LINEAR TIME-INVARIANT ROBUST SERVOMECHANISM PROBLEM:

A SELF-CONTAINED EXPOSITION

by

C. A. Desoer and Y. T. Wang

Memorandum No. UCB/ERL M77/50

30 August 1977

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

October 26, 1977
Rev. April 5, 1978

LINEAR TIME-INVARIANT ROBUST SERVOMECHANISM PROBLEM: A SELF-CONTAINED EXPOSITION

C. A. Desoer and Y. T. Wang

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Research sponsored by the Joint Services Electronics Program Contract F44620-76-C-0100.

TABLE OF CONTENTS

I. INTRODUCTION

In this chapter, we review recent development of the linear time-invariant servomechanism problem (asymptotic tracking and disturbance rejection). This problem is one of the most important subjects in control theory. For single-input single-output systems, this problem has been well-understood for about 40 years. However, it is only recently that this problem has been solved for the multi-input multi-output case. Thus it is appropriate at this time to give an overview of the state of knowledge: we present a unified self-contained treatment which employs simple derivations so that a Master's level reader will find no difficulty in understanding our development.

In section II, some notations and preliminaries are given, then the problem is stated precisely. In section III, a controller is given and shown to achieve asymptotic tracking and disturbance rejection robustly. Effects of perturbations at various data points are discussed. In section IV, characterization of a robust feedback controller is given for the lumped as well as the distributed case. The necessity of the rank condition is examined carefully in section V. The relation between transmission zeros and the servomechanism problem is further amplified in section VI. In section VII, we discuss the lumped, discrete-time case and provide a table so that one can easily translate all the previous results to the lumped discrete-time servomechanism problem. Some results available in literature are briefly reviewed in section IX. A representative list of references is given at the end of this chapter. We leave it to the science historian to describe fairly the history of the subject. We offer our apologies to any author whose work did not get the recognition that it deserves: our purpose is to present a self-contained easily understandable exposition of the main results.

II. PROBLEM FORMULATION

A. Notation and Preliminaries

Let $\mathbb{R}$ ($\mathbb{C}$) denote the field of real (complex, respectively) numbers. Let $\mathbb{C}_-$ ($\mathring{\mathbb{C}}_+$, $\mathbb{C}_+$) denote the open left (open right, closed right; respectively) half complex plane. Let $\mathbb{R}[s]$ ($\mathbb{R}(s)$) be the set of all polynomials (rational functions, respectively) in s with real coefficients. Let $\mathbb{R}[s]^{pxq}$ ($\mathbb{R}(s)^{pxq}$) be the set of all pxq matrices with elements in $\mathbb{R}[s]$ ($\mathbb{R}(s)$, respectively). Let $\partial(\phi(s))$ denote the degree of $\phi(s) \in \mathbb{R}[s]$. Let $\phi(s)$, $\psi(s) \in \mathbb{R}[s]$, then $\phi(s)|\psi(s)$ means $\phi(s)$ divides $\psi(s)$. Let $M \in \mathbb{R}^{mxn}$, then $\mathcal{R}(M)$ denotes the range space of M. Let $A \in \mathbb{R}^{nxn}$, then ψ_A denotes the minimal polynomial of A and $\sigma(A)$ denotes the spectrum of A. Let θ_n denote the zero vector in $\mathbb{C}^n$. Let [A,B,C,D] be a (not necessarily minimal) state space representation with state x, then $\chi_A(s) \triangleq \det(sI-A)$ denotes the characteristic polynomial of A with state x. Let $G(s) \in \mathbb{R}(s)^{n_0 x n_i}$ be proper and [A,B,C,D] be a minimal state space realization of G(s), then $\chi_{G(s)} \triangleq \det(sI-A)$ is said to be the characteristic polynomial of G(s). The system [A,B,C,D] is said to be exponentially stable (abbreviated, exp. stable) iff when $u = \theta_{n_i}$, for all x(0), $x(t) \to \theta_n$ exponentially as $t \to \infty$. A property is said to be robust at some data point p in some normed space (e.g. $\mathbb{R}^m$) iff it holds true throughout a (not necessarily small) neighborhood of p. Let $N_\ell(s) \in \mathbb{R}[s]^{pxq}$, $D_\ell(s) \in \mathbb{R}[s]^{pxp}$; $M(s) \in \mathbb{R}[s]^{pxp}$ is said to be a common left divisor of $N_\ell(s)$ and $D_\ell(s)$ iff there exist $N_1(s) \in \mathbb{R}[s]^{pxq}$, $D_1(s) \in \mathbb{R}[s]^{pxp}$ such that $N_\ell(s) = M(s)N_1(s)$, and $D_\ell(s) = M(s)D_1(s)$; both N_ℓ and D_ℓ are said to be right multiples of M; $L(s) \in \mathbb{R}[s]^{pxp}$ is said to be a greatest common left divisor of N_ℓ and D_ℓ iff 1) it is a common left divisor of N_ℓ and D_ℓ, and 2) it is a right multiple of every common left divisor of N_ℓ amd D_ℓ. When a greatest common left divisor L is unimodular (i.e. det L(s) = constant $\neq 0$), then N_ℓ and D_ℓ are said to be left coprime. $D_\ell^{-1}N_\ell$ is said to be a left coprime factorization of $G(s) \in \mathbb{R}(s)^{pxq}$ iff $D_\ell(s) \in \mathbb{R}[s]^{pxp}$, $N_\ell(s) \in \mathbb{R}[s]^{pxq}$ and D_ℓ, N_ℓ are left coprime. The definitions of right coprimeness and right coprime factorization are similar. Let $\mathcal{A}$ be the convolution algebra [Des. 2]: recall that f belongs to $\mathcal{A}$ iff for $t < 0$, $f(t) = 0$ and, for $t \geq 0$, $f(t) = f_a(t) + \sum_{i=0}^{\infty} f_i\, \delta(t-t_i)$, where $f_a(\cdot) \in L_1[0,\infty)$; $f_i \in \mathbb{R}$, $\forall i$; $t_i \geq 0$, $\forall i$ and $\sum_{i=0}^{\infty} |f_i| < \infty$. A pxq matrix $\mathcal{B} \in \mathcal{A}^{pxq}$ iff every element of $\mathcal{B}$ belongs to $\mathcal{A}$. Let $\hat{\mathcal{A}} \triangleq \mathcal{L}(\mathcal{A})$ $(\hat{\mathcal{A}}^{pxq} \triangleq \mathcal{L}(\mathcal{A}^{pxq}))$, the Laplace transform of $\mathcal{A}$; hence $f \in \mathcal{A}$ iff the Laplace transform of f (denoted by $\hat{f}$) belongs to $\hat{\mathcal{A}}$. Let $\mathcal{N}_\ell(s) \in \hat{\mathcal{A}}^{pxq}$, $\mathcal{D}_\ell(s) \in \hat{\mathcal{A}}^{pxp}$, then $\mathcal{N}_\ell$ and $\mathcal{D}_\ell$ are said to be pseudo-left-coprime iff there exist $\mathcal{U} \in \hat{\mathcal{A}}^{qxp}$, $\mathcal{V} \in \hat{\mathcal{A}}^{pxp}$, $\mathcal{W} \in \hat{\mathcal{A}}^{pxp}$ such that (i) det $\mathcal{W}(s) \neq 0$, for all $s \in \mathbb{C}_+$, and (ii) $\mathcal{N}_\ell(s)\,\mathcal{U}(s) + \mathcal{D}_\ell(s)\,\mathcal{V}(s) = \mathcal{W}(s)$, for all $s \in \mathbb{C}_+$. Let G be a pxq matrix-valued Laplace transformable distributions with support on $\mathbb{R}_+$, then $\mathcal{D}_\ell^{-1}\mathcal{N}_\ell$ is said to be a pseudo-left-coprime factorization of G iff (i) $\hat{G}(s) = \mathcal{D}_\ell(s)^{-1}\mathcal{N}_\ell(s)$, for all $s \in \mathbb{C}_+$; (ii) $\mathcal{N}_\ell$ and $\mathcal{D}_\ell$ are pseudo-left-coprime; and (iii) whenever $(s_i)_{i=1}^{\infty}$ is a sequence in $\mathbb{C}_+$ with $|s_i| \to \infty$, we have $\liminf_{i\to\infty} |\det \mathcal{D}_\ell(s_i)| > 0$. The definitions of pseudo-right-coprimeness and pseudo-right-coprime factorization are similar. A linear time-invariant distributed system with input u and output y is said to be $\mathcal{A}$-stable iff its transfer function $\hat{H}(s): \hat{u} \mapsto \hat{y}$ is a matrix with all its element in $\hat{\mathcal{A}}$. This implies that for any $p \in [1,\infty]$, any input $u \in L_p$ produces an output in L_p and $\|y\|_p \leq \|H\|_{\mathcal{A}} \cdot \|u\|_p$.

B. Fundamental Facts:

($\mathcal{F}$1) [A,B,C,D] is exp. stable $\Leftrightarrow \sigma(A) \subset \overset{\circ}{\mathbb{C}}_-$.

($\mathcal{F}$2) Let $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxn_i}$, then (A,B) is completely controllable $\Leftrightarrow$ rank $[sI-A \vdots B] = n$, $\forall s \in \sigma(A)$.

($\mathcal{F}$3) Let $N(s) \in \mathbb{R}[s]^{pxq}$, $D(s) \in \mathbb{R}[s]^{pxp}$, then N and D are left coprime iff $[N(s) \vdots D(s)] \in \mathbb{R}[s]^{px(p+q)}$ is full rank, $\forall s \in \mathbb{C}$.

C. Statement of Problem

Consider the following linear time-invariant lumped multi-input multi-output system:

(II.1) $\dot{x} = Ax + Bu + Ew$

(II.2) $y = Cx + Du + Fw$

(II.3) $e = r - y$

where $x(t) \in \mathbb{R}^n$ is the plant state,
$u(t) \in \mathbb{R}^{n_i}$ is the plant input,
$w(t) \in \mathbb{R}^d$ is the disturbance signal,
$r(t) \in \mathbb{R}^{n_0}$ is the reference signal to be tracked,
$y(t) \in \mathbb{R}^{n_0}$ is the output to follow the reference signal $r(\cdot)$,
$e(t) \in \mathbb{R}^{n_0}$ is the tracking error to be regulated.

Furthermore, $w(\cdot)$, $r(\cdot)$ are assumed to be modelled by the following state equations, respectively,

(II.4) $\dot{x}_w = A_w x_w$

$w = C_w x_w$

(II.5) $\dot{x}_r = A_r x_r$

$r = C_r x_r$

where $x_w(t) \in \mathbb{R}^{n_w}$, $x_r(t) \in \mathbb{R}^{n_r}$,

(C_w, A_w), (C_r, A_r) are completely observable;

and, without loss of generality,

(II.6) $\sigma(A_w) \cup \sigma(A_r) \subset \mathbb{C}_+$.

The goal is to design a feedback system with the following objectives:

($\mathcal{O}$1) the closed-loop system is exp. stable;

($\mathcal{O}$2) asymptotic tracking and disturbance rejection is achieved, i.e. for all initial states $x(0)$, $x_c(0)$, $x_w(0)$, $x_r(0)$, $e(t) \to \theta_{n_0}$, as $t \to \infty$, where $x(0)$ and $x_c(0)$ denote the initial states of the plant and the controller, respectively;

($\mathcal{O}$3) the properties ($\mathcal{O}$1), ($\mathcal{O}$2) are robust at some data point.

Throughout this paper, we will assume that the error signal $e(\cdot)$ is available and the controller to be constructed is of feedback type, i.e. it is driven by the error signal $e = r - y$(1). The eager reader may want to peek at Fig. 2 below to see the final feedback system.

Remark: For future reference, we say that for the system (II.1) ~ (II.6), asymptotic tracking holds iff with $x_w(0) = \theta_{n_w}$, for all $x(0)$, $x_c(0)$, $x_r(0)$, $e(t) \to \theta_{n_0}$ as $t \to \infty$; and asymptotic disturbance rejection holds iff with $x_r(0) = \theta_{n_r}$, for all $x(0)$, $x_c(0)$, $x_w(0)$, $y(t) \to \theta_{n_0}$ (equivalently, $e(t) \to \theta_{n_0}$, since $r(\cdot) \equiv \theta_{n_0}$). Consequently, since the system (II.1) ~ (II.6) is linear, asymptotic tracking and disturbance rejection holds iff for all $x(0)$, $x_c(0)$, $x_w(0)$, $x_r(0)$, $e(t) \to \theta_{n_0}$, as $t \to \infty$.

(1) It can be shown ([Dav. 4, Fra. 1]) that to satisfy ($\mathcal{O}$3) at certain data points, it is necessary to have the controller to be of feedback type.

III. DESIGN OF A ROBUST SERVOMECHANISM

The main result of this section is theorem III.1 which specifies the precise conditions under which one can design a controller which achieves the objectives ($\mathcal{O}1$), ($\mathcal{O}2$) and ($\mathcal{O}3$). The theorem is followed by detailed discussions of the assumptions and of the results. Following the proof, we give a brief discussion of the effect of perturbations in the controller dynamics.

A. Main Theorem:

Theorem III.1 (Design of a robust feedback controller)

Given the system described by (II.1) ~ (II.6). Suppose that $[A,B,C]$ is minimal. Let the controller be given by:

(III.1) $\dot{x}_c = A_c x_c + B_c e$

where $A_c = \text{block diag}\,\underbrace{[\Gamma,\Gamma,\ldots,\Gamma]}_{n_0\text{-tuple}} \in \mathbb{R}^{n_c \times n_c}$

$B_c = \text{block diag}\,\underbrace{[\gamma,\gamma,\ldots,\gamma]}_{n_0\text{-tuple}} \in \mathbb{R}^{n_c \times n_0}$

with

(III.2)
$$\Gamma = \begin{bmatrix} 0 & 1 & & \bigcirc \\ 0 & 0 & 1 & \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & & 1 \\ -\alpha_q & -\alpha_{q-1} & \cdots & -\alpha_1 \end{bmatrix} \in \mathbb{R}^{q \times q},$$

(III.3)
$$\gamma = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{q}$$

and $\phi_{A_w A_r} = s^q + \alpha_1 s^{q-1} + \ldots + \alpha_{q-1} s + \alpha_q$ is the least common multiple of ψ_{A_w} and ψ_{A_r}, the minimal polynomials of A_w and A_r.

