NONLINEAR NETWORK SYNTHESIS AND THE HODGE DECOMPOSITION

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Abstract

We give a general solution to a previously open problem in the decomposition of nonlinear n-ports. Any resistive (or capacitive or inductive) n-port can be decomposed into a particular interconnection of two simpler n-ports. The first is reciprocal, and the second can be further decomposed into $\binom{n}{2}$ reciprocal n-ports and $\binom{n}{2}$ linear 2n-ports.

The technique, which we believe is completely new to network theory, is based on certain algebraic properties of the Laplace operator. It is related to the Hodge theorem from differential geometry, applied to 1-forms on Euclidean space.

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I. INTRODUCTION

Our result concerns nonlinear, single-element type (resistive, capacitive, or inductive) n-ports. From now on we will refer only to the resistive case, since the others follow by substituting q for i or ϕ for v. Moreover, we will often write "n-port" for "resistive n-port."

1.1. The Problem: Decomposition of Resistive n-Ports

If $\underset{\sim}{M}$ is an nxn matrix of real numbers, then the equation

$$\underset{\sim}{M} = \frac{1}{2}(\underset{\sim}{M}+\underset{\sim}{M}^T) + \frac{1}{2}(\underset{\sim}{M}-\underset{\sim}{M}^T) \tag{1}$$

represents a way of breaking $\underset{\sim}{M}$ into its symmetric and antisymmetric parts. If M is the resistance or conductance matrix of a linear n-port R, then (1) decomposes R into its reciprocal[1] and antireciprocal parts. Since the synthesis of reciprocal and antireciprocal linear n-ports is relatively well understood [1,2], this simple technique is invaluable for reducing a general synthesis problem to two quite tractable ones.

In 1974, Chua and Lam [3] attempted to find a generalization of (1) which would work in the nonlinear case. They found that the most direct line of generalization, decomposing the Jacobian matrix of a vector-valued function according to (1), is unsuccessful because the matrix-valued function so obtained is not in general the Jacobian of any vector function.

Motivated by synthesis applications, Chua and Lam attempted to find a different decomposition technique which would allow every nonlinear n-port to be built from reciprocal n-ports and a simpler class of n-ports they called "quasi-antireciprocal." Although their method is valid for all 2-ports, it was not generally successful for $n \geq 3$, and their paper concluded with the open question of whether a generalization of (1) was possible for arbitrary n. The result reported in this paper is one such generalization.

1.2. The Approach: The Hodge Theorem, Helmholz' Theorem, and Vector Calculus

The decomposition technique that we have developed was inspired by the Hodge theorem [13,24,25]. But the Hodge theorem is applicable only to vector fields (more precisely, differential forms) on compact manifolds, although it can be extended to include vector fields on Euclidean space that vanish rapidly

[1] Reciprocity is defined carefully in part IV. For the moment we will say that an n-port is reciprocal if its constitutive relation $\underset{\sim}{v} = \underset{\sim}{f}(\underset{\sim}{i})$ or $\underset{\sim}{i} = \underset{\sim}{g}(\underset{\sim}{v})$ is a gradient map.

enough at infinity. Since our application requires us to decompose vector fields which do not vanish or even remain bounded at infinity, we have had to modify the theorem in a major way. Those readers well versed in differential geometry should be forewarned that much of the special structure of the Hodge theorem does not remain valid in our application. For example, unlike the classical result of Hodge, our decomposition is not unique. And the notion of the inner product of two vector fields, considered as points in an infinite dimensional linear space, is not defined here.

Except for occasional asides to the reader who is comfortable with ideas from differential geometry, the discussion in this paper will be conducted entirely in the language of vector calculus and matrix algebra to make it accessible to a wider audience. Therefore we will postpone further technical consideration of the Hodge theorem until the Appendix. Those interested in the mathematical origin of our result should read Appendix A first.

The starting point of our work is the standard identity from vector calculus on $\mathbb{R}^3$ [4],

$$\Delta \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}) \tag{2}$$

where $\mathbf{F}$ is a smooth vector field and Δ is the Laplace operator acting separately on each Euclidean component of $\mathbf{F}$, i.e.

$$\Delta \mathbf{F} = [\Delta F_1, \Delta F_2, \ldots, \Delta F_n]^T, \text{ where } \Delta F_i = \sum_{j=1}^{n} \frac{\partial^2 F_i}{\partial x_j^2} .$$

The Helmholz theorem, a special case of the Hodge theorem which is sometimes used in fluid mechanics [5,13], continuum mechanics [9,pp.147-150], and electromagnetic theory [4,p.222], applies (2) to the decomposition of a vector field $\mathbf{f}$ on $\mathbb{R}$ as follows. If $\mathbf{f}$ vanishes rapidly enough at infinity, then the equation

$$\Delta \mathbf{F} = \mathbf{f} \tag{3}$$

can be solved by convolution [6], i.e.

$$F_i(\mathbf{x}) = -\frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{f_i(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \, d\mathbf{y}; \quad i = 1,2,3. \tag{4}$$

Then the decomposition

$$\mathbf{f} = \Delta \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}) \tag{5}$$

breaks $\underset{\sim}{f}$ uniquely into the sum of the gradient vector field $\nabla(\nabla\cdot F)$ and the solenoidal (or divergence-free) vector field, $-\nabla\times(\nabla\times F)$.

If $\mathcal{R}$ were a 3-port resistor with the constitutive relation $i = f(v)$, and if the convolution integral (4) in the components of f converged, then Helmholz' theorem would allow us to realize $\mathcal{R}$ as the parallel connection of a reciprocal 3-port R_1 characterized by $i = \nabla(\nabla\cdot F)(v)$ and a 3-port R_2 characterized by $i = -\nabla\times(\nabla\times F)(v)$.

The significance of this second term is that a further algebraic manipulation allows us to decompose R_2 into reciprocal multiports and linear multiports. The algebraic details are somewhat lengthy when n = 3, so we postpone them until Eq. (37) and confine ourselves here to the case n = 2. In two dimensions [6] the solution to (3) is given by

$$F_i(x) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} f_i(y)\ \ln\|x-y\|\ dy, \quad i = 1,2. \tag{6}$$

and (5) can be written out in coordinates as

$$\begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} = \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_2^2} \\ \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F_1}{\partial x_1 \partial x_2} + \frac{\partial^2 F_2}{\partial x_2^2} \end{bmatrix} + \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial x_1 \partial x_2} \\ -\frac{\partial^2 F_1}{\partial x_1 \partial x_2} + \frac{\partial^2 F_2}{\partial x_1^2} \end{bmatrix} \tag{7}$$

The first term in the last line is the gradient of the scalar function $\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right)$ and the second term has zero divergence. This last term can be rewritten as the composition of a linear map and the gradient of another scalar function, $\left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1}\right)$, to produce the final decomposition

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right) \\ \frac{\partial}{\partial x_2}\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1}\right) \\ \frac{\partial}{\partial x_2}\left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1}\right) \end{bmatrix} \tag{8}$$

If $\mathcal{R}$ were a 2-port resistor characterized by $\underset{\sim}{i} = \underset{\sim}{f}(\underset{\sim}{v})$, then each of the gradient terms would represent a reciprocal 2-port, and, as we shall see, the matrix represents a linear 4-port.

Two remaining problems must be solved before we can call this result a general synthesis technique. The first is that constitutive relations of practical devices do not vanish at infinity, so the integrals (4) and (6) will normally diverge. But it will turn out that a solution to (3) exists nonetheless [7,8], provided only that f is sufficiently smooth.

The second is that there is no obvious way to generalize this result to higher dimensions, because (2) doesn't make sense for $n \geq 4$. This is where we draw on the Hodge theorem - for a generalization of the vector identity (2). The result is given in (28) in terms of standard Euclidean coordinates, and its relationship to the general form of the Hodge theorem is discussed in the appendix.

1.3. Summary of Results, and Application to Network Synthesis

In this paper we show how any C^∞ hybrid n-port resistor can be realized as an interconnection of $\binom{n}{2}$ linear nonreciprocal 2n-port resistors and $\binom{n}{2} + 1$ nonlinear reciprocal n-port resistors. The linear 2n-ports can be realized on paper using dependent sources, and it is not too difficult in practice to build them from operational amplifiers [10].

Our decomposition technique does not yield a unique synthesis: it only specifies certain constraints which the terminal characteristics of the reciprocal n-ports must satisfy. Within these constraints, a certain range of choice is possible. This is a significant practical advantage, since one version may be much more appealing than another for actual hardware construction.

The problem of synthesizing nonlinear reciprocal n-ports from 2-terminal elements is still far from solved. However some progress has been made, e.g. the recent work of Hung [11] on the synthesis of complete reciprocal n-ports. Since our result shows that reciprocal synthesis is the last roadblock in the way of a solution to the complete resistive synthesis problem, we hope it will motivate redoubled effort in that direction.

II. DECOMPOSITION OF TWO-PORT VOLTAGE-CONTROLLED RESISTORS

In this section we want to demonstrate our method by means of the simplest possible example, without striving for generality or rigor. The general version of the technique will be given in sections III and IV.

2.1. General 2-Port Voltage-Controlled Resistors

Let $\mathcal{R}$ be characterized by $i_1 = f_1(v_1,v_2)$, $i_2 = f_2(v_1,v_2)$. We first solve Poisson's equation in these two functions, i.e. $\Delta F_1 = f_1$ and $\Delta F_2 = f_2$. Solutions $F_1(v_1,v_2)$ and $F_2(v_1,v_2)$ always exist, although they are not unique [7,8].[2] Then we rearrange $\underset{\sim}{f} = (f_1,f_2)^T$ as in (8), where $x_1 = v_1$ and $x_2 = v_2$. This suggests the synthesis shown in Fig. 1. The interconnections are drawn in such a way that $v_1 = v_1' = e_1$ and $v_2 = v_2' = e_2$. And $\mathcal{L}$ will be designed so that $e_1 = e_3$ and $e_2 = e_4$, i.e. $\mathcal{L}$ "passes on" the independent variables v_1 and v_2 to the ports of R_2, so $v_1 = v_1' = v_1''$ and $v_2 = v_2' = v_2''$. R_1 is the reciprocal voltage-controlled resistor defined by the co-content function

$$\hat{G}_1(v_1',v_2') = \left(\frac{\partial F_1}{\partial v_1'} + \frac{\partial F_2}{\partial v_2'}\right) , \tag{9}$$

i.e.

$$\begin{aligned} i_1'(v_1',v_2') &= \frac{\partial \hat{G}_1}{\partial v_1'} \\ i_2'(v_1',v_2') &= \frac{\partial \hat{G}_1}{\partial v_2'} , \end{aligned} \tag{10}$$

and R_2 is defined by the co-content

$$\hat{G}_2(v_1'',v_2'') = \left(\frac{\partial F_1}{\partial v_2''} - \frac{\partial F_2}{\partial v_1''}\right) , \tag{11}$$

i.e.

$$\begin{aligned} i_1''(v_1'',v_2'') &= \frac{\partial \hat{G}_2}{\partial v_1''} \\ i_2''(v_1'',v_2'') &= \frac{\partial \hat{G}_2}{\partial v_2''} . \end{aligned} \tag{12}$$

Informally speaking, the purpose of the linear 4-port $\mathcal{L}$ is to "pass on" the independent variables v_1 and v_2 to R_2 unaltered and to "pass back" the

[2] We discuss the problem of solving Poisson's equation in Appendix B.

dependent variables i_1'' and i_2'' from R_2 so that they are "scrambled" as indicated by the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Therefore when the sign convention for currents in Fig. 1 is taken into account, the equations defining $\mathcal{L}$ will have to be those given in the figure. Note that $\mathcal{L}$ is an active element, that is, the net power flow into $\mathcal{L}$, given by $P = e_1j_1 + e_2j_2 + e_3j_3 + e_4j_4$ can be of either sign. Thus $\mathcal{L}$ cannot be synthesized from ideal transformers and gyrators alone, since these are power-conserving or "nonenergic" devices [12]. A simple circuit for realizing $\mathcal{L}$ from two dependent sources is shown in Fig. 2.

2.2. An Example: Application to Ebers-Moll Equations for Transistor

In this section we will illustrate the method by using it to synthesize the well-known d.c. Ebers-Moll circuit model for a transistor from the terminal equations. This isn't a difficult problem; an engineer could solve it almost by inspection. Its purpose is to provide a concrete example of our technique, which is of course applicable to arbitrarily complex terminal equations as well.

Example 1. We have adopted the same notation as in example 1 of [3] to simplify comparison. The low-frequency common-base Ebers-Moll equations for a pnp transistor, Fig. 3, can be written as

$$\begin{aligned} i_1 &= A_1(e^{Kv_1}-1) - B_1(e^{Kv_2}-1) = i_1(v_1,v_2) \\ i_2 &= -A_2(e^{Kv_1}-1) + B_2(e^{Kv_2}-1) = i_2(v_1,v_2) \end{aligned} \qquad (13)$$

where $i_1 = i_E$, $v_1 = v_{EB}$, $i_2 = i_C$, $v_2 = v_{CB}$, $A_1 = i_{ES}$, $B_1 = \alpha_R i_{CS} = \alpha_F i_{ES} = A_2$, $B_2 = i_{CS}$, and $K = q/kT$.

In this case we can solve Eq. (3) by inspection to get

$$F_1(v_1,v_2) = \frac{A_1}{K^2} e^{Kv_1} - \frac{A_1 v_1^2}{2} - \frac{B_1}{K^2} e^{Kv_2} + B_1 \frac{v_2^2}{2}$$

$$F_2(v_1,v_2) = -\frac{A_2}{K^2} e^{Kv_1} + \frac{A_2 v_1^2}{2} + \frac{B_2}{K^2} e^{Kv_2} - B_2 \frac{v_2^2}{2} . \tag{14}$$

Other solutions, of course, can be obtained by adding harmonic functions to F_1 and F_2. From Eqs. (9) and (11) the content functions are

$$\begin{aligned} \hat{G}_1(v_1,v_2) &= \frac{A_1}{K} e^{Kv_1} - A_1 v_1 + \frac{B_2}{K} e^{Kv_2} - B_2 v_2 \\ \hat{G}_2(v_1,v_2) &= -\frac{B_1}{K} e^{Kv_2} + B_1 v_2 + \frac{A_2}{K} e^{Kv_1} - A_2 v_1, \end{aligned} \tag{15}$$

so if we now revert to our convention that the variables for R_1 are named i_1', i_2', v_1', v_2' and the variables for R_2 are i_1'', i_2'', v_1'', v_2'', the constitutive relation for R_1 is

$$\begin{aligned} i_1' &= \frac{\partial \hat{G}_1}{\partial v_1'} = A_1(e^{Kv_1'}-1) \\ i_2' &= \frac{\partial \hat{G}_1}{\partial v_2'} = B_2(e^{Kv_2'}-1) \end{aligned} \tag{16}$$

and for R_2 we have

$$\begin{aligned} i_1'' &= \frac{\partial \hat{G}_2}{\partial v_1''} = A_2(e^{Kv_1''}-1) \\ i_2'' &= \frac{\partial \hat{G}_2}{\partial v_2''} = -B_1(e^{Kv_2''}-1), \end{aligned} \tag{17}$$

while $\mathcal{L}$ remains as in Fig. 1.

The decomposition in (16) and (17) appears in Fig. 4(a). Both R_1 and R_2 are uncoupled 2-ports, and R_1 just consists of the two diodes D_1 and D_2. But R_2 is more difficult to synthesize because the device D_4, as given in the second line of (17), is active. And since we are synthesizing the 3-terminal device of Fig. 3, we have connected the negative sides of ports 1 and 2 together to create a grounded 2-port.

