PSEUDOCLASSICAL TRANSPORT I:
THE PARTICLE AND ENERGY FLUX

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ABSTRACT

The transport of particles and energy that accompanies the trapping of electrons by a finite amplitude drift wave is calculated. Starting from the drift kinetic equation, it is shown that, in the limit of small collision frequency, the electron entropy source is stationary with respect to variations in the electron distribution function. This variational principal is employed, together with the <u>full</u> Fokker-Planck collision operators, to evaluate the electron transport coefficients and, hence, the flux of particles and energy across the magnetic field. Explicit expressions for the particle and energy flux are obtained in terms of the parameters of the plasma-wave system. These expressions should be used <u>in place of</u> the usual "quasilinear" expressions for the particle and energy flux when the autocorrelation time of the wave spectrum is sufficient to permit the trapping of electrons by individual waves. These "pseudoclassical" transport rates are found to be smaller than the quasilinear expressions that they replace. The particle flux obtained here is used in a companion paper to develop a self-consistent theory of the evolution of a finite amplitude drift wave.

1. INTRODUCTION

Low frequency drift wave instabilities such as the collisionless drift instability (Galeev et al., 1963) and the dissipative trapped electron instability (Kadomtsev and Pogutse, 1969) are of considerable current interest. These instabilities may be responsible for the anomalous transport observed in tokamaks (Dean et al., 1974).

Because the phase velocity of the low frequency drift wave is small compared to the electron thermal velocity, there are many electrons with parallel velocities near the phase velocity of this wave. These resonant electrons contribute to the linear (i.e., small amplitude) growth rate of the low frequency drift wave (Horton, 1976). After the drift wave has grown to a finite amplitude, it is possible for the resonant electrons to become trapped in the electric field of the wave. It has been suggested that this trapping is important in determining the anomalous electron transport. Pogutse (1972) has shown that the trapping of resonant electrons by a finite amplitude wave leads to "pseudoclassical" heat transport with rates which are substantially greater than the classical and neoclassical transport rates (see also Gell et al., 1975). In addition, general considerations on the nature of dissipative drift instabilities (Nevins, 1977a) lead us to expect that a new dissipative drift instability will accompany this pseudoclassical transport.

In previous work on pseudoclassical transport (Pogutse, 1972; Gell et al., 1975; Gell and Nevins, 1975) the wave responsible for the trapping of the resonant electrons was not treated self-consistently; that is, the wave amplitude was fixed. Hence, this new dissipative drift instability was not found. This inconsistency has also obscured the relation between pseudoclassical transport theory and other work on the anomalous transport associated with low frequency drift waves (e.g., Horton, 1976; Liu et al., 1976), a relation we will try to make clear.

We have extended previous work by developing a self-consistent theory of the evolution of a plasma slab in which a finite amplitude drift wave has trapped the resonant electrons. This theory is presented in two papers. In the first paper, we derive the pseudoclassical fluxes of particles and energy across the magnetic field. In a companion paper (Nevins, 1977b), hereafter refered to as II, a complete set of equations for the evolution of both the finite amplitude wave and the background plasma is developed; and the relation between pseudoclassical transport and other work on the anomalous transport associated with low frequency drift waves is clarified.

A re-derivation of the pseudoclassical fluxes is necessary to the development of this self-consistent pseudoclassical transport theory because previous work (Pogutse, 1972; Gell and Nevins, 1975) has ignored the pseudoclassical transport of particles, focusing only on the transport of ene.gy across the magnetic field. This is an important omission, as we show in II that this pseudoclassical particle flux governs the evolution the finite amplitude wave. In addition, previous calculations either employed a model collision operator to approximate the effects of electron-electron collisions, while ignoring electron-ion collisions (Pogutse, 1972); or, they modeled the drift wave with a stationary potential, which is equivelent to setting the wave frequency equal to zero (Gell and Nevins, 1975).

We include both electron-electron and electron-ion collisions, employing the full Fokker-Planck collision operators (Rosenbluth et al., 1972). We evaluate both the particle flux and the energy flux. Armed with both of these fluxes, we will obtain a complete set of equations describing the evolution of both the wave and the background plasma in II. We allow the drift wave to have a non-zero frequency, and we find that the wave frequency is important in determining the flux of particle and energy across the magnetic field.

In Section 2 a qualitative discussion of the pseudoclassical transport mechanism is presented, and a proceedure for evaluating the pseudoclassical transport coefficients is outlined.

In Sections 3-5 this proceedure is carried out, and the pseudoclassical transport coefficients are expressed as functions of the wave parameters ω, $\underline{k}$, and $(e\Phi_0/T)$. Both the dependence of the pseudoclassical diffusion coefficient on these parameters, and the numerical value calculated here have been verified in numerical simulation of the pseudoclassical diffusion process (Nevins et. al., 1977c).

In Section 6 more insight into this pseudoclassical transport mechanism is gained by examining the motion of individual particles in the field of the finite amplitude wave.

In Section 7 the electron distribution function is examined. It is found that the dominant effect of particle trapping on this distribution function is the formation of a plateau along the orbits of the trapped electrons.

2. THE MODEL

We consider a model system consisting of a plasma slab in a uniform magnetic field. A right handed coordinate system is adopted with the magnetic field parallel to the z-axis, and with the temperature and density gradients of the plasma parallel to the x-axis. The plasma slab supports a single finite amplitude electrostatic wave with a wave vector, $\underline{k}$, lying in the y-z plane (see Fig. 1). This wave is assumed to have a parallel phase velocity, $v_\phi = \omega/k_z$, in the range $v_{ti} < v_\phi < v_{te}$, where $v_{te} = (T_e/m_e)^{\frac{1}{2}}$ and $v_{ti} = (T_i/m_i)^{\frac{1}{2}}$ are the electron and ion thermal velocities respectively.

Following previous authors (Yoshikawa and Christofilos, 1972; Pogutse, 1972) we use the name "pseudoclassical" transport to describe the transport process brought about by a combination of the motion of particles in the electric field of the finite amplitude wave and Coulomb collisions. The guiding center drift in the wave electric field affects the electron transport rates in much the same way that the guiding center drifts in an inhomogeneous magnetic field give rise to "neoclassical" transport. The parameter which determines the collision frequency regime of pseudoclassical transport is $(\nu_e/k_z v_{te})$. The pseudoclassical transport theory developed by Yoshikawa and Christofilos (1972) applies in the highly collisional regime where $(\nu_e/k_z v_{te}) > 1$. This regime of pseudoclassical transport theory is closely analogous to the Pfirsch-Schluter regime of neoclassical transport theory (Pfirsch and Schluter, 1962).

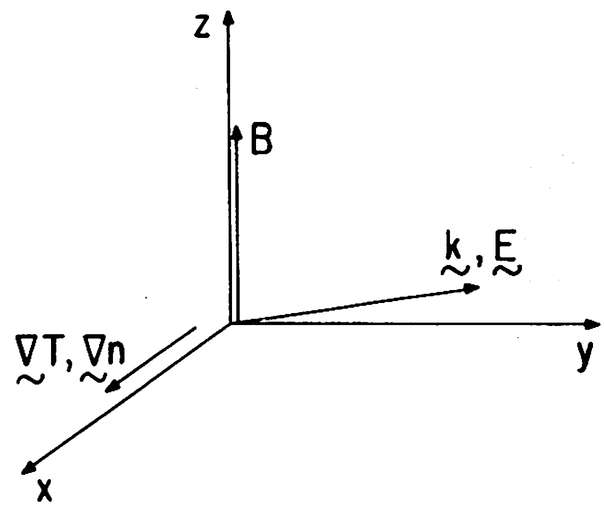

FIGURE I

The Directions of the Fields and Gradients in the System.

At the temperatures attained in many current tokamak experiments the parameter $(\nu_e/k_z v_{te})$ is small. The pseudoclassical transport theory considered by Pogutse (1972) applies in this low collision frequency regime (see also Gell et al., 1975; Gell and Nevins, 1975). This regime of pseudoclassical transport is associated with the trapping of particles by the finite amplitude wave, and bears a close analogy to the "bananna" regime of neoclassical theory (Galeev and Sagdeev, 1966).

A qualitative understanding of pseudoclassical transport in the low collision frequency regime may be gained by considering the motion of electrons trapped by the finite amplitude drift wave (Pogutse, 1972; Gell et al., 1975). Electrons with parallel velocities satisfying

$$|v_z - v_\phi| < 2^{1/2}\, v_{TRAP} \qquad (2.1)$$

may become trapped by the finite amplitude wave. v_{TRAP} is given by

$$v_{TRAP} = v_{te}\left(\frac{e\Phi_o}{T}\right)^{1/2} \qquad (2.2)$$

Φ_o is the amplitude of the wave, T is the electron temperature, and e is the electronic charge. In the wave frame these trapped electrons oscillate along the magnetic field lines at the bounce frequency

$$\omega_{BOUNCE} = k_z v_{TRAP} \; . \qquad (2.3)$$

Since $\underline{E}$ and $\underline{B}$ lie in the y-z plane, the $\underline{E} \times \underline{B}$ drift velocity is in the x direction. The electric field felt by the trapped particles oscillates at the bounce frequency, so the $\underline{E} \times \underline{B}$ drift will cause the trapped particles orbits to have a width in x of order

$$\Delta x = \frac{1}{\omega_{BOUNCE}} \frac{k_y \Phi_o}{B}$$

$$= \frac{k_y}{k_z} \left(\frac{e\Phi_o}{T}\right)^{\frac{1}{2}} \rho \qquad (2.4)$$

where ρ is a characteristic electron gyro radius, $\rho = (m_e T)^{\frac{1}{2}}/eB$. In the limit that ν_{eff}, the effective collision frequency for scattering particles out of resonance with the wave, is small compared to ω_{BOUNCE}, a random walk model with Δx as the step size may be used to estimate the transport coefficients. One obtains

$$D \simeq f(\Delta x)^2 \nu_{eff} \qquad (2.5)$$

where f is the fraction of the accessible phase space occupied by trapped particles. For those waves with parallel phase velocities small compared to the electron thermal velocity, and with amplitudes satisfying $(e\Phi_o/T) << 1$, this fraction may be estimated by

$$f \simeq \frac{v_{TRAP}}{v_{te}}$$

$$f \simeq \left(\frac{e\Phi_o}{T}\right)^{\frac{1}{2}} \qquad (2.6)$$

The effective collision frequency, ν_{eff}, is greater than ν_e, the frequency at which many small angle collisions will accumulate to produce a 90° scattering angle, because trapped electrons need only be scattered through an angle $\Delta\theta \simeq (v_{TRAP}/v_{te})$ to become passing particles. Since small angle scattering is the dominant collisional process in fully ionized plasmas, the effective collision frequency is related to the 90° collision frequency by

$$\nu_{eff} = \frac{1}{(\Delta\theta)^2}\ \nu_e$$

$$= \left(\frac{e\Phi_o}{T}\right)^{-1} \nu_e \tag{2.7}$$

The plasma - wave system will be in the low collision frequency regime of pseudoclassical theory when the bounce frequency of the particles trapped by the wave is greater than this effective collision frequency. This condition may be written as

$$(\nu_e/k_z v_{te}) < \left(\frac{e\Phi_o}{T}\right)^{3/2} \tag{2.8}$$

In this regime we expect the pseudoclassical transport coefficients to be of order

$$D = \left(\frac{k_y}{k_z}\right)^2 \left(\frac{e\Phi_o}{T}\right)^{\frac{1}{2}} \rho^2 \nu_e \tag{2.9}$$

Although $(e\Phi_o/T)$ is generally small, the ratio (k_y/k_z) can be quite large. Estimates of the magnitude of D (Pogutse, 1972; Gell et al., 1975) have shown that experimentally observed energy containment times can be explained with reasonable choices of the parameters $(e\Phi_o/T)$ and (k_y/k_z).

In our calculation of the pseudoclassical transport coefficients we use the drift kinetic equation to describe the evolution of the electron distribution function. Both electron-electron and electron-ion collisions are included using the full Fokker-Planck collision operators (Rosenbluth et al., 1972). We adopt a mathematical formalism similar to that used by Gell and Nevins (1975). This formalism makes use of the close analogy between pseudoclassical transport and neoclassical transport by adapting the variational principle of Rosenbluth et al. (1972) to the present problem.

Following the usual proceedure in transport calculations we consider two time scales: the microscopic time scale, and the macroscopic or transport time scale. The division between these two time scales is made possible by ordering in the small parameter $(\Delta x/L)$, where L is the scale length for variations in the plasma temperature and density. The assumption of local thermal equilibrium provides us with the condition that $\partial/\partial t$ cannot exceed $(\Delta x/L)\nu_e$. Within this constraint, the microscopic time scale is defined by $\frac{\partial}{\partial t} = \mathcal{O}[(\Delta x/L)\nu_e]$, while the transport time scale is defined by $\frac{\partial}{\partial t} = \mathcal{O}[(\Delta x/L)^2\nu_e]$. In the present calculation we make this expansion in $(\Delta x/L)$ about a state that includes a finite amplitude wave propagating in the y-z plane. The phase velocity of this wave is assumed to be small, $(v_\phi/v_{te})^2 = \mathcal{O}(\Delta x/L)$. The amplitude of the wave is allowed to vary slowly with both x $[\frac{1}{\Phi}\frac{\partial\Phi}{\partial x} = \mathcal{O}(1/L)]$, and t $[\frac{1}{\Phi}\frac{\partial\Phi}{\partial t} = \mathcal{O}(\Delta x/L)^2\nu_e]$.

In Sect. 3 we consider the drift kinetic equation on the microscopic time scale. Particle trapping is explicitly considered in deriving several constraints on the electron distribution function. In Sect. 4

we use these constraints to show that the electron entropy source, $\dot{S}_e$, is stationary with respect to variations in the electron distribution function. In Sect. 5 this variational principle is used together with the small parameter $(e\Phi_o/T)^{1/2}$ to evaluate the flux of particles and energy across the magnetic field. We obtain explicit expressions for these fluxes in terms of the wave parameters ω, (k_y/k_z), and $(e\Phi_o/T)$. The transport coefficients are found to scale as $(k_y/k_z)^2(e\Phi_o/T)^{1/2}\rho^2\nu_e$ as we expect from the argument preceeding Eq. (2.9).

3. SOME CONSTRAINTS ON THE DISTRIBUTION FUNCTION

In this section we derive several constraints that the steady state electron distribution function must satisfy in the presence of a finite amplitude, low frequency electrostatic wave. We consider a plasma situated in a uniform magnetic field $\underline{B} = B\,\hat{z}$. The plasma is assumed to have density and temperature profiles that depend on x. A finite amplitude electrostatic wave described by

$$\Phi(x,y,z,t) = \Phi_o(x,t)\ h(\theta) \tag{3.1}$$

is present. The wave phase, θ, is given by

$$\theta = k_y y + k_z z - \omega t \tag{3.2}$$

The function $h(\theta)$ describes the waveform. $h(\theta)$ has a magnitude of order one, and is assumed to be periodic in θ with a period of 2π. The wave amplitude, Φ_o varies slowly with both x and t. The scale length for variations of the wave amplitude with x is taken to be of order L, the scale length for variations of the temperature and the number density of the plasma with x. The wave amplitude varies with time on the transport time scale, i.e. $(\frac{1}{\Phi_o}\ \frac{\partial\Phi_o}{\partial t}) \sim (\Delta x/L)^2\ \nu_e$.

In the drift approximation the kinetic equation for the electron distribution function is given by (Hinton and Hazeltine, 1976)

$$\frac{\partial f}{\partial t} + v_z\,\frac{\partial f}{\partial z} - \frac{\nabla\Phi\, x\,\underline{B}}{B^2}\cdot\nabla_\perp f + \frac{e}{m}\,\frac{\partial\Phi}{\partial z}\,\frac{\partial f}{\partial v_z} = C_e(f) \tag{3.3}$$

where $C_e(f)$ represents the electron collision operator.

The electron distribution function f has a slow $[O(\Delta x/L)^2 \nu_e]$ variation in time due to the transport of particles and energy. In this section we are concerned only with the evolution of the electron distribution on the microscopic time scale. Hence we ignore this slow variation, keeping terms through first order in $(\Delta x/L)$.

We assume that the plasma-wave system has reached a steady state in which the temporal variation on the microscopic time scale comes only through the dependence of the distribution function and the wave amplitude on θ. This assumption rules out oscillations in the wave amplitude and the electron distribution function at the bounce frequency of the electrostatically trapped particles. Such oscillations in the wave amplitude are attenuated both by phase mixing among the trapped particle orbits (O'Neil, 1965); and by collisions, which smooth out the fine structure in the electron distribution function associated with these oscillations in a time of order ν_{eff}^{-1} (Zakharov and Karpman, 1963). The dependence of the electron distribution function on y and z is assumed to be due to the response of the electrons to the wave. Hence, the steady state electron distribution function depends on y and z only through the wave phase, θ. In this steady state the kinetic equation may be written as

$$(\omega_{BOUNCE}\, q + \frac{k_y}{B}\frac{\partial\Phi}{\partial x})\frac{\partial f}{\partial\theta} - \frac{k_y}{B}\frac{\partial\Phi}{\partial\theta}\frac{\partial f}{\partial x} + \frac{e}{m}k_z\frac{\partial\Phi}{\partial\theta}\frac{\partial f}{\partial v_z} = C_e(f) \qquad (3.4)$$

where the normalized "velocity-slip", q, is proportional to the parallel velocity in the reference frame of the electrostatic wave,

$$q \equiv (v_z - v_\phi)/v_{TRAP}. \tag{3.5}$$

The particle energy in the reference frame of the wave,

$$E = \tfrac{1}{2}mv_{TRAP}^2\, q^2 + \mu B - e\Phi \tag{3.6}$$

is a constant of the particle motion, as is μ, the magnetic moment. We make use of these constants of motion by passing from the variables (θ, x, v_z, μ) to the set $(\theta, x, E, \mu, \sigma)$, where σ is the sign of q, $\sigma \equiv q/|q|$. In this set of variables we may write the kinetic equation in the form

$$\left[q + h(\theta)\frac{\Delta x}{\Phi_o}\frac{\partial\Phi_o}{\partial x}\right]\frac{\partial f}{\partial\theta} - h'(\theta)\,\Delta x\,\frac{\partial f}{\partial x} = C_e(f)/\omega_{BOUNCE} \tag{3.7}$$

where

$$h'(\theta) = \frac{dh(\theta)}{d\theta}$$

We wish to expand Eq. (3.7) in powers of $(\Delta x/L)$ about an equilibrium that includes a finite amplitude low frequency drift wave. Low frequency drift waves arise from perturbations in the ion density caused by the self consistent $\underline{E} \times \underline{B}$ convection of ions across the zero order density gradient. The role of the electron distribution is to provide Debye shielding of these ion perturbations (Mikhailovskii, 1974; Kadomtsev, 1965).