Under these conditions, if

(III.4)
$$\text{rank} \begin{bmatrix} \lambda I - A & B \\ -C & D \end{bmatrix} = n + n_0, \quad \forall \lambda \in \sigma(A_w) \cup \sigma(A_r)$$

then the controller (III.1) is such that

(a) the composite system (plant followed by the controller (see Fig. 1))

$$\text{(III.5)}\quad \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A & 0 \\ -B_cC & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} B \\ -B_cD \end{bmatrix} u$$

is completely controllable (hence there exists a control law $u = Kx + K_cx_c$ such that the closed-loop system is exp. stable (see Fig. 2), see e.g. [And. 1], [Won. 3]);

(b) for any such control law, asymptotic tracking and disturbance rejection holds (more precisely, for all $x(0)$, $x_c(0)$, $x_w(0)$, $x_r(0)$ $e(t) \to \theta_{n_0}$, as $t \to \infty$);

(c) asymptotic tracking and disturbance rejection is robust at (A,B,C,D,B_c,K,K_c) for the class of perturbations such that

(1) B_c remains block diagonal and each block is a nonzero $\mathbb{R}^q$ vector;

(2) the closed-loop system remains exp. stable.

Comments: (i) Given $[A,B,C,D]$ together with the controller (III.1), one finds (K,K_c) such that the closed-loop system is exp. stable (see (a) above). Then it is well-known that for sufficiently small perturbations on (A,B,C,D,B_c,K,K_c), the closed-loop system remains exp. stable (even though $[A,B,C]$ may not be minimal and condition (III.4) may not be satisfied at the perturbed data point).

(ii) In section V, we will show that several considerations imply that condition (III.4) is necessary for the existence of a controller which achieves ($\mathcal{O}$1) - ($\mathcal{O}$3).

(iii) Condition (III.4) imples $n_i \geq n_0$, i.e. the number of plant inputs is greater than or equal to the number of plant outputs. Furthermore, it requires that no modes of the reference- and disturbance-signals are transmission zeros of the plant ([Dav. 9, Des. 3, Ros. 1, Wol. 1]). As we shall see, in the following proof, that these two conditions, together with complete controllability of the plant, will guarantee the complete controllability of the cascade system (plant followed by the controller, with input u and state (x,x_c)).

(iv) For the single-input single-output case ($n_i=n_0=1$), this theorem reduces to well-known results of classical control theory, e.g. an integral controller is required for tracking a step reference signal [e.g. Oga. 1, pp.184]; note that the numerator polynomial of the plant transfer function cannot have zero at the origin (see condition (III.4)).

(v) Robust asymptotic tracking and disturbance rejection is achieved by duplications of the dynamics of the reference- and disturbance-signals; this produces blocking zeros [Fer. 1,2] in the closed-loop transfer function from $\begin{bmatrix} w \\ r \end{bmatrix}$ to e at exactly the locations of the modes of the reference- and disturbance-signals, hence it completely blocks the transmission from $\begin{bmatrix} w \\ r \end{bmatrix}$ to e, the error signal. In section IV, we shall show that such duplications is necessary for robust asymptotic tracking and disturbance rejection.

(vi) It is crucial that the dynamics of the controller (represented by A_c) remains unperturbed. We will discuss the effect of perturbations in the controller dynamics in section III.B.

(vii) Although the complete controllability and observability requirements can be relaxed to stabilizability and detectability [Won. 3], we will use the notions of complete controllability and observability throughout this chapter to simplify derivations.

Proof of Theorem III.1

(a) Closed-loop exponential stability:

Let

$$M(s) = \begin{bmatrix} sI-A & 0 & B \\ B_cC & sI-A_c & -B_cD \end{bmatrix} \in \mathbb{R}[s]^{(n+n_c)x(n+n_c+n_i)}$$

Note $\det(sI-A_c) \neq 0$, $\forall s \notin \sigma(A_c)$. Now since (A,B) is completely controllable $\Leftrightarrow$ rank $[sI-A \vdots B] = n$, $\forall s \in \mathbb{C}$ (see ($\mathcal{F}$2)), we conclude that

$$\text{(III.6)} \qquad \operatorname{rank} M(s) = n + n_c, \qquad \forall s \notin \sigma(A_c)$$

Next, write

$$\text{(III.7)} \qquad M(s) = \begin{bmatrix} I_n & 0 & 0 \\ 0 & B_c & sI-A_c \end{bmatrix} \begin{bmatrix} sI-A & 0 & B \\ C & 0 & -D \\ 0 & I_{n_c} & 0 \end{bmatrix}$$

Then for all $s \in \sigma(A_c)$, the first factor has, by construction, rank $n + n_c$ and the second has, by assumption (III.4), rank $n + n_c + n_0$. Hence, by Sylvester's inequality, $\forall s \in \sigma(A_c)$

$$\text{(III.8)} \qquad n + n_c \geq \operatorname{rank} M(s) \geq (n+n_c) + (n+n_c+n_0) - (n+n_c+n_0) = n+n_c$$

Combining (III.6) and (III.8), we conclude that

$$\text{(III.9)} \qquad \operatorname{rank} M(s) = n + n_c, \qquad \forall s \in \mathbb{C},$$

and this is equivalent to the complete controllability of

$$\left(\begin{bmatrix} A & 0 \\ -B_cC & A_c \end{bmatrix}, \begin{bmatrix} B \\ -B_cD \end{bmatrix} \right).$$

(b) <u>Asymptotic tracking</u> and <u>disturbance rejection</u>:

Apply any stabilizing control law, $u = Kx + K_cx_c$, to the composite system (III.5), then the closed-loop system is given by

$$\text{(III.10)} \qquad \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \underbrace{\begin{bmatrix} A+BK & BK_c \\ -B_c(C+DK) & A_c-B_cDK_c \end{bmatrix}}_{A_L} \begin{bmatrix} x \\ x_c \end{bmatrix} + \underbrace{\begin{bmatrix} E & 0 \\ -B_cF & B_c \end{bmatrix}}_{B_L} \begin{bmatrix} w \\ r \end{bmatrix}$$

$$\text{(III.11)} \qquad e = \underbrace{[-(C+DK) \quad -DK_c]}_{C_L} \begin{bmatrix} x \\ x_c \end{bmatrix} + \underbrace{[-F \,\vdots\, I]}_{D_L} \begin{bmatrix} w \\ r \end{bmatrix}$$

and the closed-loop transfer function matrix $H(s): \begin{bmatrix} w(s) \\ r(s) \end{bmatrix} \mapsto e(s)$ is given by $C_L(sI-A_L)^{-1}B_L+D_L$. Furthermore, it is easy to show[2] that the ij th element of $H(s)$ is given by

(III.12)

$$h_{ij}(s) = \frac{1}{\det(sI-A_L)} \det \left[\begin{array}{c|c|c} sI-A-BK & -BK_c & \times \\ \hline \begin{array}{c} \gamma(C+DK)_{1\cdot} \\ \gamma(C+DK)_{2\cdot} \\ \vdots \\ \gamma(C+DK)_{n_0\cdot} \end{array} & \begin{bmatrix} sI-\Gamma & & \bigcirc \\ & sI-\Gamma & \\ & & \ddots \\ \bigcirc & & & sI-\Gamma \end{bmatrix} + \begin{bmatrix} \gamma(DK_c)_{1\cdot} \\ \gamma(DK_c)_{2\cdot} \\ \vdots \\ \gamma(DK_c)_{n_0\cdot} \end{bmatrix} & \begin{array}{c} \gamma(D_L)_{1j} \\ \gamma(D_L)_{2j} \\ \vdots \\ \gamma(D_L)_{n_0j} \end{array} \\ \hline (C+DK)_{i\cdot} & (DK_c)_{i\cdot} & (D_L)_{ij} \end{array}\right]$$

Now premultiply the big matrix in (III.12) by the elementary row matrix

(III.13)

$$R = \begin{bmatrix} I_n & & & & & \\ & I_q & & & \bigcirc & \\ & & \ddots & & & \\ & & & I_q & \cdots & -\gamma \\ & \bigcirc & & & \ddots & \vdots \\ & & & & I_q & \vdots \\ & & & & & 1 \end{bmatrix} \leftarrow (i+1)^{th} \text{ block}$$

we have

---

[2] Observe that, by definition

$h_{ij}(s): \begin{bmatrix} w \\ r \end{bmatrix}_j \mapsto e_i$, where $\begin{bmatrix} w \\ r \end{bmatrix}_j$ denotes the j^{th} component of $\begin{bmatrix} w \\ r \end{bmatrix}$ and is given by

$$h_{ij}(s) = (C_L)_{i\cdot} \cdot (sI-A_L)^{-1}(B_L)_{\cdot j} + (D_L)_{ij},$$

where $(C_L)_{i\cdot}$ ($(B_L)_{\cdot j}$) denotes the i th row (j th column) of C_L (B_L, respectively) and $(D_L)_{ij}$ denotes the ij th element of D_L. Finally (III.12) follows from Cramer's rule applied to

$$\begin{bmatrix} sI-A_L & (B_L)_{\cdot j} \\ -(C_L)_{i\cdot} & (D_L)_{ij} \end{bmatrix} \begin{bmatrix} -x_L \\ v_j \end{bmatrix} = \begin{bmatrix} \theta_{n+n_c} \\ e_i \end{bmatrix}$$

where $x_L = (x,x_c)^T$, $v_j = [(w,r)^T]_j$.

(III.14) $$h_{ij}(s) = \frac{1}{\det(sI-A_L)} \times$$

$$\det \left[\begin{array}{c|c|c}
sI-A-BK & -BK_c & \times \\
\hline
\begin{array}{c} \gamma(C+DK)_{1\cdot} \\ \gamma(C+DK)_{2\cdot} \\ \vdots \\ \bigcirc \\ \vdots \\ \gamma(C+DK)_{n_0\cdot} \end{array} &
\begin{bmatrix} sI-\Gamma & & & \bigcirc \\ & sI-\Gamma & & \\ & & \ddots \; sI-\Gamma & \\ \bigcirc & & & \ddots \; sI-\Gamma \end{bmatrix} + \begin{bmatrix} \gamma(DK_c)_{1\cdot} \\ \gamma(DK_c)_{2\cdot} \\ \vdots \\ \bigcirc \\ \vdots \\ \gamma(DK_c)_{n_0\cdot} \end{bmatrix} &
\begin{array}{c} \gamma(D_L)_{1j} \\ \gamma(D_L)_{2j} \\ \vdots \\ \bigcirc \\ \vdots \\ \gamma(D_L)_{n_0j} \end{array} \\
\hline
(C+DK)_{i\cdot} & (DK_c)_{i\cdot} & (D_L)_{ij}
\end{array}\right]$$

$$= \frac{\det(sI-\Gamma) \cdot n_{ij}(s)}{\det(sI-A_L)}$$

(III.15) $$= \frac{\phi_{A_wA_r}(s) \cdot n_{ij}(s)}{\det(sI-A_L)} \qquad \text{(by construction of } \Gamma\text{)}$$

Since the closed-loop system is exp. stable, i.e. $\sigma(A_L) \subset \overset{\circ}{\mathbb{C}}_-$, $\phi_{A_wA_r}(s)$ and $\det(sI-A_L)$ are coprime (by assumption (II.6), $\sigma(A_w) \cup \sigma(A_r) \subset \mathbb{C}_+$). Thus, for $1 \le i \le n_0$,

$$e_i(s) = \sum_{j=1}^{d} h_{ij}(s)w_j(s) + \sum_{j=d+1}^{d+n_0} h_{ij}(s)r_{j-d}(s) + \tau(s)$$

(III.16) $$= \frac{n_i(s)}{\det(sI-A_L)} + \tau(s)$$

where $n_i(s)$ is a polynomial in s which depends on $x_w(0)$, $x_r(0)$ and $\tau(s)$ represents the contribution of the initial state $(x(0),x_c(0))$; $\tau(s)$ has no $\mathbb{C}_+$-poles since, by construction, $\sigma(A_L) \subset \mathring{\mathbb{C}}_-$. The partial fraction expansion of (III.16) gives

$$\text{(III.17)} \qquad \lim_{t\to\infty} e(t) = \theta_{n_0}, \ \forall x(0),\ x_c(0),\ x_w(0),\ x_r(0),$$

i.e. asymptotic tracking and disturbance rejection holds.

(c) Robustness property:

Assume now the data point (A,B,C,D,B_c,K,K_c) is under (not necessarily small) perturbations which is such that B_c remains block diagonal with each block being a nonzero vector and the closed-loop system remains exp. stable. Denote the new data point by $(\tilde{A},\tilde{B},\tilde{C},\tilde{D},\tilde{B}_c,\tilde{K},\tilde{K}_c)$. Then instead of eqn. (III.15), we have[3]

$$\text{(III.18)} \qquad \tilde{h}_{ij}(s) = \frac{\phi_{A_wA_r}(s)\tilde{n}_{ij}(s)}{\det(sI-\tilde{A}_L)}$$

Hence, as long as the closed-loop system remains exp. stable, i.e. $\sigma(\tilde{A}_L) \subset \mathring{\mathbb{C}}_-$, the polynomial $\phi_{A_wA_r}(s)$ and $\det(sI-\tilde{A}_L)$ are still coprime and, instead of eqn. (III.16), we have

$$\text{(III.19)} \qquad e_i(s) = \frac{\tilde{n}_i(s)}{\det(sI-\tilde{A}_L)} + \tilde{\tau}(s)$$

---

(3) Note $\tilde{B}_c$ = block diag $(\tilde{\gamma}_1,\tilde{\gamma}_2,\ldots,\tilde{\gamma}_{n_0})$, so instead of (III.13), we apply elementary row matrix $\tilde{R}$ to (III.12) and then obtain (III.18), where $\tilde{R}$ is obained by replacing γ in R by $\tilde{\gamma}_i$.

where $\tilde{\tau}(s)$ represents the contribution of the initial state $(x(0),x_c(0))$; $\tilde{\tau}(s)$ has no $\mathbb{C}_+$-poles since, by assumption, $\sigma(\tilde{A}_L) \subset \mathring{\mathbb{C}}_-$. Now the partial fraction expansion of (III.19) gives

$$\lim_{t\to\infty} \tilde{e}(t) = \theta_{n_0}, \quad \forall x(0),x_c(0),x_w(0),x_r(0),$$

i.e. asymptotic tracking and disturbance rejection still holds at the perturbed data point $(\tilde{A},\tilde{B},\tilde{C},\tilde{D},\tilde{B}_c,\tilde{K},\tilde{K}_c)$. Thus the property of asymptotic tracking and disturbance rejection is <u>robust</u> at (A,B,C,D,B_c,K,K_c) under the prescribed class of perturbations.

Q.E.D.