The formal synthesis procedure ends here, but with a little engineering common sense we can greatly simplify the result obtained so far. The first step is to notice that D_4 is active only because $\mathcal{L}$ produces the constraint $j_1 = -j_4$. If we substitute for $\mathcal{L}$ the 4-port $\mathcal{L}'$ for which $j_1 = j_4$, as in Fig. 4(b), then the active device D_4 becomes the passive diode D_4'. Next we simplify and rearrange Fig. 4(b) so that it appears as in Fig. 5(a). Notice that the arrows on the dependent sources are now reversed because $i_1'' + i_2'' = -(j_3+j_4)$. The functional subunit labelled I and enclosed by dotted lines produces the current $i_1'-i_1$, which is i_2''. But $i_2'' = B_1(e^{Kv_2}-1) = (B_1/B_2)i_2'$. And similarly, subunit II produces $i_2'-i_2 = i_1''$, and $i_1'' = (A_2/A_1)i_1'$. Thus we can substitute single dependent sources for units I and II, which reduces Fig. 5(a) to Fig. 5(b), the well-known d.c. Ebers-Moll model for a transistor.

III. THE DECOMPOSITION OF AN ARBITRARY C^∞ FUNCTION $\underset{\sim}{f} : \mathbb{R}^n \to \mathbb{R}^n$

3.1. Decomposition of Functions- the Main Result

In this section we will derive the higher dimensional generalization of the 3-dimensional vector identity (2). Let $C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$ stand for the class of all functions mapping $\mathbb{R}^n \to \mathbb{R}^n$ for which partial derivatives of all orders exist at each point,[3] and $C^\infty(\mathbb{R}^n)$ be the class of C^∞ functions mapping $\mathbb{R}^n \to \mathbb{R}$. By identifying $C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$ with the class of C^∞ vector fields on $\mathbb{R}^n$, we can adopt the following standard definitions from vector calculus. If $\underset{\sim}{F} \in C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$ and $\phi \in C^\infty(\mathbb{R}^n)$, then $\text{div } \underset{\sim}{F} \triangleq \underset{\sim}{\nabla} \cdot \underset{\sim}{F} \triangleq \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$, $\text{grad } \phi \triangleq \underset{\sim}{\nabla}\phi \triangleq \left[\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_n}\right]^T$, $\Delta\phi \triangleq \underset{\sim}{\nabla}^2\phi = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2}$, and $\underset{\sim}{\Delta}\underset{\sim}{F} = [\Delta F_1, \Delta F_2, \ldots, \Delta F_n]^T$. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be harmonic if $\Delta\phi = 0$.

Lemma 1. If $\underset{\sim}{F} \in C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$, then

$$\underset{\sim}{\Delta}\underset{\sim}{F} = \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{F}) + \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} [\underset{\sim}{A}_{ij}]\, \underset{\sim}{\nabla}\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right), \tag{18}$$

where each term $[\underset{\sim}{A}_{ij}]$ represents an antisymmetric nxn matrix with only two nonzero entries, specifically

[3] It is possible, although a little awkward, to produce a C^k version of these results for finite k. The awkwardness arises from the fact that $\underset{\sim}{\Delta}\underset{\sim}{F} = \underset{\sim}{f}$ with $\underset{\sim}{f} \in C^k$ does not necessarily imply that $\underset{\sim}{F} \in C^{k+2}$ [13].

$$[\underset{\sim}{A}_{ij}]_{k,\ell} = \begin{cases} 1; & k = i,\ \ell=j \\ -1; & k = j,\ \ell=i \\ 0; & \text{otherwise.} \end{cases} \tag{19}$$

(The matrix $[\underset{\sim}{A}_{12}]$ for the case n=2 appears in (8), and the matrices $[\underset{\sim}{A}_{12}]$, $[\underset{\sim}{A}_{13}]$, and $[\underset{\sim}{A}_{23}]$ for the case n=3 can be found in (37).)

Proof. Choose m, $1 \leq m \leq n$. Then the m-th component of $\Delta \underset{\sim}{F}$ is

$$\sum_{i=1}^{n} \frac{\partial^2 F_m}{\partial x_i^2} . \tag{20}$$

The m-th component of $\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{F})$ is

$$\frac{\partial}{\partial x_m}\left(\sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}\right) = \sum_{i=1}^{n} \frac{\partial^2 F_i}{\partial x_m \partial x_i} . \tag{21}$$

Since each matrix $[\underset{\sim}{A}_{ij}]$ has nonzero entries only in locations (i,j) and (j,i), the only terms of the double sum in (18) which contribute to the m-th component of $\Delta \underset{\sim}{F}$ are those corresponding to indices (m,j) with $m < j$ or (i,m) with $i < m$. (See Eq. (37) for an example.) Thus the m-th component of

$$\sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} [\underset{\sim}{A}_{ij}]\, \underset{\sim}{\nabla}\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right) \tag{22}$$

is the sum of two terms. The first is the m-th entry in

$$\sum_{j>m} [\underset{\sim}{A}_{mj}]\, \underset{\sim}{\nabla}\left(\frac{\partial F_m}{\partial x_j} - \frac{\partial F_j}{\partial x_m}\right), \tag{23}$$

which is, by (19),

$$\sum_{j>m} \left(\frac{\partial^2 F_m}{\partial x_j^2} - \frac{\partial^2 F_j}{\partial x_j \partial x_m}\right); \tag{24}$$

and the second is

$$\sum_{i<m} [\underset{\sim}{A}_{im}] \underset{\sim}{\nabla}\left(\frac{\partial F_i}{\partial x_m} - \frac{\partial F_m}{\partial x_i}\right), \tag{25}$$

which is

$$\sum_{i<m} \left(\frac{\partial^2 F_m}{\partial x_i^2} - \frac{\partial^2 F_i}{\partial x_i \partial x_m}\right). \tag{26}$$

But the sum of (26), (24), and (21) is (20), and since m was arbitrary, this completes the proof. ¤

We say that $\underset{\sim}{g} \in C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$ is a <u>gradient map</u> if $\underset{\sim}{g} = \underset{\sim}{\nabla}\phi$ for some $\phi \in C^\infty(\mathbb{R}^n)$, and ϕ is called the <u>scalar potential</u> for $\underset{\sim}{g}$. It is a standard fact from vector calculus that $\underset{\sim}{g}$ is a gradient map $\Leftrightarrow$

$$\frac{\partial g_i}{\partial x_j}(\underset{\sim}{x}) = \frac{\partial g_j}{\partial x_i}(\underset{\sim}{x}), \quad 1 \le i,j \le n, \quad \forall \underset{\sim}{x} \in \mathbb{R}^n. \tag{27}$$

The central mathematical result of this paper is the following theorem, which allows us to write an <u>arbitrary</u> function $\underset{\sim}{f} \in C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$ as the sum of a <u>finite</u> number of terms involving only <u>linear</u> maps and <u>gradient</u> maps.

<u>Theorem 1</u>. If $\underset{\sim}{f} \in C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$, then there exists a (nonunique) function $\underset{\sim}{F} \in C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$ such that $\underset{\sim}{\Delta}\underset{\sim}{F} = \underset{\sim}{f}$. And by decomposing $\underset{\sim}{\Delta}\underset{\sim}{F}$ as in (18), we can write $\underset{\sim}{f}$ in the form

$$\boxed{\underset{\sim}{f} = \underset{\sim}{\Delta}\underset{\sim}{F} = \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{F}) + \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} [\underset{\sim}{A}_{ij}] \underset{\sim}{\nabla}\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right)}, \tag{28}$$

where the matrices $[\underset{\sim}{A}_{ij}]$ are defined in (19).

In other words, $\underset{\sim}{f}$ can be written as

$$\underset{\sim}{f} = \underset{\sim}{g}_o + \sum_{\ell=1}^{\binom{n}{2}} [\underset{\sim}{A}_\ell] \underset{\sim}{g}_\ell, \tag{29}$$

where the $\underset{\sim}{g}_\ell \in C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$, $0 \leq \ell \leq \binom{n}{2}$, are gradient maps and the $[\underset{\sim}{A}_\ell]$, $1 \leq \ell \leq n$ are nxn antisymmetric matrices.

Proof. The equation $\Delta \underset{\sim}{F} = \underset{\sim}{f}$ has a solution $\underset{\sim}{F}$ in $C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$ if $\underset{\sim}{f} \in C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$ [7,pp.80,82] [8,pp.3,128,287,355]. (In fact, since two solutions must differ by a function with harmonic components, i.e. $\underset{\sim}{\Delta}(\underset{\sim}{F}'-\underset{\sim}{F}'') = \underset{\sim}{0}$, and since all harmonic functions are C^∞, all solutions will be C^∞ if one solution is.) Then the expansion of $\underset{\sim}{\Delta}\underset{\sim}{F}$ in (18) gives the decomposition of $\underset{\sim}{f}$ in (28). In order to match up terms between (28) and (29), it is important to note that there are exactly $(n^2-n)/2$ or $\binom{n}{2}$ ordered pairs of integers (i,j) with $1 \leq i < j \leq n$. The scalar potential for each term $\underset{\sim}{g}_\ell$ in (29) is given explicitly in (28), i.e. the scalar potential for $\underset{\sim}{g}_o$ is $\underset{\sim}{\nabla}\cdot\underset{\sim}{F}$ and the scalar potential for $\underset{\sim}{g}_\ell$, where ℓ corresponds to an ordered pair (i,j), is $\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right)$. ◻

Equations (8) and (38) are special cases of Theorem 1.

3.2. Solenoidal Functions

Borrowing from the language of electromagnetic fields [4], we say that $\underset{\sim}{f} \in C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$ is solenoidal if $\underset{\sim}{\nabla}\cdot\underset{\sim}{f} = \underset{\sim}{0}$. In this case the decomposition of $\underset{\sim}{f}$ in (29) can be simplified in two ways: the term $\underset{\sim}{g}_o$ disappears entirely, and the scalar potentials for the gradient maps $\underset{\sim}{g}_\ell$, $1 \leq \ell \leq \binom{n}{2}$, can be calculated very simply in terms of line integrals of the component functions of $\underset{\sim}{f}$. The explicit formula for the decomposition of a solenoidal function, which we have adapted from a technique used in the proof of the Poincare' Lemma in differential topology [20,21,24], appears in Eq. (30).

Lemma 2. If $\underset{\sim}{f} \in C^\infty(\mathbb{R}^n \to \mathbb{R}^n)$ is solenoidal, then

$$\underset{\sim}{f} = \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} [\underset{\sim}{A}_{ij}]\underset{\sim}{\nabla}\left\{x_j\int_0^1 t^{n-2} f_i(t\underset{\sim}{x})dt - x_i\int_0^1 t^{n-2} f_j(t\underset{\sim}{x})dt\right\}, \tag{30}$$

where the matrices $[\underset{\sim}{A}_{ij}]$ are defined in Eq. (19).

Equation (30) resembles a method for reconstructing a scalar function by taking the line integral of its gradient along rays from the origin. A formal proof is in Appendix C. The importance of this special decomposition for solenoidal functions comes from the following simple lemma, which is a nonlinear generalization of the standard linear decomposition technique in Eq. (1). (The relation between (1) and the results in this part will be discussed further in section V.)

Lemma 3. If $\underset{\sim}{f} \in C^{\infty}(\mathbb{R}^n \to \mathbb{R}^n)$, then $\underset{\sim}{f}$ can be written (nonuniquely) as the sum of a gradient map and a solenoidal map as follows:

$$\underset{\sim}{f} = \underset{\sim}{\nabla}\phi + (\underset{\sim}{f} - \underset{\sim}{\nabla}\phi), \tag{31}$$

where ϕ is any solution of $\Delta\phi = \underset{\sim}{\nabla} \cdot \underset{\sim}{f}$.

Proof. We discuss in Appendix B the fact that a C^{∞} solution ϕ of $\Delta\phi = \underset{\sim}{\nabla} \cdot \underset{\sim}{f}$ always exists. And since $\underset{\sim}{\nabla} \cdot (\underset{\sim}{f} - \underset{\sim}{\nabla}\phi) = \underset{\sim}{\nabla} \cdot \underset{\sim}{f} - \Delta\phi = 0$, the second term is solenoidal as claimed. ¤

3.3. An Alternate Version of the Decomposition of Functions

The method of decomposition indicated in theorem 1 requires that we solve Poisson's equation n times, once with each of the component functions f_i on the right hand side. But lemmas 2 and 3 allow us to proceed by a different route which only requires us to solve Poisson's equation once, and therefore saves a great deal of labor when n is large.

Given $\underset{\sim}{f} \in C^{\infty}(\mathbb{R}^n \to \mathbb{R}^n)$, we first write $\underset{\sim}{f}$ as the sum of a gradient term and a solenoidal term as in (31). Then we decompose the solenoidal term as in (30). The resulting decomposition is

$$\begin{aligned} \underset{\sim}{f} &= \underset{\sim}{\nabla}\phi + (\underset{\sim}{f} - \underset{\sim}{\nabla}\phi) \\ &= \underset{\sim}{\nabla}\phi + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} [\underset{\sim}{A}_{ij}]\underset{\sim}{\nabla}\{x_j \int_0^1 t^{n-2}[f_i(t\underset{\sim}{x}) - (D_i\phi)(t\underset{\sim}{x})]dt - x_i \int_0^1 t^{n-2}[f_j(t\underset{\sim}{x}) - (D_j\phi)(t\underset{\sim}{x})]dt\} \end{aligned} \tag{32}$$

where $\Delta\phi = \nabla \cdot \underset{\sim}{f}$.

IV. THE DECOMPOSITION OF HYBRID N-PORT RESISTORS

4.1. Hybrid Representations and Reciprocal n-Ports

Definition 1. A hybrid representation of an n-port resistor is a representation of the form $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$, where

$$\underset{\sim}{x} = [i_1,\ldots,i_k,v_{k+1},\ldots,v_n]^T$$
$$\underset{\sim}{y} = [v_1,\ldots,v_k,i_{k+1},\ldots,i_n]^T, \tag{33}$$

for some value of k, $0 \le k \le n$.

We allow the values $k = 0$ and $k = n$, to indicate that the voltage-controlled and current-controlled representations are included as special cases.

Definition 2. A hybrid n-port is simply an n-port resistor characterized by a hybrid representation.

The importance of hybrid representations is that some useful n-ports have a hybrid representation but no voltage- or current-controlled representation, e.g. the ideal transformer.

Remark 1. For the geometrically inclined reader, the issue here is the following: if we label the axes of $\mathbb{R}^{2n}$ as $\{v_1,\ldots,v_n,i_1,\ldots,i_n\}$, then a resistive n-port $\mathcal{R}$ is uniquely identified as a set $C \subset \mathbb{R}^{2n}$, where C is the set of all points $\underset{\sim}{p} = [v_1,\ldots,v_n,i_1,\ldots,i_n]^T$ such that the voltages and currents represented by the components of the vector $\underset{\sim}{p}$ can simultaneously exist at the ports of $\mathcal{R}$. The set C is called the constitutive relation of $\mathcal{R}$. For any physically meaningful n-port (the technical term is "regular" [16]), C will be an n-dimensional manifold M, embedded in $\mathbb{R}^{2n}$, and in most practical cases M will be connected and globally diffeomorphic to $\mathbb{R}^n$.

At this point the attitude of circuit theorists diverges radically from that of differential topologists. In the first place, a circuit theorist considers the coordinates $\{v_1,\ldots,v_n,i_1,\ldots,i_n\}$ on $\mathbb{R}^{2n}$ to be fixed, permanent, and immutable because they represent the physically measurable variables at the ports of $\mathcal{R}$. And secondly, a circuit theorist would not consider M to be in any sense equivalent to every diffeomorphic image of M. Thus if $\mathcal{R}_1$ and $\mathcal{R}_2$ are two n-port resistors characterized by different embeddings of the same abstract n-dimensional manifold M in $\mathbb{R}^{2n}$, then $\mathcal{R}_1$ and $\mathcal{R}_2$ are completely distinct from a circuit point of view. For example, a 1 ohm resistor is characterized by $M_1 = \{(v,i) \in \mathbb{R}^2 | v = i\}$ and a 1 volt source is characterized by $M_2 = \{(v,i) \in \mathbb{R}^2 | v = 1\}$. Now M_1 and M_2 are merely two different embeddings of $\mathbb{R}^1$ in $\mathbb{R}^2$, and therefore topologically equivalent, but of course a 1 ohm resistor and a 1 volt source are completely distinct circuit elements.