The effect of the Debye shielding is included at zero order in $(\Delta x/L)$ by expanding the electron distribution function about a local Boltzmann distribution,

$$f_o = n(x) \left[\frac{m}{2\pi T(x)}\right]^{3/2} \exp\left[\frac{-1}{T} (\tfrac{1}{2}mv^2 - e\Phi)\right] \qquad (3.8)$$

where n and T are functions of x only. This choice of the zero order distribution function is consistent with the electron distribution function obtained in the linear (in the wave amplitude) analysis of low frequency drift waves.

Written in terms of our adopted set of variables $(\theta, x, \mu, E, \sigma)$, the Boltzmann distribution becomes:

$$f_o = n\left[\frac{m}{2\pi T}\right]^{3/2} \exp\left(\frac{-1}{T} (E + mqv_\phi \, v_{TRAP} + \tfrac{1}{2}mv_\phi^{\,2})\right) \qquad (3.9)$$

where $q = q(\theta, x, \mu, E, \sigma)$.

The electron distribution function may be written as

$$f = f_o \, (1 + \hat{f}) \qquad (3.10)$$

where $\hat{f} = O\,(\Delta x/L)$. This expansion of the electron distribution function is put into the kinetic equation, (3.7), and terms in like powers of $(\Delta x/L)$ are equated. The terms involving $\frac{\partial f_o}{\partial \theta}$ require special attention. From Eq. (3.9) we see that f_o depends on θ only through q. Evaluating this factor we find

$$\frac{\partial f_o}{\partial \theta} = \frac{h'(\theta)}{q} \, (a_0 \Delta x) f_o \qquad (3.11)$$

where

$$a_0 \equiv \frac{eB}{k_y T}\,\omega \tag{3.12}$$

Low frequency drift instabilities have frequencies of order $\omega_{ne} = -(k_y T/eB)\frac{1}{n}\frac{\partial n}{\partial x}$. Hence, $a_0 = \mathcal{O}(1/L)$, and $\frac{\partial f_o}{\partial \theta}$ is first order in $(\Delta x/L)$. The steady state kinetic equation is then satisfied at zero order in $(\Delta x/L)$ as

$$C_e(f_o) = 0 \; .$$

To first order in $(\Delta x/L)$ we obtain

$$q f_o \frac{\partial \hat{f}}{\partial \theta} - \Delta x \left[\frac{\partial f_o}{\partial x} - a_0 f_o\right] h'(\theta) = C_e(f)/\omega_{BOUNCE} \tag{3.13}$$

In Eq. (3.13) we require an expression for f_o valid only to zero order in $(\Delta x/L)$. To this order f_o is given by

$$f_o = n\left[\frac{m}{2\pi T}\right]^{3/2} e^{-E/T} + \mathcal{O}(\Delta x/L) \tag{3.14}$$

so $\frac{\partial f_o}{\partial x}$ may be written as

$$\frac{\partial f_o}{\partial x} = -\,(a_1 + A_2 E) f_o \tag{3.15}$$

where we have defined

$$a_1 \equiv -\frac{\partial}{\partial x}\left(\frac{M}{T}\right) = -\frac{1}{n}\frac{\partial n}{\partial x} + \frac{3}{2}\frac{1}{T}\frac{\partial T}{\partial x} \tag{3.16}$$

$$A_2 \equiv \frac{\partial}{\partial x}\left(\frac{1}{T}\right) \tag{3.17}$$

and M, the chemical potential of the electrons, is given by:

$$M = T\,\ell n\left[n(2\pi\hbar^2/mT)^{3/2}\right]. \tag{3.18}$$

These three quantities, a_0, a_1, and A_2 are all of order (1/L). a_0 is a measure of the departure of the system from thermal equilibrium because of the variations in the wave potential with time. a_1 and A_2 measure the departure from thermal equilibrium due to variations in the density and temperature with x. In Sect. 4 we find that the quantities

$$A_1 \equiv a_0 + a_1 \tag{3.19}$$

and A_2 are the thermodynamic forces acting on the plasma (De Groot and Mazur, 1962).

Using the definitions (3.17) and (3.19) we may write the kinetic equation to first order in ($\Delta x/L$) as

$$f_o[\Delta x(A_1 + A_2E)\ h'(\theta) + q\ \frac{\partial\hat{f}}{\partial\theta}] = C_e(f)/\omega_{BOUNCE} \tag{3.20}$$

We wish to consider the enhanced collisional transport by electrons due to the trapping of particles in the electrostatic wave. In the random walk picture, this enhancement is due to the enlarged fundamental step that the diffusing particle is taking, or equivently, that the orbit of the trapped particle is much larger than the Larmor orbit. This enlarged orbit results from the superposition of the x-directed drift velocity upon the motion of the trapped particles along the magnetic field lines. If the trapped particles are to complete this enlarged orbit, then the collision frequency for scattering particles out of the trapped region of velocity space should be small compared with the bounce frequency of the particles trapped in the electrostatic wave. Consequently we consider the limit

$$\nu_{eff}/\omega_{BOUNCE} << 1$$

An examination of Eq. (3.4) indicates that in the limit $\Phi_o \to 0$, the steady state kinetic equation is satisfied by $f = f_o$. Hence $\hat{f}$ describes the perturbation in the electron distribution function due to the presence of the finite amplitude drift wave. In a linear theory this perturbation diverges as velocity-slip, q, goes to zero. The divergence is avoided in this calculation by allowing the wave to trap the resonant particles. We might expect the resulting perturbation in the electron distribution function to be localized about the trapped region of phase space, where $q \lesssim 1$. In fact, we will find that only the velocity derivatives of this perturbation are localized about the trapped region. This localization allows us to make the estimate

$$\frac{\partial^2}{\partial v^2} \hat{f} \sim \frac{1}{v_{TRAP}^2} \hat{f}$$

The Fokker-Planck collision operator contains a term of the form $\nu_e \frac{T}{m} \frac{\partial^2}{\partial v^2}$. This term dominates the collision operator because of the rapid variation of $\hat{f}$ in the trapped region. Hence, we may estimate the magnitude of the collision operator in Eq. (3.20) by

$$C_e(f_o \hat{f}) \simeq \nu_{eff} f_o \hat{f}$$

We make use of the small parameter $(\nu_{eff}/\omega_{BOUNCE})$ by expanding $\hat{f}$ in the form

$$\hat{f} = \hat{f}^0 + \hat{f}^1 + \ldots$$

where $\hat{f}^1 = O(\nu_{eff}/\omega_{BOUNCE})$. The leading term in this expansion, $\hat{f}^0$ is examined in Sect. 7, where the assumption that the velocity derivatives of $\hat{f}$ are localized about the trapped region of phase space is verified.

To zero order in $(\nu_{eff}/\omega_{BOUNCE})$ we find

$$q \frac{\partial \hat{f}^0}{\partial \theta} = - \Delta x(A_1 + A_2 E)\ h'(\theta)\ , \tag{3.21}$$

while to first order in $(\nu_{eff}/\omega_{BOUNCE})$ we have

$$q f_o \frac{\partial \hat{f}^1}{\partial \theta} = C_e(f_o \hat{f}^0)\quad . \tag{3.22}$$

The θ-derivatives in Eqs. (3.21) and (3.22) are to be taken at constant E and μ. These variables, E and μ, label the orbits of particles in phase space, while θ (and for trapped particles σ) determines the position of a particle on a particular orbit. Hence, integrating Eqs. (3.21) and (3.22) over θ corresponds to the usual procedure for determing the perturbed distribution function of integrating the perturbation along particle orbits.

We first consider Eq. (3.22). This equation allows us to express the change in $\hat{f}^1$ as an integral of the collision operator along a particle orbit. We use this equation to obtain two constraints that must be satisfied by $\hat{f}^0$ if there is to be a properly behaved solution, $\hat{f}^1$, to this equation.

Passing particle orbits extend over many periods of the finite amplitude wave. In our steady state $\hat{f}$ is required to have the same periodicity as the wave potential. Hence, the net change in $\hat{f}^1$ over one wave period must vanish. This is only possible when $\hat{f}^0$ satisfies the constraint

$$\int_0^{2\pi} d\theta \, \frac{C_e(f_o \hat{f}^0)}{q} = 0 \, . \tag{3.23}$$

in the untrapped region of (E,μ).

In the trapped region of (E,μ) space the steady state electron distribution function must be continuous at the turning points. Thus we require

$$f(\theta = \theta_{1,2}, \ \sigma = +1) = f(\theta = \theta_{1,2}, \ \sigma = -1) \tag{3.24}$$

where $\theta_1(E,\mu)$ and $\theta_2(E,\mu)$ are the turning points of electrons trapped in the electrostatic wave.

Hence, the change in $\hat{f}^1$ between these turning points along the upper (i.e., $\sigma = +1$) branch of the trapped orbit must be identical to the change in $\hat{f}^1$ along the lower $(\sigma = -1)$ branch. This condition can be met only if $\hat{f}^0$ satisfies the constraint

$$\sum_{\sigma} \int_{\theta_1}^{\theta_2} d \, \frac{C_e(f_o \hat{f}^0)}{|q|} = 0 \tag{3.25}$$

in the trapped region of (E,μ) space.

From relation (3.6), recalling that E and θ are independent variables, we have

$$h'(\theta) = q \frac{\partial q}{\partial \theta}$$

Inserting this expression into Eq. (3.21) and integrating with respect to θ we find that $\hat{f}^0$ may be written in the form

$$\hat{f}^0 = - \Delta x(A_1 + A_2 E)\ q + g(\mu,\ E,\ \sigma) \qquad (3.26)$$

where the function g is independent of θ. The velocity-slip vanishes at the turning points; hence, $q(\theta_1) = q(\theta_2) = 0$. Together with Eq. (3.24) and (3.26) this implies that in the trapped region of (E,μ) space

$$g(\mu,\ E,\ \sigma = +1) = g(\mu,\ E,\ \sigma = -1) \qquad (3.27)$$

In deriving Eqs. (3.26) and (3.27) we have dropped the collision term. This term enters our calculation at first order in $(\nu_{eff}/\omega_{BOUNCE})$. Hence, we may view $\hat{f}^0$ as the perturbation in the electron distribution caused by a finite amplitude drift wave in a collisionless plasma. A similar problem, that of a finite amplitude Langmuir wave, has been studied extensively (O'Neil, 1965; Zakharov and Karpman, 1963; Bernstein et al., 1957). This problem is generally approached in a reference frame in which the Langmuir wave is stationary. In this reference frame, it is found that a steady state, in the sense described above, is attained when the distribution function is constant along the orbits traced out by particles as they move in the electric field of the finite amplitude wave.

We may interpret Eq. (3.26) in a similar manner. The first term on the right hand side, $-\Delta x(A_1 + A_2E)q$, provides the perturbation in the electron distribution function that is required to balance the variations in f_o along particle orbits. The second term, $g(\mu, E, \sigma)$, is an arbitrary function of the constants of motion E, μ, and (for untrapped particles) σ. Hence, Eq. (3.26) is the most general perturbation satisfying the condition that the over-all electron distribution function, $f = f_o(1 + \hat{f})$, be constant along particle orbits in the finite amplitude wave.

In much of the work on particle trapping in finite amplitude Langmuir waves, the collision frequency vanished identically. The function g was determined either from the waveform, together with the dispersion relation (Bernstein et al., 1957), or from the initial conditions (O'Neil, 1965). In the present calculation we are considering a weakly collisional plasma. Hence the collisions will determine the form of g (Zakharov and Karpman, 1963). In the present work, g is determined by using Eqs. (3.23) through (3.27) to show that the local entropy source, $\dot{S}_e$, is an extremum with respect to variations in g. This variational principle will be used to evaluate the perturbed electron distribution function, as well as the flux of particles and energy across the magnetic field.

4. THE ENTROPY SOURCE AND A VARIATIONAL PRINCIPLE

Several authors have shown that the steady states of various systems near thermal equilibrium are characterized by an extremum in the rate of change of a thermodynamic potential (DeGroot and Mazur, 1962; Onsager, 1931; Rayleigh, 1873). In particular, Rosenbluth et al. (1972) have shown that the rate of entropy production is minimized in the steady state of an axisymmetric, toroidal plasma confinement system. Rosenbluth et al. (1972) used this variational principle in evaluating the neoclassical transport coefficients.

This leads us to consider the rate of entropy production in our system. We first motivate our discussion of the rate of entropy production by using thermodynamic principles to relate the entropy source to the transport coefficients and the thermodynamic forces, as this relation illustrates the central role that the entropy source plays in non-equilibrium thermodynamics. We then proceed to show that the entropy source of the plasma-wave system considered here is stationary with respect to variations in the electron distribution function that satisfy the constraints derived in Section 3.

In Nevins (1977a) we used the Boltzmann definition of the entropy and the drift kinetic equation, together with the small parameter $(\Delta x/L)$ to show that the wave phase averaged entropy density, S_e, satisfies the equation

$$T \frac{\partial S_e}{\partial t} = \frac{\partial w}{\partial t} - M \frac{\partial n}{\partial t} \tag{4.1}$$

w is the wave phase averaged electron energy density, n is the wave phase averaged number density, and M is the chemical potential. Due to the averaging over the wave phase, the potential energy of the electrons is included in the electron energy density, w.

We recognize Eq. (4.1) as a fluid version of the Thermodynamic Identity (Landau and Lifshitz, 1958),

$$T\,dS = dw - M\,dn$$

Hence, we expect that, given suitable definitions of the entropy density, the number density, and the energy density, an equation similar to Eq. (4.1) can be derived for any system near thermal equilibrium.

To proceed in our discussion of the evolution of the entropy density, we require equations for the evolution of n and w. These equations are derived in Nevins (1977a) by taking the appropriate moments of the drift kinetic equation.

It is found that

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial x}\Gamma_e \qquad (4.2)$$

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial x}Q_e + T\, a_0\Gamma_e \qquad (4.3)$$

where the θ-averaged fluxes, Γ_e and Q_e are given by

$$\Gamma_e \equiv \int \frac{d\theta}{2\pi}\, d^3\underline{v}\; v_{dr}\; f \qquad (4.4)$$

$$Q_e \equiv \int \frac{d\theta}{2\pi}\, d^3\underline{v}\,(\tfrac{1}{2}mv^2 + e\Phi)\, v_{dr} f \qquad (4.5)$$

and v_{dr} is the x-directed drift velocity

$$v_{dr} \equiv \frac{-k_y}{B}\Phi_o\, h'(\theta)$$

An interesting feature of Eq. (4.3) is the energy source term, $T\, a_0\, \Gamma_e$. We show in II that this term describes the energy transfer between the wave and the particles.

Combining Eqs. (4.1) through (4.3), we find that the entropy density satisfies the equation

$$\frac{\partial S_e}{\partial t} = -\frac{\partial}{\partial x}J_s + \dot{S}_e \qquad (4.6)$$

where the entropy flux, J_s, is given by

$$J_s = \frac{1}{T}Q_e + \frac{M}{T}\Gamma_e \qquad (4.7)$$

and the entropy source, $\dot{S}_e$, is given by

$$\dot{S}_e = (a_0 + a_1)\,\Gamma_e + A_2 Q_e \tag{4.8}$$

$$= A_1\,\Gamma_e + A_2 Q_e \tag{4.9}$$

We may interpret the terms on the right hand side of Eq. (4.8) by noting that the first term, $a_0\Gamma_e$, results from heating of the plasma by the wave, the second term is the contribution to the entropy source from particle diffusion, and the third term is the contribution to the entropy source from the transport of energy across the magnetic field.

The coefficients of the fluxes in the expressions for the entropy source are the thermodynamic forces (DeGroot and Mazur, 1962). Hence, Eq. (4.9) shows that the thermodynamic forces acting on our plasma-wave system are A_1 and A_2. The force conjugate to the particle flux, A_1, differs from the thermodynamic force that one obtains in the absence of a wave, a_1, by the term a_0, which arises from the heating of the plasma by the wave.