<u>Comment</u>: The controller (III.1) actually achieves asymptotic tracking and disturbance rejection for a much larger class of reference- and disturbance-signals than the one described by (II.4) ~ (II.6); more precisely, asymptotic tracking and disturbance rejection is achieved by the controller (III.1) for <u>any</u> class of reference (disturbance) signal characterized by $(\tilde{A}_r,\tilde{C}_r)$ $((\tilde{A}_w,\tilde{C}_w))$ such that $\phi_{\tilde{A}_w\tilde{A}_r} | \phi_{A_wA_r}$, where $\phi_{\tilde{A}_w\tilde{A}_r}$ $(\phi_{A_wA_r})$ is the least common multiple of the minimal polynomials of $\tilde{A}_w$ and $\tilde{A}_r$ (A_w and A_r, respectively). As an example, let $\tilde{A}_w = 0$ (i.e. disturbance free) and $\tilde{A}_r$ be such that $\psi_{\tilde{A}_r} = \phi_{A_wA_r}$.

B. <u>Effect of Perturbations in the Controller Dynamics</u>

We have seen that the controller (III.1) provides asymptotic tracking and disturbance rejection robustly under the class of perturbations which maintains the closed-loop exp. stability, the decoupled structure of the controller and the dynamics of the controller. For engineers, it is important to know what will happen to the tracking and disturbance rejection properties if there is some <u>small</u> perturbation in the <u>dynamics</u> of the controller?

Suppose[4] that A_c is perturbed slightly into (using tildes to denote perturbed quantities)

$$\text{(III.20)} \qquad \tilde{A}_c = \begin{bmatrix} \tilde{\Gamma} & & \bigcirc \\ & \tilde{\Gamma} & \\ \bigcirc & \ddots & \tilde{\Gamma} \end{bmatrix} ,$$

then eqn. (III.18) becomes

$$\tilde{h}_{ij}(s) = \frac{\det(sI-\tilde{\Gamma})\tilde{n}_{ij}(s)}{\det(sI-\tilde{A}_L)} ,$$

where $\det(sI-\tilde{\Gamma}) = \prod_{k=1}^{q} (s-\lambda_k+\varepsilon_k)$. This equation should be compared to

$$\phi_{A_wA_r}(s) = \prod_{k=1}^{q} (s-\lambda_k).$$

In order to avoid detailed enumeration of cases, let us restrict ourselves to the case where all ε_k's are small and nonzero.

Then

$$\tilde{e}_i(s) = \sum_{j=1}^{d} \tilde{h}_{ij}(s)w_j(s) + \sum_{j=d+1}^{d+n_0} \tilde{h}_{ij}(s)r_{j-d}(s) + \tilde{\tau}(s)$$

$$\text{(III.21)} \qquad = \frac{\det(sI-\tilde{\Gamma})}{\det(sI-\tilde{A}_L)} \cdot \frac{\tilde{n}_i(s)}{\phi_i(s)} + \tilde{\tau}(s)$$

where $\phi_i(s) \mid \phi_{A_wA_r}(s)$ and $\tilde{\tau}(s)$ has the same meaning as in (III.19) above.

---

(4) To simplify notations, we assume every Γ subject to the <u>same</u> perturbation and denote the perturbed Γ by $\tilde{\Gamma}$. However, the following analysis goes through with different perturbations on each Γ.

Noting that for small perturbations, $\tilde{A}_L$ is still a stable matrix and that all the $\mathbb{C}_+$-poles of $\tilde{e}_i(s)$ are contributed by the zeros of $\phi_i(s)$ — which is unperturbed — we obtain, for $1 \leq i \leq n_0$,

$$\tilde{e}_i(s) = \sum_k \sum_{j=1}^{m_k} \frac{\delta_k^j}{(s-\lambda_k)^j} + \text{terms with } \overset{\circ}{\mathbb{C}}_-\text{-poles} \tag{III.22}$$

where the sum is taken over the $\mathbb{C}_+$-zeros of $\phi_i(s)$; the δ_k^j's depend continuously on the ε_k's and when all ε_k's are zero, all δ_k^j's are zero.

Case 1: if all λ_k lie on the $j\omega$ axis and are simple zeros of $\phi_{A_w A_r}$ (hence bounded reference- and disturbance-signals), then small perturbations on Γ (i.e. all ε_k's are small) will produce a small steady-state error, because the δ_k^j's are small.

Case 2: if some $\lambda_k \in \overset{\circ}{\mathbb{C}}_+$ or if some λ_k lies on the $j\omega$-axis and is a multiple zero of $\phi_{A_w A_r}$, then, even for small perturbations of Γ, the error signal will blow up as $t \to \infty$.

Therefore, if the class of reference- and disturbance-signals belong to case 1 (e.g. step, sinusoidal, etc.), a small perturbation in the controller dynamics may be tolerated, since it only produces small steady-state error. For case 2, the conclusion above is pure mathematical fiction because, for tracking and disturbance rejection, we are, in real life, only interested in a bounded time interval, say [0,T] (of course, to avoid saturation, the servomechanism will have to be reset at the end of the interval). Hence, by adjusting the closed-loop system poles (via K and K_c) such that the closed-loop system time-constants are very small compared to T, the magnitude of the error signal at time T will be small for sufficiently small perturbations in the controller dynamics. Therefore for a given specification on the error at time T, the ε_k's may be different from zero; but the tighter the specification and the larger T is, the smaller the ε_k's must be.

IV. CHARACTERIZATION OF A MINIMAL ORDER ROBUST FEEDBACK CONTROLLER

In this section, we first show that under certain robustness requirements, any feedback controller which achieves asymptotic tracking and disturbance rejection must have certain property; consequently, a minimal order robust feedback controller must be characterized by equation (III.1) (modulo coordinate transformation). Then a generalization of this result to the distributed case is briefly discussed in part B.

A. Lumped Case

Let us consider the linear time-invariant lumped system described by (II.1)~(II.6) and a feedback controller (with input e and state x_c). Suppose there exists a control law $u = Kx + K_c x_c$ such that the closed-loop system is exp. stable. Let $D_\ell^{-1}N_\ell$ be a left coprime factorization of $(C+DK)(sI-A-BK)^{-1}B + D$, where $D_\ell(s) \in \mathbb{R}[s]^{n_0 \times n_0}$ and $N_\ell \in \mathbb{R}[s]^{n_0 \times n_i}$. Let $N_r D_r^{-1}$ be a right coprime factorization of $K_c G_c(s)$, where $G_c(s)$ is the controller transfer function matrix (see Fig. 2 and Fig. 3).

Theorem IV.1 (characterization of a robust feedback controller)

Given the linear time-invariant system described by (II.1)~(II.6). Assume that

(i) $[A,B,C]$ is minimal;

(ii) $\begin{bmatrix} E \\ F \end{bmatrix}$ is full column rank,

then any feedback controller (with input e and state x_c) satisfying the following three conditions:

(a) the composite system is completely controllable (hence there exists a control law $u = Kx + K_c x_c$ such that the closed-loop system is exp. stable, where x and x_c are the plant and controller states, respectively);

(b) for any such control law, asymptotic tracking and disturbance rejection holds (i.e. for all $x(0)$, $x_c(0)$, $x_w(0)$, $x_r(0)$, $e(t) \to \theta_{n_0}$, as $t \to \infty$);

(c') closed-loop exp. stablility, asymptotic tracking and disturbance rejection are robust at (A,B,C),

must have the property that

(IV.1) every element of $D_r(s)$ is a multiple of $\phi_{A_w A_r}(s)$,

where $\phi_{A_w A_r}$ is the least common multiple of the minimal polynomials of A_w and A_r, and $D_r(s)$ is the denominator polynomial matrix in the left coprime factorization of $K_c G_c(s)$. (Note that any element of $D_r(s)$ which is the zero polynomial automatically satisfies condition (IV.1) above because $0 \cdot \phi_{A_w A_r}(s) = 0$).

The following corollary shows that a minimal order robust feedback controller must be characterized by (III.1) (modulo coordinate transformation):

Corollary (Characterization of a minimal order robust feedback controller)

Under the assumptions of theorem IV.1, a minimal order robust feedback controller which achieves (a)~(c') must be given by (III.1), modulo coordinate transformation.

Remarks: (1) Assumption (ii) represents no loss of generality since if it did not hold we could (a) make a change of coordinates from w to $\tilde{w}$, (b) choose this change to effect a reduction to column echelon form of $\begin{bmatrix} E \\ F \end{bmatrix}$, and (c) throw away all the identically zero columns thus generated, together with the corresponding components of $\tilde{w}$.

(2) The robustness requirement (c') is crucial to the required structure of controller: it can be shown (e.g. [Fra. 2,3]) that without robustness requirement, a feedback controller which contains one copy of the dynamics of the reference- and disturbance-signals (instead of n_0 copies in the robustness case, where n_0 is the number of reference signals) can achieve closed-loop stability, asymptotic tracking and disturbance rejection.

Now we deduce the minimal order property from the robustness requirements on closed-loop stability, asymptotic tracking and disturbance rejection:

Proof of Theorem IV.1

By condition (a), as shown on Fig. 2, closed-loop stability of the system (with respect to the state $\begin{bmatrix} x \\ x_c \end{bmatrix}$) is achieved by a state feedback law $u = Kx + K_c x_c$; refer to Fig. 3, closed-loop stability implies that [Cal. 2, Des. 4]

(IV.2) $\det(D_\ell D_r + N_\ell N_r)$ has no $\mathbb{C}_+$-zeros,

where $D_\ell^{-1} N_\ell$ is a left coprime factorization of the plant transfer function matrix $(C+DK)(sI-A-BK)^{-1}B + D$ and $N_r D_r^{-1}$ is a right coprime factorization of the controller transfer function matrix $K_c G_c(s)$.

Since the closed-loop system is linear, we can consider the effect of r and w separately:

Case 1 $w \equiv \theta_d$ (asymptotic tracking)

Perform a partial fraction expansion of r, the reference signal to be tracked, then

(IV.3) $$r(s) = C_r(sI-A_r)^{-1}x_r(0) = \sum_{i=1}^{k} \frac{r_i}{\phi_{ri}}$$

where $r_i \in \mathbb{C}^{n_0}$ and $\phi_{ri} \in \mathbb{C}[s]$ has only $\mathbb{C}_+$-zeros, by assumption (II.6). Note that $(sI-A_r)^{-1}$ has a pole of order m at p if and only if ψ_{A_r}, the minimal polynomial of A_r, has a zero of order m at p. Furthermore, since (C_r,A_r) is completely observable, the least common multiple of the ϕ_{ri}'s $(i = 1,2,\ldots,k)$ is ψ_{A_r}.

The error resulting from this input is

$$e(s) = [I+D_\ell^{-1}N_\ell N_r D_r^{-1}]^{-1} \cdot \left(\sum_{i=1}^{k} \frac{r_i}{\phi_{ri}}\right)$$

$$\text{(IV.4)} \qquad = D_r[D_\ell D_r+N_\ell N_r]^{-1}D_\ell \left(\sum_{i=1}^{k} \frac{r_i}{\phi_{ri}}\right)$$

Consider now an arbitrarily small perturbation of the plant from (A,B,C) to $(\tilde{A},\tilde{B},\tilde{C})$ which maintains the closed-loop stability, then the pair (N_ℓ,D_ℓ) becomes $(\tilde{N}_\ell,\tilde{D}_\ell)$. Note for almost all $(\tilde{A},\tilde{B},\tilde{C})$, $\det \tilde{D}_\ell(s) = \det(sI-\tilde{A}-\tilde{B}K)$ modulo a nonzero constant factor[5]. For asymptotic tracking to be robust under such perturbations, it must be that, for $i = 1,2,\ldots,k$.

$$\text{(IV.5)} \qquad D_r(\tilde{D}_\ell D_r+\tilde{N}_\ell N_r)^{-1}\tilde{D}_\ell\left(\frac{r_i}{\phi_{ri}}\right) \text{ has no } \mathbb{C}_+\text{-poles}$$

For any zero, say p, of ϕ_{ri} (hence Re $p \geq 0$ by (II.6)), by the stability requirement and for almost all perturbed plants[6]

[5]For some special $(\tilde{A},\tilde{B},\tilde{C})$, the pair $(\tilde{C}+DK,\tilde{A}+\tilde{B}K)$, $((\tilde{A}+\tilde{B}K,\tilde{B})$, resp.) may turn out to be not completely obserbable (controllable). However, some suitable arbitrarily small perturbation from the data point will restore complete observability (controllability).

[6]The first factor if (IV.6) is always nonsingular as a consequence of the stability requirement (IV.2) and $\det \tilde{D}_\ell(p) \neq 0$ for almost all perturbed plants.

(IV.6) $\quad \tilde{P} \triangleq (\tilde{D}_\ell D_r + \tilde{N}_\ell N_r)^{-1}\tilde{D}_\ell$ is nonsingular at p

Suppose that $\phi_{ri}(s) = (s-p)^m$ with $m \geq 1$, and that, for the purpose of a proof by contraposition, some element of $D_r(s)$ is not a multiple of $(s-p)^m$, then, for some perturbed data point $(\tilde{D}_\ell, \tilde{N}_\ell)$, by Taylor expansion of $\tilde{P}(s)$ about p, we obtain, from (IV.5) and (IV.6),

(IV.7) $$D_r(s) \cdot \tilde{P}(s) \frac{r_i}{(s-p)^m} = D_r(s) \cdot \left[\tilde{P}(p) \frac{r_i}{(s-p)^m} + \tilde{P}'(p) \frac{r_i}{(s-p)^{m-1}} + \ldots\right]$$

By (IV.6), $\tilde{P}(p)$ is nonsingular and by choice of $(\tilde{D}_\ell, \tilde{N}_\ell)$, any component of $\tilde{P}(p)\, r_i$ can be made non-zero,(7) hence the leading term in (IV.7) has at least a pole of first order at p. Consequently, by (IV.4), e(t) does not go to zero as $t \to \infty$, which is a contradiction. Therefore we have shown that robust asymptotic tracking requires that <u>all</u> the elements of $D_r(s)$ be multiples of ψ_{A_r}.