In fact, the only notion that circuit theory wants to borrow from differential topology at this point is the idea that it is possible to consider various choices of coordinates on M. And, as we shall see, the class of coordinate systems known as hybrid representations is extremely limited from a topologist's point of view.

Our final requirement for a hybrid n-port is that, after possibly renumbering the ports, M must be the graph of a function $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$, where $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are defined in (33). In other words, there is a vector $\underset{\sim}{x}$ of k currents and n-k voltages, all from different ports, such that $\underset{\sim}{x}$ is a set of coordinates for M under the global parametrization: $\underset{\sim}{x} \mapsto [f_1(\underset{\sim}{x}),\ldots,f_k(\underset{\sim}{x}),x_{k+1},\ldots,x_n,x_1,\ldots,x_k,f_{k+1}(\underset{\sim}{x}),\ldots,f_n(\underset{\sim}{x})]^T = [v_1,\ldots,v_n;i_1,\ldots,i_n]^T \in \mathbb{R}^{2n}$. In this case $\underset{\sim}{f}$ is called a hybrid representation of the n-port. If n=1, this amounts to assuming that M is the graph of a function $v = f(i)$ or else the graph of a function $i = f(v)$. Although this is a very special requirement from a geometric point of view, it is sufficiently general for most circuit applications. An extended discussion of these points can be found in [14,15,16].

Definition 3. A conjugate hybrid representation of an n-port is a representation of the form $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$, where

$$\begin{aligned} x &= [-i_1,-i_2,\ldots,-i_k,v_{k+1},\ldots,v_n]^T \\ y &= [v_1,v_2,\ldots,v_k,i_{k+1},\ldots,i_n]^T. \end{aligned} \tag{34}$$

Comparing (34) with (33), we see that the only difference between a hybrid representation and a conjugate hybrid representation is in the first k entries of $\underset{\sim}{x}$. The reason for introducing this awkward distinction[4] is that for hybrid representations there is generally no relation between $\mathcal{R}$ being reciprocal (see Remark 2 below) and $\underset{\sim}{f}$ being a gradient map, except in the special voltage-controlled and current-controlled cases, k = 0 and k = n. But if $\mathcal{R}$ is a C^1 hybrid n-port resistor and $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$ is a conjugate hybrid representation of $\mathcal{R}$, then $\mathcal{R}$ is reciprocal if and only if $\underset{\sim}{f}$ is a gradient map [16]. In this case the scalar potential for $\underset{\sim}{f}$ is called the hybrid content.

Remark 2. The notion of reciprocity is familiar to all circuit theorists, so there is no need to define it here in engineering terms. But in geometric terms, reciprocity is a local property of the manifold M and reflects the way the

[4] One could avoid this notation by defining the reference current direction to be out of the positively referenced terminal of each port, opposite the standard convention. Then reciprocity would be equivalent to $\underset{\sim}{f}$ being a gradient map.

manifold is embedded in $\mathbb{R}^{2n}$. Specifically, if the constitutive relation of an n-port is an n-dimensional C^1 manifold M embedded in $\mathbb{R}^{2n}$, and if the axes of $\mathbb{R}^{2n}$ are labelled as in remark #1, then the n-port is said to be reciprocal if the 2-form $\sum_{k=1}^{n} dv_k \wedge di_k$ vanishes on M [15,17,18]. It is not hard to verify that whenever the conjugate hybrid representation $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$ is a global parametrization of M, this definition is equivalent to $\underset{\sim}{f}$ being a gradient map.

If a given hybrid n-port can be characterized by several different conjugate hybrid representations, then every representation $\underset{\sim}{f}$ is a gradient map or else none of them is. Thus reciprocity is truly a property of the n-port and not of the particular choice of coordinates used to represent it. The importance of reciprocity in network synthesis lies in the fact that, (except for non-regular, i.e. pathological, cases) every 2-terminal element is reciprocal and every n-port which can be synthesized from reciprocal elements is also reciprocal [17].

4.2. n-Port Decomposition

We now want to show how the results of section III allow us to synthesize any C^∞ hybrid n-port from $\binom{n}{2}+1$ reciprocal hybrid n-ports and $\binom{n}{2}$ linear 2n-ports.

Given a C^∞ hybrid n-port, we first write its constitutive relation in the conjugate hybrid form $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$, where $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are defined in (34). We then solve $\underset{\sim}{\Delta}\underset{\sim}{F} = \underset{\sim}{f}$ for the vector function $\underset{\sim}{F}(\underset{\sim}{x})$, which allows us to decompose $\underset{\sim}{f}$ as we did in Eq. (28), i.e.

$$\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{\Delta}\underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{F}) + \sum_{\substack{i=1\\ i<j}}^{n}\sum_{j=1}^{n}[\underset{\sim}{A}_{ij}]\underset{\sim}{\nabla}\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right). \tag{35}$$

The first reciprocal n-port is characterized by $\underset{\sim}{y} = \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{F})(\underset{\sim}{x})$, and the other $\binom{n}{2}$ reciprocal n-ports are characterized by $\underset{\sim}{y} = \underset{\sim}{\nabla}\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right)(\underset{\sim}{x})$, $1 \le i < j \le n$. Each linear 2n-port, represented by a matrix $[\underset{\sim}{A}_{ij}]$, "passes forward" the independent variables $\underset{\sim}{x}$ unaltered and "passes back" the dependent variables $\underset{\sim}{y}$ after operating on $\underset{\sim}{y}$ with the matrix $[\underset{\sim}{A}_{ij}]$.

4.3. Decomposition of a General 3-Port Hybrid Resistor

Example 1 illustrated the technique for the 2-port voltage-controlled case $k = 0$, $n = 2$. The following example illustrates the hybrid case $k = 1$, $n = 3$.

Example 2. Suppose we wish to decompose a given C^{∞} 3-port resistor $\mathcal{R}$, with the hybrid representation $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$ where $\underset{\sim}{x} = [i_1, v_2, v_3]^T$ and $\underset{\sim}{y} = [v_1, i_2, i_3]^T$. We must first change the constitutive relation to the conjugate hybrid form $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$, where $\underset{\sim}{x} = [-i_1, v_2, v_3]^T$ and $\underset{\sim}{y}$ is unaltered. Next we solve Poisson's equation $\underset{\sim}{\Delta}\underset{\sim}{F} = \underset{\sim}{f}$ for the vector function $\underset{\sim}{F}(\underset{\sim}{x})$, which we can decompose according to (35).

In this case (35) takes on the special form of (5), i.e.

$$\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{\Delta}\underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{F})(\underset{\sim}{x}) - \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla}\times\underset{\sim}{F})(\underset{\sim}{x}), \tag{36}$$

since $n = 3$. Although we could simply use (35) to decompose $\underset{\sim}{F}$ and arrive at (38), we choose instead to expand and rearrange the last term on the right hand side of (36) in order to display the algebraic manipulations that led to the general formula in Theorem 1.

$$-\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla}\times\underset{\sim}{F})(\underset{\sim}{x})$$

$$= \underset{\sim}{\nabla} \times \begin{bmatrix} \frac{\partial F_2}{\partial x_3} - \frac{\partial F_3}{\partial x_2} \\ \frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial x_2 \partial x_1} - \frac{\partial^2 F_3}{\partial x_1 \partial x_3} + \frac{\partial^2 F_1}{\partial x_3^2} \\ \frac{\partial^2 F_2}{\partial x_3^2} - \frac{\partial^2 F_3}{\partial x_2 \partial x_3} - \frac{\partial^2 F_1}{\partial x_1 \partial x_2} + \frac{\partial^2 F_2}{\partial x_1^2} \\ \frac{\partial^2 F_3}{\partial x_1^2} - \frac{\partial^2 F_1}{\partial x_1 \partial x_3} - \frac{\partial^2 F_2}{\partial x_3 \partial x_2} + \frac{\partial^2 F_3}{\partial x_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial x_2 \partial x_1} \\ -\frac{\partial^2 F_1}{\partial x_1 \partial x_2} + \frac{\partial^2 F_2}{\partial x_1^2} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_3^2} - \frac{\partial^2 F_3}{\partial x_3 \partial x_1} \\ 0 \\ -\frac{\partial^2 F_1}{\partial x_1 \partial x_3} + \frac{\partial^2 F_3}{\partial x_1^2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial^2 F_2}{\partial x_3^2} - \frac{\partial^2 F_3}{\partial x_3 \partial x_2} \\ -\frac{\partial^2 F_2}{\partial x_2 \partial x_3} + \frac{\partial^2 F_3}{\partial x_2^2} \end{bmatrix} \tag{37}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_1 \partial x_2} - \frac{\partial^2 F_2}{\partial x_1^2} \\ \frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F_1}{\partial x_3 \partial x_2} - \frac{\partial^2 F_2}{\partial x_3 \partial x_1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_1 \partial x_3} - \frac{\partial^2 F_3}{\partial x_1^2} \\ \frac{\partial^2 F_1}{\partial x_2 \partial x_3} - \frac{\partial^2 F_3}{\partial x_2 \partial x_1} \\ \frac{\partial^2 F_1}{\partial x_3^2} - \frac{\partial^2 F_3}{\partial x_3 \partial x_1} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 F_2}{\partial x_1 \partial x_3} - \frac{\partial^2 F_3}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F_2}{\partial x_2 \partial x_3} - \frac{\partial^2 F_3}{\partial x_2^2} \\ \frac{\partial^2 F_2}{\partial x_3^2} - \frac{\partial^2 F_3}{\partial x_3 \partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{\nabla} \left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1} \right) + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \underset{\sim}{\nabla} \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \right)$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \underset{\sim}{\nabla} \left(\frac{\partial F_2}{\partial x_3} - \frac{\partial F_3}{\partial x_2} \right)$$

$$= \sum_{i=1}^{3} \sum_{\substack{j=1 \\ i<j}}^{3} [\underset{\sim}{A}_{ij}] \underset{\sim}{\nabla} \left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} \right) .$$

Adding this to the first term on the right hand side of (36) yields the final decomposition

$$y = \Delta F(x) = \underbrace{\nabla(\nabla\cdot F)(x)}_{R_1} + \underbrace{[A_{12}]}_{\mathcal{L}_2}\underbrace{\nabla\left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1}\right)(x)}_{R_2} \tag{38}$$

$$+ \underbrace{[A_{13}]}_{\mathcal{L}_3}\underbrace{\nabla\left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right)(x)}_{R_3} + \underbrace{[A_{23}]}_{\mathcal{L}_4}\underbrace{\nabla\left(\frac{\partial F_2}{\partial x_3} - \frac{\partial F_3}{\partial x_2}\right)(x)}_{R_4},$$

in agreement with (35).

This decomposition allows us to synthesize $\mathcal{R}$ from reciprocal 3-ports and linear 6-ports as shown in Fig. 6. The network is drawn in such a way that $i_1 = i_1' = j_1'' = j_1''' = j_1''''$, $v_2 = v_2' = e_2'' = e_2''' = e_2''''$, and $v_3 = v_3' = e_3'' = e_3''' = e_3''''$. The linear 6-ports are intended to "pass forward" the independent variables $(-i_1, v_2, v_3)$ without alteration, so they are partially characterized by the equations $j_4'' = -j_1''$, $e_5'' = e_2''$, $e_6'' = e_3''$ for $\mathcal{L}_2$; $j_4''' = -j_1'''$, $e_5''' = e_2'''$, $e_6''' = e_3'''$ for $\mathcal{L}_3$; and $j_4'''' = -j_1''''$, $e_5'''' = e_2''''$, $e_6'''' = e_3''''$ for $\mathcal{L}_4$. The outcome is simple: $i_1 = i_1' = i_1'' = i_1''' = i_1''''$, $v_2 = v_2' = v_2'' = v_2''' = v_2''''$, and $v_3 = v_3' = v_3'' = v_3''' = v_3''''$.

The resistive 3-port R_1 is characterized by the hybrid content function

$$\mathcal{H}_1(-i_1', v_2', v_3') = \frac{\partial F_1}{\partial(-i_1')} + \frac{\partial F_2}{\partial v_2'} + \frac{\partial F_3}{\partial v_3'}, \text{ i.e.}$$

$$\begin{aligned} v_1'(-i_1', v_2', v_3') &= \frac{\partial \mathcal{H}_1}{\partial(-i_1')} \\ i_2'(-i_1', v_2', v_3') &= \frac{\partial \mathcal{H}_1}{\partial v_2'} \\ i_3'(-i_1', v_2', v_3') &= \frac{\partial \mathcal{H}_1}{\partial v_3'} \end{aligned} \tag{39}$$

as required by (38). Similarly R_2, R_3 and R_4 are characterized by the hybrid content functions $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$, respectively, where

$$\mathcal{H}_2(-i_1'',v_2'',v_3'') = \frac{\partial F_1}{\partial v_2''} - \frac{\partial F_2}{\partial(-i_1'')}$$

$$\mathcal{H}_3(-i_1''',v_2''',v_3''') = \frac{\partial F_1}{\partial v_3'''} - \frac{\partial F_3}{\partial(-i_1''')} \tag{40}$$

$$\mathcal{H}_4(-i_1'''',v_2'''',v_3'''') = \frac{\partial F_2}{\partial v_3''''} - \frac{\partial F_3}{\partial v_2''''} .$$

We can complete the characterization of $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$ by noting that $\mathcal{L}_2$ must "pass back" the dependent variables (v_1'',i_2'',i_3'') "scrambled" as indicated by the matrix $[\underset{\sim}{A}_{12}]$, i.e. $\mathcal{L}_2$ must produce the transformation $i_2'' \to e_1''$, $v_1'' \to -j_2''$, $0 \to j_3''$. So the remaining equations needed to characterize $\mathcal{L}_2$ are $e_1'' = -j_5''$, $j_2'' = -e_4''$, $j_3'' = 0$. Similarly, $\mathcal{L}_3$ must satisfy $e_1''' = -j_6'''$, $j_2''' = 0$, $j_3''' = -e_4'''$; and $\mathcal{L}_4$ must satisfy $e_1'''' = 0$, $j_2'''' = -j_6''''$, $j_3'''' = j_5''''$.

In the case of arbitrarily large n and k, $0 \leq k \leq n$, it is easy to produce a network corresponding to (35) by exactly the same method used here. The details are lengthy and uninformative, so we have omitted them.

4.4. x-Solenoidal n-Ports

Definition 4. An n-port resistor characterized by the conjugate hybrid representation $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$ is said to be $\underset{\sim}{x}$-solenoidal if $\underset{\sim}{f}(\underset{\sim}{x})$ has zero divergence, i.e. $\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \equiv 0$.

We have adopted the term "$\underset{\sim}{x}$-solenoidal" rather than simply "solenoidal" because one hybrid representation of an n-port may have zero divergence while another does not. (Consider for example the current-controlled and voltage-controlled representations of the uncoupled linear 3-port consisting of three resistors with values of 1Ω, 1Ω, and -2Ω. Only the current-controlled representation is solenoidal.)

If $\mathcal{R}$, with the conjugate hybrid representation $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$, is $\underset{\sim}{x}$-solenoidal, then the decomposition in section 4.2 can be greatly simplified by using the special result given in (30). In this case we do not have to solve Poisson's equation at all, and the resulting synthesis requires $\binom{n}{2}$ reciprocal hybrid n-ports rather than $\binom{n}{2} + 1$. For reference, we summarize this result in lemma 4.

Lemma 4.

If $\mathcal{R}$ is a C^∞ $\underset{\sim}{x}$-solenoidal n-port, then $\mathcal{R}$ can be realized by an interconnection of $\binom{n}{2}$ reciprocal n-ports and $\binom{n}{2}$ linear 2n-ports.

4.5. An Alternate Version of the Decomposition of n-Ports.

We can use the conclusion in section 3.3 to propose a variation on the method of n-port decomposition given in section 4.2. Using equation (32) as the basis for our technique yields essentially the same result as (28), but requires that we solve Poisson's equation only once. In addition, it suggests the following reformulation of our decomposition result in language similar to that of the classical linear decomposition theorem.