For a system near thermal equilibrium, the particle flux and energy flux may be written as products of the thermodynamic forces and the transport coefficients:

$$J_n = \sum_m L_{nm} A_m \tag{4.10}$$

where the J's represent the fluxes and the L's represent the transport coefficients. Combining Eqs. (4.9) and (4.10), we find that the entropy source may be written as

$$\dot{S}_e = \sum_{n,m} A_n\, L_{nm}\, A_m \tag{4.11}$$

Equation (4.11) illustrates the importance of the entropy source in non-equilibrium thermodynamics. In Eq. (4.11) the entropy source has been written as a bilinear form in the thermodynamic forces. The coefficients of this bilinear form are the transport coefficients. Hence, the transport coefficients may be obtained by evaluating the entropy source in terms of the thermodynamic forces. The expression for the entropy source may then be compared with Eq. (4.11), and the transport coefficients read off. We will use this strategy in Section 5 to evaluate the transport coefficients.

We now proceed to show that the entropy source is stationary with respect to variations in the electron distribution function. In order to make the optimum use of the small parameters $(\Delta x/L)$ and $(\nu_{eff}/\omega_{BOUNCE})$, we estimate the magnitude of the entropy source. The thermodynamic forces of Eqs. (3.17) and (3.19) are of order $(1/L)$. We expect the transport coefficients to be proportional to $\Delta x^2 \nu_{eff}$. Hence, we must evaluate the entropy source through order $(\Delta x/L)^2 \nu_{eff}$.

Multiplying the drift kinetic equation through by $(1 + \ell n\ f)$ and applying the phase space averaging operator, $\int \frac{d\theta}{2\pi} \int d^3\underline{v}$, one immediately obtains an equation for the evolution of the entropy density in the form of Eq. (4.6). The source term in this equation is

$$\dot{S}_e = - \int \frac{d\theta}{2\pi} \int d^3\underline{v}\ C_e(f)\ \ell n\ f. \qquad (4.12)$$

The distribution function is decomposed as in Eq. (3.10),

$$f = f_o(1 + \hat{f})$$

and the conservation of particles and energy in collisions is used to eliminate the terms involving $\ell n\, f_o$.

Through second order in $(\Delta x/L)$, the remaining terms in $\dot{S}_e$ may be written in the form

$$\dot{S}_e = \int \frac{d\theta}{2\pi} K_e(\hat{f},\hat{f}) \tag{4.13}$$

where the local (in θ) entropy source, $K_e(\hat{f},\hat{f})$ is given by the bi-linear form

$$K_e(\hat{f},\hat{g}) \equiv - \int d^3\underline{v}\ \hat{f}\ C_e(f_o\hat{g}) \tag{4.14}$$

From Eqs. (4.13) and (4.14), it is clear that an expression for $\hat{f}$ valid to zero order in $(\nu_{eff}/\omega_{BOUNCE})$ and to first order in $(\Delta x/L)$ will be sufficient to obtain $\dot{S}_e$ to the desired order. Thus, we may use $\hat{f}^0$ of Eq. (3.26) in evaluating the local entropy source.

To proceed in the derivation of a variational principle on the rate of entropy production, it is necessary to show that the local entropy source, K_e, is self-adjoint. The local entropy source may be written as

$$K_e(\hat{f},\hat{g}) = K_{ee}(\hat{f},\hat{g}) + K_{ei}(\hat{f},\hat{g}) \tag{4.15}$$

The entropy source due to electron-electron collisions, K_{ee}, is given by

$$K_{ee}(\hat{f},\hat{g}) = - \int d^3\underline{v}\ \hat{f}\ C_{ee}(f_o\hat{g}), \tag{4.16}$$

and K_{ei}, the entropy source from electron-ion collisions, is given by

$$K_{ei}(\hat{f},\hat{g}) = - \int d^3\underline{v}\ \hat{f}\ C_{ei}(f_o\hat{g}). \tag{4.17}$$

$C_{ee}(f)$ is the linearized electron-electron collision operator and C_{ei} is the electron-ion collision operator. Taking C_{ee} to be of the Boltzmann form, and using a Lorentz model for the electron-ion collisions it may be shown that K_{ee} and K_{ei} are separately self-adjoint (see Appendix A). We make use of this fact in Appendix D, where the contribution of electron-ion collision to the particle diffusion coefficient is calculated.

We now consider the effect of variations in the electron distribution function on the entropy source, $\dot{S}_e$. As we pointed out above, it is sufficient to use $\hat{f}^0$ to describe the electron distribution function in the calculation of $\dot{S}_e$. $\hat{f}^0$ has not yet been evaluated, but it is known that $\hat{f}^0$ must satisfy the constraints derived in Sect. 3. In particular the θ dependence of $\hat{f}^0$ is determined by Eq. (3.26),

$$\hat{f}^0 = - \Delta x (A_1 + A_2 E) q + g(\mu, E, \sigma).$$

Hence, variations in $\hat{f}^0$ are equivalent to variations in the as yet unknown function $g(\mu, E, \sigma)$. Varying the rate of entropy production as given by Eq. (4.13) with respect to g, and using the self-adjointness of K_e, we find that variations in $\dot{S}_e$ satisfy

$$\delta\dot{S}_e = -2 \int \frac{d\theta}{2\pi} \int d^3\underline{v}\, \delta g\, C_e(f_o \hat{f}^0) \tag{4.18}$$

Upon changing integration variables from $(\theta, \underline{v})$ to (θ, μ, E) and using the fact that variations in $g(\mu, E, \sigma)$ must be independent of θ, Eq. (4.18) becomes

$$\delta\dot{S}_e = - \frac{4B}{m^2 v_{TRAP}} \int \frac{d\mu\, dE}{2\pi} \sum_{\sigma} \delta g(\mu, E, \sigma) \int d\theta \frac{C_e(f_o \hat{f}^0)}{|q|} \tag{4.19}$$

$\delta\dot{S}_e$ will vanish for all allowable δg if, in the untrapped region of (E, μ), $\hat{f}^0$ satisfies the condition

$$\int_0^{2\pi} d\Theta \frac{C_e(f_o \hat{f}^0)}{q} = 0$$

In the trapped region of (E, μ) space the function g, and hence δg, must be independent of σ [c.f., Eq. (3.24)]. Hence, in the trapped region, $\hat{f}^0$ must satisfy

$$\sum_{\sigma} \int_{\theta_1}^{\theta_2} d\Theta \, \frac{C_e(f_o \hat{f}^0)}{|q|} = 0.$$

In Section 3 we showed that $\hat{f}^0$ indeed satisfies these conditions [c.f., Eqs. (3.23) and (3.25)]. Hence, we conclude that

$$\delta \dot{S}_e = 0 \qquad (4.20)$$

for all allowable variations in the electron distribution function.

5. EVALUATION OF THE ENTROPY SOURCE

In this section we evaluate the entropy source, $\dot{S}_e$, to lowest order in the small parameter $(e\Phi_o/T)^{1/2}$. The particle flux and the energy flux are then obtained by comparing the resulting expression for the entropy source with Eq. (4.11).

The entropy source is evaluated in three stages. In the first stage we commit ourselves to the Fokker-Planck form of the linearized collision operator. Equations (4.13) and (4.14) are then used to write the entropy source as a functional of the perturbed electron distribution function, $\hat{f}^0$. This functional is examined, and only the leading terms in $(e\Phi_o/T)^{1/2}$ retained. This procedure yields the much simplified functional approximation to $\dot{S}_e$ given in Eq. (5.13).

In the second stage the variational principle derived in Section 4 is employed together with Eq. (5.13) to evaluate the velocity derivative of the perturbed distribution function, $\frac{\partial \hat{f}^0}{\partial v_z}$. We note that requiring this approximation to $\dot{S}_e$ to be stationary with respect to variations in $\hat{f}^0$ does not simply reproduce the constraints derived in Section 3 because Eq. (5.13) contains additional information, namely that $(e\Phi_o/T)^{1/2}$ is small.

In the final stage our expression for $\frac{\partial \hat{f}^0}{\partial v_z}$ is substituted into the simplified functional form for $\dot{S}_e$. After evaluating certain integrals we obtain, in Eq. (5.25), an algebraic expression for $\dot{S}_e$. This expression is bilinear in the thermodynamic forces. Hence, it may be compared with Eq. (4.11) to obtain algebraic expressions for both the particle flux and the energy flux.

In the first stage we require expressions for the local entropy sources, $K_{ee}(\hat{f},\hat{g},)$ and $K_{ei}(\hat{f},\hat{g})$. Rosenbluth et. al. (1972) use a Fokker-Planck expansion of the collision operator to show that these may be written as

$$K_{ee}(\hat{f},\hat{g}) = \frac{\pi e^4}{m^2} \ell n\Lambda \int d^3\underline{v}_a d^3\underline{v}_b \ f_o(v_a) \ f_o(v_b) \quad V_{\alpha\beta}\left(\frac{\partial \hat{f}_a}{\partial v_{a_\beta}} - \frac{\partial \hat{f}_b}{\partial v_{b_\beta}}\right)\left(\frac{\partial \hat{g}_a}{\partial v_{a_\alpha}} - \frac{\partial \hat{g}_b}{\partial v_{b_\alpha}}\right) \qquad (5.1)$$

and

$$K_{ei}(\hat{f},\hat{g}) = \frac{\pi e^4}{m^2} \ell n\Lambda \int d^3\underline{v}_a \ d^3 v_b \ f_o(v_a) f_{oi}(v_b) \quad V_{\alpha\beta} \frac{\partial \hat{f}_a}{\partial v_{a_\beta}} \frac{\partial \hat{g}_a}{\partial v_{a_\alpha}} \qquad (5.2)$$

where α and β label the Cartesian components of vectors (summation over repeated Greek subscripts is implied),

$$V_{\alpha\beta} \equiv (1/v^3) \ (v^2\delta_{\alpha\beta}-v_\alpha v_\beta)$$

$$\underline{v} \equiv (\underline{v}_a-\underline{v}_b) \qquad (5.3)$$

$\ell n\ \Lambda$ is the usual Coulomb logarithm and, in the limit of large mass ratio, we may take the ion distribution function to be

$$f_{oi}(\underline{v}) = n\,\delta\,(\underline{v}) \qquad (5.4)$$

Note that these expressions for K_{ee} and K_{ei} are self-adjoint as required, and that $K_{ee}(\hat{f},\hat{f})$ and $K_{ei}(\hat{f},\hat{f})$ are positive definite. Thus, the entropy source, $\dot{S}_e$, is positive as one expects.

The expression for K_{ee} may be further simplified by recalling that the derivatives of the perturbation in the electron distribution function, $\hat{f}^0$, have been assumed to be localized about the trapped region of phase space. Because of this localization, each integral of $\frac{\partial \hat{f}^0}{\partial v}$ over velocities in Eq. (5.1) reduces K_{ee} by the fraction of velocity space occupied by trapped particles. In Section 2 we estimated this fraction as $\left(\frac{e\Phi_o}{T}\right)^{\frac{1}{2}}$; hence, to lowest order in $\left(\frac{e\Phi_o}{T}\right)^{\frac{1}{2}}$, K_{ee} is given by

$$K_{ee} = \frac{2\pi e^4}{m^2} \ell n \Lambda \int d^3\underline{v}_a d^3\underline{v}_b \; f_o(v_a) \; f_o(v_b) \; V_{\alpha\beta} \frac{\partial \hat{f}_a}{\partial v_{a_\alpha}} \frac{\partial \hat{f}_a}{\partial v_{a_\beta}} \tag{5.5}$$

The component of the velocity parallel to $\underline{B}$, v_z, together with the wave phase, θ, determine the trapped region. The localization about the trapped region leads us to expect that

$$\frac{\partial \hat{f}^0}{\partial v_z} >> \frac{\partial \hat{f}^0}{\partial v_{x,y}} \tag{5.6}$$

We verify these assumptions about the derivatives of the perturbed distribution function in Sect. 7. Using Eq. (5.6), we approximate the entropy source as

$$\dot{S}_e = \frac{2\pi e^4}{m^2} \ell n \Lambda \int \frac{d\theta}{2\pi} \int d^3\underline{v} \; f_o \left(\frac{\partial \hat{f}}{\partial v_z}\right)^2 \sum_s a_s \int d^3\underline{v}_s \; f_{os} \; V_{zz} \tag{5.7}$$

where s labels the two species (electrons and ions), and

$$a_e = 1$$

$$a_i = \tfrac{1}{2}$$

At this point it is helpful to introduce some new variables. The condition that a particle is trapped is

$$E - \mu B < e\Phi_o \tag{5.8}$$

where Φ_o is the wave amplitude. This suggests that we define

$$\lambda \equiv (E - \mu B)/T \tag{5.9}$$

The trapped region of phase space is then determined by

$$\lambda < e\Phi_o/T \tag{5.10}$$

We can exploit the fact that g in Eq. (3.26) is not a function of θ by changing our variables of integration in Eq. (5.7) from $(\theta,\underline{v})$ to (θ,λ,E). The volume elements are related by

$$\int d\theta\, d^3\underline{v} = \sum_\sigma \frac{2\pi}{m} v_{te}\left(\frac{e\Phi_o}{T}\right)^{-\frac{1}{2}} \int \frac{d\theta d\lambda dE}{|q|}$$

where σ is the sign of the velocity-slip, q.

From Eq. (3.26) we see that

$$\frac{\partial \hat{f}^0}{\partial v_z} = \frac{1}{v_{TRAP}}\left[G(E) + \left(\frac{e\Phi_o}{T}\right) q \frac{\partial g}{\partial \lambda}\right] \tag{5.11}$$

where

$$G(E) \equiv - \Delta x(A_1 + A_2E) \tag{5.12}$$

The rate of entropy production is then given by

$$\dot{S}_e = \frac{4\pi^2 e^4}{m^3} v_{te}\, \ell n\Lambda \sum_\sigma \left(\frac{e\Phi_o}{T}\right)^{-\frac{1}{2}} \int \frac{d\theta d\lambda dE}{2\pi|q|} f_o(E) \left(\frac{\partial \hat{f}^0}{\partial v_z}\right)^2 \sum_s F_s(E) \tag{5.13}$$

where

$$F_s(E) = a_s \int d^3\underline{v}\, f_{os}\, V_{zz} \tag{5.14}$$

and we have used the fact that to lowest order in $(\Delta x/L)$ and $(e\Phi_o/T)^{\frac{1}{2}}$, $(E/T) \simeq \frac{1}{2}mv^2/T$.

We now evaluate $\frac{\partial g}{\partial \lambda}$ by varying the entropy source with respect to $g(\lambda, E, \sigma)$. This variation together with Eq. (4.20) gives

$$\left\{\sum_{\sigma}\right\}_{\substack{\text{trapped}\\ \text{region}}} \int \frac{d\theta}{2\pi}\left[\sigma G(E) + |q|\left(\frac{e\Phi_o}{T}\right)\frac{\partial g}{\partial \lambda}\right] = 0 \qquad (5.15)$$

The sum on σ is to be taken only in the trapped region, where $\lambda < \frac{e\Phi_o}{T}$. Solving for $\frac{\partial g}{\partial \lambda}$ in the untrapped region, we obtain

$$\frac{\partial g}{\partial \lambda} = -\left(\frac{e\Phi_o}{T}\right)^{-1} G(E)\ \frac{1}{\langle q \rangle} \qquad (5.16)$$

Where we have introduced the notation

$$\langle a(\theta) \rangle \equiv \int \frac{d\theta}{2\pi}\, a(\theta) \qquad (5.17)$$

We showed in Sect. 3 that g must be independent of σ in the trapped region of phase space. Thus, for $\lambda < \frac{e\Phi_o}{T}$, $\frac{\partial g}{\partial \lambda}$ is given by

$$\frac{\partial g}{\partial \lambda} = 0 \qquad (5.18)$$

and $\frac{\partial \hat{f}^0}{\partial v_z}$ may be written as

$$\frac{\partial \hat{f}^0}{\partial v_z} = \frac{G(E)}{v_{TRAP}}\, q\left[\frac{1}{q} - \left(\frac{1}{\langle q \rangle}\right)_u\right] \qquad (5.19)$$

where the term $(1/\langle q \rangle)_u$ contributes only in the untrapped region.

