alpha}(I-\tilde{K}^c_{\alpha\alpha}\tilde{G}^c_{\alpha\alpha})^{-1} = \tilde{G}^c_{\alpha\alpha}(I-G^c_{\alpha\alpha})^{-1}$. In other words, S^c_α can be viewed as the constant output feedback system shown in Fig. 2 with the following replacements: $\tilde{G}$ by $\tilde{G}^c_{\alpha\alpha}$, $\tilde{K}$ by $\tilde{K}^c_{\alpha\alpha}$, u by u^c_α, e by e^c_α, $\tilde{y}$ by $\tilde{y}^c_{\alpha\alpha}$. Consequently the stability definition below follows previous pattern.

The <u>SCS</u> S^c_α is said to be <u>stable</u> iff the map $\tilde{G}^c_{\alpha\alpha}(I-G^c_{\alpha\alpha})^{-1}$ is stable.

By Assumption 2 in Sec. III and block-lower-triangular structure in (32), $(I-G^c_{\alpha\alpha})^{-1}$ is a well-defined causal map. Hence every SCS is is described by a <u>well-defined causal</u> map.

For $\alpha > \beta = 1,\ldots,\mu-1$, we denote by $S^c_{\alpha\beta}$, the <u>interconnection-subsystem</u> (IS) associated with $\mathcal{C}_{\alpha\beta}$ of $\mathcal{D}_{int}$: it is obtained from the overall system by removing all the summing nodes and subsystems which do not appear in $\mathcal{C}_{\alpha\beta}$. Hence its input is u^c_β, its output is $\tilde{y}^c_{\alpha\beta}$ and it is described by the map $\tilde{G}^c_{\alpha\beta}$.

The <u>IS</u> $S^c_{\alpha\beta}$ is said to be <u>stable</u> iff the map $\tilde{G}^c_{\alpha\beta}$ is stable.

In view of (27)-(33), a little thought will reveal that the overall system $S : (u^c_\alpha)^{\mu}_{\alpha=1} \longmapsto (\tilde{y}^c_{\alpha\beta})^{\mu}_{\alpha,\beta=1}$ can be viewed as a series-parallel connection of SCS's, IS's and constant gain subsystems $\tilde{K}^c_{\alpha\beta}$ as shown in Fig. 4 for the case $\mu = 3$.

For $\beta = 1,\ldots,\mu$, we denote by S_β, the <u>column-subsystem</u> (CS) described by the maps $\tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1}$, $\alpha = \beta, \beta+1,\ldots,\mu$. These maps are the contribution of u^c_β to $(\tilde{y}^c_{\alpha\beta})^{\mu}_{\alpha=\beta}$ while neglecting the effect of all other inputs $(u^c_j)^{\beta-1}_{j=1}$.

Prior to establishing in Theorem III below, that the overall system is stable if and only if every CS is stable, we note some relationships between the stabilities of CS's, SCS's and IS's which are direct consequences of the definitions and of the structural decomposition.

<u>Fact 1</u> (NTV,LTI)

If CS S_β is stable, then SCS S^c_β is stable. ⌑

Since SCS S^c_β is stable implies that the map $(I-G^c_{\beta\beta})^{-1}$ is stable, we have

<u>Fact 2</u> (NTV,LTI)

If SCS S^c_β is stable and $\forall\alpha > \beta$, IS $S^c_{\alpha\beta}$ are stable, then CS S_β is stable. ⌑

Note that the stability of CS does not imply that of the corresponding IS's. However, in view of $\tilde{G}^c_{\alpha\beta} = \tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1}(I-G^c_{\beta\beta})$, we have

<u>Fact 3</u> (NTV,LTI)

If $G^c_{\beta\beta}$ is stable and CS S_β is stable, then $\forall\alpha > \beta$, IS $S^c_{\alpha\beta}$ are stable. ⌑

Suppose CS S_β is stable and for some $\alpha > \beta$, IS $S^c_{\alpha\beta}$ is not stable. In LTI dynamics, this implies that there exists some pole-zero cancellation between $\tilde{G}^c_{\alpha\beta}$ and $(I-G^c_{\beta\beta})^{-1}$. Clearly under independent parameter perturbations of $\tilde{G}^c_{\alpha\beta}$ and $G^c_{\beta\beta}$, such pole-zero cancellation will not be preserved. Thus for the stability of CS S_β to be robust, it is reasonable to assume that every IS $S^c_{\alpha\beta}$ is stable. Under this assumption, from Facts 1 and 2, we have

Fact 4 (NTV,LTI)

If every IS $S^c_{\alpha\beta}$, $\alpha > \beta$ is stable, then the stability of CS S_β is equivalent to the stability of SCS S^c_β. ¤

This fact emphasizes the importance of the stability study of the SCS. (see Sec. VIII below)

VI. Structural Result

Theorem III below gives a necessary and sufficient condition for the overall system stability. This result is structural in the sense that it is based on the block-lower-triangular structure obtained by SCC decomposition. It holds for NTV dynamics with $\mathcal{L}$-stability, LTI distributed dynamics with $\mathcal{A}$-stability and LTI lumped dynamics with exp-stability.

Theorem III (NTV,LTI)

Consider the overall system S described by (32) and satisfying Assumptions 1 and 2. The overall system is stable if and only if $\forall\beta = 1,\ldots,\mu$, CS S_β are stable. ¤

VII. MES Decomposition of SCS

In this section, we apply the ideas presented in [24, Sec. V] to the present formulation. In order to avoid ambiguity and to make this paper self-contained, we shall develop the concept of minimum essential set.

Throughout Sec. VII and VIII, we study the stability of a single map, namely $(I-G^c_{\alpha\alpha})^{-1}$. For convenience and to alleviate the already burdensome notation, we will drop the subscript α throughout Sec. VII and VIII. Thus we write

$$e^c = (I-G^c)^{-1}u^c \tag{34}$$

In addition to the graph theoretic terms defined in Sec. V, we will need the following terms. By definition, $U \subset V$ is called an <u>essential set</u> of a digraph $\mathcal{D} \triangleq (V,E)$ iff the section graph $\mathcal{D}(V-U)$ is acyclic. Given a digraph, an essential set with minimum number of vertices is called a <u>minimum essential set</u> (MES) of the digraph. It should be noted that our definitions allow an acyclic digraph to have self-loops; this follows from our requirement that a circuit be of length > 1.

Consider the strongly connected subsystem S^c and its interconnection digraph $\mathcal{C} \triangleq (V^c,E^c)$ which by construction is strongly connected. We now perform the MES decomposition on $\mathcal{C}$.

Step 1: Find an essential set V^2 of $\mathcal{C}$ and define $V^1 \triangleq V^c-V^2$. By construction, the section graph $\mathcal{C}(V^1)$ is acyclic.

Step 2: Relabel the vertices of $\mathcal{C}$ so that every vertex in V^1 is numbered lower than all the vertices in V^2.

Step 3: Relabel the vertices of $\mathcal{G}(V^1)$ so that its adjacency matrix A^{11} is a lower triangular matrix.[9]

A little thought reveals that the adjacency matrix A^c of $\mathcal{G}$ after Step 3 will be in the <u>bordered lower triangular</u> form:

$$A^c = \begin{array}{c c} & \begin{array}{cc} m^1 & m^2 \end{array} \\ \begin{array}{c} m^1 \\ m^2 \end{array} & \left[\begin{array}{c|c} A^{11} & A^{12} \\ \hline A^{21} & A^{22} \end{array}\right] \end{array} \tag{35}$$

where (i) for $i = 1,2, m^i$ is the number of vertices in V^i and (ii) A^{11} is a lower triangular matrix.