Lemma 5.

Every n-port characterized by a C^∞ conjugate hybrid representation $\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x})$ can be realized by the interconnection of a reciprocal n-port and an $\underset{\sim}{x}$-solenoidal n-port.

Example 3.

To illustrate this alternate procedure based on (32), we will again consider the d.c. Ebers-Moll equations for a pnp transistor,

$$\begin{aligned} i_1 &= A_1(e^{Kv_1}-1) - B_1(e^{Kv_2}-1) = i_1(v_1,v_2) \\ i_2 &= -A_2(e^{Kv_1}-1) + B_2(e^{Kv_2}-1) = i_2(v_1,v_2). \end{aligned} \tag{41}$$

With $\underset{\sim}{x} = \underset{\sim}{v}$, $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{i}(\underset{\sim}{v})$, and $n = 2$, (32) becomes

$$\begin{bmatrix} i_1(v_1,v_2) \\ i_2(v_1,v_2) \end{bmatrix} = \begin{bmatrix} \frac{\partial\phi}{\partial v_1} \\ \frac{\partial\phi}{\partial v_2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underset{\sim}{\nabla}\left\{ v_2\int_0^1 \left[i_1(t\underset{\sim}{v})-(D_1\phi)(t\underset{\sim}{v})\right]dt - v_1\int_0^1 \left[i_2(t\underset{\sim}{v})-(D_2\phi)(t\underset{\sim}{v})\right]dt\right\}, \tag{42}$$

with

$$\frac{\partial^2\phi}{\partial v_1^2} + \frac{\partial^2\phi}{\partial v_2^2} = \frac{\partial i_1}{\partial v_1} + \frac{\partial i_2}{\partial v_2}. \tag{43}$$

This will produce the type of synthesis shown in Fig. 1, but with $\hat{G}_1(v_1,v_2) = \phi(v_1,v_2)$ and $\hat{G}_2(v_1,v_2)$ equal to the expression in brackets in (42).

Substituting (41) into (43), we have

$$\frac{\partial^2\phi}{\partial v_1^2} + \frac{\partial^2\phi}{\partial v_2^2} = A_1Ke^{Kv_1} + B_2Ke^{Kv_2}, \tag{44}$$

and the algebraically simplest solution is

$$\phi(v_1,v_2) = (A_1/K)e^{Kv_1} + (B_2/K)e^{Kv_2}. \tag{45}$$

The expression in brackets in (42) becomes, upon substitution of (41) and (45),

$$v_2\int_0^1\left[-A_1-B_1(e^{Kv_2t}-1)\right]dt - v_1\int_0^1\left[-B_2-A_2(e^{Kv_1t}-1)\right]dt. \tag{46}$$

If we carry out the integrations in (46), take the gradient as indicated in (42), and substitute both that result and (45) into (42), we arrive at the final decomposition

$$\begin{bmatrix} i_1(v_1,v_2) \\ i_2(v_1,v_2) \end{bmatrix} = \begin{bmatrix} A_1e^{Kv_1} \\ B_2e^{Kv_2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} A_2(e^{Kv_1}-1) + B_2 \\ -B_1(e^{Kv_2}-1) - A_1 \end{bmatrix} . \tag{47}$$

In the language of section 2.2 and Fig. 1, the constitutive relation of R_1 is $(i_1',i_2') = (A_1e^{Kv_1'},B_2e^{Kv_2'})$ and the constitutive relation of R_2 is $(i_1'',i_2'') = \left(A_2(e^{Kv_1''}-1) + B_2, - B_1(e^{Kv_2''}-1) - A_1\right)$. As in (16) and (17), both R_1 and R_2 have turned out to be uncoupled 2-ports. But the two decompositions are very different. This time, for example, each port of R_1 and R_2 is active.

4.6. The Lack of Uniqueness

The decompositions in examples 1, 2, and 3 are very far from unique. This is potentially a great strength of the method, for it will almost certainly turn out that some network realizations will be superior to others. From the theoretical point of view, this flexibility could provide a method for generating families of equivalent nonlinear networks. This potential application is especially intriguing, because virtually nothing is known about the problem so far. And from the point of view of hardware realization, one circuit might be much easier to build than another.

Let us examine the three sources of this nonuniqueness in example 2. The first is that there will usually be many hybrid representations of $\mathcal{R}$ other than the one in which i_1, v_2, and v_3 are the independent variables. A decomposition based on one of these will yield different descriptions of R_1-R_4 and $\mathcal{L}_2$-$\mathcal{L}_4$. The second is that the equation $\underset{\sim}{f} = \Delta\underset{\sim}{F}$ has many solutions; specifically, if $\underset{\sim}{F}(-i_1,v_2,v_3)$ is a solution and $\underset{\sim}{H}(-i_1,v_2,v_3) : \mathbb{R}^3 \to \mathbb{R}^3$ has components which are harmonic functions, then $\underset{\sim}{F}+\underset{\sim}{H}$ is also a solution and yields a different specification for R_1- R_4, in general. Since the space of all harmonic functions on $\mathbb{R}^n$ is infinite dimensional for $n \geq 2$, this allows us a great deal of freedom and seems intuitively to account for most of the "slack".

The third source of freedom also affects only R_2, R_3, and R_4. Each of these resistors produces three dependent port variables, but only two of them are used in the circuit, i.e. "passed back" by the corresponding linear 6-port to contribute to the output variables (v_1,i_2,i_3). For example, $\mathcal{L}_2$ passes back v_1'' and i_2'' but "ignores" i_3''. The reason of course is that $\mathcal{L}_2$ was created to realize the transformation indicated by the matrix $[\underset{\sim}{A}_{12}]$, which has only two nonzero entries. In a similar way, $\mathcal{L}_3$ ignores i_2''' and $\mathcal{L}_4$ ignores v_1'''', so these three variables do not affect the output of $\mathcal{R}$.

We will illustrate the freedom these "ignored" port variables offer us by considering R_2. Since v_1'' and i_2'' appear in the output of $\mathcal{R}$, we will require that

$$v_1'' = \frac{\partial \mathcal{H}_2}{\partial(-i_1'')}\,(-i_1'',v_2'',v_3''), \quad i_2'' = \frac{\partial \mathcal{H}_2}{\partial v_2''}\,(-i_1'',v_2'',v_3''). \tag{48}$$

The decomposition succeeds no matter what the function $i_3''(-i_1'',v_2'',v_3'')$ may be. The only additional constraint is that if we wish R_2 to be reciprocal, we must also require that $\frac{\partial i_3''}{\partial(-i_1'')} \equiv \frac{\partial v_1''}{\partial v_3''}$ and $\frac{\partial i_3''}{\partial v_2''} \equiv \frac{\partial i_2''}{\partial v_3''}$. Observe that this still leaves the dependence of i_3'' on v_3'' to the convenience of the designer. This type of freedom is greatly expanded when $n > 3$, but disappears entirely when $n=2$.

V. RELATION TO THE STANDARD DECOMPOSITION OF A LINEAR n-PORT INTO ITS RECIPROCAL AND ANTIRECIRPOCAL PARTS

As we pointed out in the introduction, it is well-known that a linear resistive n-port R, characterized by $\underset{\sim}{y} = \underset{\sim}{M}\underset{\sim}{x}$, can be broken down into an interconnection of two n-ports R_1 and R_2, where R_1 satisfies $\underset{\sim}{y} = \frac{1}{2}\,(\underset{\sim}{M}+\underset{\sim}{M}^T)\underset{\sim}{x}$

and R_2 satisfies $\underset{\sim}{y} = \frac{1}{2}(\underset{\sim}{M}-\underset{\sim}{M}^T)\underset{\sim}{x}$. If $\underset{\sim}{x} = \underset{\sim}{i}$ and $\underset{\sim}{y} = \underset{\sim}{v}$, then the sum

$$\underset{\sim}{y} = \frac{1}{2}(\underset{\sim}{M}+\underset{\sim}{M}^T)\underset{\sim}{x} + \frac{1}{2}(\underset{\sim}{M}-\underset{\sim}{M}^T)\underset{\sim}{x} \tag{49}$$

is effected by connecting R_1 and R_2 in series; if $\underset{\sim}{x} = \underset{\sim}{v}$ and $\underset{\sim}{y} = \underset{\sim}{i}$, then they are connected in parallel. In either of these two cases, R_1 is a reciprocal device and R_2 is an antireciprocal one. (Note, however, that the equation of symmetric matrices with reciprocal elements and antisymmetric matrices with antireciprocal elements works only in the current- or voltage-controlled case. If fails when $\underset{\sim}{x}$ or $\underset{\sim}{y}$ represents a mixture of voltages and currents [17].)

The research reported in this article was motivated by a desire to find a nonlinear generalization of this classical technique. We can see that (49) is a special case of our result, since the first term on the right hand side is a gradient map and the second term has zero divergence. So our method does in fact generalize (49) although it doesn't reduce to (49) in the linear case due to the method's inherent nonuniqueness. The following example, which should make this last point clear, shows how to take advantage of the nonuniqueness to generate the classical decomposition (49) as a special case of our result.

Example 4. If we use our method to decompose the linear 2-dimensional map $\underset{\sim}{y} = \underset{\sim}{M}\underset{\sim}{x}$, i.e.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} , \tag{50}$$

we first solve the equations $\Delta F_1 = f_1$ and $\Delta F_2 = f_2$ by inspection and obtain the algebraically simplest solutions

$$\begin{aligned} F_1(x_1,x_2) &= \frac{m_{11}x_1^3}{6} + \frac{m_{12}x_2^3}{6} \\ F_2(x_1,x_2) &= \frac{m_{21}x_1^3}{6} + \frac{m_{22}x_2^3}{6} . \end{aligned} \tag{51}$$

Then decomposing (51) as in (7) and (8), we have

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_1^2} & + \frac{\partial^2 F_2}{\partial x_1 \partial x_2} \\ \frac{\partial F_1}{\partial x_1 \partial x_2} + & \frac{\partial^2 F_2}{\partial x_2^2} \end{bmatrix} + \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_2^2} & - \frac{\partial^2 F_2}{\partial x_1 \partial x_2} \\ - \frac{\partial^2 F_1}{\partial x_1 \partial x_2} + & \frac{\partial^2 F_2}{\partial x_1^2} \end{bmatrix} \tag{52}$$

$$= \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & m_{12} \\ m_{21} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ,$$

which is quite different in general from (49), although both decompositions share the feature that they yield the sum of a gradient term and a divergence-free term.

However, the solutions to Poisson's equation given in (51) are not unique, and another possibility is

$$\hat{F}_1(x_1,x_2) = \frac{m_{11}x_1^3}{6} + \frac{m_{12}x_2^3}{6} + \frac{(m_{12}+m_{21})}{4} x_1^2 x_2 - \frac{(m_{12}+m_{21})}{12} x_2^3 \tag{53}$$

$$\hat{F}_2(x_1,x_2) = \frac{m_{21}x_1^3}{6} + \frac{m_{22}x_2^3}{6} ,$$

which differs from (51) by the addition of a harmonic polynomial to the first term. If we repeat the decomposition given in (52), but this time using $\hat{F}_1$ and $\hat{F}_2$ in place of F_1 and F_2, we obtain

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Delta\hat{F}_1 \\ \Delta\hat{F}_2 \end{bmatrix} = \begin{bmatrix} m_{11} & \frac{m_{12}+m_{21}}{2} \\ \frac{m_{12}+m_{21}}{2} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{m_{12}-m_{21}}{2} \\ \frac{m_{21}-m_{12}}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \tag{54}$$

which agrees with (49).

So our method gives a variety of decompositions in the linear case — in fact, we could easily have written (50) as the sum of two nonlinear terms by adding the appropriate harmonic functions to F_1 and F_2. And, as we saw, a special choice of harmonic functions allowed us to reproduce the classical decomposition of (50) shown in (49), at least for n=2. The following lemma shows that the classical technique is a special case of our method for all n>2 as well.

Lemma 6. For any real nxn matrix $\underset{\sim}{M}$ there exists a solution $\hat{\underset{\sim}{F}}$ of the equation $\underset{\sim}{\Delta}\hat{\underset{\sim}{F}}(\underset{\sim}{x}) = \underset{\sim}{M}\underset{\sim}{x}$ such that in the decomposition

$$\underset{\sim}{M}\underset{\sim}{x} = \underset{\sim}{\Delta}\hat{\underset{\sim}{F}}(\underset{\sim}{x}) = \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\hat{\underset{\sim}{F}}) + \sum_{\substack{i=1\\i<j}}^{n}\sum_{j=1}^{n}[\underset{\sim}{A}_{ij}]\,\underset{\sim}{\nabla}\left(\frac{\partial\hat{F}_i}{\partial x_j} - \frac{\partial\hat{F}_j}{\partial x_i}\right) \tag{55}$$

the following equations hold:

$$\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\hat{\underset{\sim}{F}})(\underset{\sim}{x}) = \frac{1}{2}(\underset{\sim}{M}+\underset{\sim}{M}^T)\underset{\sim}{x} \tag{56}$$

and

$$\sum_{\substack{i=1\\i<j}}^{n}\sum_{j=1}^{n}[\underset{\sim}{A}_{ij}]\,\underset{\sim}{\nabla}\left(\frac{\partial\hat{F}_i}{\partial x_j} - \frac{\partial\hat{F}_j}{\partial x_i}\right) = \frac{1}{2}(\underset{\sim}{M}-\underset{\sim}{M}^T)\underset{\sim}{x} \ . \tag{57}$$

The proof, which is constructive and straightforward, is in Appendix C.

VI. CONCLUDING REMARKS

Our decomposition and that of Chua and Lam [3] have one thing in common. They both break up and rearrange functions mapping $\mathbb{R}^n$ to $\mathbb{R}^n$ by means of particular linear differential or integro-differential operators with constant coefficients. There are surely other operators of this type which could be useful in n-port decomposition, quite likely more useful than the ones we have investigated. So far we have been unable to isolate just what algebraic properties make an operator suitable for this purpose and therefore have been unable to undertake a systematic search. This seems to us to be a significant mathematical problem, and someone well acquainted with the algebraic properties of differential operators might be able to produce exciting results.

Second, as we mentioned in the introduction, our result highlights the importance of reciprocal resistive synthesis techniques. We do not yet know how to take advantage of the considerable latitude provided by the nonuniqueness in our method. Perhaps this sort of understanding will develop along with further progress in reciprocal synthesis.

In a more speculative vein, it seems that our result might have some applications in the qualitative theory of ordinary differential equations. Consider, for example, the autonomous planar system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \underset{\sim}{f}(\underset{\sim}{x}). \tag{58}$$

We decompose $\underset{\sim}{f}$ as in (8) and obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underset{\sim}{\nabla}\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right) + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underset{\sim}{\nabla}\left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1}\right), \tag{59}$$

which breaks $\underset{\sim}{f}$ into the sum of a gradient vector field and a Hamiltonian vector field. Keeping in mind that the matrix has no effect on the magnitude of the second term and that there are many such decompositions, suppose we could find one such that

$$\left\| \underset{\sim}{\nabla}\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right)(\underset{\sim}{x}) \right\| > \left\| \underset{\sim}{\nabla}\left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1}\right)(\underset{\sim}{x}) \right\| \tag{60}$$

everywhere except at points where both terms are zero. Then it would follow that the scalar function $\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right)$ is increasing along trajectories, a result that rules out the possibility of closed orbits and provides considerable insight into the qualitative behavior of solutions of the differential equation.

APPENDIX

APPENDIX A: THE HODGE THEOREM

The purpose of this appendix is to explain the origin of the ideas in this article and their relation to certain topics in differential geometry. Sections A.1 and A.2 are addressed to the reader with at best an undergraduate acquaintance with differential forms, while section A.3 assumes that the reader is quite familiar with the Hodge theorem.

