ON DELAYED SHARING PATTERNS

by

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ABSTRACT

A counterexample is given to a conjectured separation result for delayed sharing patterns when the delay is larger than two units. The conjecture is proved to be true for a unit delay.

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I. INTRODUCTION

We use the formal setup and, with minor modification, the notation developed by Witsenhausen [1]. The equations for the dynamics and the observations are given by

$$x_t = f_t(x_{t-1}, u_t^1, \ldots, u_t^K, v_t), \quad t = 1, \ldots, T \tag{1}$$

$$y_t^k = g_t^k(x_{t-1}, w_t^k), \quad k = 1, \ldots, K \text{ and } t = 1, \ldots, T \tag{2}$$

where at time t x_t is the state, u_t^k and y_t^k are respectively the control input and observed output at the kth "station". The primitive random variables

$$x_0; v_t, w_t^k \quad (k = 1, \ldots, K, \ t = 1, \ldots, T)$$

are all independent. The control input u_t^k must be selected from a fixed set U_t^k, and the data upon which this selection can depend is given by the collection of variables

$$(Y_{t,k}, U_{t,k}) = \{(y_s^\ell, u_s^\ell) \mid s \leq t-n, \ \ell \leq K\} \cup \{(y_s^k, u_r^k) \mid t-n < s \leq t,$$
$$t-n < r \leq t-1\} = \delta_t \cup \lambda_t^k, \text{ say.}$$

Here the delay is the fixed positive integer n. Thus at time t δ_t is the data shared by all stations whereas λ_t^k is the data available only to station k.

An (admissible) control <u>law</u> then is any set of functions $\gamma = \{\gamma_t^k; \ 1 \leq t \leq T, \ 1 \leq k \leq K\}$ such that

$$\gamma_t^k : (\delta_t, \lambda_t^k) \longmapsto u_t^k = \gamma_t^k(\delta_t, \lambda_t^k) \in U_t^k.$$

Γ is the set of all control alws. (Here and below we are ignoring many difficult technical considerations of measurability.) Γ^s is the subset of all <u>separated</u> laws, i.e., all $\{\gamma_t^k\}$ of the form

$$\gamma_t^k(\delta_t,\lambda_t^k) = \phi_t^k(F_t,\lambda_t^k),$$

where F_t, also denoted $F_t(\delta_t)$ or $F(x_{t-n}|\delta_t)$, is the conditional probability distribution of x_{t-n} given δ_t.

The cost associated with any law γ is given by

$$J(\gamma) = E \sum_1^T h_t(x_t,u_t^1,\ldots,u_t^K) \tag{3}$$

where $u_t^k = \gamma_t^k(\delta_t,\lambda_t^k)$ and E denotes expectation. It has been conjectured [1, Assertions 8,9] that in finding an optimum γ from Γ it is enough to restrict attention to Γ^S.

<u>Conjecture</u> $\inf\{J(\gamma)|\gamma \in \Gamma\} = \inf\{J(\gamma)|\gamma \in \Gamma^S\}$.

II. COUNTEREXAMPLE FOR n=2

We present an example in which n=2, T=3, k=2, the f_t,g_t are linear, h_t are quadratic and the primitive random variables are jointly Gaussian. The state $x_t = (x_t^1,x_t^2)$ is two-dimensional. Equation (1) is as follows:

$$x_0 = (x_0^1,x_0^2)$$

$$x_1 = (x_1^1,x_1^2) = (x_0^1+x_0^2,0)$$

$$x_2 = (x_2^1,x_2^2) = (x_1^1,u_2^2) = (x_0^1+x_0^2,u_2^2)$$

$$x_3 = (x_3^1,x_3^2) = (x_2^1-x_2^2-u_3^1,0) = (x_0^1+x_0^2-u_2^2-u_3^1,0).$$

Equation (2) is as follows:

$$y_t^k = x_{t-1}^k,\ k = 1,2 \text{ and } t = 1,2,3.$$

Hence the primitive random variables are (x_0^1,x_0^2) whose joint distribution will be specified later. The sets U_t^k are specified by