To exploit the structure of $\mathcal{G}$ as much as possible, it is obvious that one should use a <u>minimum</u> essential set in the decomposition. The problem of finding a minimum essential set has been studied by many researchers [39-46]. Theoretically speaking, the problem can be considered solved since it requires a finite amount of work. However the amount of work required can become potentially excessive for some large digraphs since it is a <u>NP-complete</u> problem [47]. To perform Step 1, we must first compensate for the fact that we allow self-loops, so we first remove all the self-loops in $\mathcal{G}$, then apply the algorithm given in [44] to find a minimum essential set and then put back the self-loops. Step 2 of the decomposition can be done easily. Step 3 is carried out by using the <u>topological sort</u> described in Sec. V.

[9] Recall Footnote 8.

Remark 6: Clearly by allowing A^{11} in (35) to be in block-lower-triangular form [48], we can further reduce the size of A^{22}. The tradeoff in computational efficiency between this "generalized" MES and the MES decompositions still remains an open question. ¤

Notational Convention

From now on, without loss of generality, we assume that we start out with the SCC $\mathcal{C}$ which has been relabeled after the MES decomposition. ¤

For i,j = 1,2, we define

$$V^2 \triangleq \text{the MES of } \mathcal{C} \tag{36}$$

$$V^1 \triangleq V^c - V^2 \tag{37}$$

$$u^i \triangleq \text{the } m^i\text{-vector } (u_k)_{k \in V^i} \tag{38}$$

$$e^i \triangleq \text{the } m^i\text{-vector } (e_k)_{k \in V^i} \tag{39}$$

$$G^{ij} \triangleq \text{the } m^i \times m^j\text{-matrix } [G_{k\ell}]_{k \in V^i,\ \ell \in V^j} \tag{40}$$

Equation (34) can now be written as two equations

$$(I-G^{11})e^1 - G^{12}e^2 = u^1 \tag{41}$$

$$-G^{21}e^1 + (I-G^{22})e^2 = u^2 \tag{42}$$

We define

$$\tilde{G}^{22} \triangleq G^{22} + G^{21}(I-G^{11})^{-1}\, G^{12} \tag{43}$$

The matrix signal flowgraph [49] associated with the nonlinear eq. (41), (42) is given in Fig. 5.

Remark 7: (i) When $u^1 = \theta$, from (41)

$$e^1 = (I-G^{11})^{-1}G^{12}e^2 \tag{44}$$

Substitute (44) into (42), we have

$$e^2 = (I-\tilde{G}^{22})^{-1}u^2 \tag{45}$$

Substitute (45) into (44), we have

$$e^1 = (I-G^{11})^{-1}G^{12}\ (I-\tilde{G}^{22})^{-1}\ u^2 \tag{46}$$

Thus $(I-\tilde{G}^{22})^{-1}\ :\ (\theta,u^2)\longmapsto e^2$ and $(I-G^{11})^{-1}G^{12}(I-\tilde{G}^{22})^{-1}\ :\ (\theta,u^2)\longmapsto e^1$. Hence the stability of $(I-G^c)^{-1}\ :\ (u^1,u^2)\longmapsto(e^1,e^2)$ implies that of $(I-\tilde{G}^{22})^{-1}$, $(I-G^{11})^{-1}G^{12}(I-\tilde{G}^{22})^{-1}$.

(ii) Consider Fig. 5 and calculate I minus the loop gain[10] of a SCS:

$$\begin{aligned} & I - G^{21}(I-G^{11})^{-1}G^{12}(I-G^{22})^{-1} \\ &= [I-G^{22} - G^{21}(I-G^{11})^{-1}G^{12}]\ (I-G^{22})^{-1} \\ &= (I-\tilde{G}^{22})\ (I-G^{22})^{-1} \end{aligned}$$

$$\text{Hence } (I-\tilde{G}^{22})^{-1} = (I-G^{22})^{-1}\ [I-G^{21}(I-G^{11})^{-1}G^{12}(I-G^{22})^{-1}]^{-1} \tag{47}$$

Note that (47) generalizes for the <u>nonlinear</u> case the standard expression relating closed-loop gain and the open-loop gain in the linear case. ¤

VIII. Sufficient Conditions for the $\mathcal{L}$-Stability of SCS

In this section we present two sufficient conditions for the $\mathcal{L}$-stability of an SCS based on the MES decomposition.

Using equations (41), (42) and the small Gain Theorem at the MES decomposition level, we have

[10] Note that this calculation does <u>not</u> require linearity in any of the G^{ij}'s.

Theorem IV (NTV)

Consider the map $(I-G^c)^{-1}$ described by (41)-(42) and satisfying Assumptions 1 and 2. Suppose that $\gamma[G^{12}]$, $\gamma[(I-G^{11})^{-1}]$, $\gamma[(I-G^{22})^{-1}]$, $\gamma[G^{21}(I-G^{11})^{-1}]$ are finite. Under these conditions if

$$\gamma[G^{21}(I-G^{11})^{-1}] \cdot \gamma[G^{12}] \cdot \gamma[(I-G^{22})^{-1}] < 1 \tag{48}$$

then the map $(I-G^c)^{-1}$ is $\mathcal{L}$-stable. ◻

Remark 8: (i) Theorem IV still holds if the superscripts 1 and 2 are interchanged throughout.

(ii) Note that $\gamma[G^{21}]$, $\gamma[(I-G^{11})^{-1}]$ are finite implies that $\gamma[G^{21}(I-G^{11})^{-1}]$ is finite.

(iii) We shall now check that under the assumptions of Theorem IV, the necessary conditions for the $\mathcal{L}$-stability of $(I-G^c)^{-1}$ given in Remark 7 (i), namely the $\mathcal{L}$-stabilities of $(I-\tilde{G}^{22})^{-1}$ and of $(I-G^{11})^{-1}G^{12}(I-\tilde{G}^{22})^{-1}$ are satisfied. Since $\gamma[G^{21}(I-G^{11})^{-1}G^{12}(I-G^{22})^{-1}]$ $\leq \gamma[G^{21}(I-G^{11})^{-1}] \cdot \gamma[G^{12}] \cdot \gamma[(I-G^{22})^{-1}]$, by the assumption (48) of Theorem IV and the Small Gain Theorem, $[I-G^{21}(I-G^{11})^{-1}G^{12}(I-G^{22})^{-1}]^{-1}$ is $\mathcal{L}$-stable. In view of (47), this together with $\mathcal{L}$-stability of $(I-G^{22})^{-1}$ imply that $(I-\tilde{G}^{22})^{-1}$ is $\mathcal{L}$-stable. Since $(I-G^{11})^{-1}$, G^{12} are assumed to be $\mathcal{L}$-stable, so is $(I-G^{11})^{-1}G^{12}(I-\tilde{G}^{22})^{-1}$. ◻

From Theorems I and IV, we have

Corollary IV.1 (NTV)