In section A.1 we develop the equation $\Delta = d\delta+\delta d$ for 1-forms on $\mathbb{R}^n$ and explain what these symbols mean. Lots of examples and explicit calculations are included and the relation to vector calculus on $\mathbb{R}^3$ is emphasized. Then we show how an algebraic manipulation of the Euclidean coordinate expressions for this decomposition leads to (18) and to our main result, theorem 1. The reader who is familiar with these ideas may wish to skip immediately to theorems A-1 and A-2. They restate our theorem 1 in the language of 1-forms on $\mathbb{R}^n$ and are designed to bridge the mental gap between theorem 1 and the Hodge theorem.

Section A.2 is an informal introduction to the Hodge theorem itself. The simplest possible example, the decomposition of the space of 1-forms on the circle, is worked out in complete detail.

Section A.3. is addressed to the reader who is already familiar with the Hodge theorem. In it we discuss the rather peculiar relationship that exists between that theorem and our main result, especially those features of the Hodge theorem that disappear in our version.

A.1. The Operators $*$, d, δ, Δ and the Decomposition of 1-Forms on $\mathbb{R}^n$.

This section presupposes a very elementary acquaintance with differential forms and wedge products on $\mathbb{R}^n$. The authors have found [20-23] to be excellent introductions, and any one of them will provide much more background than is needed here.

The conclusions we will draw in this section about 1-forms on $\mathbb{R}^n$ have exact natural analogs concerning vector fields on $\mathbb{R}^n$ and maps of $\mathbb{R}^n$ into itself, once we agree to identify the 1-form $\sum_{i=1}^{n} f_i(\underset{\sim}{x})dx^i$, the vector field $\sum_{i=1}^{n} f_i(\underset{\sim}{x}) \frac{\partial}{\partial x_i}$, and the function $\underset{\sim}{f} : \mathbb{R}^n \to \mathbb{R}^n$ whose component functions in the usual coordinates are $\{f_1, f_2, \ldots, f_n\}$. We will take this identification for granted in the remainder of the appendix.

Let $E^p(\mathbb{R}^n)$ be the space of smooth (i.e. C^∞) p-forms on $\mathbb{R}^n$. The next four definitions introduce basic operators that transform k-forms to ℓ-forms. The first operator is algebraic rather than differential in character and produces an (n-p)-form from a p-form.

Definition A.1. The Hodge star operator, *, is that (unique) linear operator mapping $E^p(\mathbb{R}^n)$ into $E^{n-p}(\mathbb{R}^n)$ such that, if $\omega = dx^{i_1} \wedge \ldots \wedge dx^{i_p}$, then $\omega \wedge (*\omega) = dx^1 \wedge \ldots \wedge dx^n$.

For example, on $\mathbb{R}^2$,

$$* \phi(x,y) = \phi(x,y)*(1) = \phi(x,y)dx\wedge dy$$

$$*dx = dy, \quad *dy = -dx \qquad \text{(A-1)}$$

$$*(fdx+gdy) = fdy-gdx.$$

The negative signs in (A-1) arise because of the anticommutative property of the wedge product, $dx\wedge dy = -dy\wedge dx$.

Lemma A.1. If $\alpha \in E^p(\mathbb{R}^n)$, then $**\alpha = (-1)^{p(n-p)}\alpha$.

The proof follows easily from the anticommutativity of the wedge product on $E^1(\mathbb{R}^n)$.

The next operator, which is the most fundamental of all, produces a (p+1)-form from a p-form.

Definition A.2. The exterior derivative d is that (unique) linear differential operator mapping $E^p(\mathbb{R}^n)$ into $E^{p+1}(\mathbb{R}^n)$ such that

$$d\left(f(x_1,\ldots,x_n)dx^{i_1}\wedge\ldots\wedge dx^{i_p}\right) = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i\right) \wedge dx^{i_1}\wedge\ldots\wedge dx^{i_p}. \qquad \text{(A-2)}$$

The operator d, applied to $E^o(\mathbb{R}^n)$, corresponds to the gradient operator in vector calculus, since $d\,\phi(\underset{\sim}{x}) = \sum_{i=1}^{n} \frac{\partial\phi}{\partial x_i}\, dx^i$. And d(or more precisely *d), applied to $E^1(\mathbb{R}^3)$, corresponds to the curl operator, since

$$*d\{f_1dx+f_2dy+f_3dz\} = *\left\{\frac{\partial f_1}{\partial x}\, dx\wedge dx + \frac{\partial f_1}{\partial y}\, dy\wedge dx + \frac{\partial f_1}{\partial z}\, dz\wedge dx + \frac{\partial f_2}{\partial x}\, dx\wedge dy\right.$$

$$\left. + \frac{\partial f_2}{\partial y}\, dy\wedge dy + \frac{\partial f_2}{\partial z}\, dz\wedge dy + \frac{\partial f_3}{\partial x}\, dx\wedge dz + \frac{\partial f_3}{\partial y}\, dy\wedge dz + \frac{\partial f_3}{\partial z}\, dz\wedge dz\right\} \tag{A-3}$$

$$= *\left\{\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy\wedge dz + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dz\wedge dx + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\wedge dy\right\}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dx + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dy + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dz,$$

since $dx^i \wedge dx^i = 0$. In fact, d, applied to $E^1(\mathbb{R}^n)$, is the correct extension to n dimensions of the curl, which is defined in vector calculus only for $\mathbb{R}^3$.

Lemma A.2. If $\alpha \in E^p(\mathbb{R}^n)$, then $dd\alpha = 0 \in E^{p+2}(\mathbb{R}^n)$, i.e. $d^2 = 0$.

The proof is a straightforward computation and can be found in [20-23].

If $\alpha \in E^o(\mathbb{R}^3)$, then lemma A.2 is just the familiar identity from vector calculus, $\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla}\phi) = 0$.

The next definition constructs from * and d a new operator which transforms a p-form to a (p-1)-form.

Definition A.3. The linear differential operator δ, mapping $E^p(\mathbb{R}^n)$ into $E^{p-1}(\mathbb{R}^n)$, is defined by $\delta = (-1)^{n(p+1)} {*d*}$.

For notational convenience, our definition of δ differs in sign from that given in [24]. The following calculation shows that δ, applied to $E^1(\mathbb{R}^3)$, corresponds to the divergence operator from vector calculus. (Note that $(-1)^{n(p+1)} = 1$ in this case.)

$\delta(f_1dx+f_2dy+f_3dz)$

$$f_1dx + f_2dy + f_3dz \overset{*}{\mapsto} f_1dy\wedge dz + f_2dz\wedge dx + f_3dx\wedge dy$$

$$\overset{d}{\mapsto}\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right) dx\wedge dy\wedge dz \overset{*}{\mapsto}\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right) \tag{A-4}$$

When applied to a 2-form on $\mathbb{R}^3$, δ corresponds to the negative of the curl, as we see below.

<u>$\delta(f_1 dy\wedge dz+f_2 dz\wedge dx+f_3 dx\wedge dy)$</u>

$$f_1 dy\wedge dz + f_2 dz\wedge dx + f_3 dx\wedge dy \overset{*}{\mapsto} f_1 dx + f_2 dy + f_3 dz \overset{d}{\mapsto} \left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right) dy\wedge dz$$

$$+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right) dz\wedge dx + \left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right) dx\wedge dy \overset{(-1)^{3\cdot 3}*}{\longmapsto}$$

$$\left(\frac{\partial f_2}{\partial z}-\frac{\partial f_3}{\partial y}\right) dx + \left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right) dy + \left(\frac{\partial f_1}{\partial y}-\frac{\partial f_2}{\partial x}\right) dz \tag{A-5}$$

<u>Lemma A.3</u>. If $\alpha \in E^p(\mathbb{R}^n)$, then $\delta\delta\alpha = 0 \in E^{p-2}(\mathbb{R}^n)$, i.e. $\delta^2 = 0$.

<u>Proof</u>. $\delta^2\alpha = (-1)^{np}(-1)^{n(p+1)}*d**d* = (-1)^{p(n-p)-1}*d^2* = 0$, since $d^2 = 0$. ◻

If $\alpha \in E^2(\mathbb{R}^3)$, then lemma A.3 just says $\underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\times\alpha) = 0$.

On $E^1(\mathbb{R}^3)$, the following definition reduces to Eq. (2).

<u>Definition A.4</u>. The <u>Beltrami-Laplace operator</u> $\Delta : E^p(\mathbb{R}^n) \to E^p(\mathbb{R}^n)$ is defined by $\Delta = d\delta+\delta d$.

A justification for this definition and symbol is the following lemma, which shows that Δ, when applied to a 1-form on $\mathbb{R}^n$, simply takes the Laplacian of each component function separately, in agreement with our definition in section 3.1 . In fact, this conclusion holds for general p-forms on $\mathbb{R}^n$, but we shall only be interested in the case p=1.

<u>Lemma A.4</u>.

$$\Delta\left(\sum_{i=1}^{n} f_i dx^i\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial^2 f_i}{\partial x_j^2}\right)dx^i \tag{A-6}$$

Proof. When applied to a 1-form, $\delta = (-1)^{n\cdot 2}{*d*}$, so $d\delta = d{*d*}$. And when applied to a 2-form, $\delta = (-1)^{n\cdot 3}{*d*} = (-1)^n{*d*}$, so $\delta d = (-1)^n{*d*d}$. We let dV represent the n-form $dx^1\wedge\ldots\wedge dx^n$ and dV/dx^i represent the (n-1)-form corresponding to dV but with the term dx^i missing, etc. The separate parts are computed as follows:

$d\,\delta\left(\sum_{i=1}^{n} f_i dx^i\right)$

$$\sum_{i=1}^{n} f_i dx^i \overset{*}{\mapsto} \sum_{i=1}^{n} (-1)^{i-1} f_i \frac{dV}{dx^i} \overset{d}{\mapsto} \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}\, dV \overset{*}{\mapsto} \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \overset{d}{\mapsto} \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f_i}{\partial x_j \partial x_i}\, dx^j . \tag{A-7}$$

$\delta d\left(\sum_{i=1}^{n} f_i dx^i\right)$

$$\sum_{i=1}^{n} f_i dx^i \overset{d}{\mapsto} \sum_{\substack{i=1\\ i<j}}^{n} \sum_{j=1}^{n} \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}\right) dx^i\wedge dx^j \overset{*}{\mapsto} \sum_{\substack{i=1\\ i<j}}^{n} \sum_{j=1}^{n} \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}\right) (-1)^{i+j-1} \frac{dV}{dx^i dx^j} \overset{d}{\mapsto}$$

$$\left\{\begin{aligned} &\sum_{\substack{i=1\\ i<j}}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 f_j}{\partial x_i^2} - \frac{\partial^2 f_i}{\partial x_i \partial x_j}\right) (-1)^j \frac{dV}{dx^j} \\ &+\sum_{\substack{i=1\\ i<j}}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 f_j}{\partial x_j \partial x_i} - \frac{\partial^2 f_i}{\partial x_j^2}\right) (-1)^{i-1} \frac{dV}{dx^i} \end{aligned}\right\} \overset{*}{\mapsto} \tag{A-8}$$

$$\sum_{\substack{i=1\\ i<j}}^{n} \sum_{j=1}^{n} \left\{\left(\frac{\partial^2 f_j}{\partial x_i^2} - \frac{\partial^2 f_i}{\partial x_i \partial x_j}\right) (-1)^n dx^j + \left(\frac{\partial^2 f_j}{\partial x_j \partial x_i} - \frac{\partial^2 f_i}{\partial x_j^2}\right) (-1)^{n-1} dx^i\right\} \overset{(-1)^n}{\mapsto}$$

$$\sum_{\substack{i=1\\ i<j}}^{n} \sum_{j=1}^{n} \left\{\left(\frac{\partial^2 f_j}{\partial x_i^2} - \frac{\partial^2 f_i}{\partial x_i \partial x_j}\right) dx^j + \left(\frac{\partial^2 f_i}{\partial x_j^2} - \frac{\partial^2 f_j}{\partial x_j \partial x_i}\right)\right\} dx^i .$$

We rearrange the result of (A-7) as follows:

$$d\delta \sum_{i=1}^{n} f_i dx^i = \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 f_i}{\partial x_i \partial x_j} dx^j + \frac{\partial^2 f_j}{\partial x_i \partial x_j} dx^i \right) + \sum_{\substack{i=1 \\ i=j}}^{n} \sum_{j=1}^{n} \frac{\partial^2 f_i}{\partial x_i \partial x_j} dx^j, \qquad \text{(A-9)}$$

and add this to the result of (A-8) to get

$$(d\delta + \delta d) \sum_{i=1}^{n} f_i dx^i = \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 f_j}{\partial x_i^2} dx^j + \frac{\partial^2 f_i}{\partial x_j^2} dx^i \right) + \sum_{\substack{i=1 \\ i=j}}^{n} \sum_{j=1}^{n} \frac{\partial^2 f_i}{\partial x_i \partial x_j} dx^j$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial^2 f_i}{\partial x_j^2} \right) dx^i \quad . \qquad \square$$

Lemma A.5. If $\beta \in E^1(\mathbb{R}^n)$, then there exists a solution $\alpha \in E^1(\mathbb{R}^n)$ of the equation $\Delta\alpha = \beta$, i.e. Δ maps $E^1(\mathbb{R}^n)$ onto itself.

Since lemma A.4 has established that Δ acts on each component of α separately, it is enough to show that Δ maps $C^\infty(\mathbb{R}^n)$ onto itself. Proofs of this fact can be found in [7,8].

Lemma A.5 is special to $\mathbb{R}^n$ and does not generalize to $E^1(M)$ for a compact manifold M. This is one way in which our technique differs from the Hodge decomposition.

Definition A.5. Suppose $\alpha \in E^1(\mathbb{R}^n)$. Then α is exact if there exists $\phi \in E^0(\mathbb{R}^n)$ such that $\alpha = d\phi$ (and hence $d\alpha = 0$), and α is co-exact if there exists $\beta \in E^2(\mathbb{R}^n)$ such that $\alpha = \delta\beta$ (and hence $\delta\alpha = 0$).

An exact 1-form on $\mathbb{R}^3$ corresponds to a gradient (and hence curl-free) vector field . See (A-3). And a co-exact 1-form on $\mathbb{R}^3$ corresponds to a vector field which is itself the curl of some vector field (and hence has zero divergence). See (A-5).

Theorem A.1. Every 1-form β on $\mathbb{R}^n$ can be written as the sum of an exact 1-form and a co-exact 1-form as follows: choose $\alpha \in E^1(\mathbb{R}^n)$ such that $\Delta\alpha = \beta$ and then

$$\beta = d\delta\alpha + \delta d\alpha = d\omega + \delta\eta \qquad \text{(A-10)}$$

where $\delta\alpha = \omega \in E^{o}(\mathbb{R}^{n})$ and $d\alpha = \eta \in E^{2}(\mathbb{R}^{n})$.

Theorem (A.1) follows immediately from def. A.4 and lemma A.5. In the body of this article, examples of the above decomposition appear in the language of functions mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ rather than the language of differential forms. Equations (7) and (8) demonstrate the case n=2, (37) and (38) demonstrate n=3, and the general case appears in (28).

Definition A.6. For each ordered pair of integers (i,j) with $1 \le i < j \le n$, the linear map $[\underset{\sim}{A}_{ij}] : E^{1}(\mathbb{R}^{n}) \to E^{1}(\mathbb{R}^{n})$ is defined by

$$[\underset{\sim}{A}_{ij}]\left(\sum_{k=1}^{n} f_k(\underset{\sim}{x})dx^k\right) = f_j(\underset{\sim}{x})dx^i - f_i(\underset{\sim}{x})dx^j. \tag{A-11}$$

This definition of $[\underset{\sim}{A}_{ij}]$ agrees with that given in (19).

We are finally in a position to restate theorem 1 and equation (28) in a way that clearly reveals their inheritance from the Hodge theorem.