---

(7) Without loss of generality, we may assume that D_ℓ is row proper. Consider a perturbation $\delta D_\ell \in \mathbb{R}^{n_o \times n_o}$ such that $\tilde{D}_\ell = D_\ell + \delta D_\ell$. Since $\partial[\det(D_\ell + \delta D_\ell)] = \partial[\det D_\ell]$ and for small δD_ℓ, $\det[(D_\ell + \delta D_\ell)D_r + N_\ell N_r]$ is a polynomial whose coefficients are close to those of $\det[D_\ell D_r + N_\ell N_r]$; hence for sufficiently small δD_ℓ, closed-loop stability is maintained. Thus any sufficiently small δD_ℓ is an allowable perturbation. Let $\Delta = D_\ell D_r + N_\ell N_r$. Then for this class of perturbations,

$$\begin{aligned}\tilde{P} &= (\tilde{D}_\ell D_r + N_\ell N_r)^{-1}\tilde{D}_\ell \\ &= (\Delta + \delta D_\ell \cdot D)^{-1} \cdot (D_\ell + \delta D_\ell) \\ &\approx (I - \Delta^{-1} \cdot \delta D_\ell \cdot D_r) \cdot \Delta^{-1} \cdot (D_\ell + \delta D_\ell) \\ &\approx \Delta^{-1} D_\ell + \Delta^{-1} \cdot \delta D_\ell \cdot [I - D_r \Delta^{-1} D_\ell]\end{aligned}$$

So the effect of such perturbation on $P(p)r_i$ is approximately equal to $\Delta(p)^{-1} \cdot \delta D_\ell \cdot [I - D_r(p)\Delta(p)^{-1}D_\ell(p)]r_i$. Consequently, δD_ℓ can always be chosen so that $P(p)r_i$ is pushed away from any manifold in $\mathbb{C}^{n_o}$.

Case 2. $r \equiv \theta_{n_0}$ (Asymptotic disturbance rejection)

If in (II.1), $E = 0$, the zero matrix in $\mathbb{R}^{n \times d}$, then Fig. 2 shows that the effect of w on the system is identical to a reference signal to be tracked of the form $-Fw$. Thus if $E = 0$, we are right back to the previous case.

If in (II.1), $E \neq 0$, then the disturbance w is applied to the output y through the plant (see Fig. 2). Since the plant transfer function is $D_\ell^{-1}N_\ell$, the open-loop effect of w on the output y is equivalent to a reference signal to be tracked of the form[8] $D_\ell^{-1}M_\ell w$, for some $M_\ell \in \mathbb{R}[s]^{n_o \times d}$ (see Fig. 4).

Now the closed-loop transfer function $w \mapsto y$ is given by

$$\text{(IV.9)} \qquad [I+D_\ell^{-1}N_\ell N_r D_r^{-1}]^{-1}D_\ell^{-1}M_\ell = D_r[D_\ell D_r+N_\ell N_r]^{-1}M_\ell$$

As before, plant perturbation from (A,B,C) to $(\tilde{A},\tilde{B},\tilde{C})$ will transform this transfer function to

$$\text{(IV.10)} \qquad D_r[\tilde{D}_\ell D_r+\tilde{N}_\ell N_r]^{-1}\tilde{M}_\ell .$$

Let the partial fraction expansion of w be

$$\text{(IV.11)} \qquad w = \sum_{i=1}^{\ell} \frac{w_i}{\phi_{wi}}$$

---

[8] To see this, let $D_\ell^{-1}N$ be a left coprime factorization of $(C+DK)(sI-A-BK)^{-1}$. Then $\det D_\ell(s) = \chi_{(C+DK)(sI-A-BK)^{-1}} = \chi_{(C+DK)(sI-A-BK)^{-1}B}$ modulo a nonzero constant factor, where the last equality follow since $(A+BK,B)$ is completely controllable. Therefore $D_\ell^{-1}NB$ is a left coprime factorization of $(C+DK)(sI-A-BK)^{-1}B$ (otherwise $\partial[\det D_\ell] > \partial[\chi_{(C+DK)(sI-A-BK)^{-1}B}]$); consequently $D_\ell^{-1}(NB+D_\ell D)$ is a left coprime factorization of $(C+DK)(sI-A-BK)^{-1}B+D$. Hence the open-loop effect of w on the output y can be represented by $(C+DK)(sI-A-BK)^{-1}E+F = D_\ell^{-1}(NE+D_\ell F) \triangleq D_\ell^{-1}M_\ell$. Note D_ℓ and M_ℓ are not necessarily left coprime.

where $w_i \in \mathbb{C}^d$ and $\phi_{wi}(s) \in \mathbb{C}[s]$ has only $\mathbb{C}_+$-zeros (by assumption (II.6)). Note that ψ_{A_w} is the least common multiple of ϕ_{wi}, $i = 1,2,\ldots,\ell$. Clearly, any disturbance w will cause an output y decaying exponentially if and only if

$$\text{(IV.12)} \qquad D_r(\tilde{D}_\ell D_r + \tilde{N}_\ell N_r)^{-1}\tilde{M}_\ell \sum_{i=1}^{\ell} \frac{w_i}{\phi_{wi}} \quad \text{has no } \mathbb{C}_+\text{-poles}$$

Now suppose that $\phi_{wi}(s) = (s-p)^m$ with $m \geq 1$, then by an argument similar to (IV.7) and noting that D_ℓ, N_ℓ, M_ℓ can be perturbed, we conclude that robust asymptotic disturbance rejection requires that all elements of $D_r(s)$ be multiples of ψ_{A_w}.

Putting the two cases together we conclude that: If the aymptotic tracking and disturbance rejection property (i.e. for all $x(0)$, $x_c(0)$, $x_w(0)$, $x_r(0)$, $e(t) \to \theta_{n_o}$, as $t \to \infty$) is to be maintained in the face of arbitrary plant perturbations (i.e. (A,B,C) becomes $(\tilde{A},\tilde{B},\tilde{C})$) subject only to the condition that such perturbations do not upset the closed-loop stability, then the controller must be such that every element of $D_r(s)$ is a multiple of $\phi_{A_wA_r}$, the least common multiple of ψ_{A_w} and ψ_{A_r}.

Q.E.D.

Proof of corollary

From the result of Theorem IV.1, every element of $D_r(s)$ must be a multiple of $\phi_{A_wA_r}$; hence the minimal order of a robust controller is $n_o \cdot \partial(\phi_{A_wA_r})$ (by taking $D_r(s) = \text{diag}(\underbrace{\phi_{A_wA_r}, \phi_{A_wA_r}, \ldots, \phi_{A_wA_r}}_{n_o\text{-tuple}})$.

Furthermore, we have shown, in Theorem III.1, that a robust feedback controller (III.1) such that[9] $D_r(s) = \text{diag}\underbrace{(\phi_{A_wA_r}, \phi_{A_wA_r}, \ldots, \phi_{A_wA_r})}_{n_o\text{-tuple}}$ does satisfy conditions (a)~(c'). Hence, a minimal order robust feedback controller which achieves (a)~(c') must be characterized by (III.1) (modulo coordinate transformation).

Q.E.D.

Remark: It is important to note (see eqns. (IV.4) and (IV.9)) that the poles of the closed-loop transfer function from $w \mapsto y$ and from $r \mapsto e$ belong to some subset of zeros of the closed-loop system characteristic polynomial $\det(D_\ell D_r + N_\ell N_r)$.

B. Distributed Case

Theorem IV.1 of part A can be generalized to distributed systems. The problem formulation goes as follows:

Let $\mathcal{D}_\ell^{-1}\mathcal{N}_\ell$ be a pseudo-left-coprime-factorization of the plant transfer function matrix and $\mathcal{N}_r\mathcal{D}_r^{-1}$ be a pseudo-right-coprime factorization of the controller transfer function matrix, where $\mathcal{D}_\ell(s) \in \hat{\mathcal{A}}^{n_o \times n_o}$, $\mathcal{N}_\ell(s) \in \hat{\mathcal{A}}^{n_o \times n_i}$, $\mathcal{D}_r(s) \in \hat{\mathcal{A}}^{n_o \times n_o}$, $\mathcal{N}_r(s) \in \hat{\mathcal{A}}^{n_i \times n_o}$

---

[9] It should be noted that any stabilizing (K, K_c) pair is such that (K_c, A_c) is completely observable, (otherwise, the unobservable modes of A_c which are unstable will not be stabilized by $u = Kx + K_c x_c$), hence $\det D_r(s) = \det(sI - A_c)$. Now the transfer function matrix $(sI-A_c)^{-1}B_c$ of the controller (III.1) has a right coprime factorization $N_r D_r^{-1}$, where $D_r = \text{diag}(\phi_{A_wA_r}, \phi_{A_wA_r}, \ldots, \phi_{A_wA_r}) \in \mathbb{R}[s]^{n_o \times n_o}$ and $N_r = \text{diag}(\nu(s), \nu(s), \ldots, \nu(s)) \in \mathbb{R}[s]^{qn_o \times n_o}$ with $\nu(s) = [1\ s\ s^2 \ldots s^{q-1}]^T \in \mathbb{R}[s]^q$. Since (K_c, A_c) is completely observable, $(K_c N_r) D_r^{-1}$ is a right coprime factorization of $K_c G_c(s)$.

[Des. 2]. Assume that the closed-loop system is $\mathcal{A}$-stable (see Fig. 5)[10]; this will be the case if and only if [Cal. 2, Fra. 7]

(IV.13) $$\inf_{\mathbb{C}_+} |\det(\mathcal{D}_\ell \mathcal{D}_r + \mathcal{N}_\ell \mathcal{N}_r)| > 0.$$

For technical reasons, we must assume that for some $\sigma_1 < 0$, the functions $\mathcal{D}_\ell, \mathcal{N}_\ell, \mathcal{D}_r, \mathcal{N}_r$ are analytic in Re $s > \sigma_1$.

<u>Case 1</u> $\omega \equiv \theta_d$ (<u>asymptotic tracking</u>).

For the class of reference signals described by (II.5), we have, instead of (IV.4),

(IV.14) $$e(s) = \mathcal{D}_r [\mathcal{D}_\ell \mathcal{D}_r + \mathcal{N}_\ell \mathcal{N}_r]^{-1} \mathcal{D}_\ell \cdot \left(\sum_{i=1}^{k} \frac{r_i}{\phi_{ri}} \right)$$

where as in (IV.3), the least common multiple of the ϕ_{ri}'s is ψ_{A_r}. Suppose that asymptotic tracking is robust[11] at $(\mathcal{D}_\ell, \mathcal{N}_\ell)$ subject to the condition that the closed loop system remains $\mathcal{A}$-stable[12] (i.e. (IV.13) holds at the perturbed data point $(\tilde{\mathcal{D}}_\ell, \tilde{\mathcal{N}}_\ell)$). Call the zeros of ψ_{A_r}, p_i, and their respective multiplicities m_i, $i = 1,2,\ldots,\alpha_r$.

---

(10) For closed-loop stability, we require that $H_{er} \in \hat{\mathcal{A}}^{n_o \times n_o}$, $H_{vr} \in \hat{\mathcal{A}}^{n_o \times n_i}$ $H_{yz} \in \hat{\mathcal{A}}^{n_o \times n_i}$, $H_{vr} \in \hat{\mathcal{A}}^{n_i \times n_i}$, where $H_{\beta\alpha}$ denotes the transfer function $\alpha \longmapsto \beta$.

(11) Here robustness means that asymptotic tracking is maintained for all $(\tilde{\mathcal{D}}_\ell, \tilde{\mathcal{N}}_\ell)$ such that $\|\tilde{\mathcal{D}}_\ell - \mathcal{D}_\ell\|_{\mathcal{A}} < \varepsilon$, $\|\tilde{\mathcal{N}}_\ell - \mathcal{N}_\ell\|_{\mathcal{A}} < \varepsilon$, where $\|\cdot\|_{\mathcal{A}}$ denotes the $\mathcal{A}$-norm [Des. 2].

(12) Using the continuity of the function det(·), it can easily be shown [Fra. 7] that closed-loop $\mathcal{A}$-stability is robust at $(\mathcal{D}_\ell, \mathcal{N}_\ell)$.

Recall that (II.6) $\Rightarrow p_i \in \mathbb{C}_+, \forall i$. Using results of [Cal. 3] and a theorem of Doetsch [Doe. 1: p. 488, Theorem 1], it can be shown that if, for some $i \in \{1,2,\ldots,\alpha_r\}$, some element of $\mathcal{D}_r$ has a zero at p_i of multiplicity <u>less than</u> m_i, then, for some perturbed data point $(\tilde{\mathcal{D}}_\ell, \tilde{\mathcal{N}}_\ell)$, (IV.14) implies that e(s) has a pole at p_i, say of order ν_i with $\nu_i \geq 1$, and the asymptotic representation (for $t \to +\infty$) of e(t) includes a term $p(t)\exp(p_i t)$, where p(t) is a <u>non-zero</u> polynomial of degree $\nu_i - 1$. Consequently, since $p_i \in \mathbb{C}_+$, e(t) does not go to zero as $t\to\infty$. In conclusion, the robust asymptotic tracking requirement implies that for $i = 1,2,\ldots,\alpha_r$, <u>every</u> element of $\mathcal{D}_r(s)$ must have a zero at p_i of multiplicity larger than or equal to m_i.

<u>Case 2</u> $r \equiv \theta_{n_0}$ (<u>asymptotic disturbance rejection</u>).

Reasoning as in part A, assume that we can represent the disturbance signal w by an equivalent reference signal of the form $\mathcal{D}_\ell^{-1}\mathcal{M}_\ell$ w, where $\mathcal{M}_\ell(s) \in \hat{\mathcal{Q}}^{n_0 x d}$ and $(\mathcal{D}_\ell, \mathcal{M}_\ell)$ is not necessarily pseudo-left-coprime. Call the zeros of ψ_{A_w}, p_i, and their respective multiplicities, m_i, $i = 1,2,\ldots,\alpha_w$. Then, reasoning as above, we can show that robust asymptotic disturbance rejection at $(\mathcal{D}_\ell, \mathcal{N}_\ell, \mathcal{M}_\ell)$ will require that for $i = 1,2,\ldots,\alpha_w$, <u>every</u> element of $\mathcal{D}_r(s)$ must have a zero at p_i of multiplicity larger than or equal to m_i.

In summary, under the assumption that the closed loop system is $\mathcal{Q}$-stable, asymptotic tracking and disturbance rejection being robust at the plant data-point $(\mathcal{D}_\ell, \mathcal{N}_\ell, \mathcal{M}_\ell)$ requires that <u>every</u> element of $\mathcal{D}_r(s)$ has a zero at every zero of $\phi_{A_w A_r}$ with a respective multiplicity at least as large as that of the zero of $\phi_{A_w A_r}$. Roughly speaking, we might say: every element of $\mathcal{D}_r(s)$ must be a multiple of $\phi_{A_w A_r}$.

V. NECESSITY OF THE RANK CONDITION

In this section, we illustrate, in several ways, why the rank condition

$$\text{(III.4)} \qquad \text{rank} \begin{bmatrix} \lambda I-A & B \\ -C & D \end{bmatrix} = n+n_o, \quad \forall\lambda \in \sigma(A_w) \cup \sigma(A_r)$$

is necessary:

Theorem V.1 below shows that if a feedback controller is given by (III.1) (as will be the case if the controller is to achieve the requirements (a)~(c') as shown in theorem IV.1 and its corollary), then the failure of the rank condition (III.4) will result in loss of complete controllability of the composite system (plant followed by the controller); theorem V.2 (theorem V.3) shows that the failure of the rank condition (III.4) will result in loss of robust asymptotic tracking (robust asymptotic disturbance rejection, respectively) property. These results illuminate the importance of the rank condition; recall that it is equivalent to (1) the number of plant inputs, n_i, must be greater or equal to the number of tracking outputs, n_o, and (2) no mode of the reference-and disturbance-signals can be a transmission zero of the plant.