Consider the SCS S^c after MES decomposition and satisfying Assumptions 1 and 2. If (a) $\forall i \neq j$, $i,j \in V^c$, $\gamma[G_{ij}]$ is finite; (b) $\forall i \in V^1$, $\gamma[(I-G_{ii})^{-1}]$ is finite; (c) $\forall i \in V^2$, $\gamma[G_{ii}]$ is finite or $\gamma[(I-G_{ii})^{-1}]$ is finite; (d) $\gamma[(I-G^{22})^{-1}]$ is finite; and

(e) $\gamma[G^{21}(I-G^{11})^{-1}] \cdot \gamma[G^{12}] \cdot \gamma[(I-G^{22})^{-1}] < 1$, then the SCS S^c is $\mathcal{L}$-stable. ⌑

Theorem V (NTV)

Consider the map $(I-G^c)^{-1}$ described by (41)-(42) and satisfying Assumptions 1 and 2. If $\gamma[G^{12}]$, $\gamma[(I-G^{11})^{-1}]$, $\gamma[(I-\tilde{G}^{22})^{-1}]$, $\tilde{\gamma}[G^{21}(I-G^{11})^{-1}]$ are finite, then the map $(I-G^c)^{-1}$ is $\mathcal{L}$-stable. ⌑

Remark 9: (i) Theorem V still holds if the superscripts 1 and 2 are interchanged throughout. By the MES decomposition (a) $(I-G^{11})$ has a lower-triangular structure and thus is easy to invert, (b) $(I-\tilde{G}^{22})$, which has no particular structure, has a size equal to $|V^2|$ and usually $|V^2| << |V^1|$. If we interchange the superscripts 1 and 2, we would consider $(I-\tilde{G}^{11})$ and $(I-G^{22})$: however none of these maps has any particular structure. Hence there is a definite advantage in using $\tilde{G}^{22}$ instead of $\tilde{G}^{11}$.

(ii) Note that the third assumption of Theorem V is also necessary. The first three assumptions of Theorem V together imply the $\mathcal{L}$-stability of $(I-G^{11})^{-1}G^{12}(I-\tilde{G}^{22})^{-1}$, which is a necessary condition for the $\mathcal{L}$-stability of $(I-G^c)^{-1}$.

(iii) The assumption $\tilde{\gamma}[G^{21}(I-G^{11})^{-1}] < \infty$ allows us to write $G^{21}(I-G^{11})^{-1}\ (G^{12}e^2+u^1) = G^{21}(I-G^{11})^{-1}\ G^{12}e^2 + \bar{u}^1$ where $\|\bar{u}^1\| \leq \tilde{\gamma}[G^{21}(I-G^{11})^{-1}] \cdot \|u^1\|$ and $\bar{u}^1$ can be viewed as an input equivalent to u^1 but applied at node 2 in Fig. 5.

From Theorems I and V, we have

Corollary V.1 (NTV)

Consider the SCS S^c after MES decomposition and satisfying Assumptions 1 and 2. If (a) $\forall i > j$, $i \in V^c$, $j \in V^1$, $\tilde{\gamma}[G_{ij}]$ is finite and there exists $\bar{e}_{ij} \in \mathcal{L}$ such that $G_{ij}\bar{e}_{ij} \in \mathcal{L}$; (b) $\forall i \in V^1$, $\tilde{\gamma}[(I-G_{ii})^{-1}]$ is finite and there exists $\bar{e}_{ii} \in \mathcal{L}$ such that $G_{ii}\bar{e}_{ii} \in \mathcal{L}$; (c) $\forall i \in V^2$, $\gamma[G_{ii}]$ is finite or $\gamma[(I-G_{ii})^{-1}]$ is finite; (d) $\forall i \neq j$, $i \in V^c$, $j \in V^2$, $\gamma[G_{ij}]$ is finite; and (e) $\gamma[(I-\tilde{G}^{22})]^{-1}$ is finite, then the SCS S^c is $\mathcal{L}$-stable. ¤

Remark 10: (i) The assumption that there exists $\bar{e}_{ij} \in \mathcal{L}$ such that $G_{ij}\bar{e}_{ij} \in \mathcal{L}$, is very mild. It is satisfied when G_{ij} is unbiased, i.e. $G_{ij}\theta = \theta$, which is the class of operators considered in [1,2,14-17]. (ii) Suppose that G_{ij} does not satisfy this assumption. Then for any input in $\mathcal{L}$ its output is in $\mathcal{L}_e - \mathcal{L}$. However the overall system stability requires the input e_j as well as the output $\tilde{y}_{ij}$ of every subsystem G_{ij} to be in $\mathcal{L}$ when the external inputs u_k's are in $\mathcal{L}$. Hence the overall system is not $\mathcal{L}$-stable. ¤

IX. Simplifying Characteristic Functions Using Decompositions

We shall now consider the savings in computing the characteristic function of the overall system S, given by Theorem II D(L) in Sec. IV, due to the two graph theoretic decompositions.

After the SCC decomposition, G is in block-lower-triangular form. Let $j \in V^c_\alpha$. Then $G_{\cdot j}$, the jth column of G, has all zero entries in the first $\sum_{\beta=1}^{\alpha-1} m^c_\beta$ rows. Let $(N_{\cdot j}, D_j)$ be a p.r.c.f. (r.c.f., resp.) of $G_{\cdot j}$. Then $N_{\cdot j}$ also has all zero entries in the first $\sum_{\beta=1}^{\alpha-1} m^c_\beta$ rows. Thus $N \triangleq [N_{\cdot 1} \vdots \cdots \vdots N_{\cdot m}]$ is in the same block-lower-triangular form as G.

Since $\tilde{D} \triangleq \text{diag}(D_1,\ldots,D_m)$ is diagonal, $(\tilde{D}-N)$ is also in the same block-lower-triangular form as G. Let $N^c_{\alpha\alpha}$, $\tilde{D}^c_\alpha$ denote the αth diagonal block of N, $\tilde{D}$ respectively, i.e.

$$N^c_{\alpha\alpha} = [N_{ik}]_{i,k \in V^c_\alpha} , \quad \tilde{D}^c_\alpha = \text{diag}\,((D_i)_{i \in V^c_\alpha})$$

Hence,

$$\det(\tilde{D}-N) = \prod_{\alpha=1}^{\mu} \det(\tilde{D}^c_\alpha - N^c_{\alpha\alpha})$$

Note that $\forall\alpha = 1,\ldots,\mu$, $\det(\tilde{D}^c_\alpha-N^c_{\alpha\alpha}) \in \hat{\mathcal{A}}$, so it is bounded in $\mathbb{C}_+$. Hence $\inf_{s \in \mathbb{C}_+} |\det(\tilde{D}-N)(s)| > 0$ if and only if $\forall\alpha = 1,\ldots,\mu$ $\inf_{s \in \mathbb{C}_+} |\det(\tilde{D}^c_\alpha-N^c_{\alpha\alpha})(s)| > 0$ $\Big(\forall\alpha = 1,\ldots,\mu$, $\det(\tilde{D}^c_\alpha-N^c_{\alpha\alpha}) \in \mathbb{R}[s]$, so it has no poles in $\mathbb{C}$; hence $\det(\tilde{D}-N)(s)$ has no $\mathbb{C}_+$-zero if and only if $\forall\alpha = 1,\ldots,\mu$, $\det(\tilde{D}^c_\alpha-N^c_{\alpha\alpha})(s)$ has no $\mathbb{C}_+$-zero, respectively for the lumped case$\Big)$.