Theorem A.2. For each 1-form $\beta = \sum_{i=1}^{n} \beta_i(\underset{\sim}{x})dx^i$ on $\mathbb{R}^n$, there exists a 1-form $\alpha = \sum_{i=1}^{n} \alpha_i(\underset{\sim}{x})dx^i$ such that

$$\beta = \Delta\alpha = d\delta\alpha + \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} [\underset{\sim}{A}_{ij}]\, d\left(\frac{\partial\alpha_i}{\partial x_j} - \frac{\partial\alpha_j}{\partial x_i}\right). \tag{A-12}$$

This is just Eq. (A-10) with the term $\delta d\alpha$ expanded as in Eq. (28).

Notice that we have not required β to vanish at infinity. Theorem (A.2) is exactly the extension of the Hodge theorem to $\mathbb{R}^n$ that we need for n-port decomposition. However, when restricted to differential forms on compact manifolds, the Hodge theorem acquires a great deal of beautiful structure which is lacking in our version. We have provided a painless introduction to the Hodge theorem on compact manifolds in the next section; intermediate treatments can be found in [13,25]; and a very elegant presentation with complete proofs appears in [24].

A.2. Introduction to the Hodge Decomposition of Differential Forms on a Compact Riemannian Manifold

This section is designed to provide an elementary descriptive introduction to material which is treated much more carefully in [13,24,25]. These three references are relatively readable and should be consulted for further information.

The Hodge theorem provides an orthogonal direct sum decomposition of $E^p(M)$, the space of smooth p-forms on a compact oriented Riemannian manifold M. The operators $*$, d, δ, and Δ in this new setting are straightforward generalizations of the versions defined in the previous section for p-forms on Euclidean spaces. In fact, if we now let $x_1,\ldots,x_n$ represent a set of local coordinates on M such that $dV = dx^1\wedge\ldots\wedge dx^n$, then the coordinate expressions for $*$, d, δ, and Δ on M carry over from section A.1 exactly.

The Riemannian metric on M is used to define $*$ in a coordinate independent way, and d was already coordinate independent. Therefore δ and Δ, which are defined in terms of d and $*$, are also coordinate independent. The space $E^p(M)$ is viewed as an infinite dimensional vector space, and the expression $(\omega\wedge*\eta)$ defines a pointwise inner product of any two p-forms, ω and η. The inner product on $E^p(M)$ is produced simply by integration over M, i.e. $\langle\omega,\eta\rangle \triangleq \int_M \omega\wedge*\eta dV$, so $E^p(M)$ becomes an infinite-dimensional inner product space.

In this setting a great deal of structure emerges which was lost in our application. It turns out that $\Delta = d\delta+\delta d$ is self-adjoint and that the image of $d\delta$ and the image of δd are orthogonal in the inner product defined above. Furthermore, the kernel of Δ is finite dimensional.

Since the kernel of any self-adjoint operator is orthogonal to its image, the conclusion is that Δ breaks up $E^p(M)$ into three mutually orthogonal subspaces: $\ker(\Delta)$ - the finite dimensional space of harmonic forms; image $(d\delta)$ - the space of exact forms; and image (δd) - the space of co-exact forms. The following is the simplest possible example to illustrate these ideas.

Example A.1. On the unit circle, S^1, we choose $\theta \in [0,2\pi]$ as the coordinate. If $f:[0,2\pi] \to \mathbb{R}$ is a smooth function, then $f(\theta)\, d\theta$ is a smooth 1-form on S^1 iff f" joins up" smoothly at 0 and 2π, i.e. $f^{(n)}(0) = f^{(n)}(2\pi)$, $n = 0, 1, 2, \ldots$, where $f^{(n)}$ represents the n-th (one-sided) derivative of f and $f^{(o)} = f$. So $E^1(S^1)$ is just the space of smooth functions on $[0,2\pi]$ which join up properly at the endpoints.

To decompose $E^1(S^1)$ via the Hodge theorem, we first notice that $\Delta = \frac{\partial^2}{\partial\theta^2}$, and that $\Delta = d\delta$ since $\delta d = 0$ because there are no 2-forms on S^1. So the 1-form $f(\theta)\,d\theta$ is harmonic, i.e. $\Delta[f(\theta)d\theta] = d\delta[f(\theta)d\theta] = 0$, iff $\frac{\partial^2 f}{\partial\theta^2} = 0$ or $f(\theta) = a\theta + c$. But since $f(0) = f(2\pi)$, a must be zero, so the harmonic forms on S^1 are just the "constant" forms, $c\,d\theta$. And the space of harmonic forms on S^1 is 1-dimensional.

Now $f(\theta)d\theta \in$ image $(d\delta)$, i.e. $f(\theta)d\theta$ is exact, iff $f(\theta) = \frac{\partial^2\phi(\theta)}{\partial\theta^2}$ for some smooth function ϕ on S^1. Since in that case $\phi' = \frac{\partial\phi}{\partial\theta}$ will also be a smooth function on S^1, f must satisfy

$$\int_0^{2\pi} f(\theta)d\theta = \phi'(2\pi) - \theta'(0) = 0 \tag{A-13}$$

in order for $f(\theta)d\theta$ to be exact. It is not hard to verify that (A-13), which just says that the average value of f is zero, is also a sufficient condition on f to make $f(\theta)d\theta$ exact.

In conclusion, the Hodge theorem applied to $E^1(S^1)$ breaks up any 1-form $f(\theta)d\theta$ as follows:

$$f(\theta)d\theta = \left\{\frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta\right\}d\theta + \left\{f(\theta) - \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta\right\}d\theta \tag{A-14}$$

$$= f_h(\theta)d\theta + f_e(\theta)d\theta \ ,$$

where $f_h(\theta)d\theta$ is the harmonic part (since $f_h(\theta)$ is constant), and $f_e(\theta)d\theta$ is the exact part (since it is smooth, "joins up right" at 0 and 2π, and satisfies (A-13)). Furthermore, $\langle f_h(\theta)d\theta, f_e(\theta)d\theta\rangle = \int_{S^1} f_h(\theta)d\theta\wedge * f_e(\theta)d\theta$ $= \int_0^{2\pi} f_h(\theta)f_e(\theta)d\theta = f_h(\theta)\int_0^{2\pi} f_e(\theta)d\theta = 0$, illustrating that (A-14) is in fact an orthogonal decomposition.

A.3. The Relationship Between Theorem 1 and the Hodge Theorem

The reader who is acquainted with the Hodge theorem in its normal setting in differential geometry will perhaps be puzzled by the odd partial similarity between theorem 1 or theorem (A.2) and the Hodge theorem. At first glance, Eq. (28) (or Eq. (A-12)) seems to be a purely local sort of result, a trivial generalization of the vector calculus identity (2) to Euclidean spaces of arbitrary dimension. But this impression is misleading because theorem 1 depends on a fact about Euclidean space which is far from obvious and which fails entirely for general Riemannian manifolds: the fact that the Laplace operator maps $C^\infty(\mathbb{R}^n)$ onto itself [7,8].

Our extension of the Hodge decomposition to 1-forms on $\mathbb{R}^n$ which do not vanish at infinity (the result carries over easily to general p-forms, although we are only interested in the case p=1) provides exactly the needed result in circuit theory, but destroys most of the interesting structure the theorem possessed in its original setting. In contrast to the Hodge decomposition, our result breaks every 1-form into the sum of two terms in the image of the Laplacian and no harmonic term appears; the decomposition is not unique; and there is no inner product.

We have often wondered if some useful inner product could be introduced in this context if we restricted our attention to, say, the class of 1-forms whose coefficients in the usual coordinate system on $\mathbb{R}^n$ grow no faster than polynomials at infinity. Clearly a measure μ could be defined such that the inner product $\langle \omega,\eta \rangle = \int_{\mathbb{R}^n} \omega \wedge {*\eta}\, d\mu$ is defined for all 1-forms ω, η in this class. Could such a measure be found that would also cause the decomposition to regain its orthogonality properties in this context? We hope that some reader whose mathematical skills are more equal to such a problem will find this one of interest.

APPENDIX B: SOLVING POISSON'S EQUATION

The version of our decomposition technique in section 3.1 requires that we solve Poisson's equation,

$$\Delta\phi(=\nabla^2\phi) = \psi, \tag{B-1}$$

n times, once with each of the port relations f_i substituted for ψ on the right hand side. The alternate version in sections 3.2-3.3 requires only one such solution, this time with $\nabla\cdot f$ on the right. In examples 1 and 3 we were able to solve the equations by inspection, but we won't generally be so lucky.

The standard method of solution involves a convolution integral. Specifically, if $\psi \in C^\infty(\mathbb{R}^n)$ has compact support or at least vanishes fast enough at infinity, then one solution is given by

$$\phi(x) = \iint_{\mathbb{R}^n}\cdots\int W_n(y)\ \psi(x-y)\ dy \tag{B-2}$$

where

$$W_n(y) = \begin{cases} \frac{1}{2\pi}\ \ell n\ \|y\|, & n=2 \\ \frac{-1}{(n-2)A_{n-1}}\ \|y\|^{2-n}, & n \geq 3 \end{cases} \tag{B-3}$$

and A_{n-1} is the surface area (i.e. the (n-1)-measure) of the unit (n-1)-sphere in $\mathbb{R}^n$ [6]. Equations 6 and 4 are examples of the cases n=2 and n=3, respectively.

Of course the integral (B-2) could be difficult to calculate in closed form. But for our application there is a much more serious objection. The constitutive relations of realistic network elements practically never vanish at infinity and in most cases the dependent variables actually become unbounded, as in (41), when the independent variables go to infinity. The divergence of the constitutive relation frequently exhibits this same behavior, as in (44). In this case the integral (B-2) will not converge at all! For example, if we attempt to solve (44) by using (B-2), it is easy to see that the integral

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty} \ell n\left(\sqrt{y_1^2+y_2^2}\right)\left[A_1Ke^{K(x_1-y_1)} + B_2Ke^{K(x_2-y_2)}\right]dy_1dy_2$$

$$= \frac{K}{4\pi} \iint_{-\infty}^{\infty} \ln(y_1^2+y_2^2) \left[A_1 e^{K(x_1-y_1)} + B_2 e^{K(x_2-y_2)}\right] dy_1 dy_2 \tag{B-4}$$

diverges, by considering the behavior of the integrand as $y_1 \to -\infty$, $y_2 \to -\infty$.

Of course this does not imply that (44) has no solution, and in fact we found a simple answer by inspection in (45). Fortunately, it turns out that this is what always happens. The Laplacian (in common with every other linear differential operator with constant coefficients) maps $C^\infty(\mathbb{R}^n)$ onto itself, i.e. for any C^∞ function ψ there is a C^∞ function ϕ that solves (B-1), no matter how badly ψ blows up at infinity [7,pp.80,82][8,pp.3,128,287,355]. So the existence of a solution is guaranteed, the question is how to find it.

If it is impossible to solve Poisson's equation by inspection, then the best approach is to recognize that we are not really interested in arbitrarily large values of the independent port variables $\underset{\sim}{x}$, since any physical device has some voltage and current threshold beyond which it breaks down. Instead, we can define some bounded region of interest, B, for the variables $\underset{\sim}{x}$, and attempt to find a solution of (B-1) which is valid inside B. One approach would be simply to set ψ to zero outside B and then find a solution by means of (B-2). This would work, of course, but it still leaves us with the problem that the convolution integral is frequently quite difficult to calculate in closed form.

For this reason we suggest the following approach. First expand ψ in a Fourier series on B. And since the terms of the Fourier series are eigenfunctions of Δ, we can then solve Poisson's equation by inspection for each term in the expansion of ψ. This gives us a Fourier expansion for ϕ.

To be more specific, let B be a square region of $\mathbb{R}^n$ centered at the origin, i.e. $B = \{\underset{\sim}{x} \in \mathbb{R}^n \,\big|\, |x_i| \le b,\ i = 1,\ldots,n\}$, and choose b large enough that B contains all the operating points (values of $\underset{\sim}{x}$) which are of physical interest.

The sequence of functions $\left\{\frac{1}{\sqrt{2b}}\, e^{ik\pi x/b}\right\}_{k=-\infty}^{\infty}$ is a complete orthonormal sequence in the Hilbert space of complex square integrable functions on [-b,b], and therefore if we define the family of product functions $\left\{g_{k_1,\ldots,k_n}(x)\right\}$ on B by

$$g_{k_1,\ldots,k_n}(\underset{\sim}{x}) = \left(\frac{1}{\sqrt{2b}}\right) e^{\frac{ik_1\pi x_1}{b}} \cdots \left(\frac{1}{\sqrt{2b}}\right) e^{\frac{ik_n\pi x_n}{b}} = \left(\frac{1}{2b}\right)^{n/2} \exp\left(\frac{i\pi}{b} \sum_{j=1}^{n} k_j x_j\right), \tag{B-5}$$

then $\left\{g_{k_1,\ldots,k_n}(\underset{\sim}{x})\right\}$, where each index ranges over all the integers, is a complete orthonormal family in the space of square integrable complex valued functions on B [26]. So the Fourier expansion of ψ on B is just

$$\psi = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} c_{k_1,\ldots,k_n} g_{k_1,\ldots,k_n} , \tag{B-6}$$

where

$$c_{k_1,\ldots,k_n} = \int_{-b}^{b} \cdots \int_{-b}^{b} \psi(x_1,\ldots,x_n)\; g^{*}_{k_1,\ldots,k_n}(x_1,\ldots,x_n)\,dx_1 \;\ldots\; dx_n . \tag{B-7}$$

A simple calculation shows that

$$\Delta\left(g_{k_1,\ldots,k_n}\right) = -\left[\left(\frac{\pi}{b}\right)^{2n} (k_1^2+\ldots+k_n^2)\right] g_{k_1,\ldots,k_n} \quad . \tag{B-8}$$

Equation (B-8) is the reason for our approach - it simplifies things enormously that the g's are eigenfunctions of the Laplacian. Therefore, once we have expanded ψ as in (B-6), we can solve (B-1) by inspection, term by term.

In practice, we would approximate ψ by a few terms in the expansion (B-6). Those terms for which $\max\left\{|k_1 ,\ldots, k_n|\right\}$ is small are the most important. And we would choose B somewhat larger than the actual region of interest, since (B-6) will not converge pointwise on the boundary of B.

