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THE THEORY OF ALL SUBSTRUCTURES OF A STRUCTURE: CHARACTERISATION AND DECISION PROBLEMS

by

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THE THEORY OF ALL SUBSTRUCTURES OF A STRUCTURE: CHARACTERISATION AND DECISION PROBLEMS

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Abstract

An infinitary characterisation of the first-order sentences true in all substructures of a structure $\underset{\sim}{M}$ is used to obtain partial reductions of the decision problem for such sentences to that for $Th(\underset{\sim}{M})$. For the relational structure $\langle \mathbb{R}, \leq, + \rangle$ this gives a decision procedure for the $\exists x \forall \vec{y}$-part of the theory of all substructures, yet we show that the $\exists x_1 x_2 \forall \vec{y}$-part, and hence the entire theory, is Π^1_1-complete. Applications in the philosophy of science are mentioned.

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1. Introduction

A philosophical analysis of scientific theories often shows that the structure $\underset{\sim}{M}$ described by the theory is not truly intended as a description of reality, but rather as a universal domain into which any structure which might correspond to reality can be represented by some homomorphic embedding. This is, for example, a view underlying the theory of Fundamental Measurement [6]. This situation raises logical questions: the implications of the theory for the experimenter are not so much the (first-order) theory $Th(\underset{\sim}{M})$ of the structure as the theory $Th(R(\underset{\sim}{M}))$ of all structures representable in the given structure. For $Th(R(\underset{\sim}{M}))$ contains the conditions which are necessary for representability in $\underset{\sim}{M}$, and which therefore ought to be tested before proceeding to assume such representability. Thus our problem is: given $\underset{\sim}{M}$, find $Th(R(\underset{\sim}{M}))$.

The logical metatheory associated with this problem was first investigated by Scott and Suppes [12]; similarly motivated model-theoretic investigations of special $\underset{\sim}{M}$ have been carried out recently by Adams [1] and Nahrens [10]. The problem of describing the notion of representability in model-theoretic terms was dealt with by Scott and Suppes. For recent models of measurement, their description is insufficiently general; we have given a generalization which appears adequate in [7]. For the present paper, we will use a simpler, more concrete description: A structure is representable in $\underset{\sim}{M}$ iff it is isomorphic to a substructure of $\underset{\sim}{M}$. The theory can be generalized, but the decisive features of the metatheory are most apparent in the present setting.

The metatheory given by Scott and Suppes consists of the application of the Los-Tarski Theorem, characterizing the universal part of the theory $Th(R(\underset{\sim}{M}))$, and the application to certain practical cases of Vaught's test for finite universal axiomatizability. We proceed to characterize the entire theory $Th(R(\underset{\sim}{M}))$, and to study the decision problem for this theory.

For any relational structure $\underset{\sim}{M}$, we denote the universe of $\underset{\sim}{M}$ by $|\underset{\sim}{M}|$, and the class of all (isomorphic copies of) substructures of $\underset{\sim}{M}$ by $S(\underset{\sim}{M})$. We will consider exclusively the case where $\underset{\sim}{M}$ is an infinite, purely relational structure, i.e. similarity types without any operation symbols. The basic results generalize easily to other similarity types, but the more detailed analysis does not. (The purely relational similarity types are more natural in applications.)

2. Characterization and Reduction Theorems

The task of this section is to characterize $Th(S(\underset{\sim}{M}))$, $\underset{\sim}{M}$ an infinite relational structure, as a subset of $Th\ \underset{\sim}{M}$. Clearly all universal sentences in $Th(\underset{\sim}{M})$ are true in all substructures of $\underset{\sim}{M}$. By the Los-Tarski Theorem, exactly these sentences and their logical consequences are true in $S(\underset{\sim}{E})$ for every model $\underset{\sim}{E}$ of $Th(\underset{\sim}{M})$. Thus if there are any further sentences in $Th(S(\underset{\sim}{M}))$, they must be false in some substructure of some model $\underset{\sim}{E}$ of $Th(\underset{\sim}{M})$. It is this case which requires further study.

We motivate the discussion by a simple, suggestive example. Let $\underset{\sim}{M} = \langle M,R\rangle$ be an infinite well-founded relation with arbitrarily long R-chains. By definition, this means

$$S(\underset{\sim}{M}) \models \exists x\, \forall y\, \neg Ryx \qquad (+)$$

This is not a first-order property of $\underset{\sim}{M}$. We can eliminate the model-theoretic notions, and write a second-order property of $\underset{\sim}{M}$ equivalent to (+):

$$\underset{\sim}{M} \models \forall S \neq \emptyset\ \exists x \in S\ \ \forall y \in S\ \neg Ryx \qquad (++)$$

In fact, Tarski noted [13] an infinitary equivalent

$$\underset{\sim}{M} \models \forall y_1 y_2 \cdots \bigvee_{n<\omega} -Ry_{i+1}y_i \tag{+++}$$

It is clear that for an arbitrary sentence $\psi = Q - \phi$, Q quantifier prefix, ϕ quantifier free, we have an equivalence between

$$S(\underset{\sim}{M}) \models Q - \phi \tag{*}$$

$$\underset{\sim}{M} \models \forall S \neq \emptyset \; Q^S - \phi \tag{**}$$

and we now show that a third equivalent condition analogous to (+++) can be obtained from ψ, by an effective procedure.

By the analysis of first-order quantification due to Skolem and Herbrand [5] there is, for any similarity type σ and any first-order σ-sentence χ, a sequence $\langle\phi_k : k \in \omega\rangle$ of finite quantifier-free σ-formulae with free variables from among $\langle x_j : j \in \omega\rangle$, the <u>Herbrand expansion</u> of χ, describing a canonical construction of a σ-structure satisfying χ (if one exists; otherwise $\bigwedge_{k \leq h} \phi_k$ is propositionally inconsistent for some $h \in \omega$) as the union of an extension chain of finite σ-structures. We relativize this test for satisfiability in pure first-order logic to $S(\underset{\sim}{M})$ by requiring that the construction be executed on $\underset{\sim}{M}$.

<u>Theorem 2.1</u>. Let σ be a purely relational similarity type, and $Q - \phi$ a σ-sentence, Q: quantifier prefix, ϕ quantifier-free. Then there is an effective procedure $e: \omega \to \omega^n$ (depending only on Q) such that for any σ-structure $\underset{\sim}{M}$, $S(\underset{\sim}{M}) \models Q - \phi$ iff

$$\underset{\sim}{M} \models \forall x_1 x_2 \cdots \bigvee_{k \in \omega} -\phi(x_{e(k)_1}, \ldots, x_{e(k)_n}) \,. \tag{***}$$

Proof. It is well known that for purely relational similarity type, the Herbrand expansion of $-Q\text{-}\phi$ can be taken as $\bigwedge_{k\in\omega}\phi_k$, where

$$\phi_k = \phi(x_{e(k)_1},\dots,x_{e(k)_n})$$

where e is an effectively computable function as described in the statement. Thus if $\underset{\sim}{M} \models \exists x_1x_2\cdots \bigwedge_{k\in\omega}\phi_k$, then for some $\{m_i \in |\underset{\sim}{M}|,\ i\in\omega\}$,

$$\underset{\sim}{M} \upharpoonright \{m_i:\ i\in\omega\} \models -Q\text{-}\phi \qquad \text{and} \qquad \underset{\sim}{M} \upharpoonright \{m_i:\ i\in\omega\} \in S(\underset{\sim}{M})\ .$$

Conversely, if $\underset{\sim}{A} \in S(\underset{\sim}{M})$ and $\underset{\sim}{A} \models -Q\text{-}\phi$ then $\underset{\sim}{A} \models \exists x_1x_2\cdots \bigwedge_{k\in\omega}\phi_k$ and so $\underset{\sim}{M} \models \exists x_1x_2\cdots \bigwedge_{k\in\omega}\phi_k$. Hence $S(\underset{\sim}{M}) \models Q\text{-}\phi \Leftrightarrow \underset{\sim}{M} \models -\exists x_1x_2\cdots\bigwedge_{k\in\omega}\phi_k$, and the latter condition is equivalent to (***). □

If the condition of (***) is a logical consequence of some first-order property of $\underset{\sim}{M}$, then $\psi = Q\text{-}\phi$ holds in $S(\underset{\sim}{E})$ for every model $\underset{\sim}{E}$ of $Th(\underset{\sim}{M})$, and hence ψ is a logical consequence of a universal sentence $\theta \in Th(\underset{\sim}{M})$ (and conversely). In fact we can now be more precise:

Theorem 2.2. For any purely relational similarity type σ, σ-structure $\underset{\sim}{M}$, and first-order σ-sentence $\psi = Q\text{-}\phi$, the following are equivalent:

(i) $\models \theta \rightarrow \psi$ for some universal sentence $\theta \in Th(\underset{\sim}{M})$

(ii) $\exists N\in\omega$: $\underset{\sim}{M} \models \forall x_1\cdots\bigvee_{k\le N} -\phi_k$, where ϕ_k is as specified in Theorem 2.1.