After the MES decomposition, $G^c_{\alpha\alpha} = \begin{bmatrix} G^{11}_{\alpha\alpha} & G^{12}_{\alpha\alpha} \\ G^{21}_{\alpha\alpha} & G^{22}_{\alpha\alpha} \end{bmatrix}$ is in bordered lower-triangular form. By reasoning as above, $\tilde{D}^c_\alpha - N^c_{\alpha\alpha} = \begin{bmatrix} \tilde{D}^1_\alpha-N^{11}_{\alpha\alpha} & -N^{12}_{\alpha\alpha} \\ -N^{21}_{\alpha\alpha} & \tilde{D}^2_\alpha-N^{22}_{\alpha\alpha} \end{bmatrix}$ is also in bordered lower-triangular form. Now

$$\det(\tilde{D}^c_\alpha-N^c_{\alpha\alpha}) = \det[\tilde{D}^2_\alpha-N^{22}_{\alpha\alpha}-N^{21}_{\alpha\alpha}(\tilde{D}^1_\alpha-N^{11}_{\alpha\alpha})^{-1}N^{12}_{\alpha\alpha}] \cdot \det(\tilde{D}^1_\alpha-N^{11}_{\alpha\alpha}) \tag{50}$$

where due to lower-triangular form [11] of $\tilde{D}^1_\alpha - N^{11}_{\alpha\alpha}$,

$$\det(\tilde{D}^1_\alpha - N^{11}_{\alpha\alpha}) = \prod_{i \in V^1_\alpha} \det(D_i - N_{ii}) \tag{51}$$

Thus we have

Corollary II.1 D(L) (LTI)

Consider a LTI overall system S described by (32), which satisfies Assumptions 1 and 2, and has the factorizations described above. (For the lumped case assume that G is proper).

The overall system S is $\mathcal{A}$-stable (exp-stable, resp.) if and only if $\forall \alpha = 1,\ldots,\mu$

$$\inf_{s \in \mathbb{C}_+} \left|\det[\tilde{D}^2_\alpha - N^{22}_{\alpha\alpha} - N^{21}_{\alpha\alpha}(\tilde{D}^1_\alpha - N^{11}_{\alpha\alpha})^{-1} N^{12}_{\alpha\alpha}](s) \cdot \prod_{i \in V^1_\alpha} \det(D_i - N_{ii})(s)\right| > 0. \tag{52}$$

(resp. $\det[\tilde{D}^2_\alpha - N^{22}_{\alpha\alpha} - N^{21}_{\alpha\alpha}(\tilde{D}^1_\alpha - N^{11}_{\alpha\alpha})^{-1} N^{12}_{\alpha\alpha}](s) \cdot \prod_{i \in V^1_\alpha} \det(D_i - N_{ii})(s)$ has no $\mathbb{C}_+$-zero.) ⌑

[11] Recall that we write the G_{ij}'s as if they were SISO, i.e. as if $(D_i - N_{ii})$ are scalar functions. We write here $\det(D_i - N_{ii})$ instead of $(D_i - N_{ii})$ so that the result applies for the MIMO G_{ij}'s.

X. Conclusions

This paper has treated in a very general setting the input-output stability of an arbitrary interconnection of subsystems. Four classes of results are presented: (i) for both NTV and LTI dynamics, a sufficient condition for the overall system stability without using any decomposition (Theorem I); (ii) for both NTV and LTI dynamics, the structural Theorem III stating the equivalence between the overall system stability and those of the CS's; (iii) for NTV dynamics, sufficient conditions for the $\mathcal{L}$-stability of $(I-G^c_{\alpha\alpha})^{-1}$ and of SCS S^c_α using the MES decomposition (Theorems IV and V and their corollaries) and (iv) for LTI dynamics, both lumped and distributed, characteristic function for the overall system stability (Theorem II D(L), Corollary II.1 D(L)).

Although this paper uses the same graph theoretic decompositions as were used in [24], there are considerable differences between these two papers.

(i) The problem formulation is different. In [24], there are two types of subsystems, G_i's and F_{ij}'s, and also two types of summing nodes, those with outputs fed into G_i's and into F_{ij}'s. In the present paper all the subsystems and all the summing modes are treated equally. The present formulation avoids this artificial distinction between the G_i's and the F_{ij}'s and, more importantly, leads to more transparent theorem formulations.

(ii) The definition of overall system stability has been modified in a very important way. In [24], the overall system stability is defined as the stability of the map $(u,v) \longmapsto (e,\eta)$ or equivalently, stability of the map $(u,v) \longmapsto (Ge,F\eta)$. Since $F\eta = (\sum_{j=1}^{m} F_{ij}\eta_j)_{i=1}^{m}$, two unstable terms, say $F_{i\ell}\eta_\ell$, $F_{ik}\eta_k$, may cancel each other and give a stable sum $\sum_{j=1}^{m} F_{ij}\eta_j$ (see Remark 2).[13] In the present paper, the overall system stability is defined to be the stability of the map $u \longmapsto (G_{ij}e_j)_{i,j=1}^{m}$: this is consistent with viewing the overall system as an interconnection of black-boxes because it requires the map from the inputs to each black-box output to be stable.

(iii) In the present paper, the structural results and their proofs are developed in a more systematic manner: the three types of dynamics (NTV, LTI distributed and LTI lumped) are carried together, rather than only carried out for the $\mathcal{L}$-stability case. A much more thorough understanding of the interplay of the results is achieved. The derivations highlight the key role played by the closure properties of stable maps, by the block-lower-triangular structure resulting from the SCC decomposition and by the particular features of the definition of stability.

[13] This formulation is the conventional one for single loop feedback systems, but in that case, since there is only one such subsystem output per summing node, the cancellation is impossible.

(iv) In [24], there are no counterparts of Theorems I, II, III and IV of the present paper.

(v) the computational effort required to compute the characteristic function for the overall system stability in [24] is considerably greater than in the present paper. (Theorem II and Corollary II.1) In [24], the characteristic function $\det(D_{F\ell}D_{Gr}-N_{F\ell}N_{Gr})(s)$ is obtained by treating the overall system S as a "big" feedback system (G,F). To compute it one must first find a coprime factorization of a "big" matrix F. In the present paper, the characteristic function $\det(\tilde{D}-N)(s)$ is given by Corollary II.1 D(L). To compute it one only need to find a coprime factorization for each column $G_{\cdot j}$ of G, $j = 1,\ldots,m$. This computational gain is achieved in spite of the more refined and more demanding definition of the overall system stability.

By using the same reasoning as in [50], sufficient conditions under which the exp-stability of the overall system is robust can easily be obtained. Due to space limitation, we do not present them in this paper.

The main challenge in <u>decentralized</u> control of large scale systems is due to informational constraints: the controller at subsystem i has no knowledge of the information at subsystem j, (e.g. the output observed, the control applied etc.), and vice versa. However in the preliminary analysis of a large system one should first concentrate on the qualitative system <u>properties</u> (e.g. stability, controllability, observatility, etc.), as opposed to the <u>control</u> of the large scale system; at that stage information contraints do not come into the picture. For such preliminary study graph theoretic decomposition techniques, not necessarily restricted to SCC and MES decompositions, should be used to analyze the structure of the large scale system. It is clear that the structure emerging from such an analysis will play a crucial role in the control problem.