Theorem V.I (Controllability).

Given the linear time-invariant system described by (II.1)~(II.6). Assume that [A,B,C] is minimal. Let the feedback controller be given by (III.1). Under these conditions, if the composite system (see Fig. 1)

$$\text{(V.I)} \qquad \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A & 0 \\ -B_cC & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} B \\ -B_cD \end{bmatrix} u$$

is completely controllable, then

(III.4) $\quad \text{rank}\begin{bmatrix} \lambda I-A & B \\ -C & D \end{bmatrix} = n+n_o, \quad \forall\lambda \in \sigma(A_w) \cup \sigma(A_r).$

To prove this theorem, we need the following lemma:

Lemma ([Cal. 1])

Given the cascade linear time-invariant dynamic system shown in Fig. 6, where $N_p \in \mathbb{R}[s]^{pxm}$ and $D_p \in \mathbb{R}[s]^{mxm}$ are right coprime; $N_c \in \mathbb{R}[s]^{qxp}$ and $D_c \in \mathbb{R}[s]^{qxq}$ are left coprime; x and x_c denote the state of a minimal state space realization of each subsystem. Then the cascade system of Fig. 6 (with state $\begin{bmatrix} x \\ x_c \end{bmatrix}$) is completely controllable (by u) iff N_cN_p and D_c are left coprime.

Proof of theorem V.1: Proof by contradiction. Suppose that

(V.2) $\quad \text{rank}\begin{bmatrix} \lambda I-A & B \\ -C & D \end{bmatrix} < n+n_o,$ for some $\lambda \in \sigma(A_w) \cup \sigma(A_r)$,

then we show that the composite system (plant followed by the feedback controller (III.1)) is not completely controllable. Let $N_pD_p^{-1}$ be a right coprime factorization of the plant transfer function matrix. Now we claim that

(V.3) $\quad \text{rank } N_p(\lambda) < n_o$

Case 1. $n_o \leq n_i$. Then, since (V.2) holds, λ is a transmission zero of the plant [A,B,C,D] and consequently ([Des. 3])

$$\text{rank } N_p(\lambda) < n_o$$

Case 2. $n_o > n_i$. Then $\forall s \in \mathbb{C}$, rank $N_p(s) < n_o$, since $N_p(s) \in \mathbb{C}^{n_o x n_i}$. Combining case 1 and 2, the inequality (V.3) is established. Now let $D_c(s) = sI-A_c \in \mathbb{R}[s]^{qn_o x qn_o}$, $N_c = B_c \in \mathbb{R}^{qn_o x n_o}$ where A_c, B_c are defined in (III.1) then $D_c^{-1}N_c$ is a left coprime factorization of the controller transfer function matrix $(sI-A_c)^{-1}B_c$, since (A_c,B_c) is completely controllable. Now we show that N_cN_p and D_c are not left coprime, hence by lemma, the composite system of Fig. 1 (plant followed by the feedback controller (III.1)) is not completely controllable.

By (V.3), there exists a non-zero vector

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n_o} \end{bmatrix} \in \mathbb{C}^{n_o} \text{ such that}$$

$$\text{(V.4)} \qquad \alpha^* N_p(\lambda) = [\alpha_1^* \; \alpha_2^* \; \cdots \; \alpha_{n_o}^*] \begin{bmatrix} -N_{p1}(\lambda)- \\ -N_{p2}(\lambda)- \\ \vdots \\ -N_{pn_o}(\lambda)- \end{bmatrix} = \theta_{n_i}^T$$

where $N_{pi}(\lambda)$ denotes the ith row of the matrix $N_p(\lambda)$ and * denotes the complex conjugate transpose. Also since $\lambda \in \sigma(\Gamma)$ (by construction of (III.1)), there exists a nonzero vector

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_q \end{bmatrix} \in \mathbb{C}^q \text{ such that}$$

$$\text{(V.5)} \qquad \beta^*(\lambda I - \Gamma) = \theta_q^T$$

Therefore, there exists a non-zero vector $\ell \in \mathbb{C}^{qn_o}$ such that

$$\ell^*[N_cN_p(\lambda) \,\vdots\, D_c(\lambda)]$$

$$\text{(V.6)} \qquad = \ell^* \left[\begin{array}{c|cccc} 0 & & & & \\ -N_{p1}(\lambda)- & \lambda I - \Gamma & & & 0 \\ \hline 0 & & & & \\ -N_{p2}(\lambda)- & & \lambda I - \Gamma & & \\ \vdots & & & \ddots & \\ 0 & & 0 & & \\ -N_{pn_o}(\lambda)- & & & & \lambda I - \Gamma \end{array}\right] = \theta_{n_i+qn_o}^T$$

where $\ell^* = [\ell_1^* \quad \ell_2^* \quad \dots \quad \ell_{n_o}^*]$, with $\ell_i \in \mathbb{C}^{n_o}$, $1 \le i \le n_o$, given by

$$\ell_i^* \triangleq \begin{cases} [\beta_1^* \quad \beta_2^* \dots \beta_{q-1}^* \quad \beta_q^*] \dfrac{\alpha_i^*}{\beta_q^*} & \text{if } \beta_q \neq 0 \\ [\beta_1^* \quad \beta_2^* \dots \beta_{q-1}^* \quad 0] & \text{if } \beta_q = 0 \end{cases}$$

So, by fundamental fact ($\mathcal{F}$3), N_cN_p and D_c are not left coprime and consequently the composite system is not completely controllable which is a contradiction.

Q.E.D.

The following two theorems show that the rank condition (III.4) also results from some robustness requirements:

Theorem V.2 (Asymptotic tracking)

Given the linear time-invariant system described by (II.1)~(II.6). Assume that rank $[C \vdots D] = n_o$. Under these conditions, if there exists a controller such that

($\mathcal{O}$1) the closed-loop system is exp. stable;

($\mathcal{O}$2') asymptotic tracking holds (i.e. with $x_w(0) = \theta_{n_w}$, for all $x(0)$, $x_c(0)$, $x_r(0)$, $e(t) \to \theta_{n_o}$, as $t \to \infty$, where $x_c(0)$ denotes the initial state of the controller);

($\mathcal{O}$3') asymptotic tracking is robust at (A,B),

then

$$\text{(V.7)} \qquad \text{rank}\begin{bmatrix} \lambda I-A & B \\ -C & D \end{bmatrix} = n+n_o, \quad \forall \lambda \in \sigma(A_r).$$

proof: equations (II.1)~(II.3) imply

$$\text{(V.8)} \qquad \begin{cases} (sI-A)\hat{x} - B\hat{u} = E\hat{w} \\ -C\hat{x} - D\hat{u} + \hat{y} = F\hat{w} \\ \hat{y} + \hat{e} = \hat{r} \end{cases}$$

Hence by eliminating $\hat{y}$, (V.8) becomes

(V.9) $$\begin{cases}(sI-A)\hat{x} - B\hat{u} = E\hat{w} \\ -C\hat{x} - D\hat{u} - \hat{e} = F\hat{w} - \hat{r}\end{cases}$$

Let $x_w(0) = \theta_{n_w}$ (so $w(\cdot) = \theta_d$). Pick any $\lambda \in \sigma(A_r)$, then choose $x_{n_r}(0) \in \mathbb{C}^{n_r}$ such that $r(t) = r_\infty e^{\lambda t}$, where $\theta_{n_o} \neq r_\infty \in \mathbb{C}^{n_o}$. Then, as $t \to \infty$,

(V.10) $$\begin{cases} y(t) \longrightarrow r_\infty e^{\lambda t}, & \text{by assumption } (\mathcal{O}2') \\ x(t) \longrightarrow x_\infty e^{\lambda t}, \text{ for some } x_\infty \in \mathbb{C}^n & \\ u(t) \longrightarrow u_\infty e^{\lambda t}, \text{ for some } u_\infty \in \mathbb{C}^{n_i} & \text{by assumption } (\mathcal{O}1) \text{ and } (\mathcal{O}2') \end{cases}$$

Using equation (V.10) in (V.9) (with $\hat{w} = \theta_d$), we obtain

(V.11) $$\begin{bmatrix}\lambda I-A & B \\ -C & D\end{bmatrix}\begin{bmatrix}x_\infty \\ -u_\infty\end{bmatrix} = \begin{bmatrix}\theta_n \\ -r_\infty\end{bmatrix}, \quad \text{for the chosen } \lambda \in \sigma(A_r).$$

Note that assumption ($\mathcal{O}3'$) requires that, for a given $r_\infty \neq \theta_{n_0}$, eqn. (V.11) has a solution at the perturbed data point $(A+\delta A, B+\delta B)$.

Now, for the sake of contradiction, assume

(V.12) $$\text{rank}\begin{bmatrix}\lambda I-A & B \\ -C & D\end{bmatrix} < n + n_o, \quad \text{for the chosen } \lambda \in \sigma(A_r).$$

Hence, there exist $p_1 \in \mathbb{C}^n$, $p_2 \in \mathbb{C}^{n_o}$ not both zero such that

(V.13) $$[p_1^* \,\vdots\, p_2^*]\begin{bmatrix}\lambda I-A & B \\ -C & D\end{bmatrix} = \theta^T_{n+n_i}$$

$$\left(\text{i.e. } \begin{bmatrix}p_1 \\ p_2\end{bmatrix} \perp \mathcal{R}\begin{bmatrix}\lambda I-A & B \\ -C & D\end{bmatrix}\right)$$

Note rank $[C \vdots D] = n_o \Leftrightarrow \text{rank}[-C \vdots D] = n_o$; this implies that $p_1 \neq \theta_n$ (if $p_1 = \theta_n$, then $p_2 = \theta_{n_o}$ which contradicts the fact that p_1, p_2 are not both zero). Therefore there is a component of p_1, say the i th component, $p_{1i} \neq 0$. Also, since $r_\infty \neq \theta_{n_o}$, there exists a component of r_∞, say the j th component, $r_{\infty j} \neq 0$.

Now we construct a perturbation $(\delta A, \delta B)$ and show that, for a given $r_\infty \neq \theta_{n_0}$, eqn (V.11) no longer has a solution at the perturbed data point $(A+\delta A, B+\delta B, C, D)$, which contradicts assumption ($\mathcal{O}3'$):

Let

$$\text{(V.14)} \qquad \delta A = \begin{bmatrix} & \bigcirc & \\ \cdots & -\varepsilon C_{j.} & \cdots \\ & \bigcirc & \end{bmatrix} \leftarrow i\text{ th row and } \delta B = \begin{bmatrix} & \bigcirc & \\ \cdots & \varepsilon D_{j.} & \cdots \\ & \bigcirc & \end{bmatrix} \leftarrow i\text{ th row}$$

where $C_{j.}$ $(D_{j.})$ is the j th row of C (D, resp.) and $\varepsilon > 0$ is arbitrarily small. This corresponds to the elementary row operation that adding ε times the (n+j)th row to the i th row in the matrix of (V.11). Therefore, (V.13) implies

$$\text{(V.15)} \qquad [\tilde{p}_1^* \,\vdots\, \tilde{p}_2^*] \begin{bmatrix} \lambda I-A-\delta A & B+\delta B \\ -C & D \end{bmatrix} = \theta^T_{n+n_i}$$

where $[\tilde{p}_1^* \,\vdots\, \tilde{p}_2^*]$ is obtained by applying elementary column operation (which is the inverse of the above elementary row operation) to $[p_1^* \,\vdots\, p_2^*]$, hence $\tilde{p}_1^* = p_1^*$

$$\tilde{p}_2^* = p_2^* + [0,\ldots,0 \underset{\substack{\uparrow \\ j\text{ th component}}}{-\varepsilon p_{1i}},\ldots 0]$$

(V.15) means that

$$\text{(V.16)} \qquad \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix} \perp \mathcal{R}\begin{bmatrix} \lambda I-A-\delta A & B+\delta B \\ -C & D \end{bmatrix}.$$

However, by (V.13),

$$\text{(V.17)} \qquad [p_1^* \;\; p_2^*] \begin{bmatrix} \theta_n \\ -r_\infty \end{bmatrix} = 0$$

Thus from (V.17), the selection of $r_{\infty j} \neq 0$ and the definition of $\tilde{p}_2$, we have

(V.18) $$[\tilde{p}_1^* \quad \tilde{p}_2^*] \begin{bmatrix} \theta_n \\ -r_\infty \end{bmatrix} = \varepsilon\, p_{1i}^*\, r_{\infty j} \neq 0.$$

(V.16) and (V.18) mean that

$$\begin{bmatrix} \theta_n \\ -r_\infty \end{bmatrix} \notin \mathcal{R}\begin{bmatrix} \lambda I-A-\delta A & B+\delta B \\ -C & D \end{bmatrix}.$$

i.e., for the chosen $\lambda \in \sigma(A_r)$, eqn. (V.11) no longer has a solution at perturbed data point $(A+\delta A, B+\delta B, C, D)$, which contradicts the assumption that asymptotic tracking is robust at (A,B).

Q.E.D.

<u>Theorem V.3</u>. <u>(Asymptotic disturbance rejection)</u>

Given the linear time-invariant system described by (II.1)~(II.6). If there exists a controller such that

($\mathcal{O}$1) the closed-loop system is exp. stable;

($\mathcal{O}$2") asymptotic disturbance rejection holds (i.e. with $x_r(0) = \theta_{n_r}$, for all $x(0), x_c(0), x_w(0)$, $y(t) \to \theta_{n_0}$, as $t \to \infty$, where $x_c(0)$ denotes the initial state of the controller);

($\mathcal{O}$3") asymptotic disturbance rejection is <u>robust at</u> (E,F),

then

(V.18) $$\operatorname{rank} \begin{bmatrix} \lambda I-A & B \\ -C & D \end{bmatrix} = n + n_o \quad , \quad \forall \lambda \in \sigma(A_w)$$

<u>Proof</u>:

Let $x_r(0) = \theta_{n_r}$ (so $r(\cdot) \equiv \theta_{n_o}$) and for $\lambda \in \sigma(A_w)$, choose $\theta_{n_w} \neq x_w(0) \in \mathbb{C}^{n_w}$ such that $w(t) = w_\infty e^{\lambda t}$, where $\theta_d \neq w_\infty \in \mathbb{C}^d$.