Proof. (ii) $\Rightarrow$ (i) trivially, as $\forall x_1\cdots\bigvee_{k\le N} -\phi_k$ satisfies the conditions on θ.

-(ii) $\Rightarrow$ -(i) by compactness: arbitrarily large partial models of $-\psi$ on $\underset{\sim}{M}$ entail the existence of a countable sequence $x_1x_2\cdots$ on some model $\underset{\sim}{E}$ of $Th(\underset{\sim}{M})$ with $(\underset{\sim}{E},x_1x_2\cdots) \models \bigwedge_{k\in\omega}\phi_k$, contradicting $S(\underset{\sim}{E}) \models \psi$. □

This means that if we can determine N, we can effectively produce θ of (i) from ψ. With this in mind, we attempt to find conditions which imply the conditions of Theorem 2.2. This is of interest in applications in measurement theory because the universal theory of $\underset{\sim}{M}$ is usually well understood, and considerable philosophical analysis is available concerning the empirical testability of universal sentences; in contrast, the methodology of testing sentences of more complicated quantifier form is poorly understood, and as we shall see in the next section, membership in $Th(S(\underset{\sim}{M}))$ of sentences not satisfying the conditions of Theorem 2.2 is very difficult to determine. In fact, even the existence of such sentences, satisfying

$$S(\underset{\sim}{M}) \models \psi, \quad \text{and not} \quad \models \theta \rightarrow \psi \quad \text{for any universal sentence} \quad \theta \in Th(\underset{\sim}{M}) \qquad (****)$$

in familiar contexts related to the archimedean property of $\langle \mathbb{R}, \leq, + \rangle$ was not recognized until recently [1].

The first result of this type can be obtained by relativizing to the universe of $\underset{\sim}{M}$ a construction of Herbrand concerning the validity problem of pure logic ([4] Thm. 9.1, p. 95 in [5]).

<u>Theorem 2.3</u>. Let σ be a purely relational similarity type, $\underset{\sim}{M}$ a σ-structure, and $\psi = \forall\vec{x}\exists\vec{y}\phi$, ϕ quantifier-free. Then

$$S(\underset{\sim}{M}) \models \psi \;\Leftrightarrow\; \underset{\sim}{M} \models \forall x_1 \cdots x_m \bigvee_{\vec{y}\in\{x_1,\ldots,x_m\}^n} \phi(x_1,\ldots,x_m,\vec{y})$$

where $n = \text{length}(\vec{y})$.

<u>Proof</u>. If $S(\underset{\sim}{M}) \models \psi$, then for any $x_1,\ldots,x_m \in \underset{\sim}{M}$, $\underset{\sim}{M} \upharpoonright \{x_1,\ldots,x_m\} \in S(\underset{\sim}{M})$ so $\underset{\sim}{M} \upharpoonright \{x_1,\ldots,x_m\} \models \exists\vec{y}\ \phi(x_1,\ldots,x_m,\vec{y})$ so $\underset{\sim}{M} \upharpoonright \{x_1,\ldots,x_m\} \models$ $\bigvee_{\vec{y}\in\{x_1,\ldots,x_m\}^n} \phi(x_1,\ldots,x_m,\vec{y})$. Thus $\underset{\sim}{M} \models \forall x_1 \cdots x_m \bigvee_{\vec{y}\in\{x_1,\ldots,x_m\}^n} \phi(\vec{x},\vec{y})$. Conversely, if $S(\underset{\sim}{M}) \not\models \psi$, then there is some $\underset{\sim}{A} \in S(\underset{\sim}{M})$ such that $\underset{\sim}{A} \models \exists\vec{x}\forall\vec{y}\ \neg\phi$.

Hence

$$\exists x_1 \cdots x_m \in \underset{\sim}{A}\colon\ \underset{\sim}{A} \restriction \{x_1 \cdots x_m\} \in S(\underset{\sim}{M}) \quad \text{and} \quad \underset{\sim}{A} \restriction \{x_1 \cdots x_m\} \models \forall \vec{y}\ -\phi(x_1 \cdots x_m, \vec{y}) ,$$

or

$$\exists x_1 \cdots x_m \in \underset{\sim}{M}\colon\ \underset{\sim}{A} \restriction \{x_1 \cdots x_m\} \models \bigwedge_{\vec{y} \in \{x_1 \cdots x_m\}^n} -\phi(x_1 \cdots x_m, \vec{y})$$

so that the condition on the right must also fail. □

From this argument, it follows that any $\forall\exists$ sentence ψ such that $S(\underset{\sim}{M}) \models \psi$ satisfies the conditions of Theorem 2.1, with the reduced universal form just the first $N = m^n$ conjuncts of the Herbrand expansion for ψ. The class of $\forall\exists$-sentences is the largest prefix class for which such a uniform result holds. The sentence $\exists x \forall y\text{-}Ryx$ satisfies (****) for any well-founded relation $\langle M,R\rangle = \underset{\sim}{M}$ with arbitrarily long R-chains.

Conditional results of the type of Thm 2.3 can obtained for somewhat more complicated quantifier prefix classes, now depending on properties of $\underset{\sim}{M}$. For a precise formulation of these results we consider the Herbrand expansions associated with the prefixes to be studied in a bit more detail than before. In considering whether $S(\underset{\sim}{M}) \models \exists\vec{x}\forall\vec{y}\text{-}\phi$ we consider attempts to construct a substructure of $\underset{\sim}{M}$ satisfying $\forall\vec{x}\exists\vec{y}\phi$. Such an attempt can be viewed as a process organized in stages: Let $X_K \subseteq |\underset{\sim}{M}|$ be the finite set of points accumulated by stage $K \in \omega$. Stage K+1 is reached iff for any $\vec{x}$ from X_K, there is a $\vec{y}$ from $\underset{\sim}{M}$ such that $\underset{\sim}{M} \models \phi(\vec{x},\vec{y})$, and X_{K+1} is the set of all points in all these $\vec{x}$ and some choice of one $\vec{y}$ per $\vec{x}$. The assertion that stage K+1 can be reached by using a sequence of solutions $\vec{y}$, from a finite nonempty initial set X_0, is a conjunction of variable substitution instances of $\phi(\vec{x},\vec{y})$ (each of which was some ϕ_k under our earlier description), and we will define ϕ_K to be the negation of this conjunction. In the following this convention will allow us to give bounds on the number of reachable stages in the process, rather than explicitly compute the bounds on the number of substitution instances of ϕ.

Noting that for ϕ quantifierfree, $\vec{x}$ of length ℓ,

$$S(\underset{\sim}{M}) \models \forall\vec{x}\exists\vec{y}\forall\vec{z}\phi \Leftrightarrow \forall\vec{s}\in|\underset{\sim}{M}|^{\ell}\ S((\underset{\sim}{M},\vec{s})) \models\exists\vec{y}\forall\vec{z}\phi(\vec{s},\vec{y},\vec{z})$$

we obtain a description of counterexample construction processes for $\forall\exists\forall$ sentences similar to the above, augmented by an initial step consisting of a choice of $\vec{s}\in|\underset{\sim}{M}|^{\ell}$, and the requirement that X_0 contains all elements of $|\underset{\sim}{M}|$ involved in $\vec{s}$. Then the ϕ_K, $K\in\omega$, are defined as above.