SCC decomposition is particularly useful for stability study because it decomposes the overall system into a series-parallel connection of SCS's and IS's and the stability is preserved under such connection (Lemma 1). Hence the stability question of the overall system becomes decentralized in the sense that to ensure overall system stability, we only need to make each SCS and each IS stable. As noted in introducing Fact 4, for the overall system stability to be robust, it is reasonable to assume every IS is stable. Under this assumption, the overall system is stable if and only if every SCS is stable, and hence stabilization of the overall system is <u>decentralized</u> into the stabilization of each SCS.

Even though this paper considered mostly interconnection of <u>nonlinear</u> subsystems, a key feature of its analysis was that the interconnection occured through <u>summing</u> nodes which are <u>linear</u> elements. (This type of additive nonlinear interaction is also found in most works based on the Lyapunov function technique [13].) It is not clear at this time what will be the most successful model for truly nonlinear interaction among subsystems. Some work has already been reported for the feedback case. [38,51]

Proof of Theorem I

Let $U \triangleq \{(i,j) | i = j \text{ and } G_{ii} \text{ is unstable}\}$. By assumption, $(I-G)^{-1} : u \longmapsto e = (e_j)_{j=1}^{m}$ is stable. Since $\forall (i,j) \notin U$, G_{ij} is stable, the map

$$u \longmapsto G_{ij} e_j \text{ is stable } \forall (i,j) \notin U \tag{A.1}$$

Now consider some $(i,i) \in U$. Substituting e_i by (10), we have

$$G_{ii} e_i = G_{ii}(I-G_{ii})^{-1} (u_i + \sum_{\substack{j=1 \\ j \neq i}}^{m} G_{ij} e_j) \tag{A.2}$$

By (A.1) and Lemma 1,

the map

$$u \longmapsto (u_i + \sum_{\substack{j=1 \\ j \neq i}}^{m} G_{ij} e_j) \text{ is stable.} \tag{A.3}$$

By (13) the assumed stability of $(I-G_{ii})^{-1}$ is equivalent to the map

$$G_{ii}(I-G_{ii})^{-1} \text{ is stable.} \tag{A.4}$$

In view of (A.2), (A.3) and (A.4) together imply that

$$u \longmapsto G_{ii} e_i \text{ is stable.} \tag{A.5}$$

(A.1) and (A.5) establish the overall system stability.

Q.E.D.

Proof of Lemma 2D(L)

We first prove the lemma for the distributed case. Recall the well-established identity for matrices M,N,

$$M(I-NM)^{-1} = (I-MN)^{-1}M. \tag{A.6}$$

Substituting for H and using (A.6) we have

$$H(I-KH)^{-1} = N_r D^{-1} N_\ell (I-KN_r D^{-1} N_\ell)^{-1}$$

$$= N_r D^{-1} (I-N_\ell K N_r D^{-1})^{-1} N_\ell$$

$$= N_r (D-N_\ell K N_r)^{-1} N_\ell \tag{A.7}$$

Hence the equivalence of the lemma will be established once we show that the three matrices in (A.7) form a p.r.l.c.f. of $H(I-KH)^{-1}$. By assumption (N_r, D, N_ℓ) is a p.r.l.c.f. of H, hence there exist $\hat{\mathcal{A}}$-matrices U_r, U_ℓ, V_r, V_ℓ such that

$$\det[U_r N_r + V_r D](s)$$

$$= \det[(U_r + V_r N_\ell K) N_r + V_r (D - N_\ell K N_r)](s) \neq 0 \qquad \forall s \in \mathbb{C}_+ \tag{A.8}$$

and

$$\det[N_\ell U_\ell + D V_\ell](s)$$

$$= \det[N_\ell (U_\ell + K N_r V_\ell) + (D - N_\ell K N_r) V_\ell](s) \neq 0 \qquad \forall s \in \mathbb{C}_+ \tag{A.9}$$

Since K is a constant matrix, and $U_r, V_r, N_\ell, U_\ell, N_r, V_\ell$ are $\hat{\mathcal{A}}$-matrices, by closure property of algebra $\hat{\mathcal{A}}$, $(U_r + V_r N_\ell K)$, $(U_\ell + K N_r V_\ell)$ are also $\hat{\mathcal{A}}$-matrices. Now (A.10)

$$\det(D - N_\ell K N_r) = \det D \cdot \det(I - N_\ell K N_r D^{-1})$$

$$= \det D \cdot \det(I - K N_r D^{-1} N_\ell)$$

$$= \det D \cdot \det(I - KH) \tag{A.11}$$

By definition of p.r.l.c.f., $\forall$ sequences $(s_i)_{i=1}^{\infty} \subset \mathbb{C}_+$ where $|s_i| \to \infty$, $\liminf_{i \to \infty} |\det D(s_i)| > 0$ and by assumption $\liminf_{i \to \infty} |\det(I-KH)(s_i)| > 0$

Hence by (A.11), for all such sequences,

$$\liminf_{i \to \infty} |\det(D - N_\ell K N_r)(s_i)| > 0.$$

This together with (A.7)-(A.10) imply that $(N_r, D-N_\ell K N_r, N_\ell)$ is a p.r.l.c.f. of $H(I-KH)^{-1}$.

For the lumped case, the proof follows similarly. In view of (A.11) the assumption $\det(I-KH)(\infty) \neq 0$ implies that $\det(D-N_\ell K N_r) \not\equiv 0$.

Q.E.D.

Proof of Theorem II D(L)

Let $\tilde{N} \triangleq \mathrm{diag}(N_{.1},\ldots,N_{.m})$

Thus $\tilde{G} = \tilde{N}\tilde{D}^{-1}$ and since $(N_{.j}, D_j)$ is a p.r.c.f. of $G_{.j}$, for $j = 1,\ldots,m$, it is easy to see that $(\tilde{N},\tilde{D})$ is a p.r.c.f. of $\tilde{G}$, or equivalently $(\tilde{N},\tilde{D},I)$ is a p.r.l.c.f. of $\tilde{G}$. From (17), $G = \tilde{K}\tilde{G}$. Thus Assumption 2 is equivalent to $\forall$ sequences $(s_i)_{i=1}^{\infty} \subset \mathbb{C}_+$ where $|s_i| \to \infty$, $\liminf_{i\to\infty} |\det(I-\tilde{K}\tilde{G})(s_i)| > 0$. Now we apply Lemma 2D to the overall system S (Fig. 2) with $H \leftarrow \tilde{G}$, $K \leftarrow \tilde{K}$, $N_r \leftarrow \tilde{N}$, $D \leftarrow \tilde{D}$, $N_\ell \leftarrow I$ and with $N = \tilde{K}\tilde{N}$, Theorem IID follows.

For the lumped case, the proof follows similarly.

Q.E.D.