$$U_t^k = \begin{cases} R & \text{if } (k,t) = (1,3) \text{ or } (2,2) \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence a control law $\gamma = \{\gamma_t^k\}$ in Γ is essentially characterized by, and hence identifiable with, the pair $\{\gamma_3^1, \gamma_2^2\}$ since $\gamma_t^k \equiv 0$ for the remaining (k,t). From the above we get

$$\delta_3 = \{y_1^1, y_1^2\} = \{x_0^1, x_0^2\}, \ \lambda_3^1 = \{y_2^1, y_3^1\} = \{x_0^1+x_0^2, x_0^1+x_0^2\} = \{x_0^1+x_0^2\},$$

$$\delta_2 = \phi, \ \lambda_2^2 = \{y_1^2, y_2^2\} = \{x_0^2, x_1^2\} = \{x_0^2\} \text{ (since } x_1^2 \equiv 0),$$

so that that conditional distribution $F(x_1 = \xi | \delta_3)$ is degenerate:

$$F(x_1^1 = \xi^1, x_1^2 = \xi^2 | x_0^1, x_0^2) = \delta(\xi^1 - x_0^1 - x_0^2)\delta(\xi^2)$$

where δ is the "delta function".

Therefore Γ consists of all $\{\gamma_3^1, \gamma_2^2\}$ of the form

$$\gamma_3^1: (x_0^1, x_0^2) \longmapsto u_3^1 \in R, \ \gamma_2^2: x_0^2 \longmapsto u_2^2 \in R$$

whereas Γ^s consists of all $\{\gamma_3^1, \gamma_2^2\}$ of the form

$$\gamma_3^1: (x_0^1+x_0^2) \longmapsto u_3^1 \in R, \ \gamma_2^2: x_0^2 \longmapsto u_2^2 \in R.$$

Observe that in Γ^s γ_3^1 can depend only on $x_0^1+x_0^2$ whereas in Γ it can depend on x_0^1 and x_0^2. Finally the cost function is taken to be

$$J(\gamma) = \frac{1}{2} E\{(x_3^1)^2 + (u_3^1)^2\} = \frac{1}{2} E\{(x_0^1+x_0^2-u_2^2-u_3^1)^2 + (u_3^1)^2\}.$$

To see the situation more clearly set

$$w = x_0^1+x_0^2, \ x_0^2 = y, \ \gamma_3^1(\cdot) = v(\cdot), \ \gamma_2^2(\cdot) = u(\cdot).$$

Then the optimal law in Γ is the solution of

$$\min \frac{1}{2} E\{(w-u-v)^2 + v^2\}$$

$$\text{s.t. } u = u(y), \ v = v(y,w), \tag{4}$$

and the optimal law in Γ^s is the solution of

$$\min \frac{1}{2} E\{(w-u-v)^2+v^2\}$$

$$\text{s.t. } u = u(y),\ v = v(w). \tag{5}$$

We assume that the primitive random variables (w,y) are jointly Gaussian with zero mean and covariance matrix

$$\Sigma_{wy} = \begin{bmatrix} \sigma_w^2 & \sigma_{wy} \\ \sigma_{yw} & \sigma_y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$

The problem (4) has the classical information pattern. Its optimal solution is <u>unique</u> and can be calculated in a straightforward manner and turns out to be

$$u^*(y) = \frac{1}{2} y,\ v^*(y,w) = \frac{1}{2} w - \frac{1}{4} y.$$

Observe that v^* depends on y and so it is <u>not</u> in Γ^s. Since the optimal law in Γ is unique we conclude that for the example

$$\min\{J(\gamma) | \gamma \in \Gamma\} < \min\{J(\gamma) | \gamma \in \Gamma^s\}, \tag{6}$$

and so the conjecture is false for delay $n \geq 2$.[1]

With this example before us we can see why the conjecture is false. Essentially $F_t(\delta_t)$ consists of all the information about x_{t-n} contained in δ_t but δ_t also contains information about the <u>controls</u> selected between t-n and t and this latter information may not be "captured" by

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[1] The problem (5) is very close to the example treated by Witsenhausen [2]. An analysis similar to the one given there shows that the optimal separated law in Γ^s is <u>nonlinear</u>. Strictly speaking we have not shown that an optimum law in Γ^s exists. This can be done along the same lines as in [2]. Alternatively we can give an example very similar to the one given here but in which all variables take on finitely many values so that we can compute $J(\gamma)$ for each γ.