In the reduction theorems to be shown, we must consider the action of the automorphism group G of $\underset{\sim}{M}$. For any $\ell\in\omega$, $\vec{s}\in|\underset{\sim}{M}|^{\ell}$, we denote by $G(\vec{s})$ the group of automorphisms of $(\underset{\sim}{M},\vec{s})$. For any set S and group G of permutations of S, the <u>orbits of</u> S <u>under</u> G are the equivalence classes under the relation $S_1 \sim S_2 \Leftrightarrow \exists g\in G\ S_1 = g(S_2)$. For any $m\in\omega$, $[S]^m$ denotes the set of m-element subsets of S. Any permutation g of S induces a permutation on $[S]^m$, which we identify with g.

<u>Lemma 2.4</u>. Let G be a group of permutations of an infinite set S, $m\geq 1$. If $[S]^m$ consists of a single orbit under G, then for any ℓ, $1\leq\ell\leq m$, $[S]^{\ell}$ consists of a single orbit under G.

<u>Proof</u>. Let $\theta\in[S]^m$, and assume $\{O_\alpha\}$ is the set of orbits of $[S]^{\ell}$, and not a singleton. A certain finite set $\{O_i\}$ of orbits of $[S]^{\ell}$ are represented by $[\theta]^{\ell}$, but because $[S]^m$ forms <u>one</u> orbit under G, exactly the same configuration of orbits of $[S]^{\ell}$ is represented by any $\theta'\in[S]^m$, hence $\{O_\alpha\}$ is finite. By Ramsey's Theorem, get $S'\subseteq S$ of cardinality

at least m, homogeneous for one of the O_α. Then this must have been the configuration of orbits of $[S]^\ell$ represented by each $\theta \in [S]^m$, i.e. it is the unique orbit of $[S]^\ell$. □

Theorem 2.5. Let $\psi = \exists\vec{x}\forall\vec{y}\phi(\vec{x},\vec{y})$, ϕ quantifierfree formula, $\vec{x} = (x_1,\ldots,x_m)$. If $[|\underset{\sim}{M}|]^m$ consists of a single orbit under the action of the automorphism group G of $\underset{\sim}{M}$, then $S(\underset{\sim}{M}) \models \psi \Leftrightarrow\ \models \theta \rightarrow \psi$ for some universal sentence $\theta \in Th(\underset{\sim}{M})$.

Supplement. For θ we can take, for some $N < m$

$$\forall x_1 \cdots \vee_{K \leq N} \phi_K .$$

Proof. The implication from right to left is trivial. Assume thus that $S(\underset{\sim}{M}) \models \psi$. By Theorem 2.2, we only need show that $(\exists N \in \omega)\, \underset{\sim}{M} \models \forall x_1 \cdots \vee_{K \leq N} \phi_K$. Let k be the minimal number of elements of $|\underset{\sim}{M}|$ occurring in any m-tuple $\underset{\sim}{x}$ such that

$$\underset{\sim}{M} \models \forall\vec{y}\ \phi(\vec{x},\vec{y})$$

and let $\vec{x}_0$ be such an m-tuple in which exactly k distinct elements of $|\underset{\sim}{M}|$ occur. We will see that $N \leq k-1$, whence certainly $N < m$.

By induction, if $\underset{\sim}{M} \models \exists x_1 \cdots \wedge_{K \leq t} -\phi_K$, any instantiation $x_1,\ldots$ involves at least $t+1$ distinct elements of $|\underset{\sim}{M}|$: Clearly this is true for $t = 0$. Assume we have an instantiation for t_0 of no more than t_0 elements, and that the result has been shown for $t < t_0$. Then the subformula $\exists x_1 \cdots \wedge_{K \leq t_0 - 1} -\phi_K$ inherits an instantiation, which must contain t_0 elements of $|\underset{\sim}{M}|$. Thus our assumption means that we can extend the instantiation to $-\phi_{t_0}$ without adding any new elements of $|\underset{\sim}{M}|$. If X is the set of elements involved in the instantiation, then by the definition of the

ϕ_K and this last observation,

$$\underset{\sim}{M} \upharpoonright X \models \forall \vec{x} \exists \vec{y} -\phi(\vec{x},\vec{y})$$

contradicting $S(\underset{\sim}{M}) \models \psi$.

Applying this for $t = k-1$, any instantiation of

$$\underset{\sim}{M} \models \exists x_1 \cdots \wedge_{K \leq t} -\phi_K$$

must involve k distinct elements of $|\underset{\sim}{M}|$ in some ϕ_I, $I \leq K-1$. Again by the structure of the ϕ_I, all m-tuples formed from these elements must occur in ϕ_I in some substitution instance of $-\phi(x_1 \cdots x_m, \vec{y})$. But now one such m-tuple $\vec{x}_1$ must be an automorphic image of $\vec{x}_0$, for $[|\underset{\sim}{M}|]^m$ and hence $[|\underset{\sim}{M}|]^k$ by the lemma consists of a single orbit under the action of the automorphism group of $\underset{\sim}{M}$. Then $\underset{\sim}{M} \models \forall \vec{y}\ \phi(\vec{x}_1,\vec{y})$, when $-\phi(\vec{x}_1,\vec{y})$ occurs in ϕ_I (with some other variables substituted for $\vec{y}$); thus the instantiation considered cannot satisfy all ϕ_K, $K \leq t = k-1$. □

<u>Lemma 2.6</u>. Let a group G act on a set S, $\ell \geq 1$ fixed. If $\forall \vec{s} \in S^\ell$, S consists of finitely many orbits of $G(\vec{s})$, then S^ℓ consists of finitely many orbits of G.

<u>Proof</u>. Pick representatives of the orbits of S under $G(\vec{s})$ for each $\vec{s} \in \bigcup_{k=0}^{\ell} S^k$. We assign an orbit representative in S^ℓ to an arbitrary $\vec{s} \in S^\ell$: Let $\sigma_0 = \vec{s}$. At stage n, $1 \leq n \leq \ell$, given

$$\sigma_{n-1} = \langle r_1,\ldots,r_{n-1},s_n^{(n-1)},\ldots,s_\ell^{(n-1)}\rangle$$

let $r_n = \lambda_n(s_n^{(n-1)})$ be the representative of the orbit of $s_n^{(n-1)}$ under the action of $G(\vec{r})$, where $\vec{r} = \langle r_1,\ldots,r_{n-1}\rangle$, i.e. $\lambda_n \in G(\vec{r})$. Set

$$G_n = \lambda_n(\sigma_{n-1}) = \langle r_1,\ldots,r_n,s_{n+1}^{(n)},\ldots,s_\ell^{(n)}\rangle \quad .$$

Then $\sigma_\ell = \langle r_1 \cdots r_\ell \rangle$ is the representative of the orbit of $\vec{s}$ under G.

At each stage n, r_n was chosen from among finitely many candidates (dependent on $r_1 \cdots r_{n-1}$), so all possible r_i together form a finitely branching tree of finite depth ℓ. Thus only finitely many σ_ℓ are possible.

□

Theorem 2.7. Let $\psi = \forall\vec{x}\exists y\forall\vec{z}\ \phi(\vec{x},\vec{y},\vec{z})$, ϕ quantifier-free, $\vec{x} = \langle x_1 \cdots x_\ell \rangle$, $\ell \geq 0$. If $(\forall \vec{s} \in |\underset{\sim}{M}|^\ell)$ $|\underset{\sim}{M}|$ consists of finitely many orbits under the action of $G(\vec{s})$, then $S(\underset{\sim}{M}) \models \psi \Leftrightarrow\ \models \theta \rightarrow \psi$ for some universal sentence $\theta \in Th(\underset{\sim}{M})$.