Proof of Theorem III

By definition, $\forall\beta = 1,\ldots,\mu$, CS S_β are stable if and only if

$$\forall\alpha \geq \beta = 1,\ldots,\mu,\ \tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1} \text{ are stable.} \tag{A.21}$$

By definition, the overall system is stable if and only if

$$\forall\alpha \geq \beta = 1,\ldots,\mu,\ (u^c_i)_{i=1}^{\beta} \mapsto \tilde{y}^c_{\alpha\beta} \text{ are stable.} \tag{A.22}$$

Recall

$$\tilde{y}^c_{\alpha\beta} = \tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1}\,(u^c_\beta + \sum_{j=1}^{\beta-1} \tilde{K}^c_{\beta j}\,\tilde{y}^c_{\beta j}) \tag{33}$$

We prove (A.21) $\Rightarrow$ (A.22) by induction on β. Suppose (A.21) holds. From (33), $\forall\alpha \geq 1$,

$$\tilde{y}^c_{\alpha 1} = \tilde{G}^c_{\alpha 1}(I-G^c_{11})^{-1}\ u^c_1$$

i.e. $\tilde{G}^c_{\alpha 1}(I-G^c_{11})^{-1} : u^c_1 \longmapsto \tilde{y}^c_{\alpha 1}$. Thus (A.21) implies that (A.22) holds $\forall\alpha \geq \beta = 1$.

Suppose (A.22) holds $\forall\alpha \geq \beta = 1,\ldots,\gamma-1$. (A.23)

From (33), $\forall\alpha \geq \gamma$, we have

$$\tilde{y}^c_{\alpha\gamma} = \tilde{G}^c_{\alpha\gamma}(I-G^c_{\gamma\gamma})^{-1}\ (u^c_\gamma + \sum_{j=1}^{\gamma-1} \tilde{K}^c_{\gamma j}\ \tilde{y}^c_{\gamma j}).$$

Note that by (A.21) $\tilde{G}^c_{\alpha\gamma}(I-G^c_{\gamma\gamma})^{-1}$ is stable $\forall\alpha \geq \gamma$, and by (A.23) the map $(u^c_i)^{\gamma}_{i=1} \longmapsto (\tilde{y}^c_{\gamma j})^{\gamma-1}_{j=1}$ is stable. Thus (A.22) holds $\forall\alpha \geq \beta = 1,\ldots,\gamma$. Hence, by induction, (A.22) holds $\forall\alpha \geq \beta = 1,\ldots,\mu$.

We prove (A.22) $\Rightarrow$ (A.21) by contradiction. Suppose (A.22) holds. Suppose, for sake of contradiction, that for some α,β with $\alpha \geq \beta$, $\tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1}$ is not stable.

For LTI dynamics, consider the input $(\theta,\ldots,\theta,u^c_\beta,\ \theta,\ldots,\theta)$. By (33) and linearity, the corresponding output is

$$\hat{y}^c_{\alpha\beta} = \tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1}\ u^c_\beta$$

Thus (A.22) implies that $\tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1}$ is stable. Hence we have reached a contradiction.

For NTV dynamics, (A.22) implies that there exist $\bar{\gamma},\bar{\beta}$ such that $\forall u^c_i \in \mathcal{L}^{m^c_i}_e$, $i = 1,\ldots,\beta$, $\forall T \in \mathcal{T}$,

$$\|\tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1}(u^c_\beta + \sum_{j=1}^{\beta-1} \tilde{K}^c_{\beta j}\ \tilde{y}^c_{\beta j})\|_T \leq \bar{\gamma}(\sum_{i=1}^{\beta}\|u^c_i\|_T) + \bar{\beta} \tag{A.24}$$

Let $(\hat{y}^c_{\beta j})^{\mu}_{j=1}$ be the output corresponding to the input $(\theta,\ldots,\theta,u^c_\beta,\theta,\ldots,\theta)$,

Let $f \triangleq \sum_{j=1}^{\beta-1} \tilde{K}^c_{\beta j}\, \hat{y}^c_{\beta j}$. Observing that $(\hat{y}^c_{\beta j})^{\beta-1}_{j=1}$ depends only on the first $(\beta-1)$ components of that input, namely $(\theta,\ldots,\theta)$, by the assumed $\mathcal{L}$-stability of the overall system,

$$f \in \mathcal{L}^{m^c_\beta} \tag{A.25}$$

For that input (A.24) becomes

$$\|\tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1}\,(u^c_\beta+f)\|_T \leq \bar{\gamma}\|u^c_\beta\|_T + \bar{\beta} \tag{A.26}$$

Letting $\hat{u}^c_\beta \triangleq u^c_\beta + f$ and recalling that $\mathcal{L}^{m^c_\beta}_e$ is closed under addition, we conclude from (A.26) that there exist $\bar{\gamma},\bar{\beta}$ such that $\forall \hat{u}^c_\beta \in \mathcal{L}^{m^c_\beta}_e$, $\forall T \in \mathcal{T}$,

$$\begin{aligned}\|\tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1}\,\hat{u}^c_\beta\|_T &\leq \bar{\gamma}(\|\hat{u}^c_\beta - f\|_T) + \bar{\beta}\\ &\leq \bar{\gamma}\|\hat{u}^c_\beta\|_T + \bar{\gamma}\|f\|_T + \bar{\beta}\\ &\leq \bar{\gamma}\|\hat{u}^c_\beta\|_T + (\bar{\gamma}\|f\| + \bar{\beta})\end{aligned}$$

where by (A.25), $\|f\| < \infty$. Thus $\tilde{G}^c_{\alpha\beta}(I-G^c_{\beta\beta})^{-1}$ is $\mathcal{L}$-stable. Hence we have reached a contradiction.

Q.E.D.

<u>Proof of Theorem IV</u>

From the SCS equation (42),

$$\begin{aligned}e^2 &= (I-G^{22})^{-1}\,(u^2+G^{21}e^1)\\ &= (I-G^{22})^{-1}\,[u^2+G^{21}(I-G^{11})^{-1}(u^1+G^{12}e^2)]\end{aligned}$$

Hence $\forall u^1 \in \mathcal{L}_e^{m^1}$, $\forall\ u^2 \in \mathcal{L}_e^{m^2}$, $\forall T \in \mathcal{T}$,

$$\begin{aligned}\|e^2\|_T &\le \gamma[(I-G^{22})^{-1}]\ \|u^2+G^{21}(I-G^{11})^{-1}\ (u^1+G^{12}e^2)\|_T + \beta[(I-G^{22})^{-1}]\\ &\le \gamma[(I-G^{22})^{-1}] \cdot \{\gamma[G^{21}(I-G^{11})^{-1}]\ \|u^1+G^{12}e^2\|_T + \beta[G^{21}(I-G^{11})] +\\ &\quad \|u^2\|_T\} + \beta[(I-G^{22})^{-1}]\\ &\le \gamma[(I-G^{22})^{-1}] \cdot \Big(\gamma[G^{21}(I-G^{11})^{-1}] \cdot \{\gamma[G^{12}]\|e^2\|_T + \beta[G^{12}] +\\ &\quad \|u^1\|_T\} + \beta[G^{21}(I-G^{11})^{-1}] + \|u^2\|_T\Big) + \beta[(I-G^{22})^{-1}]\end{aligned}$$