Then, as $t \to \infty$,

(V.19) $$\begin{cases} y(t) \longrightarrow \theta_{n_o}, & \text{by assumption } (\mathcal{O}2") \\ x(t) \longrightarrow x_\infty e^{\lambda t}, \text{ for some } x_\infty \in \mathbb{C}^n, & \\ u(t) \longrightarrow u_\infty e^{\lambda t}, \text{ for some } u_\infty \in \mathbb{C}^{n_i} & \end{cases} \text{by assumptions } (\mathcal{O}1) \text{ and } (\mathcal{O}2")$$

Using eqn (V.19) in (V.9) (with $\hat{r} = \theta_{n_o}$), we obtain

$$\text{(V.20)} \qquad \begin{bmatrix} \lambda I-A & B \\ -C & D \end{bmatrix} \begin{bmatrix} x_\infty \\ -u_\infty \end{bmatrix} = \begin{bmatrix} E \\ F \end{bmatrix} w_\infty \triangleq k, \qquad \lambda \in \sigma(A_w)$$

By assumption ($\mathcal{O}$3"), asymptotic disturbance rejection is robust at (E,F), so k is not constrained to any manifold of $\mathbb{C}^{n+n_o}$, because k can always be moved slightly away from any such manifold by some small suitable perturbation ($\delta E, \delta F$). Hence, (V.20) is solvable implies that

$$\mathcal{R}\begin{bmatrix} \lambda I-A & B \\ -C & D \end{bmatrix} = \mathbb{C}^{n+n_o}, \quad \forall\lambda \in \sigma(A_w); \text{ i.e.}$$

$$\text{rank} \begin{bmatrix} \lambda I-A & B \\ -C & D \end{bmatrix} = n + n_o, \quad \forall\lambda \in \sigma(A_w)$$

Q.E.D.

Remarks: (i) It should be noted that in theorem V.2 and V.3, we don't impose any constraint on the controller structure (in contrast to theorem V.1, where the controller is assumed to be of feedback type).

(ii) Suppose (A,B) is completely controllable, then it can be shown, by method analogus to the one used in the proof of theorem V.2, that if $\lambda \in \sigma(A_w)$ is real, then under the assumptions of theorem V.2, asymptotic disturbance rejection achieved robustly at (A,B) implies that the rank condition (V.18) holds. When $\lambda \in \sigma(A_w)$ is complex, this method failed because it leads to $\delta A \in \mathbb{C}^{nxn}$, $\delta B \in \mathbb{C}^{nxn_i}$, whereas the allowable perturbations are such that $\delta A \in \mathbb{R}^{nxn}$, $\delta B \in \mathbb{R}^{nxn_i}$.

VI. ASYMPTOTIC TRACKING/DISTURBANCE REJECTION AND TRANSMISSION ZEROS

In section V, we have investigated several aspects of the rank condition (III.4) in the robust asymptotic tracking and disturbance rejection problem. Recall that rank condition (III.4) is equivalent to (1) the number of plant inputs, n_i, must be greater or equal to the number of tracking outputs, n_o, and (2) no mode of the reference- and disturbance-signals may be a transmission zero of the plant. In fact, this property arises from fundamental considerations applied to a general feedback system modeled by transfer function matrices. Theorem VI.1 below states this fact precisely. Next Theorem VI.1's corollary specialized the conclusions to the robust asymptotic tracking and disturbance rejection problem stated in section II.

Consider the unity feedback system shown on Fig. 7 where $r(t), e(t), y(t) \in \mathbb{R}^{n_o}$, $u(t) \in \mathbb{R}^{n_i}$, $K(s) \in \mathbb{R}(s)^{n_i x n_o}$ and $G(s) \in \mathbb{R}^{n_o x n_i}$. Let $N_r(s)D_r(s)^{-1}$ be a right coprime factorization of the controller $K(s)$ and $D_\ell(s)^{-1}N_\ell(s)$ be a left coprime factorization of the plant $G(s)$. The theorem below states precisely an algebraic fact whose interpretation is detailed in the remarks following its statement.

<u>Theorem VI.1</u> (<u>Transmission zeros and robust tracking feedback systems</u>)

Consider the unity feedback system shown on Fig. 7. Let $\phi(s) \in \mathbb{R}[s]$ have only $\mathbb{C}_+$-zeros. Suppose that

(i) $\phi(s)|D_r(s)$;

(ii) the closed-loop system is exponentially stable (thus $\det[D_\ell D_r+N_\ell N_r]$ has no $\mathbb{C}_+$-zeros), then

(1) $n_i \geq n_o$;

(2) neither the plant G(s) nor the controller K(s) have transmission zeros at $\lambda \in$ {zeros of $\phi(s)$}.

Remarks. (i) That the controller K(s) has no transmission zero at any $\lambda \in$ {zeros of $\phi(s)$} follows directly from the right coprimeness of the pair (D_r,N_r). Indeed, the right coprimeness of (D_r,N_r) is equivalent to

$$\text{rank}\begin{bmatrix} N_r(s) \\ \cdots\cdots \\ D_r(s) \end{bmatrix} = n_o \qquad \forall s \in \mathbb{C}$$

Now, for each $\lambda \in$ {zeros of $\phi(s)$}, $D_r(\lambda) = 0$, hence for all such λ, rank $N_r(\lambda) = n_o$. Since $N_r(\lambda) \in \mathbb{C}^{n_i x n_o}$, this requires that $n_i \geq n_o$ and that $K(s) = N_r(s)\,D_r(s)^{-1}$ has no transmission zeros at such λ's.

(ii) Let [A,B,C,D] be any minimal realization of G(s). Then the conclusion that $n_i \geq n_o$ and that G(s) has no transmission zero at $\lambda \in$ {zeros of $\phi(s)$} is equivalent to $\text{rank}\begin{bmatrix} \lambda I-A & B \\ -C & D \end{bmatrix} = n+n_o$, $\forall\lambda \in$ {zeros of $\phi(s)$}.

(iii) A straightforward calculation shows that the feedback system shown on Fig. 7 with assumptions (i), (ii) will track asymptotically the class of reference signals $r(\cdot)$ such that $r(s) = \bar{r}(s)/\phi(s)$ with $\bar{r}(s) \in \mathbb{R}(s)^{n_o}$. Note that this asymptotic tracking property is robust under the class of perturbations such that assumptions (i) and (ii) are maintained at the perturbed data point.

Proof of Theorem VI.1:

The identity

$$(D_\ell D_r+N_\ell N_r)\;(D_\ell D_r+N_\ell N_r)^{-1} = I_{n_o}, \qquad \forall s \in \mathbb{C}$$

implies that

$$\text{(VI.1)} \qquad D_\ell D_r(D_\ell D_r+N_\ell N_r)^{-1} + N_\ell N_r(D_\ell D_r+N_\ell N_r)^{-1} = I_{n_o}, \quad \forall s \in \mathbb{C}$$

Evaluating (VI.1) at $s = \lambda \in \{\text{zeros of } \phi(s)\} \subset \mathbb{C}_+$ and noting that $D_r(\lambda) = 0$ (by assumption (i)) and that $[D_\ell D_r+N_\ell N_r]^{-1}(\lambda)$ is nonsingular (by assumption (ii)), we obtain

$$\text{(VI.2)} \qquad N_\ell(\lambda)N_r(\lambda)\ [D_\ell D_r+N_\ell N_r]^{-1}(\lambda) = I_{n_o}.$$

Therefore, for all such λ's,

$$\text{(VI.3)} \qquad N_\ell(\lambda)N_r(\lambda) \in \mathbb{C}^{n_o x n_o} \text{ is nonsingular}$$

and, since $N_\ell \in \mathbb{R}[s]^{n_o x n_i}$ and $N_r \in \mathbb{R}[s]^{n_i x n_o}$, (VI.3) implies that rank $N_\ell(\lambda)$ = rank $N_r(\lambda) = n_o$. Thus $n_o \leq n_i$ and, by [Des. 3], G(s) and K(s) have no transmission zero at $\lambda \in \{\text{zeros of } \phi(s)\}$.

Q.E.D.

To see the implications of theorem VI.1 on the asymptotic tracking and disturbance rejection problem we considered in previous sections, we follow the notation used in section IV.A: recall that theorem VI.1 asserts that under certain robustness requirements, any <u>feedback</u> controller which achieves asymptotic tracking and disturbance rejection must be such that (see eqn. (IV.1))

$$\phi_{A_w A_r}(s) \,\big|\, D_r(s)$$

If we compare Fig. 3 with Fig. 7 and identify $K(s) = K_c G_c(s) = K_c(sI-A_c)^{-1}B_c$, $G(s) = (C+DK)(sI-A-BK)^{-1}B+D$, then we have the following corollary.

<u>Corollary</u> (<u>Asymptotic tracking/disturbance rejection and transmission zeros</u>)

Consider the linear time-invariant system described by (II.1) ~ (II.6).

Assume that

(i) $[A,B,C]$ is minimal;

(ii) $\begin{bmatrix} E \\ F \end{bmatrix}$ is full column rank,

then any feedback controller (with input e and state x_c) satisfying the conditions (a), (b), (c') stated in Theorem IV.1 must be such that

(VI.4) $K(s) = K_c(sI-A_c)^{-1}B_c$ has no transmission zero at $\lambda \in \sigma(A_w) \cup \sigma(A_r)$.

Furthermore, the plant is such that

$$\text{rank}\begin{bmatrix} \lambda I-A & B \\ -C & D \end{bmatrix} = n+n_o, \quad \forall\lambda \in \sigma(A_w) \cup \sigma(A_r).$$

<u>Proof of Corollary</u>:

From theorem IV.1, we have that

$$\phi_{A_wA_r} \mid D_r(s)$$

By the closed-loop stability condition (a) of Theorem IV.1, we have that (see eqn. (IV.2))

$\det(D_\ell D_r+N_\ell N_r)$ has no $\mathbb{C}_+$-zeros.

Hence, by theorem VI.1, we conclude that $n_i \geq n_o$ and that neither $G(s)$ nor $K(s)$ have transmission zeros at $\lambda \in \{\text{zeros of } \phi_{A_wA_r}\} = \sigma(A_w) \cup \sigma(A_r)$ (so VI.4 follows). The rank condition (III.4) follows because that it is equivalent to (1) $n_i \geq n_o$, and (2) the transfer function matrix $C(sI-A)^{-1}B+D$ has no transmission zero at $\lambda \in \sigma(A_w) \cup \sigma(A_r)$ and that the transmission zeros of $C(sI-A)^{-1}B+D$ are invariant under constant state feedback (i.e. $C(sI-A)^{-1}B+D$ and $(C+DK)(sI-A-BK)^{-1}B+D$ have the same transmission zeros).

Q.E.D.

VII. THE DISCRETE-TIME CASE

All the results above are stated for continuous-time case. For lumped systems, all the proofs above are purely algebraic and are based on simple properties of rational functions, determinants and matrices, hence the results above apply equally well to the discrete-time case with modifications indicated in the following table, where $D(\theta,1)$ and $D(\theta,1)^c$ denote the open unit disk centered at θ in $\mathbb{C}$ and its complement in $\mathbb{C}$, respectively.

VIII. CONCLUSION

This chapter has given a self-contained comprehensive treatment of the linear time-invariant robust servomechanism problem for multi-input multi-output systems. Theorem III.1 exhibits a feedback controller (III.1) which achieves asymptotic tracking and disturbance rejection robustly. Tolerance of perturbations on the controller dynamics is disussed in section III.B. Theorem IV.1 and its corollary show that, under some robustness requirements, any minimal order robust feedback controller which achieves asymptotic tracking and disturbance rejection must be given by (III.1); hence it must contain n_o copies of the reference- and disturbance-signal dynamics, where n_o is the number of reference signals. This result shows that this n_o-fold replication is necessary in order to have robustness: this agrees with the intuitive engineering idea that robustness requires redundancy. A similar characterization of a robust feedback controller for the distributed case is derived in section IV.B. Finally, the rank condition (III.4) is examined carefully in section V: it is shown that lack of the rank condition (III.4) will result in either loss of controllability of the composite system (theorem V.1), or loss of robust asymptotic tracking (theorem V.2) or loss of robust asymptotic disturbance rejection (theorem V.3). Theorem VI.1 investigates the relation between transmission zeros and a general robust tracking feedback system. Then theorem VI.1's corollary specializes the conclusions to the robust servomechanism problems stated in section II. Generalizations to large-scale decentralized systems and distributed systems have already been reported [Dav. 6,7; Fra. 7]. Some progress on the robust nonlinear servomechanism problem has been made [Des. 6].

IX. NOTES ON LITERATURE

For single-input single-output systems, the servomechanism problem (asymptotic tracking and disturbance rejection) has been well understood for about 40 years [Bro. 1]. It took a lot of work to develop insight and design philosophies for the multi-input multi-output case. In the course of this search, several approaches were proposed:

1) Optimal Control Approach [Joh. 1,2,3]

The problem formulation is basically the same as eqns. (II.1)~(II.6). A non-robust controller has been obtained via the construction of the so called disturbance-accommodating controller which is made up by a copy of the system and a copy of the disturbance signal dynamics. Control law is then obtained by minimizing some chosen quadratic cost functionals. Robustness was not discussed. The duplication of disturbance signal dynamics arises from the idea of reconstructing the disturbance "state". This approach has provided one way of solving this problem. However, whether this duplication in the controller is essential is not established (actually, this is a major drawback in optimal control approach: the relation between the cost functional and the control structure is not obvious). For a complete discussion of this approach which includes disturbance rejection, one should refer to [Joh. 3].

2) Geometric approach ([Bha. 1,2; Fra. 1,2,3,4,5,6; Pea. 1,2; Seb. 1, Sta. 1, Won. 1,2,3,4]

The problem formulation here is slightly more general (except that only the case D = 0 is considered); here interaction between disturbance and reference signals is allowed (however, for engineering purposes, we feel that the formulation (II.1)~(II.6) is general enough). At first, the non-robust case is considered: construction of controller is based on the notions of observer theory (e.g. [Fra. 2,3]) (or state space extension techniques (e.g. [Pea. 1,2; Sta. 1; Won. 3]) and internal stability of the whole system (plant together with the "exogenous" system which is uncontrollable), hence duplication of the exogenous system dynamics is not surprising (by observer theory); the far more interesting fact is that under certain conditions, any non-robust feedback controller must contain a copy of the exogenous system dynamics [Fra. 2,3]. Then the robust (or structurally stable) controller is treated by the same approach. However, to show that under certain assumptions, the robust controller must contain n_o copies of the exogenous system dynamics (the so called "internal model principle"), where n_o = number of outputs to be regulated, turns out to be very involved [Fra. 1]. In summary, this approach has achieved essentially the same conclusions as the above treatment.