Supplement. For θ we can take $\forall x_1 \cdots \vee_{K \leq N} \phi_K$, where N is the maximal number of orbits of $|\underset{\sim}{M}|$ under $G(\vec{s})$, $\vec{s} \in |\underset{\sim}{M}|^\ell$.

Proof. We view the attempted constructions of a substructure of $\underset{\sim}{M}$ satisfying

$$-\psi = \exists\vec{x}\forall y\exists\vec{z}\ -\phi(\vec{x},\vec{y},\vec{z})$$

as reflected in the Herbrand expansion of ψ as a family of trees, each with root some $\vec{s} \in |\underset{\sim}{M}|^\ell$, nodes elements of $|\underset{\sim}{M}|$, and such that for any node y its successors are exactly some $z_1,\ldots,z_m \in |\underset{\sim}{M}|$ such that

$$\underset{\sim}{M} \models -\phi(\vec{s},y,z_1 \cdots z_m) .$$

A node y is terminal iff $\underset{\sim}{M} \models \forall\vec{z}\ \phi(\vec{s},y,\vec{z})$. We must show that $\exists N \in \omega$: $(\forall \vec{s} \in |\underset{\sim}{M}|^\ell)(\forall$ tree with root $\vec{s})$ some node of level at most N is terminal; this is exactly the θ given in the supplement.

First fix $\vec{s} \in |\underset{\sim}{M}|^\ell$. We show by induction that of level K in any tree with root $\vec{s}$ representatives are to be found of at least K orbits of $|\underset{\sim}{M}|$ under the action of $G(\vec{s})$. This is clear for levels 0, 1. Assume that it first fails at level K, i.e. in extending the tree from level $K-1$ to level K no representatives of new orbits are added. Then we can extend from level K to level $K+1$ without adding representatives of new orbits. If y is any node at level K, there is $\lambda \in G(\vec{s})$ such that $\lambda(y)$ is a node of level $K-1$; if $\lambda(y)$ has successors $z_1 \cdots z_m$, then y can be given successors $\lambda^{-1}(z_1) \cdots \lambda^{-1}(z_m)$. We thus extend the given tree, restricted to level K, to a tree of depth $K+1$ again without obtaining representatives of new orbits, and subsequently to a complete tree of infinite depth, giving a counterexample to the hypothesis that $S(\underset{\sim}{M}) \models \psi$.

Now $|\underset{\sim}{M}|$ consists of finitely many orbits under the action of $G(\vec{s})$; it follows that trees with root $\vec{s}$ can only be complete up to this level, say $N(\vec{s})$. Next we consider $|\underset{\sim}{M}|^{\ell}$ under the action of G. It is clear that if $\lambda \in G$, $\lambda(\vec{s}) = \vec{t} \Rightarrow$ the λ-image of any tree with root $\vec{s}$ is a tree with root $\vec{t}$, and vice versa; hence $N(\vec{s}) = N(\vec{t})$. Because according to the lemma there are only finitely many orbits of $|\underset{\sim}{M}|^{\ell}$ under G, we find that there are at most finitely many distinct values $N(\vec{s})$, $\vec{s} \in |\underset{\sim}{M}|^{\ell}$. Then a uniform upper bound N clearly exists. To estimate N, note that $N(\vec{s})$ is bounded by the number of orbits of $|\underset{\sim}{M}|$ under $G(\vec{s})$, for any $\vec{s} \in |\underset{\sim}{M}|^{\ell}$.

□

Theorems 2.5 and 2.7 only apply to very special quantifier prefixes. Especially because of the similarity of the proofs, one would expect to find a common and more elegant generalization. Surprisingly, this does not seem to be the case. Theorems 2.5 and 2.7 seem to be the strongest possible of their type; this has been confirmed by the study of numerous examples arising in measurement theory [7]. We consider an example which demonstrates that the conclusion of the theorems does not in general hold for larger quantifier prefixes.

<u>Example</u>. Let $\underset{\sim}{M}$ be the structure $\langle \mathbb{R}, \leq, + \rangle$, in relational similarity type. The automorphisms of $\underset{\sim}{M}$ are exactly $x \mapsto \alpha x$; $\alpha \in \mathbb{R}$, $\alpha > 0$. For no $m > 0$ do we have $[|\underset{\sim}{M}|]^m$ consisting of a single orbit under the action of this group, so Theorem 2.5 does not apply. For $\ell = 0$, $|\underset{\sim}{M}|$ consists of ≤ 3 orbits under the action of the group $G(\vec{s})$, $\vec{s} \in |\underset{\sim}{M}|^\ell$, and for $\ell > 1$, $|\underset{\sim}{M}|$ consists of uncountably many orbits. Thus Theorem 2.7 applies for $\ell \leq 1$, and shows that '$S(\underset{\sim}{M}) \models \psi \Leftrightarrow \exists \theta$ universal $\in Th(\underset{\sim}{M})$: $\models \theta \rightarrow \psi$' holds for ψ of the form $\exists y \forall \vec{z}\ \phi(x,y,\vec{z})$. The simplest quantifier forms for which this might fail are $\forall x \exists y \forall z$, $\exists x_1 x_2 \forall z$, $\exists x \forall y \exists z$. Nothing is known about the case $\exists x \forall y \exists z$. We give examples of the other two cases. Set

$$\psi_1 = \forall x \exists y \forall z\ -[(y=x\ \&\ z<x) \vee (y=z+x\ \&\ y<z<x)]$$

$$\psi_2 = \exists x_1 x_2 \forall y\ -\phi(x_1 x_2 y)$$

where $\phi(x_1 x_2 y) =$

(1.1)	$x_2 < x_1 \rightarrow [[y+y=y\ \&\ y \leq x_1\ \&\ y \neq x_2]$	0 $\in X_2$ (at the latest) no two distinct negative numbers in X_k, $\forall k \in \omega$
	$\vee\ [x_2+x_2=x_2\ \&\ x_2<y<x_1]]$	no number is ultimately the minimal number > 0 in X_k, $k \rightarrow \infty$
	&	
(2)	$x_1 = x_2 \rightarrow y \neq x_1$ &	at least two distinct numbers (i.e. in X_1)
(3.1)	$x_1 < x_2 \rightarrow [[x_1+x_1=x_1\ \&\ y<x_1]$	X_3 (at latest) contains $y < 0$
(3.2)	$\vee\ [x_2+x_2=x_2\ \&\ x_2<y]$	X_3 (at latest) contains $y > 0$
(3.3)	$\vee\ [x_1+x_2=y\ \&\ y \neq x_1$ $\&\ y \neq x_2]]$	If $x_1, x_2 \neq 0$, then "shift x_2 left by $-x_1$"

The annotations refer to the analysis of the processes associated with the Herbrand expansion of ψ_2 (as described above, preceding Lemma 2.4).

Proposition 2.8. For ψ_1 and ψ_2 above, $S(<\mathbb{R},\leq,+>) \models \psi_1$, but neither ψ_1 nor ψ_2 is a logical consequence of a universal sentence in $Th(<\mathbb{R},\leq,+>)$.

Proof. The proof depends on an analysis of attempted constructions of counterexample substructures, i.e. in which ψ_1 (or ψ_2) fails, which will not be given here in detail. (See [7], pp. 52-56.) In both cases, the constructions necessarily involve approaching a barrier with a fixed step size from an initially chosen starting point. In the archimedean case, i.e. when the construction is executed on $\mathbb{R}$, one must hit the barrier after a number of steps which is finite, but arbitrarily large depending on the initial choices. On the other hand, if we work in a nonarchimedean model of $Th(<\mathbb{R},\leq,+>)$, we can choose the step size infinitesimal compared to the distance to the barrier, and will produce a substructure in which the sentence (ψ_1,ψ_2) is false.

We conclude this section with a more radical but less constructive case of reduction to the universal theory of $\underset{\sim}{M}$.