Hence it is easy to choose positive numbers $\bar{\gamma},\bar{\beta} \ni \forall u^1 \in \mathcal{L}_e^{m^1}$, $\forall u^2 \in \mathcal{L}_e^{m^2}$, $\forall T \in \mathcal{T}$,

$$\{1-\gamma[(I-G^{22})^{-1}] \cdot \gamma[G^{21}(I-G^{11})^{-1}] \cdot \gamma[G^{12}]\}\|e^2\|_T$$

$$\le \bar{\gamma} \cdot (\|u^1\|_T + \|u^2\|_T) + \bar{\beta}$$

where by assumption (48) the scalar multiplying $\|e^2\|_T$ is greater than zero. Thus,

$$\|e^2\|_T \le \{1-\gamma[(I-G^{22})^{-1}] \cdot \gamma[G^{21}(I-G^{11})^{-1}] \cdot \gamma[G^{12}]\}^{-1}\cdot$$
$$[\bar{\gamma} \cdot (\|u^1\|_T + \|u^2\|_T) + \bar{\beta}].$$

Hence $(u^1,u^2) \longmapsto e^2$ is $\mathcal{L}$-stable. Now from (41)

$$e^1 = (I-G^{11})^{-1}\ (u^1+G^{12}e^2)$$

Since (i) $(u^1,u^2) \longmapsto e^2$ is $\mathcal{L}$-stable, and (ii) G^{12}, $(I-G^{11})^{-1}$ are assumed to be $\mathcal{L}$-stable, by Lemma 1, $(u^1,u^2) \longmapsto e^1$ is $\mathcal{L}$-stable.

Hence $(I-G)^{-1} : (u^1,u^2) \longmapsto (e^1,e^2)$ is $\mathcal{L}$-stable. Q.E.D.

Proof of Corollary IV.1:

Assumption (a) implies that

$$\gamma[G^{12}],\ \gamma[G^{21}] \text{ are finite.} \tag{A.27}$$

Assumptions (a), (b) and the lower-triangular form of $(I-G^{11})$, together imply that

$$\gamma[(I-G^{11})^{-1}] \text{ is finite.} \tag{A.28}$$

By Theorem IV, (A.27), (A.28), Assumptions (d) and (e) together imply that

$$(I-G^{c})^{-1} \text{ is } \mathcal{L}\text{-stable.} \tag{A.29}$$

Using Theorem I with G replaced by G^c, Assumptions (a), (b), (c) and (A.29) together imply that the SCS S^c is $\mathcal{L}$-stable.

Q.E.D.

Proof of Theorem V

From the SCS equations (41) and (42)

$$(I-G^{22})e^2 = u^2 + G^{21}(I-G^{11})^{-1}\ (u^1+G^{12}e^2)$$

$$[I-G^{22}-G^{21}(I-G^{11})^{-1}G^{12}]e^2 = u^2+G^{21}(I-G^{11})^{-1}\ (u^1+G^{12}e^2) - G^{21}(I-G^{11})^{-1}G^{12}e^2$$

$$e^2 = (I-\tilde{G}^{22})^{-1}\ (u^2+\tilde{u}^1) \tag{A.30}$$

where $\tilde{u}^1 \triangleq G^{21}(I-G^{11})^{-1}\ (u^1+G^{12}e^2) - G^{21}(I-G^{11})^{-1}G^{12}e^2$

satisfies $\|\tilde{u}^1\| \leq \tilde{\gamma}[G^{21}(I-G^{11})^{-1}] \cdot \|u^1\| \quad \forall u^1 \in \mathcal{L}^{m^1}$.

Since (i) $(u^1,u^2)\longmapsto(u^2+\tilde{u}^1)$ is $\mathcal{L}$-stable and (ii) $(I-\tilde{G}^{22})^{-1}$ is assumed to be $\mathcal{L}$-stable, by Lemma 1, $(u^1,u^2)\longmapsto e^2$ is $\mathcal{L}$-stable.

From (41) and (A.30)

$$e^1 = (I-G^{11})^{-1}\ (u^1+G^{12}e^2)$$

Since (i) $(u^1,u^2) \longmapsto e^2$ is $\mathcal{L}$-stable, and (ii) G^{12} and $(I-G^{11})^{-1}$ are assumed to be $\mathcal{L}$-stable, by Lemma 1, $(u^1,u^2) \longmapsto e^1$ is $\mathcal{L}$-stable.

Hence $(I-G)^{-1} : (u^1,u^2) \longmapsto (e^1,e^2)$ is $\mathcal{L}$-stable.

Q.E.D.

<u>Proof of Corollary V.1</u>:

It follows from Assumption 2 that $(I-G_{ii})^{-1}$ is a well-defined map; by Assumption (b) of the corollary, $\forall i \in V^1$, there exists $\bar{e}_{ii} \in \mathcal{L}$ such that $G_{ii}\bar{e}_{ii} \in \mathcal{L}$, hence $\bar{u}_{ii} \triangleq (I-G_{ii})\bar{e}_{ii} \in \mathcal{L}$ and consequently $(I-G_{ii})^{-1}\,\bar{u}_{ii} \in \mathcal{L}$. Thus, by Remark 1(iv), Assumptions (a) and (b) respectively imply that

$$\forall i > j,\ i \in V^c,\ j \in V^1,\ \gamma[G_{ij}] \text{ is finite,} \tag{A.31}$$

$$\forall i \in V^1,\ \gamma[(I-G_{ii})^{-1}] \text{ is finite.} \tag{A.32}$$

Assumption (a) implies that

$$\tilde{\gamma}[G^{21}] \text{ is finite.} \tag{A.33}$$

Assumptions (a), (b) and the lower-triangular form of $(I-G^{11})$, together imply that

$$\tilde{\gamma}[(I-G^{11})^{-1}] \text{ is finite.} \tag{A.34}$$

Similarly, (A.31), (A.32) and the lower-triangular form of $(I-G^{11})$, together imply that

$$\gamma[(I-G^{11})^{-1}] \text{ is finite.} \tag{A.35}$$

Assumption (d) implies that

$$\gamma[G^{12}] \text{ is finite.} \tag{A.36}$$

By Theorem V, (A.33)—(A.36) and Assumption (e) together imply that

$$(I-G^c)^{-1} \text{ is } \mathcal{L}\text{-stable.} \tag{A.37}$$

By Theorem I, (A.31), (A.37), Assumptions (c) and (d) together imply that the SCS S^c is $\mathcal{L}$-stable.

Q.E.D.

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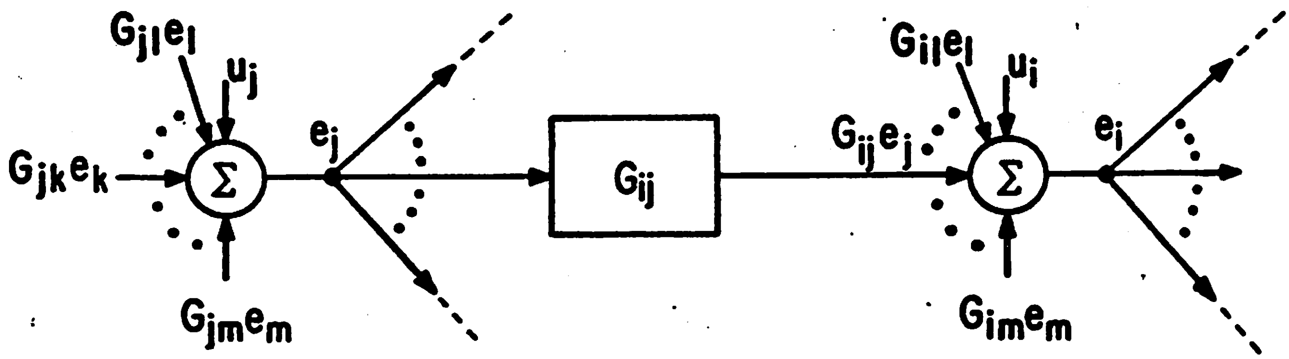

Fig. 1 A typical subsystem G_{ij}.