Combining Eqs. (5.13) and (5.19) we find that the entropy source is given by

$$\dot{S}_e = \frac{3\pi^{3/2} T}{n\, m^2}\, \nu_{eff}\, I \int dE\, f_o(E)\, G^2(E) \sum_s F_s(E) \qquad (5.20)$$

where ν_{eff} is defined as

$$\nu_{eff} \equiv \nu_e/(e\Phi_o/T)$$

and following Braginskii (1965) we take

$$\nu_e = \frac{4(2\pi)^{\frac{1}{2}} n e^4}{3m^{\frac{1}{2}} T^{3/2}} \ln\Lambda \tag{5.21}$$

I is defined by

$$I \equiv \left(\frac{e\Phi_o}{T}\right)^{-\frac{1}{2}} \int d\lambda \frac{d\theta}{2\pi} |q| \left[\frac{1}{q} - \left(\frac{1}{\langle q \rangle}\right)_u\right]^2 \tag{5.22}$$

The integral I is a measure of the fraction of the available phase space that contributes to the entropy source. The value of this integral depends somewhat on the choice of the waveform, $h(\theta)$. We have evaluated this integral numerically using the waveform $h(\theta) = \cos\theta$ (see Appendix B). We find that, to lowest order in $(e\Phi_o/T)^{\frac{1}{2}}$, I is given by

$$I = 2^{3/2}(0.69)\ (e\Phi_o/T)^{\frac{1}{2}} \tag{5.23}$$

We now define the family of integrals

$$I_k \equiv \frac{8\pi}{n^2} \frac{T}{m^2} \int dE \left(\frac{E}{T}\right)^k f_o(E) \sum_s F_s(E) \qquad k = 0, 1, 2\ . \tag{5.24}$$

With these definitions, the entropy source may be written as:

$$\dot{S}_e = 1.30\, n \left(\frac{e\Phi_o}{T}\right)^{\frac{1}{2}} \Delta x^2\, \nu_{eff} \left[I_0 A_1^{\ 2} + 2TI_1 A_1 A_2 + T^2 I_2 A_2^{\ 2}\right] \tag{5.25}$$

Equation (5.25) is the explicit evaluation of the entropy source that we require. Comparing Eq. (5.25) with Eq. (4.11), and using the Onsager reciprocity relations (Onsager, 1931) we may read off the particle and energy fluxes:

$$\Gamma_e = -1.30\ D\ I_0 (1 - \frac{\omega}{\omega_{ne}}) \frac{\partial n}{\partial x} + 1.30\ D\ (\frac{3}{2} I_0 - I_1) \frac{n}{T} \frac{\partial T}{\partial x} \tag{5.26}$$

$$Q_e = -1.30\ D\ I_1 (1 - \frac{\omega}{\omega_{ne}})\ T \frac{\partial n}{\partial x} + 1.30\ D (\frac{3}{2} I_1 - I_2)\ n \frac{\partial T}{\partial x} \tag{5.27}$$

where D is a characteristic magnitude of the trapped particle transport coefficients,

$$D \equiv \left(\frac{e\Phi_o}{T}\right)^{1/2} \Delta x^2\ \nu_{eff} \tag{5.28}$$

and ω_{ne} is the electron diamagnetic drift frequency:

$$\omega_{ne} \equiv -\frac{k_y T}{eB} \frac{1}{n} \frac{\partial n}{\partial x} \tag{5.29}$$

The energy integrals, I_k, may be evaluated analytically (see Appendix C). We find

$$I_0 = 1.73$$

$$I_1 = 1.14$$

$$I_2 = 1.89$$

Hence, the pseudoclassical fluxes are given by

$$\Gamma_e = -2.25\ D\ (1- \frac{\omega}{\omega_{ne}}) \frac{\partial n}{\partial x} + 1.89\ D \frac{n}{T} \frac{\partial T}{\partial x} \tag{5.30}$$

$$Q_e = -1.48\ D\ (1- \frac{\omega}{\omega_{ne}})\ T \frac{\partial n}{\partial x} - 0.23\ D\ n \frac{\partial T}{\partial x} \tag{5.31}$$

The heat flux, $\tilde{Q}_e$, is simply related to the energy flux, Q_e. Using the conventional definition (Braginskii, 1965),

$$\tilde{Q}_e = Q_e - \frac{5}{2} T \Gamma e, \qquad (5.32)$$

we find

$$Q_e = 4.15\, D\left(1 - \frac{\omega}{\omega_{ne}}\right) T \frac{\partial n}{\partial x} - 4.96\, D\, n \frac{\partial T}{\partial x} \qquad (5.33)$$

The characteristic value of the pseudoclassical transport coefficients, D , may be written as

$$D = D_{QL}(\nu_{eff}/\omega_{BOUNCE}) \qquad (5.34)$$

where

$$D_{QL} \equiv \frac{1}{k_z v_{te}} \left(\frac{k_y \Phi_o}{B} \right)^2 \qquad (5.35)$$

is characteristic of the magnitude of the "quasilinear" transport coefficients (Horton, 1976). Hence, the pseudoclassical fluxes derived here are smaller than the corresponding quasilinear transport rates by the factor $(\nu_{eff}/\omega_{BOUNCE})$.

This analytic calculation of the pseudoclassical transport coefficients has been verified in numerical simulations of this transport mechanism (Nevins et. al., 1977c). These numerical simulations include only electron-ion collisions. The contribution of electron-ion collisions to the pseudoclassical diffusion coefficient, $\mathcal{D}^{e-i}$, is obtained in Appendix D, where we find

$$\mathcal{D}^{e-i} = 1.30\, D \qquad (5.36)$$

The diffusion observed in these computer simulations was correctly described by Eq. (5.36) when inequality (2.8) was well satisfied (i.e., $\nu_{eff}/\omega_{BOUNCE} \lesssim 1/20$) and $(e\Phi_o/T)^{\frac{1}{2}} \lesssim .5$.

6. A MICROSCOPIC INTERPRETATION OF THE PARTICLE FLUX

A rather unusual feature of the expression for the particle flux derived in Sect. 5 is that the particle flux can be directed towards regions of <u>higher</u> density and temperature. One usually expects the particle flux to be directed away from these regions. This behavior is a collisional analogue of the phenomenea of plasma "pump-in" described by Stix (1967) using quasilinear theory.

We can obtain a qualitative understanding of the expression for the particle flux, Eq. (5.30), by considering the motion of a particle in x and v_z. In the absence of both collisions and the wave, v_z and x are constants of motion. The collisions cause particles to perform a random walk in velocity space. One might also expect collisions to affect the guiding center position, x. In our model, collisions leave x unchanged. This is because the drift kinetic equation is an equation for the evolution of the gyro-phase averaged distribution function (Hinton and Hazeltine, 1976; Hazeltine, 1973). Through first order in $(\Delta x/L)$, the collision operator in this equation is the usual Fokker-Planck collision operator. This operator leaves the particle position unchanged. Hence, the collision term in Eq. (3.3) does not affect the guiding center position. As a consequence, our model does not include classical transport, which arises from the effect of collisions on the gyro phase dependent part of the distribution function (Hazeltine, 1973; Rosenbluth and Kaufman, 1958).

In our model the guiding center position is only affected by the guiding center drift in the electric field of the finite amplitude electrostatic wave. In calculating the particle flux (5.30) and the energy flux (5.31), we have kept only the first order terms in the small parameter $\left(\frac{e\Phi_o}{T}\right)^{1/2}$. For non-resonant particles the change in the guiding center position due to interaction with the wave is proportional to Φ_o, and thus contributes to the transport at higher order than $\left(\frac{e\Phi_o}{T}\right)^{1/2}$. Hence, the ordering in $\left(\frac{e\Phi_o}{T}\right)^{1/2}$ corresponds to neglecting the effect of the wave on the non-resonant particles and considering only the interaction between the wave and the trapped or nearly trapped particles. These particles move along orbits described by*

$$\xi = v_z - \frac{k}{k_y}\,\Omega x = \text{constant} \tag{6.1}$$

Figure 2 shows the trajectory of a typical particle in $(x,\ v_z)$ including both collisions and the effect of the wave on resonant particles. The particle performs a random walk in parallel velocities until it reaches the resonant region. In the resonant region the particle oscillates along a line of constant ξ until it is scattered out of the resonant region by the collisions. Then the random walk in parallel velocities resumes.

Thus, particles that enter the resonant region near A will on the average be transported to the right, while particles that enter the resonate region near A' will be transported to the left. The net transport

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* ξ is easily seen to be constant of the guiding center equations of motion, including the effect of the wave. Recall that

$$\dot{v}_z = \frac{e}{m}\frac{\partial\Phi}{\partial z} = \frac{e}{m}\,k_z\,\Phi_o h'(\theta)$$

$$\dot{x} = -\frac{1}{B}\frac{\partial\Phi}{\partial y} = -\frac{k_y}{B}\,\Phi_o h'(\theta)$$

so

$$\dot{\xi} = \dot{v}_z - \frac{k_z}{k_y}\,\Omega\dot{x} = 0$$

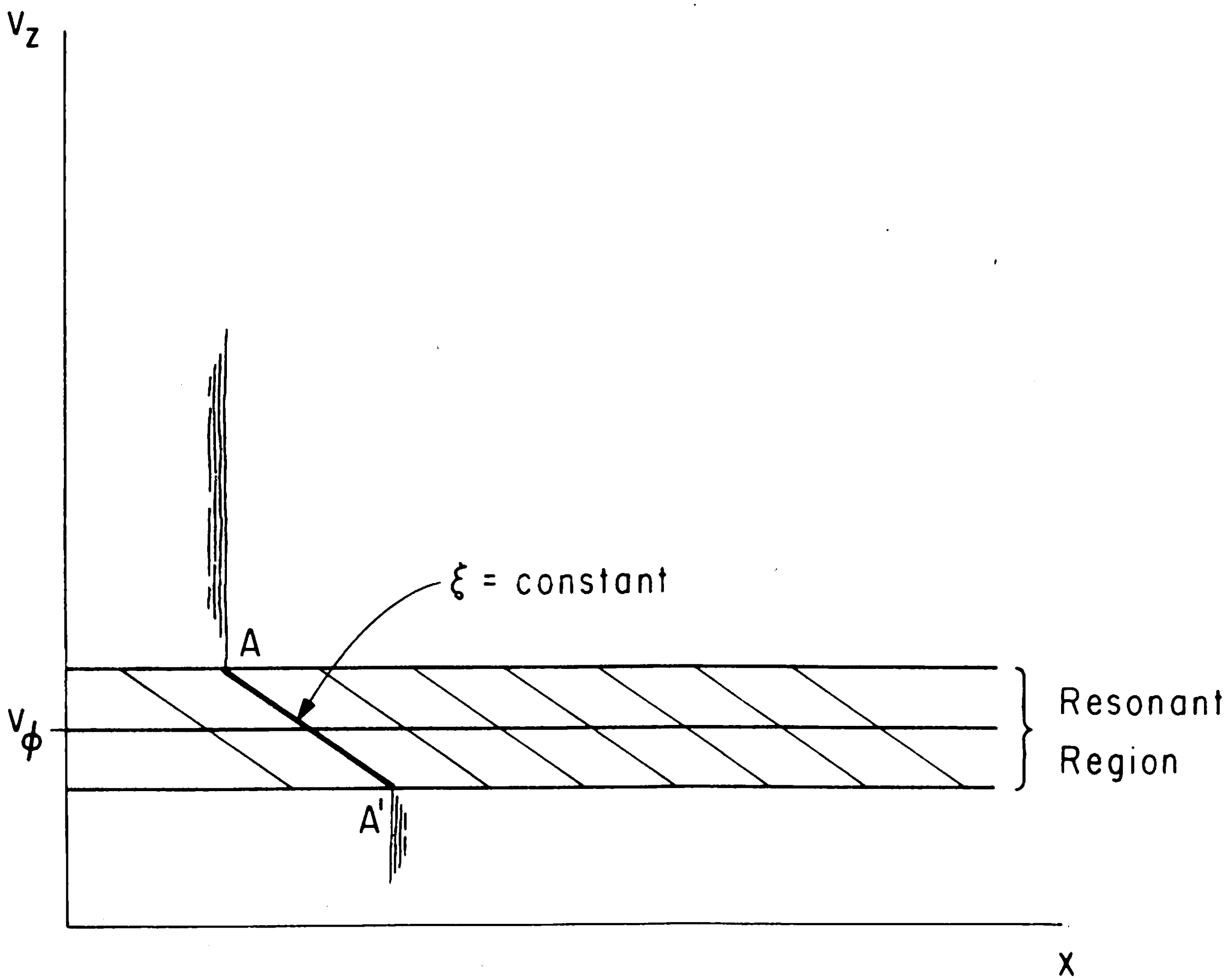

FIGURE 2

This figure shows the motion of a typical particle in the (x,v_z) plane. While the particle is in the resonant region it oscillates along a line of constant ξ. When the particle leaves the resonant region, its guiding center position remains essentially constant, while the parallel velocity changes due to collisions with other particles.

will be determined by the difference between the rates at which particles are scattering into the resonant region at A and A'. These rates are proportional to the value of the distribution function in the region of A and A'.

For the purpose of gaining a qualitative understanding of the pseudoclassical particle flux we may approximate the distribution function by a local Maxwellian. In Fig. 3 we show a contour plot of this distribution in the (x,v_z) plane.

We have isolated the dependence of the particle flux on the density gradient by choosing the number density to be decreasing with x while the temperature is chosen independent of x. If the particle orbit in the wave is oriented like the segment (A, A') in Fig. 3, then there will be more particles at A than A' and the net particle flux will be down the density gradient. On the other hand, if the resonant particle orbits are oriented like (B, B') there will be more particles at B' than at B and the net particle flux will be up the density gradient.

Evidently the direction of the particle flux due to a gradient in the number density is determined by the relationship between the slope of the lines of constant phase space density, f, in the resonant region of (x,v_z) and slope of the particle orbits in the wave. When the contours of constant f in the resonant region have a steeper slope than the particle orbits, the flux is directed down the density gradient.

The lines of constant phase space density are defined by

$$n(x)\, e^{-mv_z^2/2T} = \text{constant}. \tag{6.2}$$

Differentiating we obtain

$$\frac{1}{n}\frac{dn}{dx}\,dx - \frac{mv_z}{T}\,dv_z = 0 \tag{6.3}$$

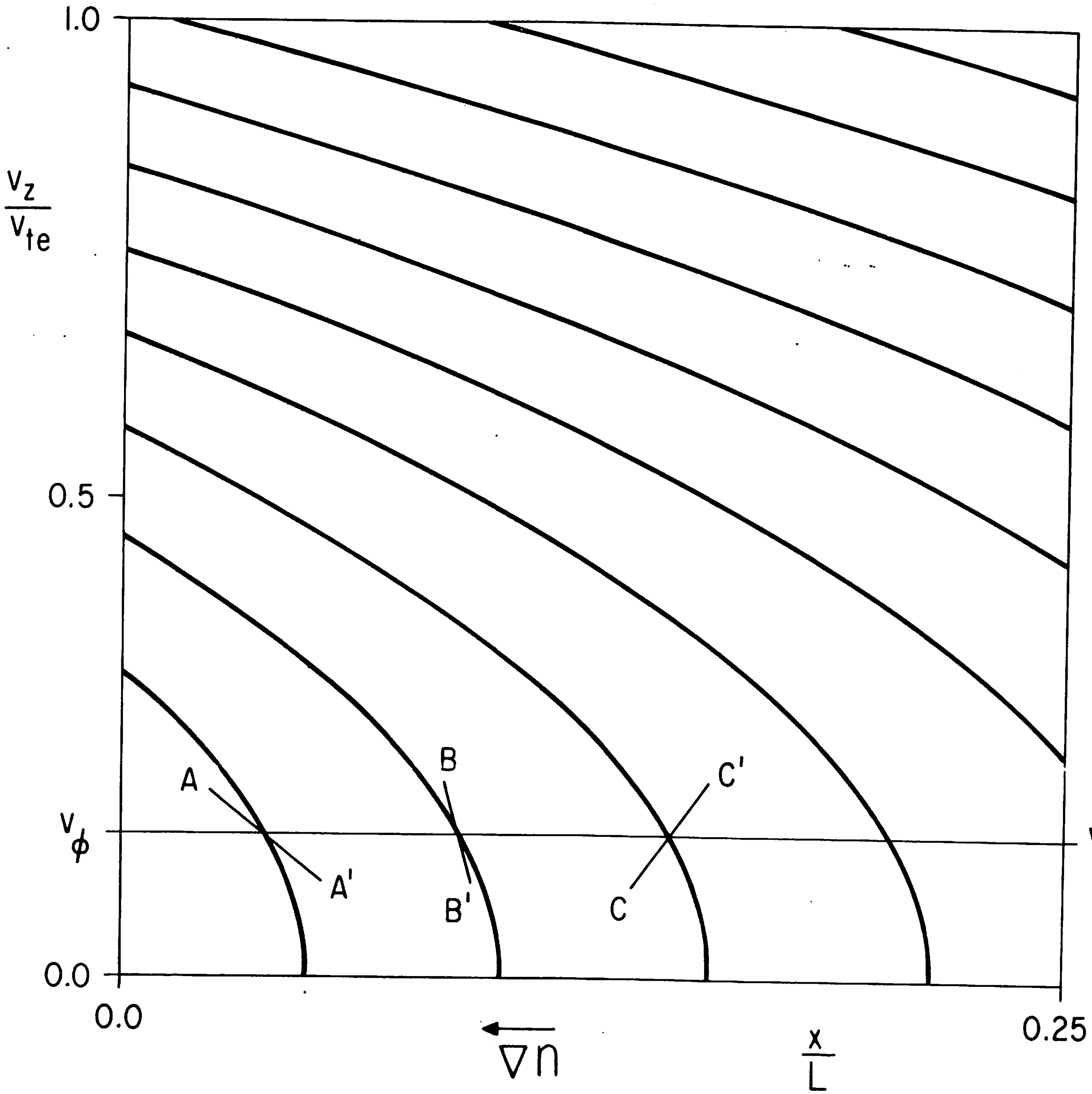

FIGURE 3

A contour plot of a local Maxwellian distribution function in the (x,v_z) plane. The density falls off like $e^{-x/L}$. Hence the distribution function takes on its maximum value in the lower left hand corner and falls off toward the upper right hand corner. The line segments (A,A'), (B,B') and (C,C') illustrate three possible orientations of the resonant particle orbits relative to the contours of constant density.

Solving for the slope in the resonant region (where $v_z \approx v_\phi$), we find that

$$\left.\frac{dv_z}{dx}\right|_{\substack{\text{constant}\\ \text{density}}} = \frac{k_z v_{te}^2}{\omega} \frac{1}{n} \frac{dn}{dx} . \qquad (6.4)$$

The resonant particles satisfy Eq. (6.1). Hence, the slope of the particle orbits is given by

$$\left.\frac{dv_z}{dx}\right|_{\substack{\text{particle}\\ \text{orbit}}} = \frac{k_z}{k_y} \Omega . \qquad (6.5)$$

Comparing Eqs. (6.4) and (6.5), we find that the particle flux will be directed down the density gradient when the wave frequency satisfied the condition

$$\frac{\omega}{\omega_{ne}} < 1 \qquad (6.6)$$

The particle flux is directed up the density gradient when this condition, (6.6), is not satisfied.