Theorem 2.9. If the first-order theory of $\underset{\sim}{M}$ is $\aleph_0$-categorical, then for any sentence ψ, $S(\underset{\sim}{M}) \models \psi \Leftrightarrow \models \theta \rightarrow \psi$ for some universal sentence $\theta \in Th(\underset{\sim}{M})$.

Proof. Any sentence $\phi \in Th(\underset{\sim}{M})$ which is false in some substructure $\underset{\sim}{S}$ of a model $\underset{\sim}{N}$ of $Th(\underset{\sim}{M})$ is false in a countable substructure $\underset{\sim}{S}'$ of $\underset{\sim}{N}$, but then $\underset{\sim}{S}'$ is a substructure of a countable model $\underset{\sim}{N}'$ of $Th(\underset{\sim}{M})$, all by the Löwenheim-Skolem-Tarski Theorem. By $\aleph_0$-categoricity, and again LST, $\underset{\sim}{N}'$ and hence $\underset{\sim}{S}'$ are substructures of $\underset{\sim}{M}$, so $\phi \notin Th(S(\underset{\sim}{M}))$. Hence $Th(S(\underset{\sim}{M}))$ is just the set of logical consequences of the universal theory of $\underset{\sim}{M}$. □

Example 2.10. Let $\underset{\sim}{M} = \langle \mathbb{R}, \leq \rangle$. (This application corresponds exactly to "ordinal measurement" [6].) It is a classical result that this theory is $\aleph_0$-categorical. In fact, $S(\underset{\sim}{M})$ contains all countable linear orders, so that $Th(S(\underset{\sim}{M}))$ is just the theory of linear order.

3. The Decision Problem

It does not seem possible to treat the decision problem for the theory of $S(\underset{\sim}{M})$ in full generality: the choice of the structure $\underset{\sim}{M}$ will strongly influence the theory of $S(\underset{\sim}{M})$. One approach is to consider the complexity of the theory of $S(\underset{\sim}{M})$ relative to the theory of $\underset{\sim}{M}$. The effective reduction methods of the previous section give rise to such relative decision procedures.

Theorem 3.0. For any infinite structure $\underset{\sim}{M}$:

(a) the following fragments of the theory of $S(\underset{\sim}{M})$ are recursive in the universal theory of $\underset{\sim}{M}$:

(i) All $\forall\exists$ (and simpler) sentences.

(ii) All $\exists x_1 \cdots x_m \forall \vec{y}$-sentences, where m is such that $[|\underset{\sim}{M}|]^m$ consists of a single orbit under the action of the automorphism group of $\underset{\sim}{M}$.

(iii) All $\forall x_1 \cdots x_\ell \exists y \forall \vec{z}$-sentences, where $|\underset{\sim}{M}|$ consists of finitely many orbits under the action of $G(\vec{s})$, for any $\vec{s} \in |\underset{\sim}{M}|^\ell$.

(b) The decision procedures of (ii) and (iii) can be taken uniform in m and ℓ respectively, except possibly in (iii) if the finiteness condition holds for all $\ell \in \omega$. In this case, however, $Th(\underset{\sim}{M})$ is $\aleph_0$-categorical, and (iv) All of $Th(S(\underset{\sim}{M}))$ is r.e. in the universal theory of $\underset{\sim}{M}$ if $Th(\underset{\sim}{M})$ is $\aleph_0$-categorical.

Proof. The assertions (i)-(iv) of the theorem are immediate consequences of Theorems 2.3, 2.5, 2.7, 2.9 respectively. The uniformity of (ii) in m is because the search bound in Theorem 2.5 is recursive in the input sentence ψ_1 namely m. In (iii) no analogous statement holds: the search bound depends on the orbit structure of $\underset{\sim}{M}$ under $G(\vec{s})$, $\vec{s} \in |\underset{\sim}{M}|^\ell$. If there is a maximal ℓ such that the finiteness condition of (iii) holds, this gives a uniform search bound for the procedure. On the other hand, if the finiteness condition holds for all $\ell \in \omega$, then by Lemma 2.6, $|\underset{\sim}{M}|^\ell$ consists of finitely many orbits under the action of the automorphism group G of $\underset{\sim}{M}$, for all $\ell \in \omega$. Then $\underset{\sim}{M}$ has only finitely many ℓ-types, for any $\ell \in \omega$, and by completeness of $Th(\underset{\sim}{M})$, $Th(\underset{\sim}{M})$ has at most finitely many ℓ-types, for any $\ell \in \omega$. As $Th(\underset{\sim}{M})$ is complete and $\underset{\sim}{M}$ infinite, $Th(\underset{\sim}{M})$ is $\aleph_0$-categorical by the Ryll-Nardjewski Theorem ([11], p. 91). □

In order to obtain further insight into the nature of the decision problem, we will consider a single example in detail. It will become clear from the results obtained that the relative recursiveness results of Theorem 3.0 arethe strongest relative complexity results that can be obtained in general. Our case study concerns the relational structure $\underset{\sim}{M} = \langle \mathbb{R}, \leq, + \rangle$. The first-order theory of this structure is decidable; in fact quite efficient decision procedures are known [2]. Applying the reduction results above, we have

Corollary 3.1. The following fragments of $Th(S(\langle \mathbb{R}, \leq, + \rangle))$ are decidable:

(i) All $\forall\exists$-sentences.

(ii) All $\exists x \forall \vec{y}$-sentences.

This follows from (i) and (iii) above, by the analysis of orbit structure on $<\mathbb{R},\leq,+>$ given earlier; that analysis also shows that no stronger results can be obtained from Theorem 3.0 in this case. In fact we show below that the problem of deciding whether an $\exists x_1 x_2 \forall \vec{y}$-sentence belongs to $Th(S(<\mathbb{R},\leq,+>))$ is Π_1^1-complete. Thus the complexity of this problem stands in no relationship to that of deciding $Th(<\mathbb{R},\leq,+>)$. We first give an upper bound on the complexity of the theory of all substructures which applies to many cases of practical interest in measurement theory:

Theorem 3.2. Let $\underset{\sim}{M} = <\mathbb{R},\leq,+,\cdot>$, in relational similarity type. There is an effective many-one reduction of the set of sentences true in all substructures of $\underset{\sim}{M}$, to the set of all true Π_1^1-sentences of arithmetic.

Proof. By Theorem 2.1, we have that for any n-ary quantifier prefix Q there is an effective procedure $e\colon \omega \to \omega^n$ such that for any sentence $Q-\phi(x_1\cdots x_n)$, ϕ quantifierfree,

$$S(\underset{\sim}{M}) \models Q\text{-}\phi \Leftrightarrow (\forall f\colon \omega \to |\underset{\sim}{M}|)(\exists i \in \omega)\, \underset{\sim}{M} \models -\phi(f(e(i)_1),\ldots,f(e(i)_n)) .$$

Now the expression on the right hand side can be reduced to an equivalent Π_1^1-sentence of arithmetic, using the standard set-theoretic construction of the ordered real field from the natural numbers via Cauchy sequences of rational numbers. □

The strongest known undecidability result is for the theory of $S(<\mathbb{R},\leq,+>)$ in relational similarity type:

Theorem 3.3. Let σ be the relational similarity type of $<\mathbb{R},\leq,+>$. Then $Th(S(<\mathbb{R},\leq,+>))$ is a Π_1^1-complete set of sentences. In fact, the set Σ of σ-sentences with quantifier prefix $\exists x_1 x_2 \forall \vec{y}$ which are true in $S(<\mathbb{R},\leq,+>)$ is Π_1^1-complete.

Proof (initial comments). By Theorem 3.2, Σ and $\mathrm{Th}(S(<\mathbb{R},\leq,+>))$ are effectively reducible to the set of true Π^1_1-sentences of arithmetic. Below we give an effective many-one reduction of the set of true Π^1_1-sentences of arithmetic to Σ. Some preliminary definitions and lemmas are needed.