$F_t(\delta_t)$. In the example: if δ_3 is known then u_2^2 is also known but if only $F(x^1|\delta_3)$ is known then u_2^2 cannot be determined.

III. CASE n=1

The conjecture is true when there is only a unit delay and has been proved for the LQG case [3], but no proof appears to have been published for the more general case. We take this opportunity to sketch a proof using dynamic programming. To ease the notational burden we assume K=2. For any random variables α,β let $P(\alpha|\beta)$ denote the conditional distribution of α given β. However we preserve the notation $F_t(\delta_t) = P(x_{t-1}|\delta_t)$.

We begin with the observation [1,p.1562] that $F_t(\delta_t)$ does <u>not</u> depend upon the choice of γ in Γ. From (2) we can conclude then that

$$P(x_{t-1},y_t^1,y_t^2)|\delta_t) = P(y_t^1,y_t^2|x_{t-1})F(x_{t-1}|\delta_t). \tag{7}$$

Here P does not depend upon γ. Also note that

$$\delta_{t+1} = \delta_t \cup \{y_t^1,u_t^1,y_t^2,u_t^2\}$$

From these facts and (1), (2), a standard use of Bayes' formula shows the existence of an "updating" function Φ_t, again not depending on γ, such that

$$F_{t+1}(\delta_{t+1}) = \Phi_t(F_t(\delta_t),y_t^1,u_t^1,y_t^2,u_t^2). \tag{8}$$

In obtaining the dynamic programming equation the next result will be useful. Let $\Gamma_t = \{(\gamma_t^1,\gamma_t^2)\}$ be the set of admissible control laws at t, and let Γ_t^s be its subset consisting of separated laws. Let η, ξ be real-valued functions of their arguments and for each (γ_t^1,γ_t^2) define $V(\delta_t;\gamma_t^1,\gamma_t^2)$ by

$$V(\delta_t;\gamma_t^1,\gamma_t^2) = E\{\eta(x_{t-1},\gamma_t^1(\delta_t,y_t^1),\gamma_t^2(\delta_t,y_t^2),v_t) + \xi(F_t(\delta_t),y_t^1,$$
$$\gamma_t^1(\delta_t,y_t^1),y_t^2,\gamma_t^2(\delta_t,y_t^2))|\delta_t\}. \qquad (9)$$

Here v_t is the disturbance term entering in (1). V is well-defined by virtue of (7) and the assumption that v_t is independent of (δ_t,y_t^1,y_t^2). From (7) it follows that if $(\gamma_t^1,\gamma_t^2) \in \Gamma_t^s$, then $V(\delta_t;\gamma_t^1,\gamma_t^2)$ is a function of $F_t(\delta_t)$, and so we write it as $V_s(F_t(\delta_t);\gamma_t^1,\gamma_t^2)$. Let,

$$V^*(\delta_t) = \inf\{V(\delta_t;\gamma_t^1,\gamma_t^2)|(\gamma_t^1,\gamma_t^2) \in \Gamma_t\}, \; V_s^*(F_t(\delta_t)) = \inf\{V_s(F_t(\delta_t);$$
$$\gamma_t^1,\gamma_t^2)|(\gamma_t^1,\gamma_t^2) \in \Gamma_t^s\}.$$

<u>Lemma 1</u> Let $\varepsilon > 0$. There exists (β_t^1,β_t^2) in Γ_t^s such that

$$V_s(F_t(\delta_t);\beta_t^1,\beta_t^2) \leq V_s^*(F_t(\delta_t)) + \varepsilon \text{ for all } \delta_t.$$