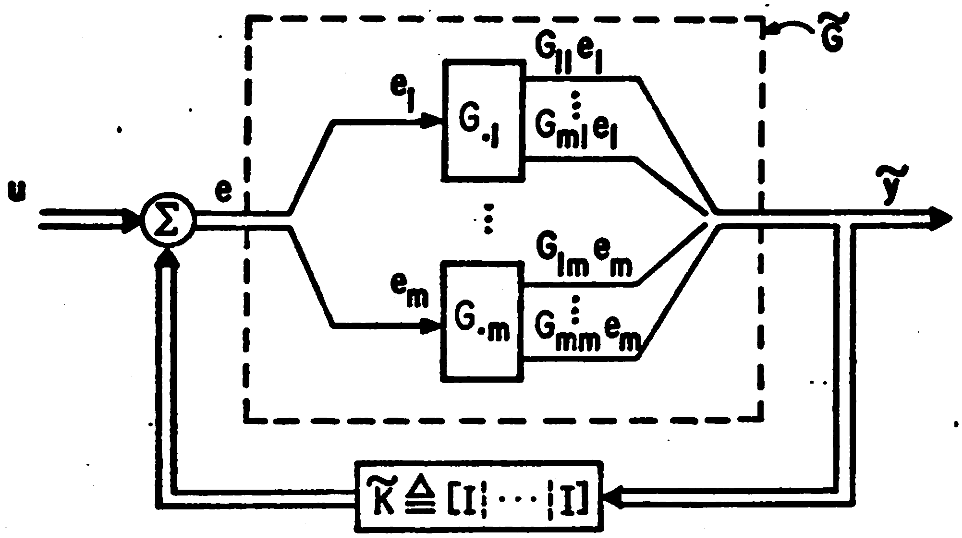

Fig. 2 Overall system S : $u \longmapsto \tilde{y}$ viewed as a constant output feedback system.

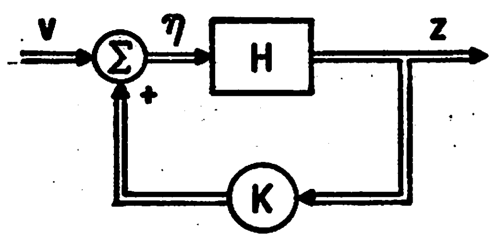

Fig. 3 Constant output feedback system.

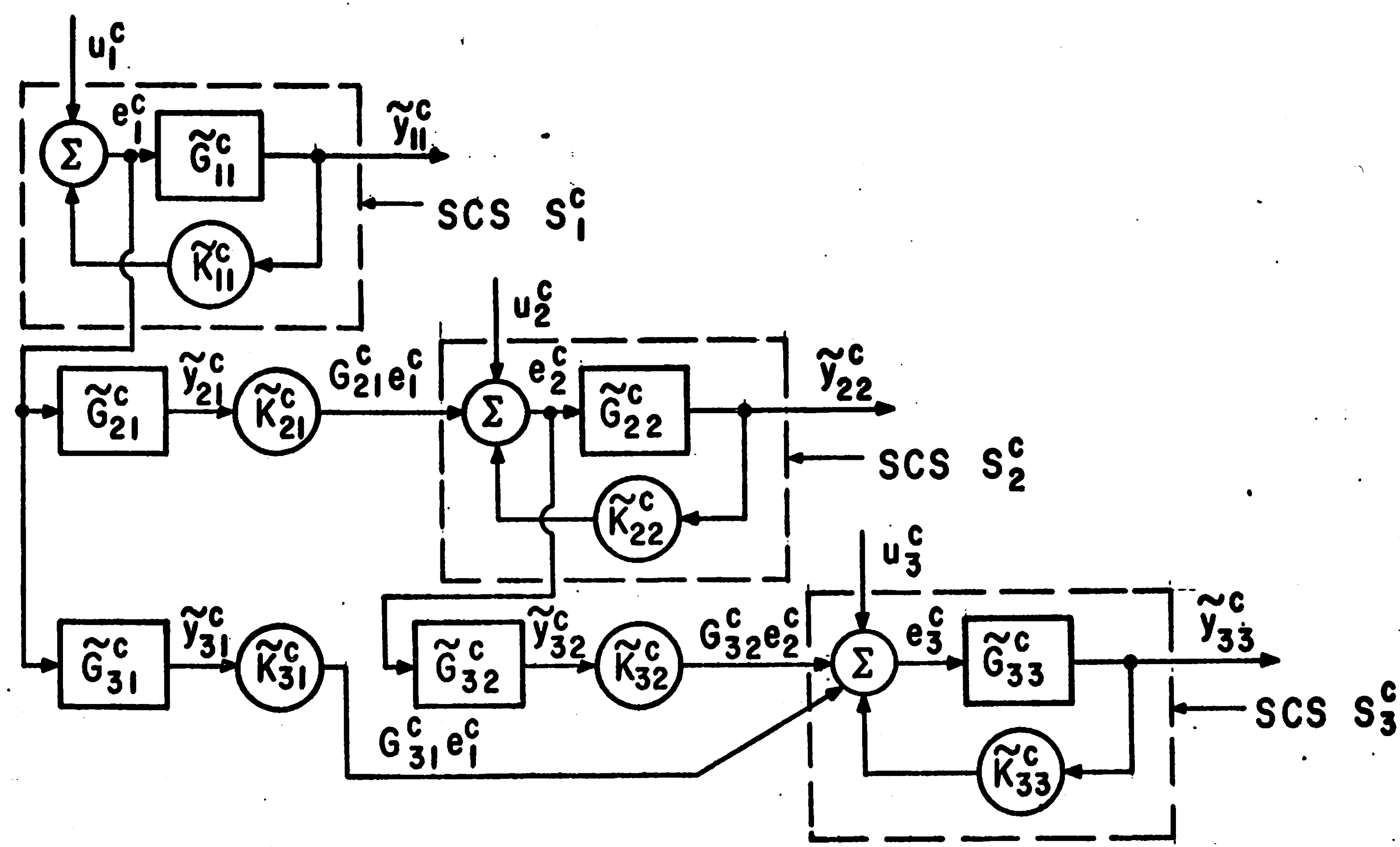

Fig. 4 Overall system S viewed as a series-parallel connection of SCS's, IS's and $\tilde{K}_{\alpha\beta}^c$'s.

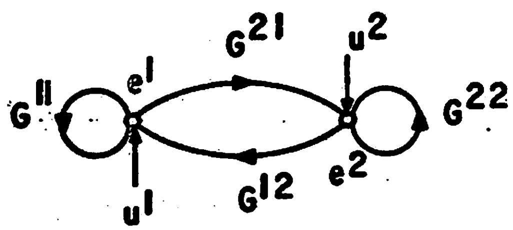

Fig. 5 Flow graph associated with (41), (42).