Let $p(x,y,z)$ denote the standard pairing relation on the natural numbers, i.e. $p(x,y,z) \leftrightarrow z = x + \frac{1}{2}(x+y)(x+y+1)$. We denote by $\underset{\sim}{N}$ the structure $<\omega,<,+,\cdot,p>$; call its similarity type τ. A function $\alpha \in {}^{\omega}\omega$ is an initial segment coding function (abbreviated: $cd(\alpha)$) iff $\alpha(0) = 0$ & $\forall n \in \omega$ $\exists m \in \omega$ $p(f(n),m,f(n+1))$. The classical results relevant to our reduction can be summarized by

Lemma 3.4. There is an algorithm A which transforms any Π^1_1-sentence s in similarity type τ into a Π^1_1-sentence $A(s)$ such that

(i) $\underset{\sim}{N} \models s \Leftrightarrow \underset{\sim}{N} \models A(s)$

(ii) $A(s)$ is of the form

$$\forall\alpha[cd(\alpha) \rightarrow \exists x \forall y_1 \cdots y_k \; \psi(x,y_1 \cdots y_k,\alpha)]$$

where ψ is a boolean combination of atomic relational formulas of the forms $a < b$, $a+b = c$, $a\cdot b = c$, $p(a,b,c)$, $\alpha(a) = b$, and constants from at most among 0, 1. (Call this similarity type τ'.)

Proof. By a normal-form theorem of recursion theory ([11], p. 175), any Π^1_1-sentence of this similarity type can be effectively transformed into a sentence of the form

$$\forall\alpha[cd(\alpha) \rightarrow \exists xy[\alpha(x) = y \;\&\; R(x,y)]]$$

$R(x,y)$ recursive relation not involving α;

equivalent on $\underset{\sim}{N}$ to the original sentence. We can easily get an $\exists\forall$-formula in the desired relational similarity type (τ') for the recursive relation R (even an $\exists$-formula, using the techniques of Matijasevic [8],[3]). Using the pairing relation, all the existential quantifiers except the first can be converted to universal quantifiers, giving the desired result. □

Lemma 3.5. For the formula $\phi(x,y)$ and the system of definitions (9.1)-(14.2) given below:

(i) for each pair of existential and universal definitions ψ_e, ψ_u given below

$$S(<\mathbb{R},\leq,+>) \models \forall xy\ \phi(x,y) \rightarrow (\psi_e \leftrightarrow \Psi_u)$$

(ii) there is a 1-1 correspondence between isomorphism classes of models of $\forall xy\ \phi(x,y)$ in $S(<\mathbb{R},\leq,+>)$ and expansions $(\underset{\sim}{N},\alpha)$ of $\underset{\sim}{N}$ by initial segment coding functions α, given by the interpretation J:

$\omega \leftrightarrow nat(\cdot)$	i.e. (9.1) or (9.2)
$0 \leftrightarrow x+x=x$	
$1 \leftrightarrow c$	i.e. the (unique) solution for the existential quantifier '$\exists c$' in $\phi(x,y)$
$+ \leftrightarrow x+y=z$	
$\cdot \leftrightarrow x\cdot y=z$	(13.1) or (13.2)
$p \leftrightarrow p(x,y,z)$	(11.1) or (11.2)
$\alpha \leftrightarrow \alpha(x)=y$	(14.1) or (14.2)

of $(\underset{\sim}{N},\alpha)$ in any structure $\underset{\sim}{S}$ in $S(<\mathbb{R},\leq,+>)$ satisfying $\forall xy\ \phi(x,y)$. This interpretation is faithful, i.e. for any initial segment coding function, and any (not necessarily first order) sentence $\phi(\alpha)$ of similarity type τ' with interpretation $\phi^J(\alpha)$,

$$(\underset{\sim}{N},\alpha) \models \phi(\alpha) \Leftrightarrow \underset{\sim}{S} \models \phi^J(\alpha) .$$

Remark. Because the interpreting formulas contain parameters, the structure $\underset{\sim}{S}$ in the above is understood to be of similarity type expanded by these parameters. Alternatively, the parameters can be eliminated, as they are in fact definable. For this see the proof below of Theorem 3.3.

$\phi(x,y) =_{def.} \exists tabcdef\{$

(1.1) $2a = a \ \& \ t < a \ \& \ (x < a \to y \geq x)$

(1.2) $\& \ t+b = a \ \& \ b+b = c \ \& \ d+d = b \ \& \ e+e = d \ \& \ f+f = e$

(2.0) $\& \ \exists y_1 y_2 y_3 \ \ y_1+b = y_2 \ \& \ y_2+b = y_3 \ \&$

(2.1) $y_1 < x < y_2 \vee y_2 < x < y_3 \vee \exists z[x+b = y_1 \ \&$

(2.2) $[(y_1 < z < y_2 \ \& \ -(y_2 < y < y_3))$

(2.3) $\vee \ (-(y_1 < y < y_2) \ \& \ y_2 < z < y_3)]]$

(3.1) $\& \ natu(x) \to \exists y_1 y_2 y_3 \ \ y_1 = x+e \ \& \ y_2 = x+d \ \& \ y_3 = y_2+e$

(3.2) $\& \ (c < x < y \ \& \ \forall z \ -(x < z < y)) \longrightarrow (Col(x) \vee Col(y))$

(3.3) $\& \ a \leq x \longrightarrow \exists uz[Col(u) \ \& \ a \leq z < e \ \& \ (x = u+z \vee u = x+z)]$

(4.1) $\& \ (a < x < e \longrightarrow \exists x_1 x_2 \ \ 2x_1 = x \ \& \ 2x = x_2) \ \& \ -(f < x < e)$

(4.2) $\& \ natu(x) \longrightarrow \exists vv_1 x_1 zz_1[a < z < e \ \& \ z = 2z_1 \ \& \ x_1 = x+c \ \& \ v = x+z \ \& \ v_1 = x_1+z_1$
$\& \ (x = a \longrightarrow 2z_1 = f)]$

(5.1) $\& \ \exists v_1 v_2 w[v_1+f = d \ \& \ v_2 = d+f \ \& \ w = d+e \ \& \ (e < y < w \longrightarrow (y = v_1 \vee y = d \vee y = v_2))]$

(5.2) $\& \ natu(x) \longrightarrow \exists uu'[u = x+d \ \& \ u' = u+c$
$\& \ \exists z_1 z_2 v_1 v_2[a < z_1 < e \ \& \ a < z_2 < e \ \& \ v_1+z_1 = u \ \& \ u+z_2 = v_2$
$\& \ \exists z_1' z_2' v_1' v_2'[(v_1'+z_1' = u' \ \& \ u'+z_2' = v_2')$
$\& \ [(z_1' = z_2 = f \ \& \ 2z_2' = z_1)$
$\vee \ (z_2 < f \ \& \ z_2' = 2z_2 \ \& \ 2z_1' = z_1)]]]]$

(6.1) $\& \ \exists uv[u+e = b \ \& \ v+f = b \ \& \ (u < y < b \longrightarrow y = v)]$

(6.2) $\& \ natu(x) \longrightarrow \exists x'ww'v[x' = x+c \ \& \ w =_e \alpha(x) \ \& \ w' =_e \alpha(x') \ \& \ p_e(w,v,w')]\}$.