<u>Proof</u> For each fixed $F_t(\delta_t)$ "select" $\beta_t^1(F_t(\delta_t),y_t^1)$, $\beta_t^2(F_t(\delta_t),y_t^2)$ such that

$$E\{\eta(x_{t-1},\beta_t^1(F_t(\delta_t),y_t^1),\beta_t^2(F_t(\delta_t),y_t^2),v_t)$$
$$+ \xi(F_t(\delta_t),y_t^1,\beta_t^1(F_t(\delta_t),y_t^1),y_t^2,\beta_t^2(F_t(\delta_t),y_t^2))|\delta_t\} \leq V_s^*(F_t(\delta_t)) + \varepsilon.$$

This is possible by the definition. ¤

We emphasize again that a rigorous proof would require showing that the selection of (β_t^1,β_t^2) can be done in such a way as to guarantee measurability.

<u>Lemma 2</u> $V^*(\delta_t) = V_s^*(F_t(\delta_t))$.

<u>Proof</u> Let (γ_t^1,γ_t^2) in Γ be arbitrary. Fix $\delta_t = \bar{\delta}_t$. Then the control law (β_t^1,β_t^2) given by

$$\beta_t^1(\delta_t,y_t^1) = \gamma_t^1(\bar\delta_t,y_t^1),\ \beta_t^2(\delta_t,y_t^2) = \gamma_t^2(\bar\delta_t,y_t^2) \text{ for all } \delta_t,y_t^1,y_t^2$$

does not depend on δ_t at all, hence it is separated, and so

$$V(\delta_t;\beta_t^1,\beta_t^2) \geq V_s^*(F_t(\delta_t)) \text{ for all } \delta_t.$$

In particular, for $\delta_t = \bar\delta_t$ we get

$$V(\bar\delta_t;\beta_t^1,\beta_t^2) = V(\bar\delta_t;\gamma_t^1,\gamma_t^2) \geq V_s^*(F_t(\bar\delta_t)).$$

But $\bar\delta_t$ is arbitrary. Hence $V^*(\delta_t) \geq V_s^*(F_t(\delta_t))$. ◻

Next, by backward induction, we define

$$W_t(F_t(\delta_t)) = \inf_{\Gamma_t^s} E\{h_T(f_T(x_{T-1},\gamma_T^1,\gamma_T^2,v_T)|\delta_T\} \tag{10}$$

and for $t = T-1,\ldots,1$,

$$W_t(F_t(\delta_t)) = \inf_{\Gamma_t^s} E\{h_t(f_t(x_{t-1},\gamma_t^1,\gamma_t^2,v_t)) + W_{t+1}(\Phi_t(F_t(\delta_t),y_t^1,\gamma_t^1,y_t^2,\gamma_t^2))|\delta_t\}. \tag{11}$$

<u>Theorem</u> Let $\gamma \in \Gamma$. Let $x_t^\gamma,u_t^\gamma,y_t^\gamma$ be the state, control and observation processes induced by γ. Then, for all t,

$$E\{\sum_t^T h_\tau(x_\tau^\gamma,u_\tau^{1\gamma},u_\tau^{2\gamma})|\delta_t\} \geq W_t(F_t(\delta_t)) \tag{12}$$

and

$$\inf_{\Gamma^s} E\{\sum_t^T h_\tau(x_\tau^\gamma,u_\tau^{1\gamma},u_\tau^{2\gamma})|\delta_t\} = W_t(F_t(\delta_t)). \tag{13}$$

Finally if (β_t^1,β_t^2), $t = 1,\ldots,T$ is a separated control law which achieves the infimum in (11) then it is optimal relative to Γ.

<u>Proof</u> From (10) and Lemma 2 we can see that (12), (13) hold for T. From (11) and Lemma 2, we can see that (12), (13) hold for t if they hold for t+1, and so by induction they hold for all t. To prove the final assertion we again use induction to show that

$$E\{\sum_{t}^{T} h_\tau(x_\tau^\beta, u_\tau^{1\beta}, u_\tau^{2\beta}) | \delta_t\} = W_t(F_t(\delta_t)) \qquad (14)$$

for all t. Setting t=1 in (14) and (12) then shows that β is optimal relative to Γ. ⌑

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