3) Algebraic approach ([Dav. 1,2,3,4,5,8; Por. 1; Smi, 1; You. 1])

The most general case has been considered by Davison and Goldberg [Dav. 5] from which we adopt the present problem formulation (II.1)~(II.6). The derivation evolved through a sequence of papers ([Dav. 1,2,3,4,5]): a simple case was first considered [Dav. 1], duplication of reference-and disturbance-signal dynamics and the robustness result is immediate, since every component of the reference-and disturbance-signals posess the same modes. The general case is then resolved through reductions and some "equivalence" transformations. Conditions are expressed in terms of rank conditions which can be shown to be equivalent to some conditions posed in the geometric approach. The results are appealing, but some of the derivations are by no means easy.

4) Frequency domain approach [Des. 5; Fer. 1,2; Wil. 1; Wolf. 1; Wol.2]

The problem formulation here is basically the same as eqns. (II.1)~(II.6). Design of a robust controller is then easily justified by showing a set of blocking zeros [Fer. 1,2] is generated by appropriated choice of controller. The proof of theorem III.1 is based on [Fer. 1]. The derivation of Theorem IV.1 is a modified version of [Des. 5]. Our treatment of section IV and theorem V.1 use the frequency domain approach and exhibit rigorously what are the necessary characteristics of a minimal-order robust feedback controller and why the rank condition is necessary.

Acknowledgement

Professor F. M. Callier made some judicious comments which considerably improved the manuscript.

REFERENCES

[And. 1] Anderson, B.D.O. and Moore, J.B., Linear Optimal Control, Prentice Hall Inc., Englewood Cliffs, 1971.

[Bha. 1] Bhattacharyya, S.P., Pearson, J.B. and Wonham, W.M., "On zeroing the output of a linear system," Information and Control, Vol. 20 (2), pp. 135-142, 1972.

[Bha. 2] Bhattacharyya, S.P. and Pearson, J.B., "On Error systems and the servomechanism problem," Int. J. Control, Vol. 16 (6) pp. 1041-1062.

[Bha. 3] Bhattacharyya, S.P. "Robustness analysis of dynamic observers," Sixth IFAC World Congress, paper 29.6, Aug. 1975.

[Bha. 4] Bhattacharyya, S.P. "The structure of robust observers," IEEE Trans. Aut. Control, Vol. AC-21 (4), pp. 581-588, Aug. 1976.

[Bra. 1] Brasch, Jr., F.M. and Pearson, J.B. "Pole assignment using dynamic compensator," IEEE Trans. Aut. Control, Vol. AC-15 (1), pp. 34-43, Jan. 1970.

[Bro. 1] Brown, G.S. and Campbell D.P., Principles of Servomechanisms, John Wiley & Sons, Inc., New York, 1948.

[Cal. 1] Callier, F.M. and Nahum, C.D., "Necessary and sufficient conditions for the complete controllability and observability of systems in series using the coprime factorization of a rational matrix," IEEE Trans. Circuits and Systems, Vol. CAS-22 (2), pp. 90-95, Feb. 1975.

[Cal. 2] Callier, F.M. and Desoer, C.A., "Open-loop unstable convolution feedback systems with dynamical feedbacks," Automatica, Vol. 12, pp. 507-518, 1976.

[Cal. 3] Callier, F.M. and Desoer, C.A., work in progress.

[Dav. 1] Davison, E.J. "The output control of linear time-invariant multivariable systems with unmeasurable arbitrary disturbances," IEEE Trans. Aut. Control, Vol. AC-17 (5) pp. 621-630, Oct. 1972.

[Dav. 2] Davison, E.J. "Correction to "The output control of linear time-invariant multivariable systems with unmeasurable arbitrary disturbances"," IEEE Trans. Aut. Control, Vol. AC-20 (6), pp. 824, Dec. 1975.

[Dav. 3] Davison, E.J., "A generalization of the output control of linear multivariable systems with unmeasurable arbitrary disturbances," IEEE Trans. Aut. Control, Vol. AC-20 (6), pp. 788-792, Dec. 1975.

[Dav. 4] Davison, E.J. "The robust control of a servomechanism problem for linear time-invariant multivariable systems," IEEE Trans. Aut. Control, Vol. AC-21 (1) pp. 25-34, Feb. 1976.

[Dav. 5] Davison, E.J. and Goldenberg, A., "Robust control of a general servomechanism problem: The servo compensator," Automatica, Vol. 11 (5), pp. 461-471, 1975.

[Dav. 6] Davison, E.J. "The robust decnetralized control of a general servomechanism problem," IEEE Trans. Aut. Control, Vol. AC-21 (1), pp. 14-24, Feb. 1976.

[Dav. 7] Davison, E.J., "The robust decentralized servomechanism problem with extra stabilizing control agents," IEEE Trans. Aut. Control, Vol. AC-22 (2), pp. 256-258, Apr. 1977.

[Dav. 8] Davison, E.J. and Smith, H.W., "A note on the design of industrial regulators: Integral feedback and feedforward controllers," Automatica, Vol. 10, pp. 329-332, 1974.

[Dav. 9] Davison, E.J. and Wang, S.H., "Properties and calculation of transmission zeros of linear multivariable systems," Automatica, Vol. 10 (6), pp. 643-658, 1974.

[Des. 1] Desoer, C.A., _Notes for a Second Course on Linear Systems_, D. Van Nostrand, New York, 1970.

[Des. 2] Desoer, C.A. and Vidyasagar, M. _Feedback Systems: Input-Output Properties_, Academic Press, New York, 1975.

[Des. 3] Desoer, C.A. and Schulman, J.D., "Zeros and poles of matrix transfer functions and their dynamic interpretation," _IEEE Trans. Circuits and Systems_, Vol. CAS-21 (1), pp. 1-8, Jan. 1974.

[Des. 4] Desoer, C.A. and Chan, W.S., "The feedback interconnection of lumped linear time-invariant systems," _J. of the Franklin Institute_, Vol. 300, Nos. 5 and 6, pp. 335-351, Nov.-Dec. 1975.

[Des. 5] Desoer, C.A. and Wang, Y.T., "On the minimum order of a robust servocompensator," _IEEE Trans. Aut. Control_, vol. AC-23(1), pp. 70-73, Feb. 1978.

[Des. 6] Desoer, C.A. and Wang, Y.T., "The robust nonlinear servo-mechanism problem," submitted to _1978 CDC_ and _Int. J. Control_.

[Doe. 1] Doetsch, G., Handbuch der Laplace-transformation, Verlag Birkhauser, Basel, 1970.

[Fer. 1] Ferreira, P.G., "The servomechanism problem and the method of the state-space in the frequency domain," Int. J. Control, Vol. 23 (2), pp. 245-255, 1976.

[Fer. 2] Ferreira, P.G. and Bhattacharyya, S.P., "On blocking zeros," IEEE Trans. Aut. Control, Vol. AC-22 (2), pp. 258-259, Apr. 1977.

[Fra. 1] Francis, B.A., "The foundation of linear multivariable regulation: The internal model principle," Ph.D. Dissertation, Univ. of Toronto, 1975.

[Fra. 2] Francis, B.A., "The linear multivariable regulator problem," ERL Memo. ERL-M560, U.C. Berkeley, 1975.

[Fra. 3] Francis, B.A., "The linear multivariable regulator problem," SIAM J. Control, Vol. 15 (3), pp. 486-505, May 1977.

[Fra. 4] Francis, B.A., Sebakhy, O.A. and Wonham, W.M., "Synthesis of multivariable regulators: the internal model principle," Appl. Math. Optimization, Vol. 1, pp. 64-86, 1974.

[Fra. 5] Francis, B.A. and Wonham, W.M., "The role of transmission zeros in linear multivariable regulators," Int. J. Control, Vol. 22 (5), pp. 657-681, 1975.

[Fra. 6] Francis, B.A. and Wonham, W.M., "The internal model principle of control theory," Automatica, Vol. 12, pp. 457-465, 1976.

[Fra. 7] Francis, B.A., "The multivariable servomechanism problem from the input-output viewpoint," IEEE Trans. Aut. Control, Vol. AC-22 (3), pp. 322-328, June 1977.

[Gra. 1] Grasselli, O.M. and Nicolo, F., "Steady-state invariant control systems under disturbances satisfying differential equations," J. of the Franklin Institute, Vol. 301 (3), pp. 287-305, 1976.

[Joh. 1] Johnson, C.D., "Optimal control of the linear regulator with constant disturbances," IEEE Trans. Aut. Control, Vol. AC-13 (4), pp. 416-421, Aug. 1968.

[Joh. 2] Johnson, C.D., "Accommodation of external disturbances in linear regulator and servomechanism problems," IEEE Trans. Aut. Control, Vol. AC-16, (6), pp. 635-644, Dec. 1971.

[Joh. 3] Johnson, C.D., "Theory of disturbance-accommodating controllers," Control and Dynamic Systems, Vol. 12, pp. 287-489, ed. by C. T. Leondes, Academic Press, New York, 1976.

[Oga. 1] Ogata, K., Modern Control Engineering, Prentice-Hall, Inc. Englewood Cliffs, 1970.

[Pea. 1] Pearson, J.B. and Staats, Jr., P.W., "Robust controllers for linear regulators," IEEE Trans. Aut. Control, Vol. AC-19 (3), pp. 231-234, June 1974.

[Pea. 2] Pearson, J.B., Shields, R.W. and Staats, Jr., P.W., "Robust solutions to linear multivariable control problems," IEEE Trans. Aut. Control, Vol. AC-19 (5), pp. 508-517, Oct. 1974.

[Por. 1] Porter, B. and Bradshaw, A., "Design of linear multivariable continuous-time tracking systems," Int. J. System Science, Vol. 5 (12) pp. 1155-1164, 1974.

[Ros. 1] Rosenbrock, H.H., State-Space and Multivariable Theory, Wiley-Interscience, New York, 1970.

[San. 1] Sandell, Jr., N.R., "Optimal linear tracking systems," Electron Syst. Lab., ESL-R-456, MIT, Cambridge, Sept. 1971.

[San. 2] Sandell, Jr., N.R. and Athans, "On type L multivariable linear systems," Automatica, Vol. 9, pp. 131-136, Jan. 1973.

[Seb. 1] Sebakhy, O.A. and Wonham, W.M., "A design procedure for multivariable regulators," Automatica, Vol. 12, pp. 467-478, 1976.

[Smi. 1] Smith, H.W. and Davison, E.J., "Design of industrial regulators: Internal feedback and feedforward control," Proc. IEE, Vol. 119, pp. 1210-1216, 1972.

[Sta. 1] Staats, Jr., P.W. and Pearson, J.B., "Robust solution of the linear servomechanism problem," Technical Report No. 7401, Elec. Eng. Dept., Rice Univ., Sept. 1974.

[Wil. 1] Willems, J.L., "Complete disturbances isolation in linear feedback systems," Electron. Letters, Vol. 10 (15), pp. 319-320, July 1974.

[Wolf. 1] Wolfe, C.A. and Meditch, J.S., "Theory of system type for linear multivariable servomechanmisms," IEEE Trans. Auto. Control, Vol. AC-22 (1), pp. 36-48, Feb. 1977.

[Wol. 1] Wolovich, W.A., "On determining the zeros of state-space systems," IEEE Trans. Aut. Control, Vol. AC-18 (5), pp. 542-544, Oct. 1973.

[Wol. 2] Wolovich, W.A., "Multivariable system synthesis with step disturbances rejection," IEEE Trans. Aut. Control, Vol. AC-19 (1), pp. 127-130, Feb. 1974.

[Won. 1] Wonham, W.M., "Tracking and regulation in linear multivariable systems," SIAM J. Control, Vol. 11, pp. 424-437, 1973.

[Won. 2] Wonham, W.M. and Pearson, J.B., "Regulation and internal stabilization in linear multivariable systems," SIAM J. Control, Vol. 12, pp. 5-8, 1974.

[Won. 3] Wonham, W.M., Linear Multivariable Control — A geometric Approach, Springer-Verlag, New York, 1974.

[Won. 4] Wonham, W.M., "Towards an abstract internal model principle," IEEE Trans. Syst. Man and Cybernetics, Vol. SMC-6 (11), pp. 735-740, Nov. 1976.

[You. 1] Young, P.C. and Willems, J.C., "An approach to the linear multivariable servomechanisms problem," Int. J. Control, Vol. 15 (5), pp. 961-979, 1972.

FOOTNOTES

(1) It can be shown ([Dav. 4, Fra. 1]) that to satisfy ($\mathcal{O}$3) at certain data points, it is necessary to have the controller to be of feedback type.

(2) Observe that, by definition

$h_{ij}(s)$: $\begin{bmatrix} w \\ r \end{bmatrix}_j \mapsto e_i$, where $\begin{bmatrix} w \\ r \end{bmatrix}_j$ denotes the j^{th} component of $\begin{bmatrix} w \\ r \end{bmatrix}$ and is given by

$$h_{ij}(s) = (C_L)_{i\cdot} \cdot (sI-A_L)^{-1}(B_L)_{\cdot j} + (D_L)_{ij},$$

where $(C_L)_{i\cdot}$ ($(B_L)_{\cdot j}$) denotes the i th row (j th column) of C_L (B_L, respectively) and $(D_L)_{ij}$ denotes the ij th element of D_L. Finally (III.12) follows from Cramer's rule applied to

$$\begin{bmatrix} sI-A_L & (B_L)_{\cdot j} \\ -(C_L)_{i\cdot} & (D_L)_{ij} \end{bmatrix} \begin{bmatrix} -x_L \\ v_j \end{bmatrix} = \begin{bmatrix} \theta_{n+n_c} \\ e_i \end{bmatrix}$$

where $x_L = (x, x_c)^T$, $v_j = [(w,r)^T]_j$.

(3) Note $\tilde{B}_c$ = block diag $(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_{n_0})$, so instead of (III.13), we apply elementary row matrix $\tilde{R}$ to (III.12) and then obtain (III.18), where $\tilde{R}$ is obained by replacing γ in R by $\tilde{\gamma}_i$.

(4) To simplify notations, we assume every Γ subject to the same perturbation and denote the perturbed Γ by $\tilde{\Gamma}$. However, the following analysis goes through with different perturbations on each Γ.

(5) For some special $(\tilde{A}, \tilde{B}, \tilde{C})$, the pair $(\tilde{C}+DK, \tilde{A}+\tilde{B}K)$, ($(\tilde{A}+\tilde{B}K, \tilde{B})$, resp.) may turn out to be not completely obserbable (controllable). However, some suitable arbitrarily small perturbation from the data point will restore complete observability (controllability).