(7) $2x = y =_{def.} \ x+x = y$

(8) $Col(x) =_{def.} \ \exists uv[nate(u) \ \& \ x = u+v \ \& \ (v = a \vee v = c \vee v = d \vee v = d+e \vee v = b)]$

(9.1) $\text{nate}(x) =_{\text{def.}} \exists y_1 y_2 z[y_1+b = x \;\&\; x+b = y_2 \;\&\; x < z < y_2]$

(9.2) $\text{natu}(x) =_{\text{def.}} \forall x_1 x_2 x_3 \; -(x_1+b = x_2 \;\&\; x_2+b = x_3 \;\&\; x_1 < x < x_3 \;\&\; x \neq x_2)$
$\qquad \& \; -(x+b = x_1 \;\&\; x_1+b = x_2 \;\&\; x_1 < x_3 < x)$

(10.1) $y =_e [x] =_{\text{def.}} x = a \;\&\; y = f$
$\qquad \vee \; x > a \;\&\; \text{nate}(x) \;\&\; a < y < e \;\&\; \exists v \;\; v = x+y$

(10.2) $y =_u [x] =_{\text{def.}} x = a \;\&\; y = f$
$\qquad \vee \; x > a \;\&\; \text{natu}(x) \;\&\; a < y < e \;\&\; \forall uv[(u = x+e$
$\qquad \& \; x < v < u) \rightarrow v = x+y]$

(11.1) $p_e(w,x,y) =_{\text{def.}} \text{nate}(y) \;\&\; \exists u v_1 v_2 z_1 z_2 \;\; u = y+d \;\&\; v_1+z_1 = u \;\&\; u+z_2 = v_2$
$\qquad \& \; z_1 =_e [w] \;\&\; z_2 =_e [x]$

(11.2) $p_u(w,x,y) =_{\text{def.}} \text{natu}(y) \;\&\; \forall u v_1 v_2 z_1 z_2[(u = y+d \;\&\; v_1+z_1 = u \;\&\; u+z_1 = v_2$
$\qquad \& \; a < z_1 < e \;\&\; a < z_2 < e)$
$\qquad \rightarrow (z_1 =_u [w] \;\&\; z_2 =_u [x])]$

(12.1) $x =_e y^2 =_{\text{def.}} \exists zuv[p_e(yyz) \;\&\; u = 2x \;\&\; v = 2y \;\&\; z = u+v]$

(12.2) $x =_u y^2 =_{\text{def.}} \forall zuv[u = 2x \;\&\; v = 2y \;\&\; z = u+v \rightarrow p_u(y_1 y_1 z)]$

(13.1) $x \cdot y =_e z =_{\text{def.}} \exists u_1 u_2 u_3 v_1 v_2 w[v_1 = x+y \;\&\; u_1 =_e v_1^2 \;\&\; u_2 =_e x^2 \;\&\; u_3 =_e y^2$
$\qquad \& \; w = u_1+u_2 \;\&\; v_2 = 2z \;\&\; u_1 = v_2+w]$

(13.2) $x\, y =_u z =_{\text{def.}} \forall u_1 u_2 u_3 v_1 v_2 w[(v_1 = x+y \;\&\; u_1 =_e v_1^2 \;\&\; u_2 =_e x^2 \;\&\; u_3 =_e y^2$
$\qquad \& \; w = u_1+u_2 \;\&\; u_1 = v_2+w) \rightarrow v_2 = 2z]$

(14.1) $\alpha(x) =_e y =_{\text{def.}} \text{nate}(x) \;\&\; \exists uvz[u = x+b \;\&\; a < z < e \;\&\; v+z = u \;\&\; z =_e [y])$

(14.2) $\alpha(x) =_u y =_{\text{def.}} \text{natu}(x) \;\&\; \forall u_1 u_2 vz[(u_1 = x+b \;\&\; u_2+e = u_1 \;\&\; u_2 < v < u_1$
$\qquad \& \; z+v = u_1) \rightarrow z =_u [y]]$

Proof. Let $\underset{\sim}{M} \in S(\langle \mathbb{R}, \leq, + \rangle)$, $\underset{\sim}{M} \models \forall xy\, \phi$. We determine $\underset{\sim}{M}$.

By clause (1.1), $a = 0 \in |\underset{\sim}{M}|$, and $\underset{\sim}{M}$ contains at most one negative element. By (1.2), $\underset{\sim}{M}$ contains an element $t < 0$, which is therefore unique, and determines other elements $b = -t$, $c = 2b$, ... (In constructing the interpretation, we will identify $a = 0$, $c = 1$.) We will show that aside from the actual magnitude of $c \in \mathbb{R}$, $\underset{\sim}{M}$ is completely determined by $\forall xy\ \phi$ and the choice of an arbitrary $\alpha \in {}^{\omega}\omega$.

Clauses (2.0)-(2.3) determine the periodic structure used to define the natural numbers. Set $t_n = t + nb$, $n \in \omega$. (We have not yet shown that $t_n \in |\underset{\sim}{M}|$, $n > 3$.) For $n = 0$ we know by (1.1), (1.2) that

(i) $\underset{\sim}{M}$ contains t_n, t_{n+1}, t_{n+2}.

(ii) $\underset{\sim}{M}$ contains a point z: $t_{n+1} < z < t_{n+2}$

(iii) $\underset{\sim}{M}$ contains no z: $t_n < z < t_{n+1}$

Assume we know (i)-(iii) for $n > 0$, n even. We show that (i)-(iii) hold for $n+2$. Considering $\forall xy\ \phi$ for $x = t_n$, the option of (2.1) is excluded by (iii), and for $y = z$ as given in (ii), the option of (2.3) is excluded. Thus (2.2) must hold, and

(i)' $\underset{\sim}{M}$ contains t_{n+1}, t_{n+2}, t_{n+3}

(ii)' $\underset{\sim}{M}$ contains no z: $t_{n+2} < z < t_{n+3}$

(iii)' $\underset{\sim}{M}$ contains no point z: $t_n < z < t_{n+1}$

(iv)' $\underset{\sim}{M}$ contains a point z: $t_{n+1} < z < t_{n+2}$

Then, setting $x = t_{n+1}$, the option (2.1) is excluded by (iii)', and (2.2) by (ii)', so that (2.3) must hold. This gives exactly (i)-(iii) for $n' = n+2$. Inductively, we have (i)-(iii) for n even, and (i)'-(iv)' for $n+1$ odd.

Now the set of t_n, n odd is defined on $\underset{\sim}{M}$ by nate(x) (9.1) and natu(x) (9.2): In (9.1), $y_1 + b = x$ & $x + b = y_2$ exclude (2.1) for x, so that $x = t_n$ for some n; then $x < z < y_2$ conflicts with (iii), so that n is odd. Similarly, (9.2) excludes (2.1) and (2.2), so that (2.3) must hold, hence n is odd.

(3.1) gives repetition of "column markers" in each integral period (t_{2n+1}, t_{2n+3}); (8) abbreviates $\mathrm{Col}(x)$ for these markers and t_{2n+1}, t_{2n+2}, t_{2n+3}. (3.2) asserts that of any two consecutive points of $\underset{\sim}{M}$, at least one is a column marker, so that as long as the set of points of $\underset{\sim}{M}$ between any two consecutive column markers is discrete it contains at most one point. By (3.3), any point differs from a column marker by some $z \in [a,e)$. By (4.1) the set $[a,e)$ is discrete, so that (3.2) does imply that there is a unique point between any consecutive column markers.

(4.1) entails that the elements of $[a,e) = [0, c\cdot 2^{-3})$ are $c\cdot 2^{-3-n}$, $n \in \omega$; (4.2) gives a recursion to ensure that the column $(t_{2n+1}, t_{2n+1}+e)$ contains (exactly) the point $t_{2n+1} + c\cdot 2^{-3-n}$. This gives the equivalence of the definitions (10.1) and (10.2) of '$y = [x]$', i.e. '$y \in (a,e)$ is a code for $x \in \omega$'.

(5.1) gives the basis and (5.2) the recursive clause of the definition of the pairing relation $p(x,y,z)$, which is coded in $\underset{\sim}{M}$ by: $\forall x,y,z \in \omega$:

$$p(x,y,z) \leftrightarrow t_{2z+1} + 2e - c\cdot 2^{-3-x} \in (t_{2z+1}+e, t_{2z+1}+2e)$$
$$\& \; t_{2z+1} + 2e + c\cdot 2^{-3-y} \in (t_{2z+1}+2e, t_{2z+1}+3e) \; .$$

From this, we have the equivalent definitions (11.1) and (11.2) of $p(x,y,z)$. It is now a purely arithmetical matter to obtain definitions of $y = x^2$ and $z = x\cdot y$ on ω from the pairing and addition: $p(y,y,z) \leftrightarrow z = 2y^2 + 2y$, and $(x+y)^2 - x^2 - y^2 = 2xy$.