This conclusion about the effect of the wave frequency on the direction of the particle flux does not depend on the relative signs chosen for k_y, k_z, ω, or $\frac{dn}{dx}$. In plotting Figures 2 and 3, we have taken k_y and k_z to be positive. $\frac{dn}{dx}$ was chosen to be negative so that the electron diamagnetic drift frequency, ω_{ne}, would be positive [cf. Eq. (5.29)]. The sign of the coefficient of $\frac{dn}{dx}$ in our expression for the particle flux, Eq. (5.30), is only in doubt when the wave frequency has the same sign as the drift frequency. Thus, we have chosen ω positive in order to gain a qualitative understanding of a process that leads to the transport of particles up the density gradient. A different choice of the sign of k_y or k_z would result in similar conclusions, although one might have to look at a different quadrant of the (x, v_z) plane. If the wave parameters are chosen such that $\frac{\omega}{\omega_{ne}}$ is negative, then we obtain an orbit like the segment (C,C'), shown in Figure 3. Since there are more particles at C than C', the net particle flux will be directed down the density gardient in agreement with condition (6.6).

Referring back to our expression for the particle flux, we see that the expected dependence of the direction of the particle flux on the magnitude of the wave frequency is contained in Eq. (5.30) through the factor of $(1 - \frac{\omega}{\omega_{ne}})$ multiplying $\frac{dn}{dx}$.

The effect of a temperature gradient on the particle flux may be isolated by considering a distribution with a temperature gradient, but no density gradient. Since the electron diamagnetic drift frequency, ω_{ne}, goes to zero with the density gradient, it is convenient to include the dependence of the particle flux on the wave frequency by writing the particle flux in the form:

$$\Gamma_e = -2.25\, D \frac{\partial n}{\partial x} + 2.25\, D \frac{n}{T} (0.84 + \frac{\omega}{\omega_{Te}}) \frac{\partial T}{\partial x} \tag{6.7}$$

where

$$\omega_{Te} \equiv - \frac{k_y T}{eB} \frac{1}{T} \frac{\partial T}{\partial x} \tag{6.8}$$

In Figs. 4 and 5 we have plotted distribtuions in which the temperature, T, decreases with x, while the number density, n, remains constant. The shape of the contours in these figures is affected by the choice of the perpendicular energy. Figure 4 shows the distribution in x and v_z at a relatively low perpendicular energy, $\frac{1}{2}mv_\perp^2 = \frac{1}{2}T$. There are fewer low energy particles in the region of higher temperature than

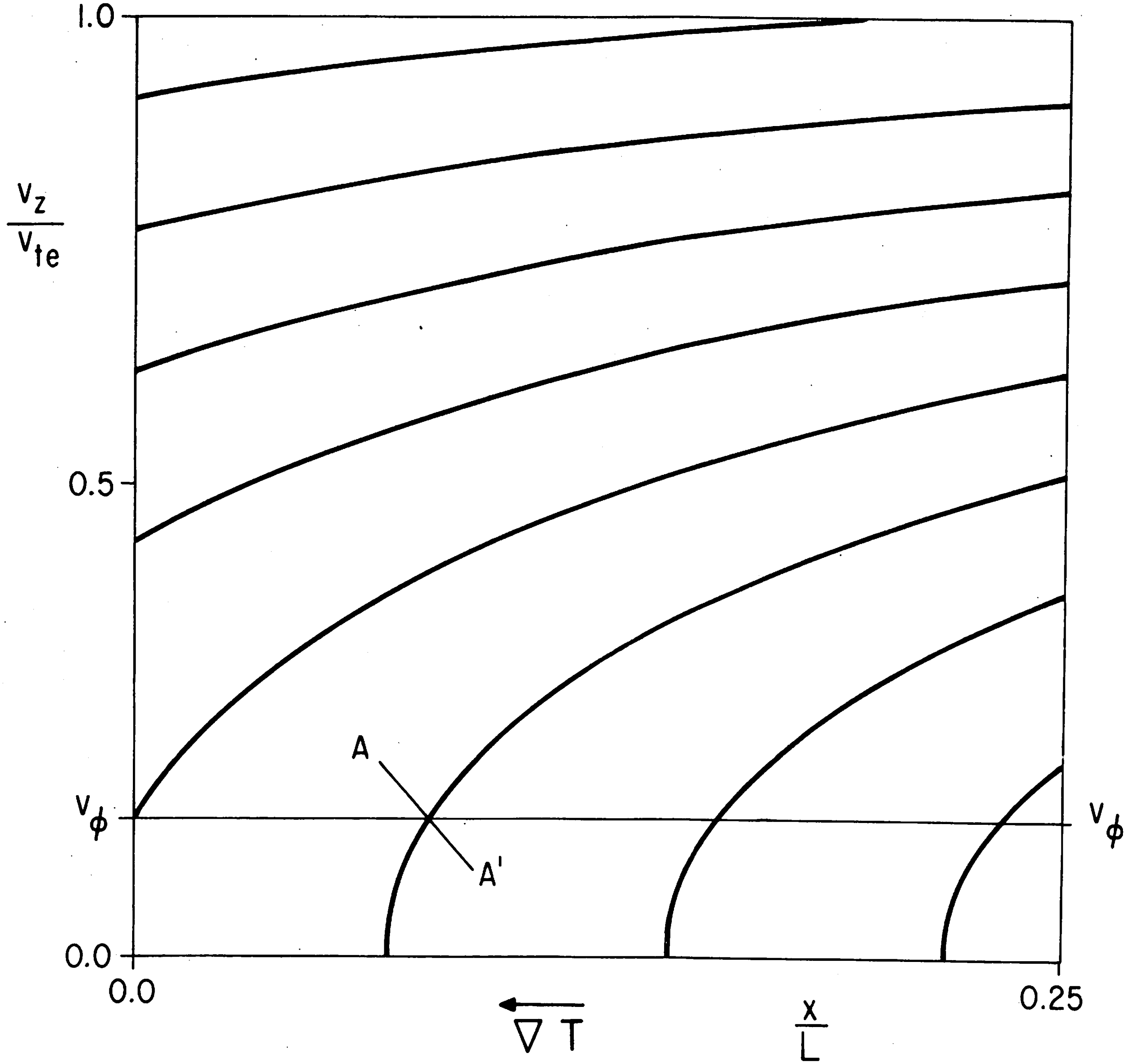

FIGURE 4.

A contour plot of a local Maxwellian distribution function in the (x,v_z) plane. The temperature gradient is directed to the left, while the density gradient has been set equal to zero. We have taken $\frac{1}{2}mv_\perp^2 = \frac{1}{2}T$ in making this plot. The maximum density contour is in the lower right hand corner of this figure.

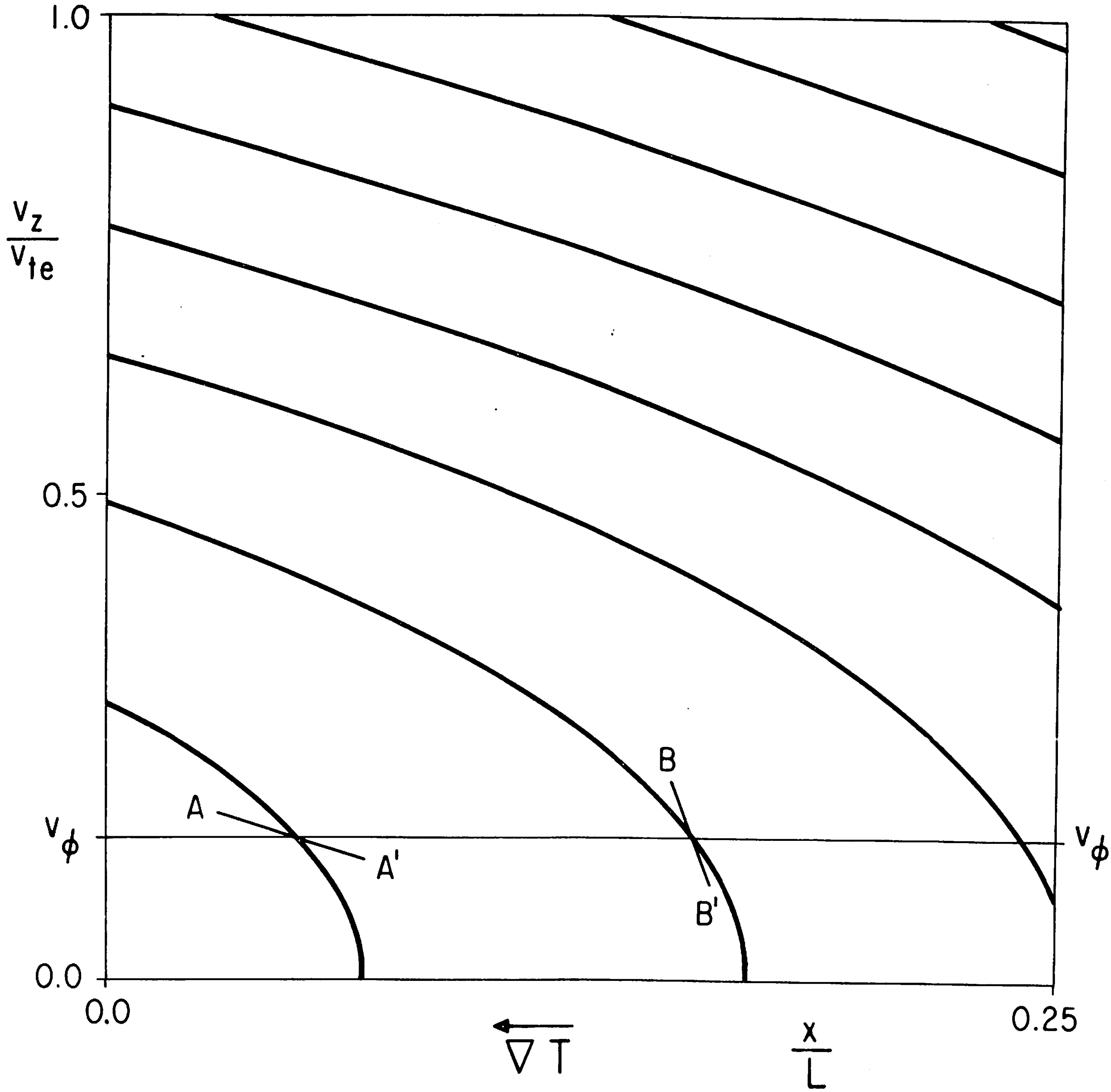

FIGURE 5.

Contour plot of a local Maxwellian distribution function in the (x,v_z) plane. The temperature gradient is directed to the left. The density gradient has been set equal to zero. We have taken $\frac{1}{2}mv_\perp^2 = 2T$. The maximum density contour is now in the lower left hand corner.

in the region of lower temperature. The segment (A,A') in Fig. 4 shows the orbit of a resonant particle in a wave with frequency ω satisfying $\left(\frac{\omega}{\omega_{Te}}\right) > 0$. There are more particles at A' than at A, so the flux of low energy particles will be directed up the temperature gradient.

In Figure 5 we have chosen a relatively high perpendicular energy $\frac{1}{2}mv_{\perp}^2 = 2T$. There are more high energy particles in the region of high temperatures than in the low temperature region. For resonant particle orbits oriented like (A,A'), the flux of high energy particles is directed down the temperature gradient. If the resonant particle orbits are oriented like (B,B'), then the flux of particles with this perpendicular energy will be directed up the temperature gradient. We again determine the direction of the particle flux by comparing the slope of the contours of constant density to the slope of the resonant particle orbits. The density contours are defined by

$$T^{-3/2} \exp\left[-\frac{1}{T}\left(\tfrac{1}{2}mv_{\perp}^2 + \tfrac{1}{2}mv_z^2\right)\right] = \text{constant}$$

Thus, the slope of the contours of constant density in the resonant region is given by

$$\left.\frac{dv_z}{dx}\right|_{\substack{\text{constant}\\ \text{density}}} = \frac{k_z v_{te}^2}{\omega}\left(\frac{\frac{1}{2}mv_{\perp}^2}{T} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial x} \tag{6.9}$$

Comparing Eq. (6.9) with Eq. (6.5) we see that the particle flux will be directed down the temperature gradient provided

$$\frac{\omega}{\omega_{Te}} < \left(\frac{\frac{1}{2}mv_{\perp}^2}{T} - \frac{3}{2}\right) \tag{6.10}$$

To determine the direciton of the net particle flux we must replace the perpendicular velocity in Eq. (6.10) by some sort of average over the distribution of perpendicular velocities. In this average the slower particles should be given a larger weight than the faster particles because the collision frequency is a decreasing function of velocity. Thus it is not surprising that the replacement for $\frac{1}{2} m v_{\perp}^2$ in Eq. (6.10) by $\frac{2}{3}$ T reproduces the dependence of the direction of the particle flux on the wave frequency given by Eq. (6.7).

We have shown that collisional plasma "pump-in" can be understood by examining the orbits of resonant particles in the (x, v_z) plane. In II we will show that a particle flux directed up the density gradient (pump in) is accompanied by a transfer of energy from the finite amplitude wave to the electron distribution, while a particle flux directed down the density gradient (pump out) is accompanied by a transfer of energy from the electron distribution to the wave. Low frequency drift waves have positive energy. Hence,collisional plasma "pump-in" can only be sustained if the wave is driven by something other than the resonant electrons; e.g., an external signal, another linear instability mechanism, or a nonlinear decay process.

7. THE PERTURBED DISTRIBUTION FUNCTION

In this section we will find the steady state electron distribution function that minimizes the entropy source. This distribution function describes the response of the electrons to the finite amplitude drift wave, and includes the effect on the distribution function of trapping of the resonant electrons by the wave. We have three objectives in this examination. The first is to show that the assumptions made in Sect. 5 about the velocity derivatives of $\hat{f}^0$ are in fact satisfied. The second objective is to obtain a qualitative picture of the distortion in the electron distribution function caused by the trapping of the resonant electrons. Finally we calculate the electron susceptibility, $\chi^{(e)}$, and find that Im $\chi^{(e)}$ vanishes through zero order in $(\nu_{eff}/\omega_{BOUNCE})$. The implications of this to the stability of the wave will be investigated in II.

We begin this examination by considering the expression for $\frac{\partial \hat{f}^0}{\partial v_z}$ that was obtained by minimizing the entorpy source. In Sect. 5 the entropy source was evaluated to first order in the small parameter $(e\Phi_o/T)^{\frac{1}{2}}$. To first order in $(e\Phi_o/T)^{\frac{1}{2}}$, the entropy source is unaffected by the addition of a term of the form $d(\theta)$ to $\partial \hat{f}^0/\partial v_z$, provided that

$$d(\theta) \lesssim \frac{1}{v_{te}} (e\Phi_o/T)^{\frac{1}{2}} (\Delta x/L) \tag{7.1}$$

Hence, $\partial \hat{f}^0/\partial v_z$ has only been determined to within a term satisfying (7.1).

Referring to Eq. (5.19), we write $\partial\hat{f}^0/\partial v_z$ as

$$\frac{v_{TRAP}}{G(\frac{1}{2}\, mv_\perp^2)}\frac{\partial\hat{f}^0}{\partial v_z} = q\left[\frac{1}{q} - \left(\frac{1}{\langle q\rangle}\right)_u\right] + O\left(\frac{e\Phi_0}{T}\right) \tag{7.2}$$

where we have used the fact that, to lowest order in $(\Delta x/L)$ and $(e\Phi_0/T)^{\frac{1}{2}}$, E may be replaced by $\frac{1}{2}\, mv_\perp^2$ as the argument of G. The term $(1/\langle q\rangle)_u$ contributes to $\partial\hat{f}^0/\partial v_z$ only in the untrapped region of phase space. The wave phase average of q may be expressed on terms of elliptic integrals (see Appendix B). Using the definition

$$r \equiv \left[\frac{4}{q^2 + 4\sin^2(\theta/2)}\right]^{\frac{1}{2}} \tag{7.3}$$

we may write $\frac{\partial\hat{f}^0}{\partial v_z}$ in the form

$$\frac{v_{TRAP}}{G(\frac{1}{2}\, mv_\perp^2)}\ \frac{\partial\hat{f}^0}{\partial v_z} = 1 - \left[\frac{|q|\, r}{\frac{4}{\pi}\, E(r)}\right]_u \tag{7.4}$$

where the untrapped region of velocity space is that region in which $r < 1$. $E(r)$ is a complete elliptic integral of the second kind (Gradshteyn and Ryzhik, 1965).

In evaluating the entropy source, $\frac{\partial\hat{f}^0}{\partial v_z}$ was assumed to be localized about the trapped region of velocity space. In Fig. 6 $\frac{\partial\hat{f}^0}{\partial v_z}$ is displayed as a function of the velocity-slip, $q = (v_z - v_\phi)/v_{TRAP}$, for several values of the wave phase, θ. We see that $\frac{\partial\hat{f}^0}{\partial v_z}$ is indeed localized about the trapped region, $q \lesssim 1$.