(6) The first factor if (IV.6) is always nonsingular as a consequence of the stability requirement (IV.2) and det $\tilde{D}_\ell(p) \neq 0$ for almost all perturbed plants.

(7) Without loss of generality, we may assume that D_ℓ is row proper. Consider a perturbation $\delta D_\ell \in \mathbb{R}^{n_o x n_o}$ such that $\tilde{D}_\ell = D_\ell + \delta D_\ell$. Since $\partial[\det(D_\ell+\delta D_\ell)] = \partial[\det D_\ell]$ and for small δD_ℓ, $\det[(D_\ell+\delta D_\ell)D_r + N_\ell N_r]$ is a polynomial whose coefficients are close to those of $\det[D_\ell D_r+N_\ell N_r]$; hence for sufficiently small δD_ℓ, closed-loop stability is maintained. Thus any sufficiently small δD_ℓ is an allowable perturbation. Let $\Delta = D_\ell D_r + N_\ell N_r$. Then for this class of perturbations,

$$\tilde{P} = (\tilde{D}_\ell D_r+N_\ell N_r)^{-1}\tilde{D}_\ell$$

$$= (\Delta+\delta D_\ell\cdot D)^{-1}\cdot(D_\ell+\delta D_\ell)$$

$$\approx (I-\Delta^{-1}\cdot\delta D_\ell\cdot D_r)\cdot\Delta^{-1}\cdot(D_\ell+\delta D_\ell)$$

$$\approx \Delta^{-1}D_\ell + \Delta^{-1}\cdot\delta D_\ell\cdot[I-D_r\Delta^{-1}D_\ell]$$

So the effect of such perturbation on $P(p)r_i$ is approximately equal to $\Delta(p)^{-1}\cdot\delta D_\ell\cdot[I-D_r(p)\Delta(p)^{-1}D_\ell(p)]r_i$. Consequently, δD_ℓ can always be chosen so that $P(p)r_i$ is pushed away from any manifold in $\mathbb{C}^{n_o}$.

(8) To see this, let $D_\ell^{-1}N$ be a left coprime factorization of $(C+DK)(sI-A-BK)^{-1}$. Then $\det D_\ell(s) = \chi_{(C+DK)(sI-A-BK)^{-1}} = \chi_{(C+DK)(sI-A-BK)^{-1}B}$ modulo a nonzero constant factor, where the last equality follow since $(A+BK,B)$ is completely controllable. Therefore $D_\ell^{-1}NB$ is a left coprime factorization of $(C+DK)(sI-A-BK)^{-1}B$ (otherwise $\partial[\det D_\ell] > \partial[\chi_{(C+DK)(sI-A-BK)^{-1}B}]$); consequently $D_\ell^{-1}(NB+D_\ell D)$ is a left coprime factorization of $(C+DK)(sI-A-BK)^{-1}B+D$. Hence the open-loop effect of w on the output y can be represented by $(C+DK)(sI-A-BK)^{-1}E+F = D_\ell^{-1}(NE+D_\ell F) \triangleq D_\ell^{-1}M_\ell$. Note D_ℓ and M_ℓ are not necessarily left coprime.

(9) It should be noted that any stabilizing (K,K_c) pair is such that (K_c,A_c) is completely observable, (otherwise, the unobservable modes of A_c which are unstable will not be stabilized by $u = Kx + K_c x_c$), hence $\det D_r(s) = \det(sI-A_c)$.

(10) For closed-loop stability, we require that $H_{er} \in \hat{\mathcal{A}}^{n_o x n_o}$, $H_{vr} \in \hat{\mathcal{A}}^{n_o x n_i}$ $H_{yz} \in \hat{\mathcal{A}}^{n_o x n_i}$, $H_{vr} \in \hat{\mathcal{A}}^{n_i x n_i}$, where $H_{\beta\alpha}$ denotes the transfer function $\alpha \mapsto \beta$.

(11) Here robustness means that asymptotic tracking is maintained for all $(\tilde{\mathcal{D}}_\ell, \tilde{\mathcal{N}}_\ell)$ such that $\|\tilde{\mathcal{D}}_\ell - \mathcal{D}_\ell\|_{\mathcal{A}} < \varepsilon$, $\|\tilde{\mathcal{N}}_\ell - \mathcal{N}_\ell\|_{\mathcal{A}} < \varepsilon$, where $\|\cdot\|_{\mathcal{A}}$ denotes the $\mathcal{A}$-norm [Des. 2].

(12) Using the continuity of the function $\det(\cdot)$, it can easily be shown [Fra. 7] that closed-loop $\mathcal{A}$-stability is robust at $(\mathcal{D}_\ell, \mathcal{N}_\ell)$.

TABLE VII

Continuous-time	Discrete-time
Laplace transform	Z-transform
$\mathbb{C}$	$D(\theta,1)$
$\mathbb{C}_+$	$D(\theta,1)^c$
$\mathbb{R}(s)^{pxq}$	$\mathbb{R}(z)^{pxq}$
$\dot{x}$	$x(k+1)$
x	$x(k)$
complete controllability	complete reachability

Dexter Wang

Table VI.

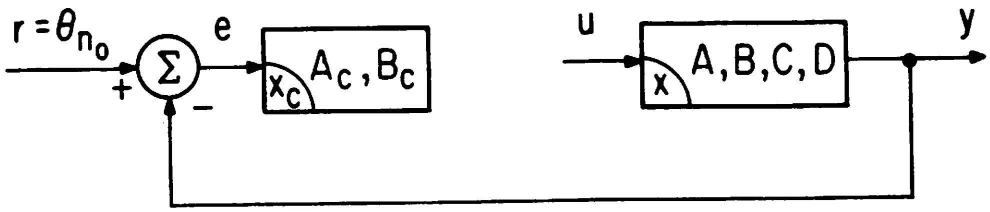

Fig. 1. Composite system under consideration: condition (III.4) implies the complete controllability of the state $\begin{bmatrix} x \\ x_c \end{bmatrix}$ by the input u.

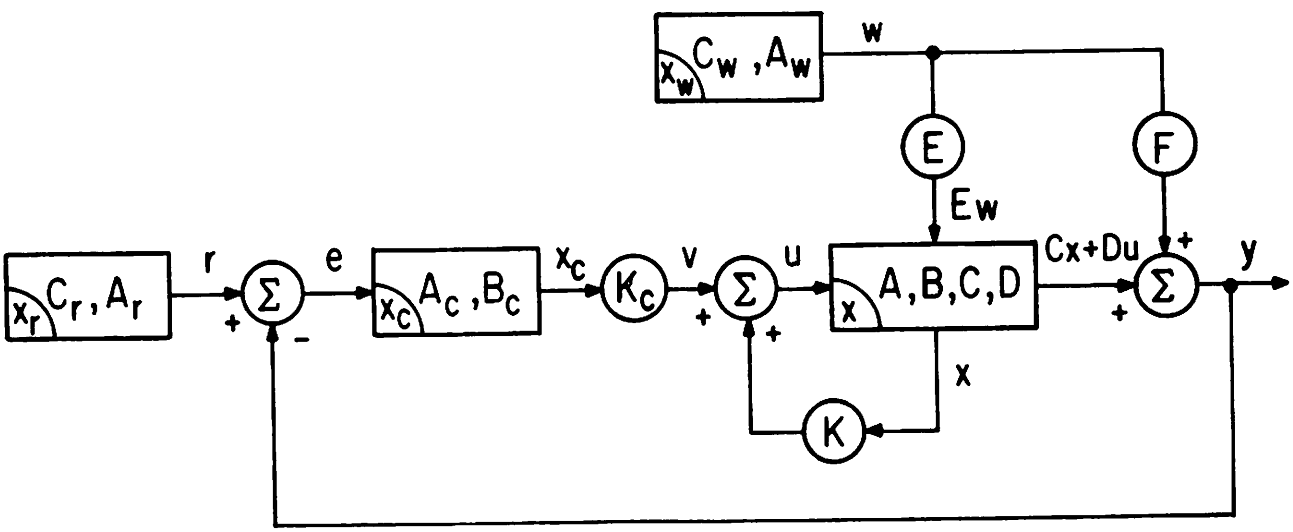

Fig. 2. Feedback system under consideration: x_c is the state of the controller; w is the disturbance and r is the reference signal to be tracked.

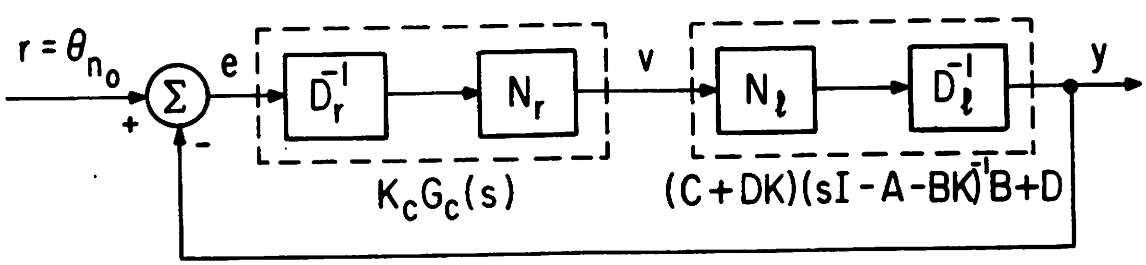

Fig. 3. To study the closed-loop stability of the feedback system in Fig. 2, we set $r \equiv \theta_{n_o}$, $w \equiv \theta_d$ and represent the subsystem transfer function matrices by coprime factorizations $N_r D_r^{-1}$ and $D_\ell^{-1} N_\ell$.

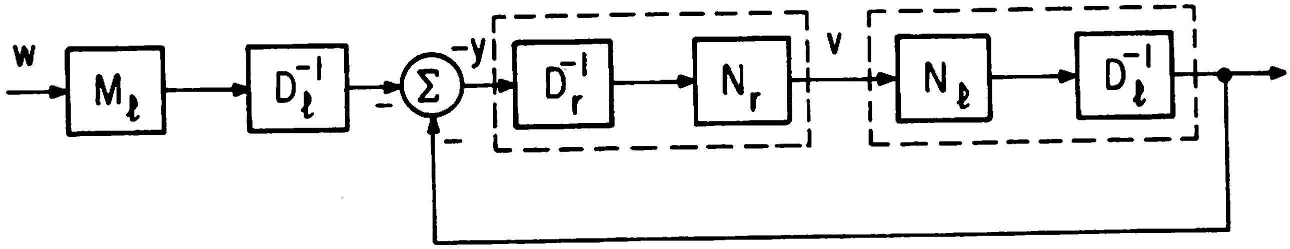

Fig. 4. The effect of the disturbance w is replaced by an equivalent reference signal of $D_\ell^{-1}M_\ell w$.

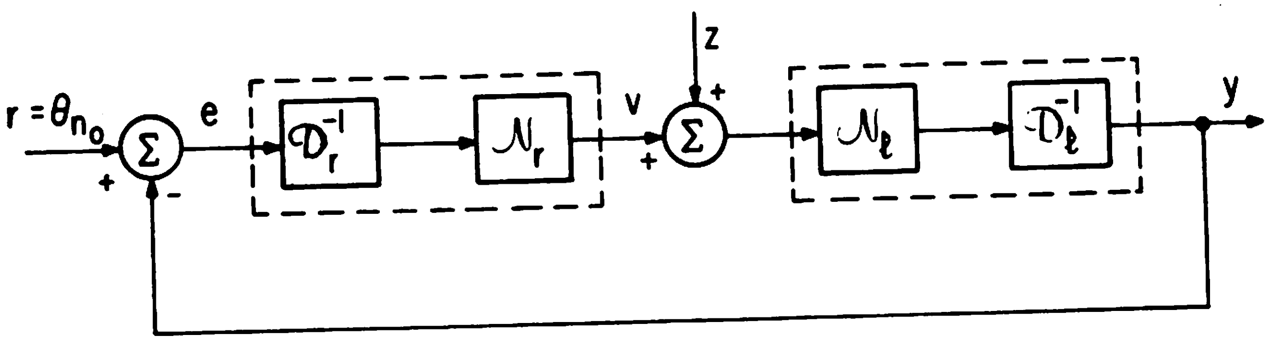

Fig. 5. Counterpart of Fig. 3 in distributed case.

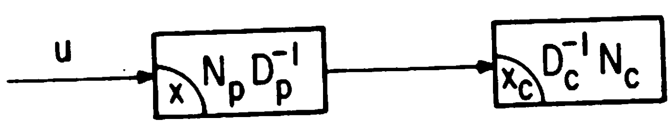

Fig. 6. Cascade system under consideration: N_p and D_p are right coprime; N_c and D_c are left coprime; x and x_c denote the states of any minimal realization of each subsystem.

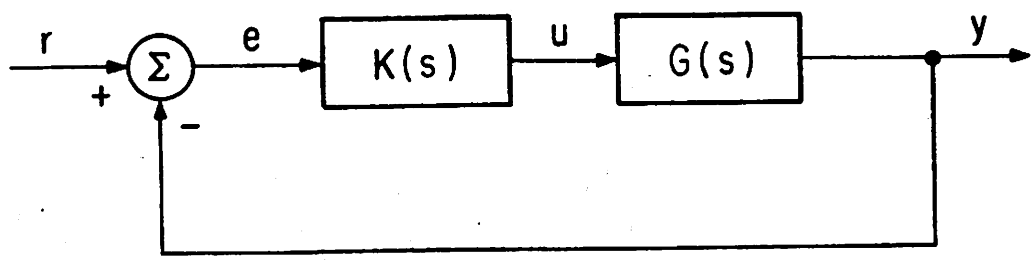

Fig. 7. Unity feedback system considered in theorem VI.1.

FIGURE CAPTIONS

Fig. 1. Composite system under consideration: condition (III.4) implies the complete controllability of the state $\begin{bmatrix} x \\ x_c \end{bmatrix}$ by the input u.

Fig. 2. Feedback system under consideration: x_c is the state of the controller; w is the disturbance and r is the reference signal to be tracked.

Fig. 3. To study the closed-loop stability of the feedback system in Fig. 2, we set $r \equiv \theta_{n_o}$, $w \equiv \theta_d$ and represent the subsystem transfer function matrices by coprime factorizations $N_r D_r^{-1}$ and $D_\ell^{-1} N_\ell$.

Fig. 4. The effect of the disturbance w is replaced by an equivalent reference signal of $D_\ell^{-1} M_\ell w$.

Fig. 5. Counterpart of Fig. 3 in distributed case.

Fig. 6. Cascade system under consideration: N_p and D_p are right coprime; N_c and D_c are left coprime; x and x_c denote the states of any minimal realization of each subsystem.

Fig. 7. Unity feedback system considered in theorem VI.1.