Finally, (6.1) gives the basis and (6.2) the inductive clause for the definition of an initial segment coding function α:

$$\forall xy \in \omega \quad \alpha(x) = y \leftrightarrow t_{2x+1} + b - c\cdot 2^{-3-y} \in (t_{2x+1}+b-e, t_{2x+1}+b) \; .$$

The variable 'v' existentially quantified over in (6.2) is the only one in $\phi(x,y)$ whose value is not uniquely determined by c; in fact the possible values are just the 'natural numbers' t_{2n+1}, $n \in \omega$. Thus α is an arbitrary initial segment coding function.

We claim that $\underset{\sim}{M}$ is completely determined by the choice of positive $c \in \mathbb{R}$ and initial segment coding function $\alpha \in {}^{\omega}\omega$. For $\underset{\sim}{M}$ only contains those elements explicitly specified by existential quantification in $\phi(x,y)$: the condition that there be at most one point between any two column markers, and the explicit specification of all points in the interval (t,c) gaurantee this. On the other hand, it is also clear that any $c > 0$ in $\mathbb{R}$ can be chosen, and any α; models with distinct α are clearly nonisomorphic and models with the same α are isomorphic.

That the interpretation J is faithful follows because (1) conditions (9.1), (9.2) determine a set nat on $\underset{\sim}{M}$ such that $\langle nat,\leq,+\rangle$, where $\leq$, $+$ are inherited from $\langle \mathbb{R},\leq,+\rangle$, is isomorphic to $\langle \omega,\leq,+\rangle$, and (2) the inductive definition of p, the arithmetic definition of $\cdot$, and the description of $cd(\alpha)$ are obviously correct. (What is crucial in the present context is that $\langle nat,\leq\rangle \simeq \langle \omega,\leq\rangle$; this follows from the archimedean property of $\langle \mathbb{R},\leq,+\rangle$. For most nonstandard models of $\mathbb{R}$ the definition given would not necessarily yield a well-order $\langle nat,\leq\rangle$. This is as it should be, for as we consider more and more saturated models $\underset{\sim}{M}'$ of the theory of $\langle \mathbb{R},\leq,+\rangle$, the theory of $S(\underset{\sim}{M}')$ becomes smaller and smaller; for recursively saturated $\underset{\sim}{M}'$ we have $\mathrm{Th}\ S(\underset{\sim}{M}')$ = the universal theory of $\langle \mathbb{R},\leq,+\rangle$.) $\square$

Using Lemmas 3.4 and 3.5, we give the transformation required to complete the proof of Theorem 3.3. Let $\exists x \forall y_1 \cdots y_k\ \psi(x,y_1 \cdots y_k,\alpha)$ be the consequent formula of the sentence (say: s) obtained by the transformation of Lemma 3.4. Let ψ^* be the formula obtained from ψ by substituting the defining formulas associated with claim (ii) of Lemma 3.5, where we always choose the universal defining formula of a pair for substitution in a positive context within ψ and the existential defining formula for substitution in a negative context. Thus ψ^* is a universal formula. Using the abbreviation

$$\Delta(tabcdef) =_{\text{def.}} a = 2a \ \& \ t < a \ \& \ t+b = a \ \& \ 2b = c \ \& \ 2d = b \ \& \ 2e = d \ \& \ 2f = e$$

we output

$$B(s) = \forall xy\ \phi(x,y) \rightarrow \exists x \forall tabcdef\, \forall y_1 \cdots y_k[(\Delta \ \&_{i}^{k} \ nate(y_i)) \rightarrow natu(x) \ \& \ \psi^*(x,y_1 \cdots y_k)] \ .$$

Then $B(s)$ is clearly (logically equivalent to) an $\exists x_1 x_2 \forall \vec{y}$-sentence of the similarity type of $<\mathbb{R},\leq,+>$. We must show that

$$S(<\mathbb{R},\leq,+>) \models B(s) \Leftrightarrow \underset{\sim}{N} \models \forall\alpha[cd(\alpha) \rightarrow \exists x \forall y_1 \cdots y_k\, \psi(x,y_1 \cdots y_k)] \quad (\text{i.e. } \underset{\sim}{N} \models s) \ .$$

Assume that $\underset{\sim}{N} \models s$. Let $\underset{\sim}{S} \in S(<\mathbb{R},\leq,+>)$. If $\underset{\sim}{S} \models -\forall xy\ \phi(x,y)$, then certainly $\underset{\sim}{S} \models B(s)$. If $\underset{\sim}{S} \models \forall xy\ \phi(x,y)$, then $\underset{\sim}{S}$ codes $(\underset{\sim}{N},\alpha)$ for some initial segment coding function α. Because $\underset{\sim}{N} \models s$, $(\underset{\sim}{N},\alpha) \models \exists x \forall y_1 \cdots y_k\ \psi(x,y_1 \cdots y_k)$. Because the interpretation J of Lemma 3.5 is faithful, $\underset{\sim}{S} \models B(s)$. Conversely, if $\underset{\sim}{N} \models -s$, then for some initial segment coding function α, $(\underset{\sim}{N},\alpha) \models -\exists x \forall y_1 \cdots y_k\, \psi(x,y_1 \cdots y_k,\alpha)$. By Lemma 3.5, let $\underset{\sim}{S} \in S(<\mathbb{R},\leq,+>)$ be such that $\underset{\sim}{S} \models \forall xy\ \phi$ and $\underset{\sim}{S}$ codes α. Then, by the faithfulness of the interpretation J, the (major) consequent of $B(s)$ is false in $\underset{\sim}{S}$, and the antecedent $\forall xy\ \phi$ is true in $\underset{\sim}{S}$, i.e. $\underset{\sim}{S} \models -B(s)$. □

This completes the proof of Theorem 3.3. The remarkable complexity of $Th(\langle \mathbb{R},\leq,+\rangle)$ is explained by the fact that this is in fact equivalent to a fragment of the monadic second-order theory of $\langle \mathbb{R},\leq,+\rangle$, by the construction (**) of section 2. Viewed in this way, the argument of Theorem 3.3 gives a method for showing undecidability of monadic second-order theories. This method is interesting because, when it works, it shows undecidability of very small fragments of the monadic theory. Examination of the construction of Theorem 3.3 shows that models of $\forall xy\ \phi(x,y)$ are not only substructures of $\mathbb{R}$, but of the dyadic rationals $D = \{n/2^k : n,k \in \omega\}$. Thus we have Π^1_1-completeness of $Th(S(\underset{\sim}{M}))$ for $\underset{\sim}{M} = \langle D,\leq,+\rangle$ and $\underset{\sim}{M} = \langle \mathbb{Q},\leq,+\rangle$ by the same argument.

A number of questions related to Theorem 3.3 are still open. With $\underset{\sim}{M} = \langle \mathbb{R},\leq,+\rangle$, what is the complexity of the $\forall x\exists y\forall\vec{z}$ and $\exists x\forall y\exists\vec{z}$-theories of $\underset{\sim}{M}$? If $\underset{\sim}{M} = \mathbb{R}$ in other weak similarity types, what is the complexity of $Th(S(\underset{\sim}{M}))$? Are there any general methods for showing undecidability for broad classes of $\underset{\sim}{M}$, comparable to the reduction principles of section 2?

Similar decision problems have rarely been studied in the literature. One example is the undecidability of the theory of the subalgebras of $\langle N,+\rangle$, due to McKenzie [9]. One could ask the analogous question for $Th(S\langle \mathbb{R},\leq,+\rangle)$ in the similarity type with + operation. Our construction above seems to depend strongly on consideration of substructures which are not subsemigroups. This suggests that this theory might be more like that studied by McKenzie, i.e. r.e. nonrecursive rather than Π^1_1-complete.

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