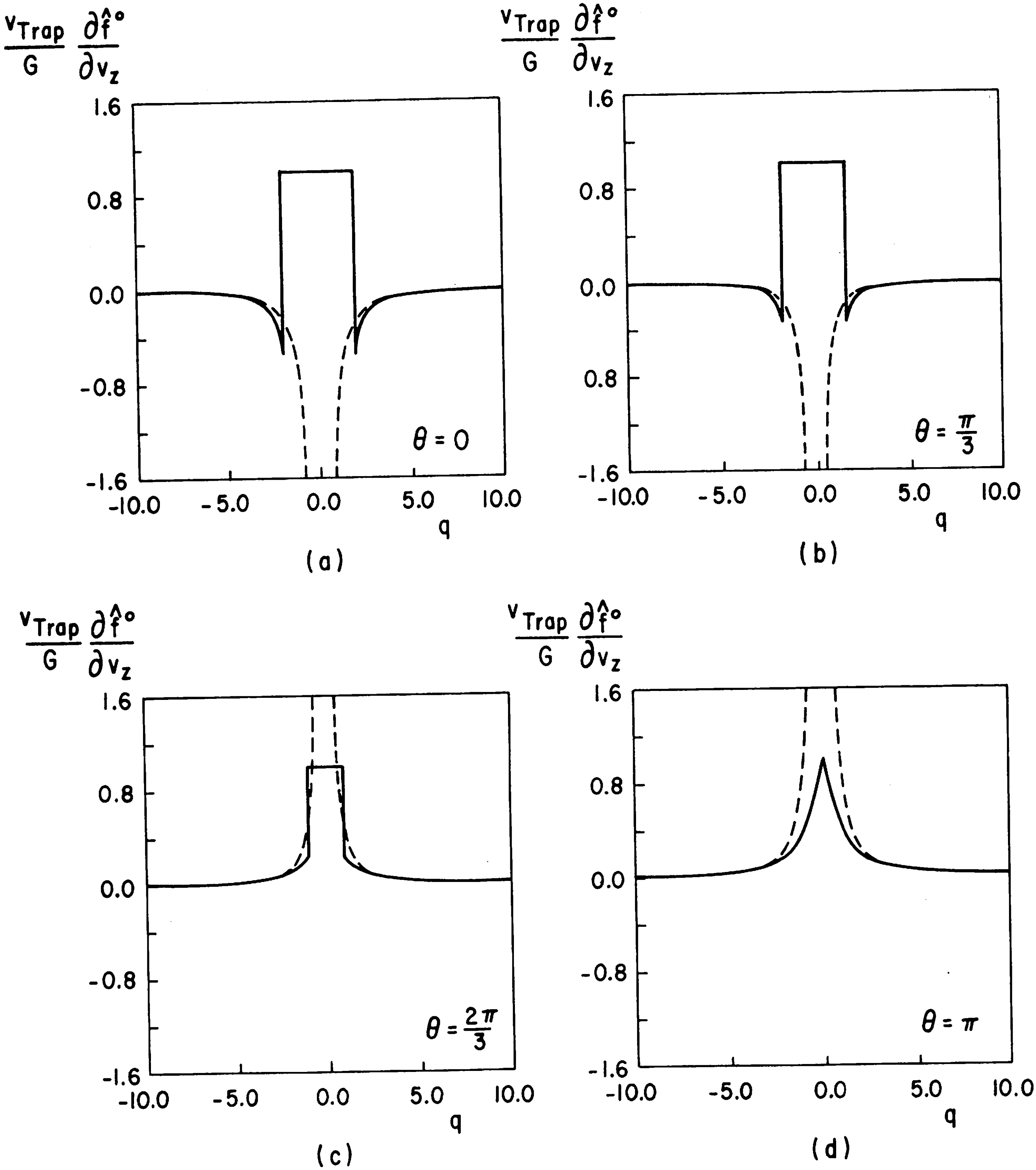

FIGURE 6. $\partial\hat{f}^o/\partial v_z$ is displayed as a function of the dimensionless velocity-slip, q. The discontinuity in $\partial\hat{f}^o/\partial v_z$ occurs along the separatrix between trapped and passing particle orbits. The dashed line shows the approximation $(\partial\hat{f}^o/\partial v_z)_{linear}$ obtained by using linear theory.

The perturbed electron distribution function, $\hat{f}^0$, may be obtained by integrating Eq. (7.4) with appropriate boundary conditions. $\hat{f}^0$ is the perturbation in the electron distribution function due to the finite amplitude wave. The orbits of electrons with parallel velocities far from the trapped region (i.e., with $q >> 1$) are only slightly perturbed by this wave. Hence, the linear procedure of integrating along unperturbed orbits will yield the correct perturbed distribution function in the limit of large q, $q >> 1$. This approximation, $\left(\frac{\partial \hat{f}^0}{\partial v_z}\right)_{\text{linear}}$, is shown in Fig. 6 as a dashed line.* We see that $\left(\frac{\partial \hat{f}^0}{\partial v_z}\right)_{\text{linear}}$ indeed converges to Eq. (7.4) in the limit of large q.

The dependence of $\hat{f}^0$ on v_z may be determined by integrating Eq. (7.4) numerically. The dependence of $\hat{f}^0$ on v_x, v_y, and θ is determined by requiring as our boundary condition, in this numerical integration, that $\hat{f}^0$ match smoothly on to the perturbed distribution function of linear wave theory in the limit of large q. The resulting perturbation in the electron distribution is shown in Fig. 7 together with the corresponding linear perturbation.

---

* Linear theory gives

$$\left(\frac{\partial \hat{f}^0}{\partial v_z}\right)_{\text{linear}} = \frac{G(\frac{1}{2} m v_\perp^2)}{v_{TRAP}} \frac{\cos \theta}{q^2} .$$

This result may be obtained by integrating Eq. (3.13) over θ, while ignoring the θ-variation of q.

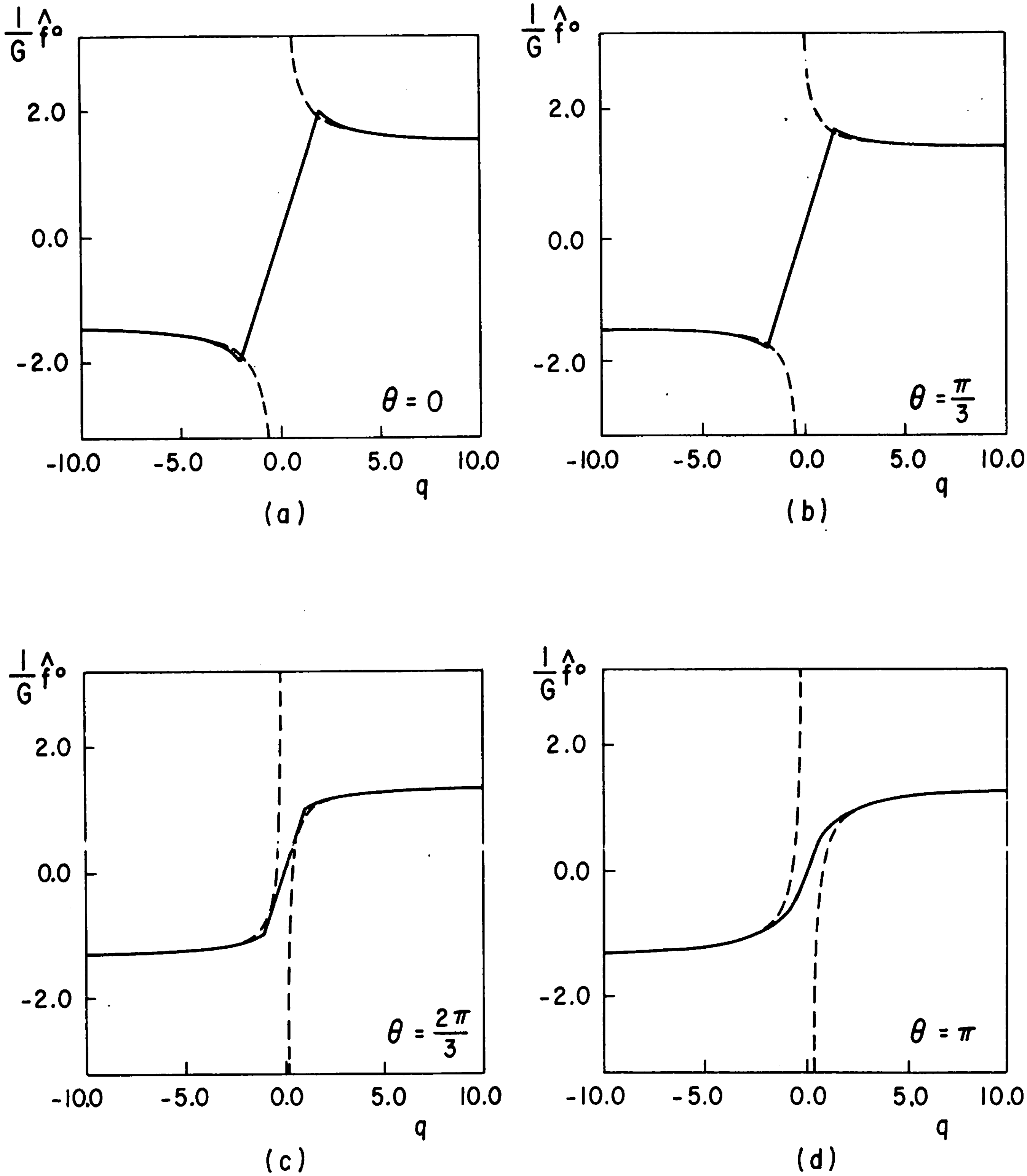

FIGURE 7. $\hat{f}^0$ is displayed as a function of the dimensionless parallel velocity, q. The asymptotic dependence of $\hat{f}^0$ obtained from linear theory is indicated by the dashed line. The linear procedure of integrating along unperturbed orbits determines the distribution function to within a constant. We have chosen this constant as $\sigma 1.38\ G(\frac{1}{2}mv_\perp^2)$, where $\sigma = q/|q|$. This σ dependence is necessary because a direct numerical integration of Eq. (7.4) shows $\hat{f}^0(q=+\infty) - \hat{f}^0(q=-\infty) = 2.76\ G(\frac{1}{2}mv_\perp^2)$.

In matching $\hat{f}^0$ with the perturbed distribution function from linear theory, we find that $\hat{f}^0$ depends on v_x and v_y only through the factor $G(\frac{1}{2} mv_\perp^2)$. Hence the derivatives of $\hat{f}^0$ with respect to v_x and v_y are given by

$$\frac{\partial \hat{f}^0}{\partial v_{x,y}} = \hat{f}^0 \frac{1}{G(\frac{1}{2} mv_\perp^2)} \frac{\partial}{\partial v_{x,y}} G(\frac{1}{2} mv_\perp^2)$$

Figure 7 shows that $\hat{f}^0/G(\frac{1}{2} mv_\perp^2) \leq 2$. Using Eq. (5.12) we may estimate the magnitude of these velocity derivatives as

$$\frac{\partial \hat{f}^0}{\partial v_{x,y}} \simeq \frac{\Delta x}{L} \frac{1}{v_{te}} . \tag{7.5}$$

From Eq. (7.4) we may estimate the magnitude of $\frac{\partial \hat{f}^0}{\partial v_z}$ as

$$\begin{aligned} \frac{\partial \hat{f}^0}{\partial v_z} &\simeq \frac{G(\frac{1}{2} mv_\perp^2)}{v_{TRAP}} \\ &\simeq \left(\frac{e\Phi_0}{T}\right)^{-\frac{1}{2}} \frac{\Delta x}{L} \frac{1}{v_{te}} \end{aligned} \tag{7.6}$$

Comparing Eqs. (7.5) and (7.6) we see that $\frac{\partial \hat{f}^0}{\partial v_{x,y}}$ is smaller than $\frac{\partial \hat{f}^0}{\partial v_z}$ by a factor of $(e\Phi_0/T)^{\frac{1}{2}}$. Hence, the assumptions made about the velocity derivatives of $\hat{f}^0$ in Sect. 5 are all satisfied provided that $(e\Phi_0/T)^{\frac{1}{2}} << 1$.

We may reconstruct the electron distribution function through first order in $(\Delta x/L)$ and zero in $(\nu_{eff}/\omega_{BOUNCE})$ by using $\hat{f}^0$ together with Eq. (3.10). The form of the resulting electron distribution function depends on the three parameters (v_ϕ/v_{te}), $(e\Phi_o/T)$, and $(\hat{\omega}^*/k_z v_{te})$, where following Kadomtsev and Pogutse (1970), we have defined

$$\hat{\omega}^* \equiv - \frac{k_y T}{eB} \left[\frac{1}{n} \frac{\partial n}{\partial x} - \frac{1}{2} \frac{1}{T} \frac{\partial T}{\partial x} \right] \tag{7.7}$$

Figure 8 shows the electron distribution function averaged over both the perpendicular velocities and the wave phase. We have chosen the parameters (v_ϕ/v_{te}), and $(\hat{\omega}^*/k_z v_{te})$ such that the linear instability condition of the collisionless drift mode (Kadomtsev, 1965), $\omega < \hat{\omega}^*$, is satisfied.

This figure shows that the trapping of the resonant particles by the drift wave causes the distribution function to become steeper near the phase velocity of the drift wave. This qualitative feature of the distribution function is a signature of particle trapping by an unstable drift wave. Steepening of the electron distribution function near the phase velocities of unstable drift waves has been observed in computer simulations of the nonlinear saturation of the collisionless drift mode (Lee and Okuda, 1976; Cheng and Okuda, 1977). The distribution function observed at saturation in these computer simulations is similar to the distribution function shown in Fig. 8.

Quasilinear diffusion may also cause the distribution function to become steeper in the resonant region of velocity space (Sagdeev and Galeev, 1969). It is possible to determine which of these mechanisms is responsible for the steepening of the electron distribution function

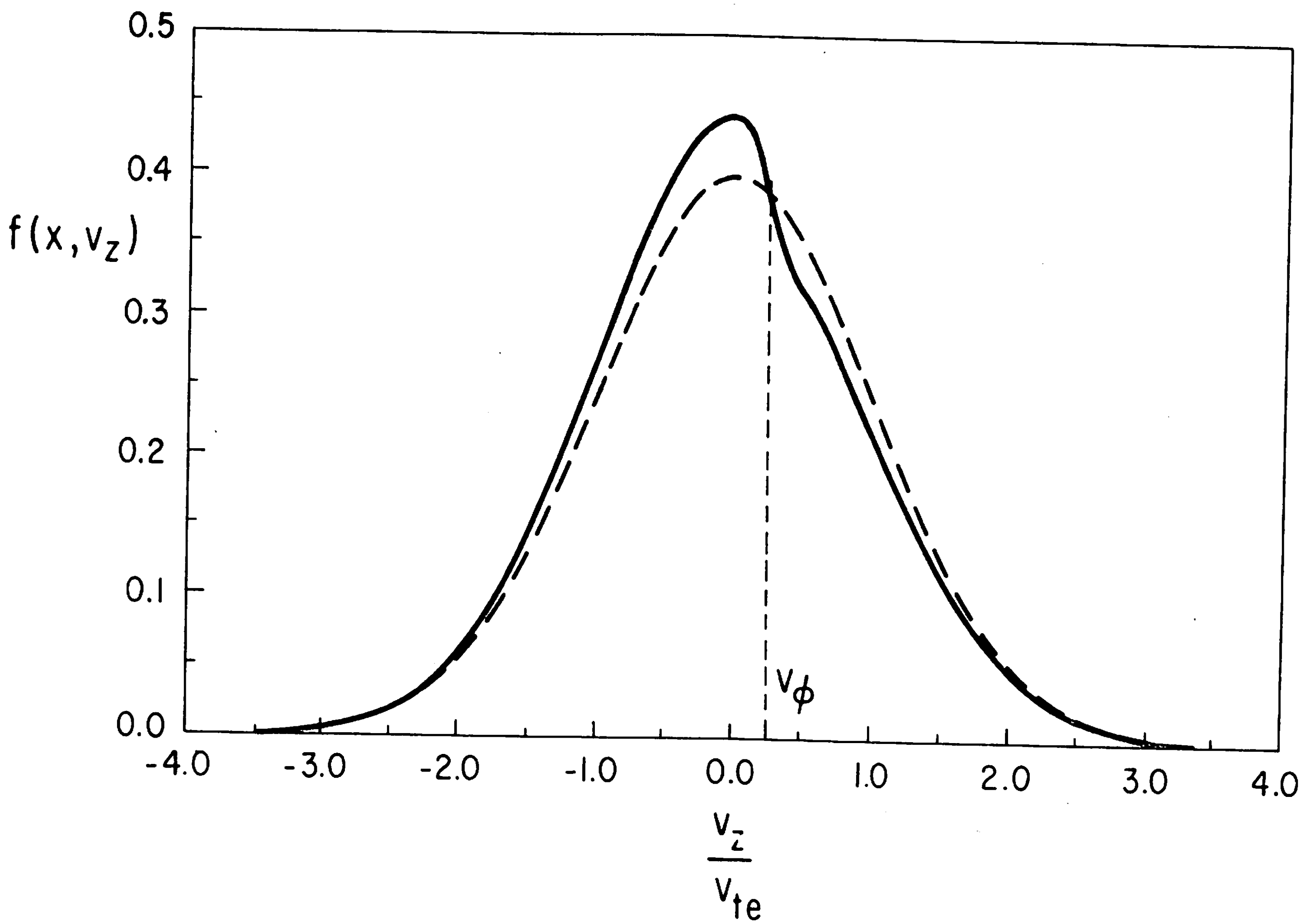

FIGURE 8. The electron distribution function at constant x. It has been averaged over v and θ. In this figure (v_ϕ/v_{te}), $(\hat{\omega}^*/k_z v_{te})$, and $(e\phi_o/T)^{\frac{1}{2}}$ have been chosen as 0.25, 1.0, and 0.1 respectively. The distribution function is distorted due to the trapping of the resonant electrons by the wave. This results in a steepening of the distribution function near the phase velocity of the drift wave. The unperturbed Maxwellian distribution is shown by the dashed line.

by examining the fluctuation spectrum of the drift waves. The evolution of the distribution function is determined by quasilinear theory when the auto-correlation time, τ_{AC}, is shorter that the bounce period of particles trapped in the waves, i.e., when $\omega_{BOUNCE}\,\tau_{AC} << 1$. Particle trapping occurs in the opposite limit, $\omega_{BOUNCE}\,\tau_{AC} >> 1$. Hence, the auto-correlation time of the drift wave fluctuation spectrum will determine which mechanism is responsible for the distortion in the electron distribution function.