APPENDIX C: PROOFS

Proof of Lemma 2. We need to show that for any value of m, $1 \leq m \leq n$, the m^{th} component of the right hand side of Eq. (30) is just $f_m(\underset{\sim}{x})$. Since by Eq. (19) each matrix $[\underset{\sim}{A}_{ij}]$ has nonzero entries only in locations (i,j) and (j,i), the only terms in the double sum in (30) which contribute to the m^{th} component are those with indices (m,j), m < j or (i,m), i < m. Thus the m^{th} component of

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} [\underset{\sim}{A}_{ij}] \underset{\sim}{\nabla} \{x_j \int_0^1 t^{n-2} f_i(t\underset{\sim}{x})dt - x_i \int_0^1 t^{n-2} f_j(t\underset{\sim}{x})dt\} \tag{C-1}$$

is the sum of two terms. The first is the m^{th} component of

$$\sum_{j=m+1}^{n} [\underset{\sim}{A}_{mj}] \underset{\sim}{\nabla} \{x_j \int_0^1 t^{n-2} f_m(t\underset{\sim}{x})dt - x_m \int_0^1 t^{n-2} f_j(t\underset{\sim}{x})dt\}, \tag{C-2}$$

and the second is the m^{th} component of

$$\sum_{i=1}^{m-1} [\underset{\sim}{A}_{im}] \underset{\sim}{\nabla} \{x_m \int_0^1 t^{n-2} f_i(t\underset{\sim}{x})dt - x_i \int_0^1 t^{n-2} f_m(t\underset{\sim}{x})dt\}. \tag{C-3}$$

Recalling the definition of $[\underset{\sim}{A}_{ij}]$, we can write (C-2) as

$$\sum_{j=m+1}^{n} D_j \{x_j \int_0^1 t^{n-2} f_m(t\underset{\sim}{x})dt - x_m \int_0^1 t^{n-2} f_j(t\underset{\sim}{x})dt\}. \tag{C-4}$$

Carrying out the differentiation in (C-4) yields three terms:

$$(n-m) \int_0^1 t^{n-2} f_m(t\underset{\sim}{x})dt \tag{C-5}$$

$$+$$

$$\sum_{j=m+1}^{n} x_j \int_0^1 t^{n-1} D_j f_m(t\underset{\sim}{x})dt \tag{C-6}$$

$$+$$

$$\sum_{j=m+1}^{n} (-x_m \int_0^1 t^{n-1} D_j f_j(t\underset{\sim}{x})dt). \tag{C-7}$$

And similarly, (C-3) can be written as the sum of the following three terms:

$$\sum_{i=1}^{m-1} (-x_m \int_0^1 t^{n-1} D_i f_i(t\underset{\sim}{x})dt) \tag{C-8}$$

$$+$$

$$(m-1)\int_0^1 t^{n-2} f_m(t\underset{\sim}{x})\,dt \tag{C-9}$$

$$+$$

$$\sum_{i=1}^{m-1} x_i \int_0^1 t^{n-1} D_i f_m(t\underset{\sim}{x})\,dt. \tag{C-10}$$

Collecting terms, the sum of (C-6) and (C-10) is

$$\sum_{\substack{i=1\\ i\neq m}}^{n} x_i \int_0^1 t^{n-1} D_i f_m(t\underset{\sim}{x})\,dt. \tag{C-11}$$

Recalling that $\underset{\sim}{\nabla} \cdot \underset{\sim}{f} = 0$, the sum of (C-7) and (C-8) is

$$-x_m \int_0^1 t^{n-1} \sum_{\substack{i=1\\ i\neq m}}^{n} D_i f_i(t\underset{\sim}{x})\,dt = x_m \int_0^1 t^{n-1} D_m f_m(t\underset{\sim}{x})\,dt, \tag{C-12}$$

and adding (C-11) and (C-12) yields

$$\int_0^1 \sum_{i=1}^{n} t^{n-1} x_i (D_i f_m)(t\underset{\sim}{x})\,dt = \int_0^1 t^{n-1}\left(\frac{d}{dt}\, f_m(t\underset{\sim}{x})\right)dt. \tag{C-13}$$

The sum of (C-5) and (C-9) is

$$(n-1)\int_0^1 t^{n-2} f_m(t\underset{\sim}{x})\,dt, \tag{C-14}$$

and (C-13) and (C-14) add up to

$$\int_0^1 \frac{d}{dt}\left(t^{n-1} f_m(t\underset{\sim}{x})\right)dx = f_m(\underset{\sim}{x}) \tag{C-15}$$

¤

Proof of Lemma 6. Since example 4 exhibits a general solution for the case n = 2, and since (55) and (56) imply (57), we need only prove (56) for the case $n \geq 3$. If we define

$$\underset{\sim}{F}(\underset{\sim}{x}) \triangleq \begin{bmatrix} \\ \\ & \underset{\sim}{M} & \\ \\ \\ \end{bmatrix} \begin{bmatrix} x_1^3/6 \\ x_2^3/6 \\ \cdot \\ \cdot \\ \cdot \\ x_n^3/6 \end{bmatrix} \tag{C-16}$$

and $\underset{\sim}{H}(\underset{\sim}{x}) \triangleq [H_1(\underset{\sim}{x}),\ldots,H_n(\underset{\sim}{x})]^T$ for any choice of harmonic functions $H_1,\ldots,H_n$, then $\hat{\underset{\sim}{F}} \triangleq \underset{\sim}{F} + \underset{\sim}{H}$ satisfies $\Delta\hat{\underset{\sim}{F}}(\underset{\sim}{x}) = \underset{\sim}{M}\underset{\sim}{x}$. Since for any nxn matrix $\underset{\sim}{A}$

$$\underset{\sim}{\nabla}\{\underset{\sim}{x}^T(\underset{\sim}{A}+\underset{\sim}{A}^T)\underset{\sim}{x}\} = 2(\underset{\sim}{A}+\underset{\sim}{A}^T)\underset{\sim}{x}, \tag{C-17}$$

all we have to do to ensure that Eq.(56) holds is to choose $\underset{\sim}{H}$ so that

$$\underset{\sim}{\nabla} \cdot \hat{\underset{\sim}{F}}(\underset{\sim}{x}) = \frac{\underset{\sim}{x}^T(\underset{\sim}{M}+\underset{\sim}{M}^T)\underset{\sim}{x}}{4}. \tag{C-18}$$

From (C-16),

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{F}(\underset{\sim}{x}) = \frac{m_{11}x_1^2}{2} + \ldots + \frac{m_{nn}x_n^2}{2} = \underset{\sim}{x}^T \operatorname{diag}\left[\frac{(\underset{\sim}{M}+\underset{\sim}{M}^T)}{4}\right]\underset{\sim}{x}, \tag{C-19}$$

so (C-18) will be satisfied if we choose $\underset{\sim}{H}$ so that

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{H}(\underset{\sim}{x}) = \underset{\sim}{\nabla} \cdot (\hat{\underset{\sim}{F}}-\underset{\sim}{F})(\underset{\sim}{x}) = \underset{\sim}{x}^T\left[\left(\frac{\underset{\sim}{M}+\underset{\sim}{M}^T}{4}\right) - \operatorname{diag}\left(\frac{\underset{\sim}{M}+\underset{\sim}{M}^T}{4}\right)\right]\underset{\sim}{x}. \tag{C-20}$$

Define

$$\underset{\sim}{B} \triangleq \left(\frac{\underset{\sim}{M}+\underset{\sim}{M}^T}{4}\right) - \operatorname{diag}\left(\frac{\underset{\sim}{M}+\underset{\sim}{M}^T}{4}\right). \tag{C-21}$$

Then $\underset{\sim}{B}$ is a symmetric matrix with zeros along the diagonal. One solution $\underset{\sim}{H}(\underset{\sim}{x})$ of (C-20), i.e. of

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{H}(\underset{\sim}{x}) = \underset{\sim}{x}^T\underset{\sim}{B}\underset{\sim}{x}, \tag{C-22}$$

is obtained by choosing the components of $\underset{\sim}{H}(\underset{\sim}{x})$ as follows

$$H_1(\underset{\sim}{x}) = x_1\left(\underset{\sim}{x}^T\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & B_{22} & B_{23} & \cdots & B_{2n} \\ 0 & B_{32} & B_{33} & \cdots & B_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & B_{n2} & B_{n3} & \cdots & B_{nn} \end{bmatrix}\underset{\sim}{x}\right),$$

$$H_2(\underset{\sim}{x}) = x_2\left(\underset{\sim}{x}^T \begin{bmatrix} B_{11} & 0 & B_{13} & \cdots & B_{1n} \\ 0 & 0 & 0 & \cdots & 0 \\ B_{31} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ B_{n1} & 0 & 0 & \cdots & 0 \end{bmatrix} \underset{\sim}{x}\right), \tag{C-23}$$

$$H_3(\underset{\sim}{x}) = x_3(B_{12}+B_{21})x_1x_2,$$

and

$$H_i(\underset{\sim}{x}) = 0, \quad 3 < i \leq n.$$

It is easy to verify that $\underset{\sim}{H}(\underset{\sim}{x})$ satisfies (C-22), so $\hat{\underset{\sim}{F}}(\underset{\sim}{x}) = \underset{\sim}{F}(\underset{\sim}{x}) + \underset{\sim}{H}(\underset{\sim}{x})$ satisfies Eq. (56).

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Figure Captions

Fig. 1. Decomposition of the 2-port voltage-controlled resistor $\mathcal{R}$ into reciprocal 2-ports R_1 and R_2 and linear 4-port $\mathcal{L}$.

Fig. 2. One possible synthesis of $\mathcal{L}$ from linear dependent sources.

Fig. 3. A pnp transistor in common base configuration.

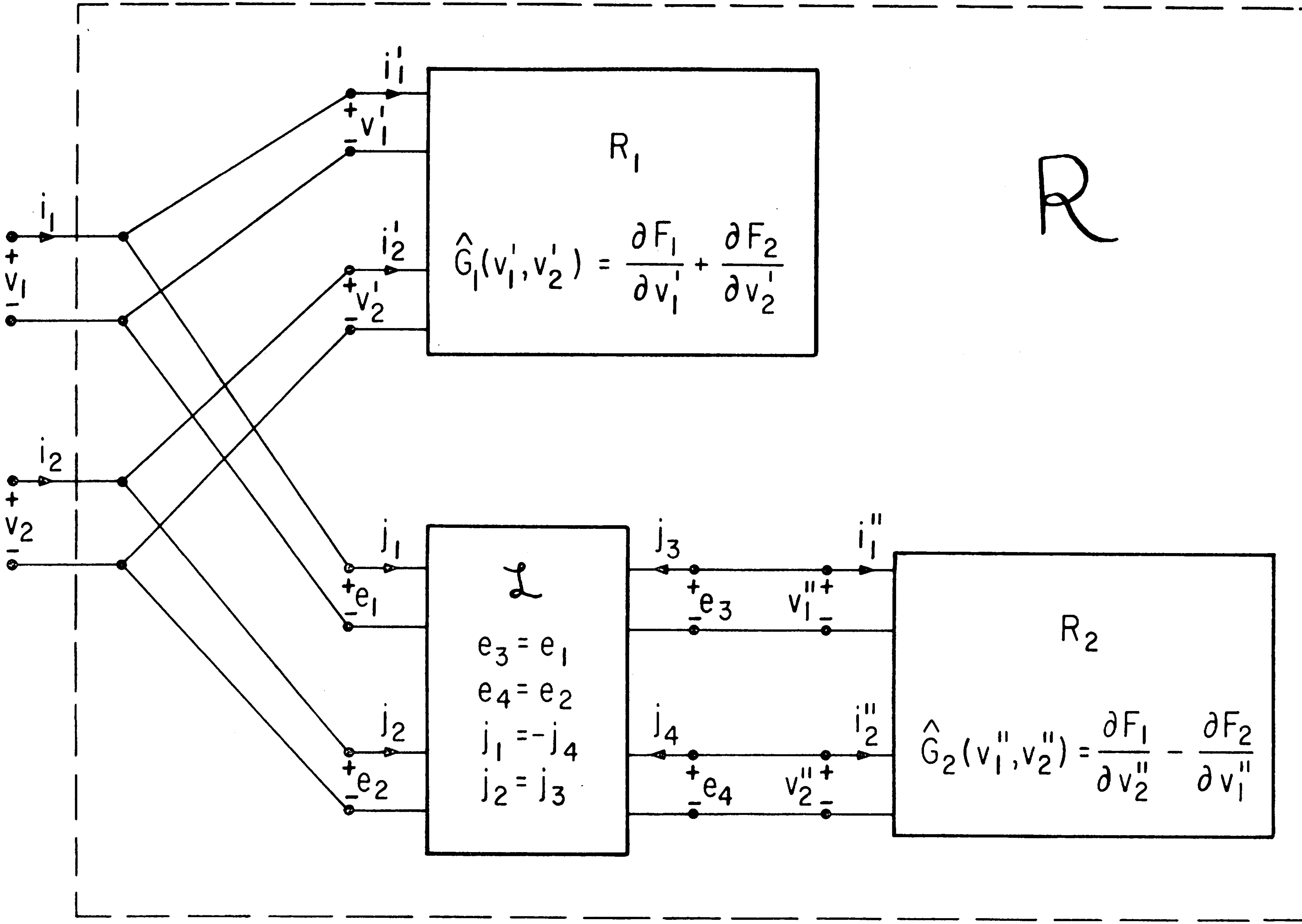
Fig. 4a. Synthesis of a 2-port characterized by the d.c. Ebers-Moll equations for a transistor, using the general method.

Fig. 4b. The circuit of Fig. 4(a) after a current source has been altered so that the passive diode D_4' can replace D_4.

Fig. 5a. The circuit of Fig. 4(b) redrawn to emphasize the important features.

Fig. 5b. The d.c. Ebers-Moll circuit model of a pnp transistor.

Fig. 6. Decomposition of the 3-port hybrid resistor $\mathcal{R}$ into reciprocal 3-ports R_1, R_2, R_3, R_4 and linear 6-ports $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$.



R
R₁
$\hat{G}_1(v_1', v_2') = \frac{\partial F_1}{\partial v_1'} + \frac{\partial F_2}{\partial v_2'}$
R₂
$\hat{G}_2(v_1'', v_2'') = \frac{\partial F_1}{\partial v_2''} - \frac{\partial F_2}{\partial v_1''}$
$e_3 = e_1$
$e_4 = e_2$
$j_1 = -j_4$
$j_2 = j_3$

Fig. 1.

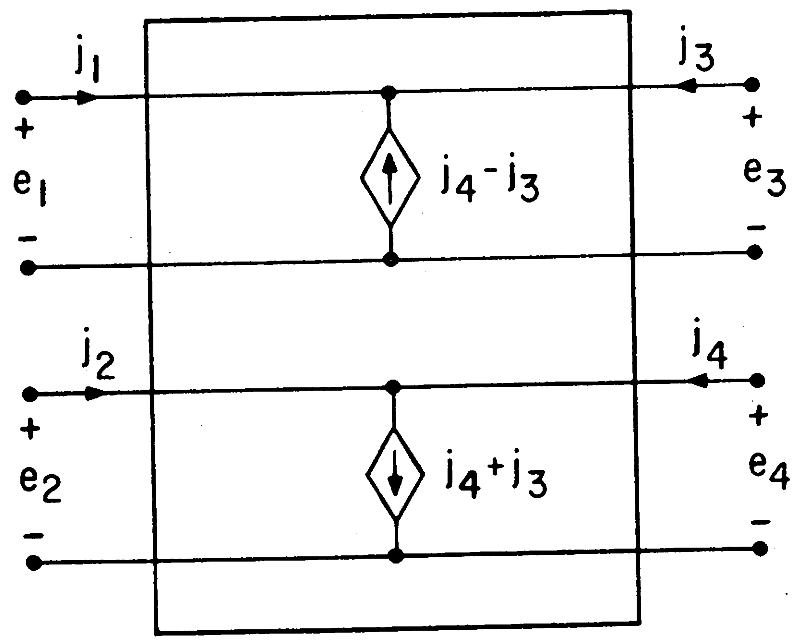


Fig. 2.

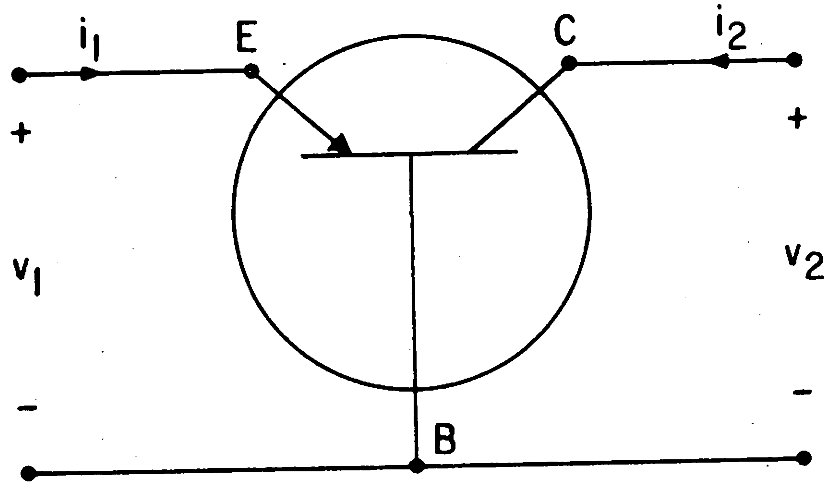
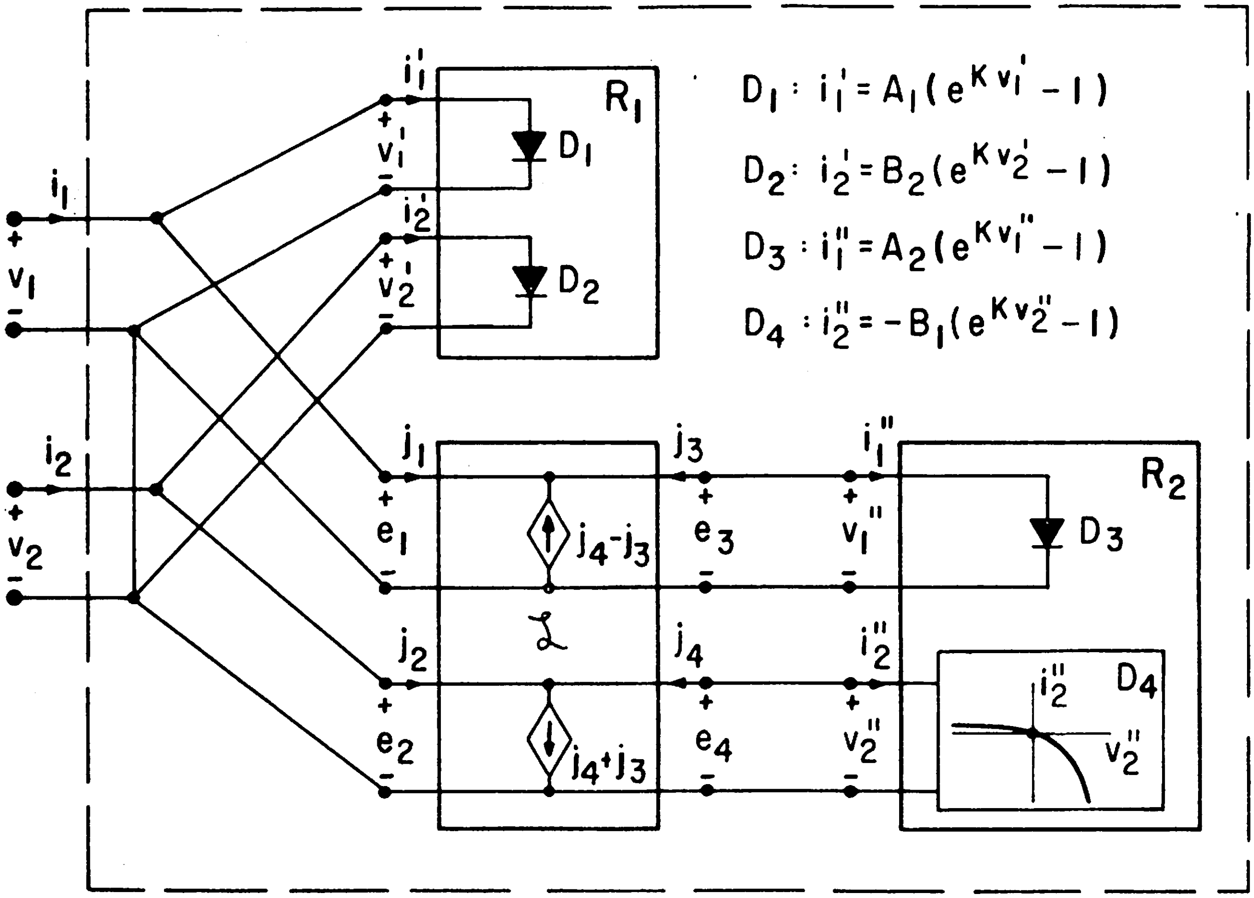
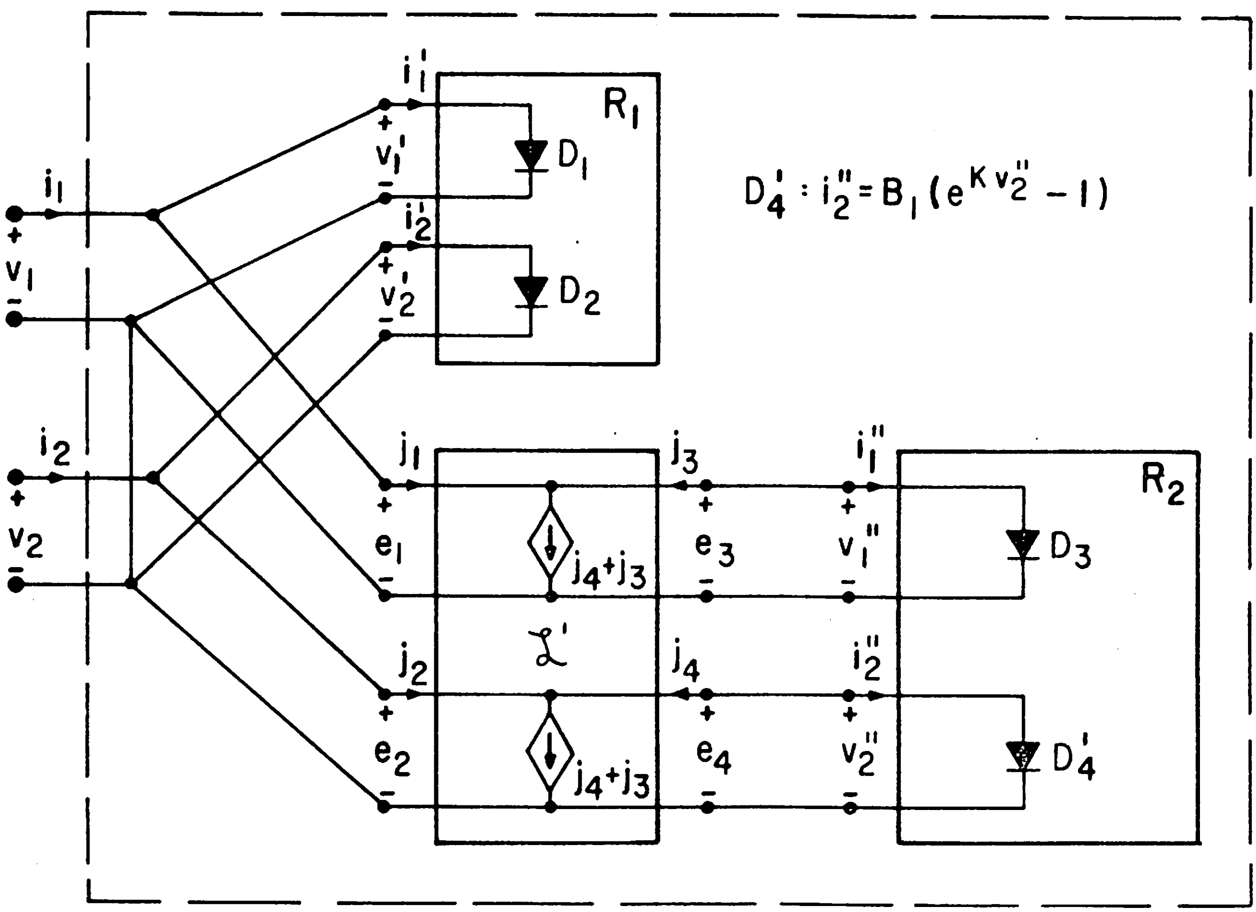


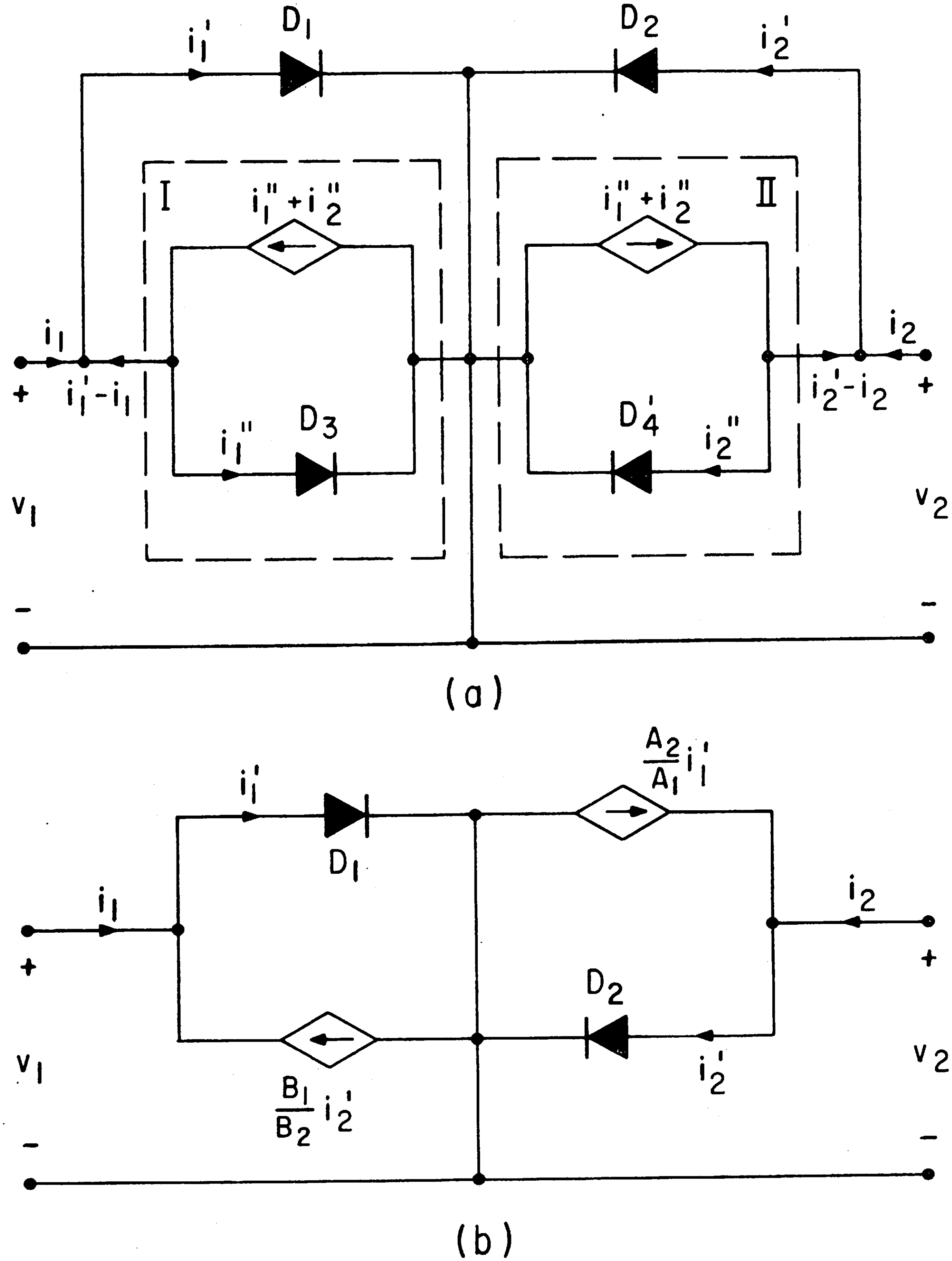
Fig. 3.


$D_1 : i_1' = A_1(e^{Kv_1'} - 1)$
$D_2 : i_2' = B_2(e^{Kv_2'} - 1)$
$D_3 : i_1'' = A_2(e^{Kv_1''} - 1)$
$D_4 : i_2'' = -B_1(e^{Kv_2''} - 1)$
R_1
R_2
D_1
D_2
D_3
D_4
i_1
i_2
v_1
v_2
i_1'
v_1'
i_2'
v_2'
j_1
j_2
j_3
j_4
e_1
e_2
e_3
e_4
$j_4 - j_3$
$j_4 + j_3$
$\mathcal{L}$
i_1''
v_1''
i_2''
v_2''

(a)


$D_4' : i_2'' = B_1(e^{Kv_2''} - 1)$
R_1
R_2
D_1
D_2
D_3
D_4'
i_1
i_2
v_1
v_2
i_1'
v_1'
i_2'
v_2'
j_1
j_2
j_3
j_4
e_1
e_2
e_3
e_4
$j_4 + j_3$
$j_4 + j_3$
$\mathcal{L}'$
i_1''
v_1''
i_2''
v_2''

(b) Fig. 4.



D1
D2
i1'
i2'
I
II
i1" + i2"
i1" + i2"
i1
i2
+
+
i1'-i1
i2'-i2
D3
D4'
i1"
i2"
v1
v2
-
-
(a)
A2/A1 i1'
i1'
D1
i1
i2
+
+
D2
v1
v2
i2'
B1/B2 i2'
-
-
(b)

Fig. 5.

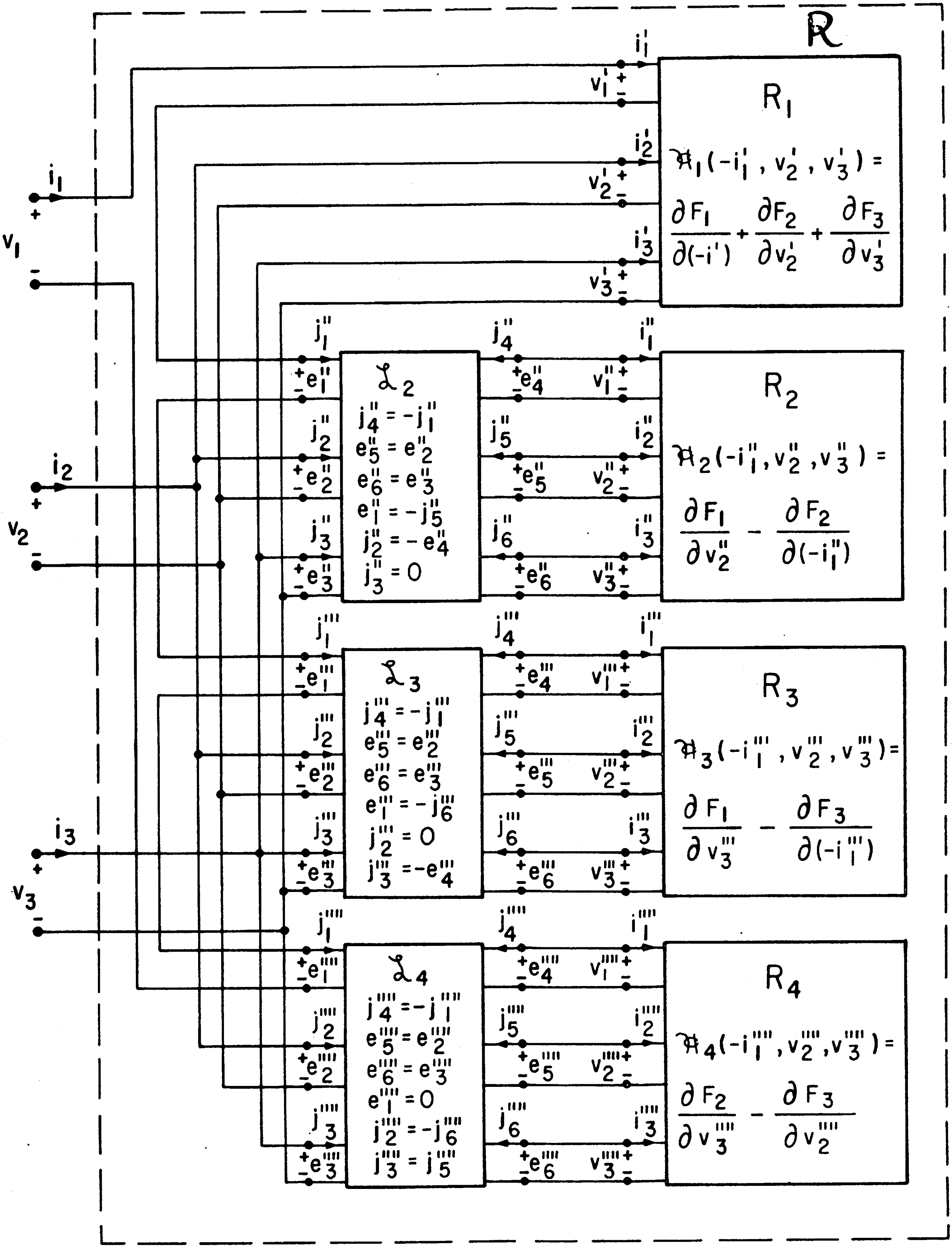


Fig. 6.