The steepening of the electron distribution function in the resonant region is in conflict with the intuitive notion that both particle trapping and quasi-linear diffusion should cause the formation of a plateau in the velocity distribution function. This conflict may be resolved by considering the orbits of resonant particles. Both particle trapping and quasi-linear diffusion (Sagdeev and Galeev, 1969) affect the distribution function by displacing particles along these resonant orbits. In Section 6 we showed that these orbits are described by

$$\xi \equiv v_z - \frac{k_z}{k_y}\,\Omega x = \text{constant}$$

Hence, the flattening of the distribution function due to particle trapping may be observed by plotting the distribution function versus v_z at constant ξ, rather than at constant x.

Figure 9 shows the velocity distribution at constant ξ. The center of the distribution is displaced in velocity by an amount $\hat{v}^* = \hat{\omega}^*/k_z$. This displacement results from taking the particle distribution to be a function of (v_z, ξ) rather than (v_z, x). The phase velocities of waves satisfying the linear instability condition, $\omega < \hat{\omega}^*$, now fall to the left

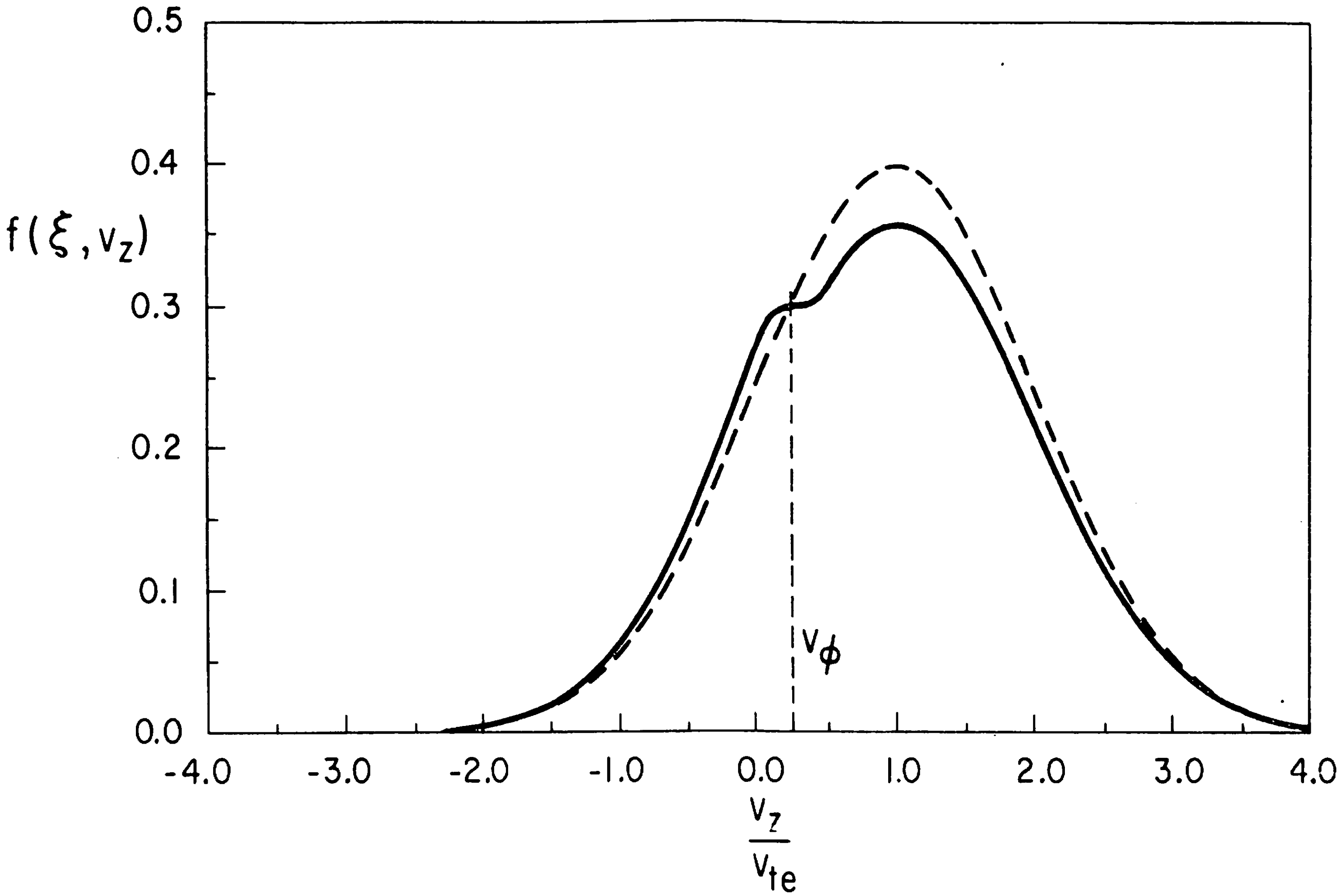

FIGURE 9. The electron distribution function at constant ξ, averaged over $v_\perp$ and θ. The parameters (v_ϕ/v_{te}), $(\hat{\omega}^*/k_z v_{te})$ and $(e\Phi_o/T)^{1/2}$ have again been chosen as 0.25, 1.0, and 0.1 respectively. The trapping of resonant electrons results in the formation of a plateau in the distribution function about the phase velocity of the drift wave. The corresponding Maxwellian distribution is shown by the dashed line. It is interesting to note that the linear growth rate of the drift wave is proportional to the derivative of this Maxwellian distribution, taken at constant ξ.

of the central maximum in the velocity distribution function. Figure 9 clearly shows that the dominant effect of particle trapping on the distribution function is the formation of a plateau along trapped particle orbits.

We have shown above (cf. Fig. 7) that linear theory yields a good approximation to the perturbed electron distribution function everywhere except in a narrow band about the trapped region of velocity space. Since this region is small, we expect that the real part of the electron susceptibility, $\mathrm{Re}\,\chi^{(e)}$ will be reasonably approximated by linear wave theory. We have investigated the dependence of $\mathrm{Re}\,\chi^{(e)}$ on the wave amplitude by numerically integrating the perturbed distribution function over both v_z and θ. We find that the percentage error introduced by using linear wave theory to approximate the real part of the electron susceptibility is of order $(e\Phi_o/T)^2$.

These numerical investigations of the electron susceptibility also show that the imaginary part of the electron susceptibility, $\mathrm{Im}\,\chi^{(e)}$, vanishes through zero order in $(\nu_{eff}/\omega_{BOUNCE})$. This result may be obtained analytically be noting that the θ dependence of $\hat{f}^0$ is given by Eq. (3.26). When the waveform is chosen as $h(\theta) = \cos\theta$, $\hat{f}^0$ will be an even function of θ. Hence, at zero order in $(\nu_{eff}/\omega_{BOUNCE})$ the electron charge density,

$$\rho_e = -e \int d^3\underline{v}\, f_o\, (1 + \hat{f}^0)$$

is also even in θ. $\mathrm{Im}\,\chi^{(e)}$ is proportional to the coefficient of $\sin\theta$ in the Fourier expansion of the electron charge density. Since $\sin\theta$ is an odd function of θ, this Fourier coefficient must vanish identically.

We now summarize this investigation of the electron response to a finite amplitude drift wave. Note that the dominant effect of particle trapping on the electron susceptibility is the vanishing of $\mathrm{Im}\,\chi^{(e)}$ at zero order in $(\nu_{eff}/\omega_{BOUNCE})$. In linear theory (Kadomtsev, 1965) $\mathrm{Im}\,\chi^{(e)}$ results from the Landau resonance between the drift wave and the electrons. The physical mechanism for the wave growth that results from this Landau resonance is the transfer of energy between the resonant electrons and the wave (Jackson, 1960). We have seen that the trapping of particles by the wave causes the distribution function to flatten along resonant particle orbits. The energy transfer between the electrons and the wave ceases when this flattening occurs. Hence, it is not surprising that the imaginary part of the electron susceptibility vanishes through first order in $(\nu_{eff}/\omega_{BOUNCE})$ when the resonant electrons become trapped by the finite amplitude drift wave. In II we will show that when collisional effects are included these trapped electrons give rise to a nonlinear $\mathrm{Im}\,\chi^{(e)}$ proportional to the electron collision frequency. This nonlinear $\mathrm{Im}\,\chi^{(e)}$ replaces the Landau resonance of linear theory when the resonant electrons become trapped and the wave reaches a steady state.

8. CONCLUSION

This concludes the derivation of the enhanced fluxes of particles and energy due to the trapping of resonant electrons by a finite amplitude drift wave. Our main results are Eqs. (5.30) and (5.31), which give expressions for these pseudoclassical fluxes in terms of the wave parameters ω, $\underline{k}$, and $(e\Phi_o/T)$. These expressions for the pseudoclassical particle and energy flux are analogous to the "quasilinear" expressions for the particle and energy flux (Stix, 1967) in that they provide a prescription for relating the drift wave spectrum to the transport rates.

Previous authors (Pogutse, 1972; Gell and Nevins, 1975) have used expressions similar to our Eq. (5.31) together with order of magnitude estimates of (k_y/k_z) and $(e\Phi_o/T)$ in order to compare the energy transport rates given by this theoretical model to the energy loss rates observed in tokamak plasmas. While these estimates demonstrate that the transport mechanism described above may be important in determining the energy containment time, they do not provide useful transport coefficients because the estimates of (k_y/k_z) and $(e\Phi_o/T)$ are <u>very</u> uncertain. These estimates do not even provide reliable scaling laws because it is not at all obvious how the drift wave spectrum, which determines (k_y/k_z) and $(e\Phi_o/T)$, will be affected by changes in the magnetic field strength, the number density, the temperature, or other plasma parameters.

Rather than estimating (k_y/k_z) and $(e\Phi_o/T)$, a complete treatment of pseudoclassical transport must provide a means of calculating these quantities. Such a treatment of pseudoclassical transport requires knowledge of the drift wave fluctuation spectrum. The calculation of this fluctuation spectrum is the central problem in determining the anomalous transport associated with low frequency drift waves. Hence, it is important to understand the relation between the pseudoclassical transport theory considered here, and other work on the anomalous transport associated with low frequency drift waves (e.g., Horton, 1976; Liu et al., 1976).

In the companion paper, II, we extend the previous work on pseudoclassical transport by deriving a complete set of equations describing the evolution of both the wave and the background plasma. We find that the pseudoclassical transport theory considered here is associated with the nonlinear development of the "collisionless" drift instability (Galeev et al., 1963), and we clarify the relation between pseudoclassical transport theory and other work on low frequency drift waves.

ACKNOWLEDGMENTS

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Appendix A - SELF-ADJOINTNESS OF THE LOCAL ENTROPY SOURCE (Sect. 4)

In deriving our variational principle on the rate of entropy production, we have used the fact that the local entropy sources, $K_{ee}(\hat{f},\hat{g})$ and $K_{ei}(\hat{f},\hat{g})$ are self-adjoint. In K_{ee} this property follows directly from the nature of two body collisions, while the local entropy source due to electron-ion collisions, $K_{ei}(\hat{f},\hat{g})$, is self adjoint in the limit of small mass ratio, when the ions may be approximated by massive, fixed scattering centers.

The effect of two body collisions on the velocity distribution function is described by the Boltzman collision operator:

$$C_{ab}(g_a) = -\int d^3\underline{v}_b \, d\Omega_a \, \sigma_{ab}(\Omega_a) |\underline{v}_a - \underline{v}_b| \, (g_a g_b - g_a'' g_b'') \qquad \text{(A.1)}$$

The subscripts a and b label the colliding particles. $\underline{v}_a$ and $\underline{v}_b$ are the particle velocities before the collision. $\underline{v}_a''$ and $\underline{v}_b''$ are the particle velocities after the collision. Ω_a describes the angle between the initial and final velocities of particle a. $\sigma_{ab}(\Omega_a)$ is the scattering cross section. g_a and g_b are the velocity distribution functions evaluated at the initial velocities, while g_a'' and g_b'' are the distribution functions evaluated at the final velocities. We assume that these velocity distribution functions are nearly Maxwellian, and write them in the form

$$g = g_o(1 + \hat{g}) \qquad \text{(A.2)}$$

where g_o is a Maxwellian distribution.

Linearizing the Boltzmann collision operator in $\hat{g}$ yields

$$C_{ab}(g_a) = -\int d^3 v_b \, d\Omega_a \, \sigma_{ab}(\Omega_a) \, |\underline{v}_a - \underline{v}_b| \, [g_{oa} g_{ob} - g_{oa}'' g_{ob}'' + g_{oa} g_{ob}(\hat{g}_a + \hat{g}_b) - g_{oa}'' g_{ob}''(\hat{g}_a'' + \hat{g}_b'')] \qquad \text{(A.3)}$$

In Eqs. (A.1) and (A.3) the collisions are described by the initial velocities, $(\underline{v}_a,\underline{v}_b)$, and the scattering angle, Ω_a. The final velocities, $(\underline{v}_a'',\underline{v}_b'')$ are to be viewed as functions of $(\underline{v}_a,\underline{v}_b,\Omega_a)$. The relations between $(\underline{v}_a,\underline{v}_b,\Omega_a)$ and $(\underline{v}_a'',\underline{v}_b'')$ are determined by the kinematics of elastic, two body collisions. It follows from these kinematic considerations that

$$|\underline{v}_a''-\underline{v}_b''| = |\underline{v}_a-\underline{v}_b| \tag{A.4}$$

$$\tfrac{1}{2}m_a v_a''^2 + \tfrac{1}{2}m_b v_b''^2 = \tfrac{1}{2}m_a v_a^2 + \tfrac{1}{2}m_b v_b^2 \tag{A.5}$$

The collisions may also be described by the final velocities, $(\underline{v}_a'',\underline{v}_b'')$ and Ω_a'', the angle between the final and the initial velocities of particle a. It can be shown that the Jacobian of the transformation between the variable set $(\underline{v}_a,\underline{v}_b,\Omega_a)$ and the set $(\underline{v}_a'',\underline{v}_b'',\Omega_a'')$ is equal to one. We also use the relation

$$\sigma_{ab}(\Omega_a) = \sigma_{ab}(\Omega_a'') \tag{A.6}$$

which follows from the principle of detailed balance.

Using Eqs. (4.16), (A.3) and (A.5) the entropy source due to electron-electron collisions may be written in the form

$$\begin{aligned} K_{ee}(\hat{f},\hat{g}) = &-\int d^3\underline{v}_a d^3\underline{v}_b\, d\Omega_a\, \sigma_{ee}(\Omega_a)\, |\underline{v}_a-\underline{v}_b|\, f_{oe}(v_a)f_{oe}(v_b)\hat{f}_a\hat{g}_a \\ &-\int d^3\underline{v}_a d^3\underline{v}_b\, d\Omega_a\, \sigma_{ee}(\Omega_a)\, |\underline{v}_a-\underline{v}_b|\, f_{oe}(v_a)f_{oe}(v_b)\hat{f}_a\hat{g}_b \\ &+\int d^3\underline{v}_a d^3\underline{v}_b\, d\Omega_a\, \sigma_{ee}(\Omega_a)\, |\underline{v}_a-\underline{v}_b|\, f_{oe}(v_a)f_{oe}(v_b)\hat{f}_a\hat{g}_a'' \\ &+\int d^3\underline{v}_a d^3\underline{v}_b\, d\Omega_a\, \sigma_{ee}(\Omega_a)\, |\underline{v}_a-\underline{v}_b|\, f_{oe}(v_a)f_{oe}(v_b)\hat{f}_a\hat{g}_b'' \end{aligned} \tag{A.7}$$

where f_{oe} is the Maxwellian distribution function that relates the arguments of the functional $K_{ee}(\hat{f},\hat{g})$ to both f and g. i.e.,

$$f = f_{oe}(1 + \hat{f})$$

and

$$g = f_{oe}(1 + \hat{g})$$

We make the following changes in the integration variables:

1^{st} term	no change
2^{nd} term	$(\underline{v}_a, \underline{v}_b, \Omega_a) \rightarrow (\underline{v}_a, \underline{v}_b, \Omega_b)$
3^{rd} term	$(\underline{v}_a, \underline{v}_b, \Omega_a) \rightarrow (\underline{v}_a'', \underline{v}_b'', \Omega_a'')$
4^{th} term	$(\underline{v}_a, \underline{v}_b, \Omega_a) \rightarrow (\underline{v}_a'', \underline{v}_b'', \Omega_b'')$

where Ω_b describes the angle between $\underline{v}_b$ and $\underline{v}_b''$, while Ω_b'' describes the angle between $\underline{v}_b''$ and $\underline{v}_b$. We use the fact that for like particle collisions

$$\sigma(\Omega_a) = \sigma(\Omega_b)$$

and

$$d\Omega_a = d\Omega_b$$

together with Eqs. (A.4) thru (A.6) to obtain

$$\begin{aligned} K_{ee}(\hat{f},\hat{g}) = &-\int d^3\underline{v}_a d^3\underline{v}_b d\Omega_a \;\; \sigma_{ee}(\Omega_a) \;\; |\underline{v}_a-\underline{v}_b| \;\; f_{oe}(v_a) f_{oe}(v_b)\hat{g}_a\hat{f}_a \\ &-\int d^3\underline{v}_b d^3\underline{v}_a d\Omega_b \;\; \sigma_{ee}(\Omega_b) \;\; |\underline{v}_a-\underline{v}_b| \;\; f_{oe}(v_b) f_{oe}(v_a)\hat{g}_b\hat{f}_a \\ &+\int d^3\underline{v}_a'' d^3\underline{v}_b'' d\Omega_a'' \;\; \sigma_{ee}(\Omega_a'') \;\; |\underline{v}_a''-\underline{v}_b''| \;\; f_{oe}(v_a'') f_{oe}(v_b'')\hat{g}_a''\hat{f}_a \\ &+\int d^3\underline{v}_b'' d^3\underline{v}_a'' d\Omega_b'' \;\; \sigma_{ee}(\Omega_b'') \;\; |\underline{v}_a''-\underline{v}_b''| \;\; f_{oe}(v_b'') f_{oe}(v_a'')\hat{g}_b''\hat{f}_a \end{aligned} \qquad (A.8)$$

Comparing (A.8) term by term with (A.7) we find

$$K_{ee}(\hat{f},\hat{g}) = K_{ee}(\hat{g},\hat{f}). \qquad (A.9)$$

In calculating the entropy source due to electron-ion collisions we take the ions to be fixed scattering centers. Then the ion distribution function is given by

$$f_i(v) = n_i \,\delta(v) \qquad (A.10)$$

and relations (A.4) and (A.5) become

$$|v_a''| = |v_a| \qquad (A.4')$$

$$\tfrac{1}{2} m_a v_a''^2 = \tfrac{1}{2} m_a v_a^2 \qquad (A.5')$$

Using Eq.(4.17) together with (A.10) and (A.5') the local entropy source due to electron-ion collisions may be written as

$$K_{ei}(\hat{f},\hat{g}) = -n_i\int d^3\underline{v}_a d\Omega_a \ \sigma_{ei}(\Omega_a) \ |\underline{v}_a| \ f_{oe}(v_a)\hat{f}_a\hat{g}_a + n_i\int d^3\underline{v}_a d\Omega_a \ \sigma_{ei}(\Omega_a) \ |\underline{v}_a| \ f_{oe}(v_a)\hat{f}_a\hat{g}_a'' \qquad (A.11)$$

Changing variables in the second term from $(\underline{v}_a,\Omega_a)$ to the set $(\underline{v}_a'',\Omega_a'')$ and using (A.4'), (A.5') and (A.6), we find

$$K_{ei}(\hat{f},\hat{g}) = -n_i\int d^3\underline{v}_a d\Omega_a \ \sigma_{ei}(\Omega_a) \ |\underline{v}_a| \ f_{oe}(v_a)\hat{g}_a\hat{f}_a + n_i\int d^3\underline{v}_a'' d\Omega_a \ \sigma_{ei}(\Omega_a'') \ |\underline{v}_a''| \ f_{oe}(v_a'')\hat{g}_a''\hat{f}_a \qquad (A.12)$$

Comparing Eqs. (A.12) and (A.11) we see that $K_{ei}(\hat{f},\hat{g})$ is indeed self adjoint.

Appendix B - EVALUATION OF THE INTEGRAL I (Section 5)

I is defined as

$$I \equiv 2^{\frac{1}{2}}\, \delta^{-\frac{1}{2}} \int_{-\delta}^{E/T} d\lambda \int_{0}^{2\pi} \frac{d\theta}{2\pi}\ |q| \left[\frac{1}{q} - \frac{1}{\langle q \rangle_u}\right]^2 \tag{B.1}$$

where $\delta \equiv \dfrac{e\Phi_o}{T}$ (B.2)

and the term in $\langle q \rangle_u$ only contributes in the untrapped region. We take the wave form to be

$$h(\theta) = \cos\theta \tag{B.3}$$

It then follows from Eqs. (2.2), (3.6) and (5.9) that q may be written as

$$q = \sigma\, 2^{\frac{1}{2}} u \tag{B.4}$$

where

$$u \equiv \left(\frac{\lambda}{\delta} + \cos\theta\right)^{\frac{1}{2}} \tag{B.5}$$

We first calculate the contribution to I from the trapped particles,

$$I_t \equiv 2^{\frac{1}{2}}\, \delta^{-\frac{1}{2}} \int_{-\delta}^{\delta} d\lambda \int_{0}^{2\pi} \frac{d\theta}{2\pi} \frac{1}{|q|}. \tag{B.6}$$

Using Eqs. (B.4) and (B.5), I_t may be written as,

$$I_t = \delta^{\frac{1}{2}} \int_{0}^{2\pi} \frac{d\theta}{2\pi} \int_{-\cos\theta}^{1} \frac{d\chi}{(\chi + \cos\theta)^{\frac{1}{2}}}.$$

Performing the integral over χ we have,

$$= 2\ \delta^{\frac{1}{2}} \int_{0}^{2\pi} \frac{d\theta}{2\pi}\ (1 + \cos\theta)^{\frac{1}{2}}.$$

Using Gradshteyn and Ryzhik (1965) Eq. (2.576.1) we find,

$$I_t = 2^{3/2}\, \delta^{\frac{1}{2}} \left(\frac{2}{\pi}\right). \tag{B.7}$$

The contribution to I from the untrapped particles is given by

$$I_u = \delta^{-\frac{1}{2}} \int_{\delta}^{E/T} d\lambda \left[\langle \frac{1}{u} \rangle_{\theta} - \frac{1}{\langle u \rangle_{\theta}} \right] . \tag{B.8}$$

Thus, we must calculate $\langle \frac{1}{u} \rangle_{\theta}$ and $\langle u \rangle_{\theta}$.

$$\langle u \rangle_{\theta} \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \left(\frac{\lambda}{\delta} + \cos\theta \right)^{\frac{1}{2}}$$

Using Gradshteyn and Ryzhik (1965) Eq. (2.576.1) we write $\langle u \rangle_{\theta}$ as

$$\langle u \rangle_{\theta} = \frac{2^{3/2}}{\pi} r^{-1} E(r) \tag{B.9}$$

where E(r) is a complete elliptic integral of the second kind and r is given by:

$$r \equiv \left[\frac{2\delta}{\lambda+\delta} \right]^{\frac{1}{2}} . \tag{B.10}$$

Similarly we have

$$\langle \frac{1}{u} \rangle_{\theta} \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{\left(\frac{\lambda}{\delta} + \cos\theta \right)^{\frac{1}{2}}} \tag{B.11}$$

Using Gradshetyn and Ryzhik (1965) Eq. (2.571.4) this becomes

$$\langle \frac{1}{u} \rangle_{\theta} = \frac{2^{\frac{1}{2}}}{\pi} r K(r) \tag{B.12}$$

where K(r) is a complete elliptic integral of the first kind. Combining Eqs. (B.8), (B.9) and (B.12) we write the contribution of the untrapped particles to I as

$$I_u = \frac{1}{(2\delta)^{\frac{1}{2}}} \int_{\delta}^{E/T} d\lambda \left[\frac{2}{\pi} r K(r) - \frac{r}{\frac{2}{\pi} E(r)} \right] . \tag{B.13}$$

Changing our integration variable from λ to r we find

$$I_u = 2^{3/2} \delta^{\frac{1}{2}} \int_0^1 \frac{dr}{r^2} \left[\frac{2}{\pi} K(r) - \frac{1}{\frac{2}{\pi} E(r)} \right] + \Delta \tag{B.14}$$

where

$$\Delta \equiv 2^{3/2}\,\delta^{\frac{1}{2}} \int_0^{\varepsilon} \frac{dr}{r^2} \left[\frac{2}{\pi}\,K(r) - \frac{1}{\frac{2}{\pi}\,E(r)}\right] \tag{B.15}$$

and

$$\varepsilon \equiv \left[\frac{2\delta}{(E/T) + \delta}\right]^{\frac{1}{2}} . \tag{B.16}$$

Over most of the occupied region of phase space $(E/T) \sim 1$, so that $\varepsilon = O(\delta)^{\frac{1}{2}}$. Hence, we may estimate the magnitude of Δ by replacing the elliptic integrals $K(r)$ and $E(r)$ by their small argument expansions. The lowest order nonvanishing contribution to Δ is then found to be

$$\Delta \simeq 2^{3/2}\,\delta^{\frac{1}{2}} \int_0^{\varepsilon} \frac{dr}{r^2} \left[\frac{3}{32}\,r^4\right] = O(\delta)^2$$

The contribution of Δ to I is fourth order in our expansion parameter $\delta^{\frac{1}{2}}$. Hence, this term may be neglected.

The remaining term in (B.14) may be integrated numerically. We find

$$I_u = 2^{3/2}\,\delta^{\frac{1}{2}}\,(0.05276\ldots) \tag{B.17}$$

Finally we combine Eqs. (B.7) and (B.17) to obtain

$$I = 2^{3/2}\,(e\Phi_o/T)^{\frac{1}{2}}\,(0.69) \tag{B.18}$$

It is interesting to note that the contribution of the trapped particles to I is an order of magnitude larger than the contribution of the untrapped particles. Hence, the trapped particles provide the dominant contribution to the entropy source.

Appendix C - EVALUATION OF THE INTEGRALS I_k (Section 5)

We wish to evaluate integrals of the form:

$$I_k = \frac{8\pi}{n^2}\frac{T}{m_e^2} \int dE \left(\frac{E}{T}\right)^k f_o(E) \sum_s F_s(E) \qquad (C.1)$$

$k = 0,1,2$

It is shown by Rosenbluth et. al. (1972) that the functions F_s may be written as:

$$F_s(E) = \frac{a_s}{4} 2^{\frac{1}{2}} \frac{nT}{m_s} \left(\frac{E}{m_e}\right)^{-3/2} * \\ * \left[\left(\frac{m_s}{m_e}\frac{E}{T}\right)^{\frac{1}{2}} \mathrm{Erf}'\left(\frac{m_s}{m_e}\frac{E}{T}\right)^{\frac{1}{2}} + \left(2\frac{m_s}{m_e}\frac{E}{T} - 1\right) \mathrm{Erf}\left(\frac{m_s}{m_e}\frac{E}{T}\right)^{\frac{1}{2}} \right] \qquad (C.2)$$

where a_s is given by:

$$a_e = 1, \qquad a_i = \tfrac{1}{2}. \qquad (C.3)$$

Thus $F_e(E)$ may be written as:

$$F_e(E) = \tfrac{1}{4}\, 2^{\frac{1}{2}}\, n \left(\frac{T}{m_e}\right)^{-\frac{1}{2}} x^{-3} \left[x\, \mathrm{Erf}'(x) + (2x^2 - 1)\, \mathrm{Erf}(x)\right] \qquad (C.4)$$

where

$$x \equiv \left(\frac{E}{T}\right)^{\frac{1}{2}}$$

and Erf(x) is the probability integral defined by

$$\mathrm{Erf}(x) \equiv \frac{2}{\pi^{\frac{1}{2}}} \int_0^x e^{-t^2}\, dt. \qquad (C.5)$$

To lowest order in $\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}$, $F_i(E)$ may be written as:

$$F_i(E) = \tfrac{1}{4}\, 2^{\frac{1}{2}}\, n \left(\frac{T}{m_e}\right)^{-\frac{1}{2}} x^{-1}. \qquad (C.6)$$

We find it useful to treat the electrons and ion terms separately. Thus we write

$$I_k = I_k^{e-e} + I_k^{e-i} \qquad (C.7)$$

where

$$I_k^{e-e} \equiv \frac{8\pi}{n^2} \frac{T}{m_e^2} \int dE \left(\frac{E}{T}\right)^k f_o(E)\ F_e(E) \tag{C.8}$$

and

$$I_k^{e-i} \equiv \frac{8\pi}{n^2} \frac{T}{m_e^2} \int dE \left(\frac{E}{T}\right)^k f_o(E)\ F_i(E) \tag{C.9}$$

I_k^{e-e} involves only the electron-electron collision operator, while I_k^{e-i} involves only the electron-ion collision operator. In evaluting these integrals it is helpful to express $f_o(E)$ in terms of the probability integral Erf(x) as:

$$f_o(E) = \frac{n}{4\pi 2^{\frac{1}{2}}} \left(\frac{m_e}{T}\right)^{3/2} \mathrm{Erf}'(x), \tag{C.10}$$

$$x \equiv \left(\frac{E}{T}\right)^{\frac{1}{2}}.$$

A. Evaluation of I_k^{e-e}

Using Eqs.(C.4), (C.8), and (C.10), I_k^{e-e} may be written as:

$$I_k^{e-e} = \int_0^\infty dx\ x^{2k-2}\ \mathrm{Erf}'(x) \left\{ x\ \mathrm{Erf}'(x) + (2x^2 - 1)\ \mathrm{Erf}(x) \right\} \tag{C.11}$$

$k = 0,1,2.$

In evaluating (C.11) we need integrals of the form

$$J_\ell \equiv \int_0^\infty dx\ x^{2\ell-1}\ \mathrm{Erf}'^2(x) \tag{C.12}$$

$\ell = 1,2$

$$K_\ell \equiv \int_0^\infty dx\ x^{2\ell}\ \mathrm{Erf}'(x)\ \mathrm{Erf}(x) \tag{C.13}$$

$\ell = 0,1,2$

$$L \equiv \int_0^\infty dx\ x^{-2}\ \mathrm{Erf}'(x) \left[x\ \mathrm{Erf}'(x) - \mathrm{Erf}(x) \right] \tag{C.14}$$

The definition of the probability integral, (C.5), may be used to express the integrals J_ℓ as

$$J_\ell = \frac{4}{\pi} \int_0^\infty dx\ x^{2\ell-1}\ e^{-2x^2} \tag{C.15}$$

$$= \frac{1}{2^{\ell-1}\pi} \int_0^\infty du\, u^{\ell-1}\, e^{-u}$$

$$J_\ell = \frac{1}{2^{\ell-1}\pi} \quad \Gamma(\ell) = \frac{(\ell-1)!}{2^{\ell-1}\pi} \qquad \text{(C.16)}$$

$$\ell = 1,2$$

where the definition of the gamma function (Gradshteyn and Ryzhik, 1965), $\Gamma(\ell)$ has been used in obtaining (C.16).

In evaluating the integrals K_ℓ the definition of the probability integral is used to express $\mathrm{Erf}'(x)$ in terms of $\mathrm{Erf}''(x)$,

$$\mathrm{Erf}'(x) = -\frac{1}{2x}\,\mathrm{Erf}''(x). \qquad \text{(C.17)}$$

Thus (C.12) may be written as:

$$K_\ell = -\frac{1}{2}\int_0^\infty dx\, x^{2\ell-1}\, \mathrm{Erf}(x)\, \mathrm{Erf}''(x). \qquad \text{(C.18)}$$

Equation (C.18) may be integrated by parts. The boundary terms vanish, leaving

$$K_\ell = \frac{1}{2}\int_0^\infty dx\, \mathrm{Erf}'(x)\, \frac{d}{dx}\left[x^{2\ell-1}\mathrm{Erf}(x)\right]. \qquad \text{(C.19)}$$

Performing the differentiation and comparing the resulting terms with Eqs.(C.12) and (C.13), we find that the integrals K_ℓ obey the recursion relation:

$$K_\ell = \frac{1}{2} J_\ell + \left(\frac{2\ell-1}{2}\right) K_{\ell-1} \qquad \text{(C.20)}$$

It remains to evaluate K_0 and L. From Eq. (C.13) we find

$$K_0 = \int_0^\infty dx\, \mathrm{Erf}'(x)\, \mathrm{Erf}(x). \qquad \text{(C.21)}$$

This integral is easily performed giving

$$K_0 = \frac{1}{2} \qquad \text{(C.22)}$$

The integral L may be evaluted by re-writing (C.14) in the form:

$$L = \int_0^\infty dx\, \mathrm{Erf}'(x)\, \frac{d}{dx}\left[\frac{\mathrm{Erf}(x)}{x}\right] \qquad \text{(C.23)}$$

integrating by parts we obtain

$$L = \left[\frac{\mathrm{Erf}(x)\ \mathrm{Erf}'(x)}{x}\right|_0^\infty - \int_0^\infty dx\ \frac{\mathrm{Erf}(x)\ \mathrm{Erf}''(x)}{x} . \tag{C.24}$$

Using (C.17) we may write (C.24) as

$$L = \left[0 - \frac{4}{\pi}\right] + \int_0^\infty dx\ \frac{d}{dx}\left[\mathrm{Erf}^2(x)\right]$$

giving

$$L = -0.273 \tag{C.25}$$

Using Eqs.(C.11) through (C.14) we may write the integrals I_k^{e-e} as

$$I_0^{e-e} = L + 2K_0 \tag{C.26a}$$

$$I_1^{e-e} = J_1 + 2K_1 - K_0 \tag{C.26b}$$

$$I_2^{e-e} = J_2 + 2K_2 - K_1 \tag{C.26c}$$

The value of the integrals J_ℓ, K_ℓ, and L given by Eqs. (C.16), (C.25), (C.22), and the recursion relation (C.20) are summarized in Table C-1, while values for I_k^{e-e} are given in Table C-2.

B. Evaluation of I_k^{e-i}

Using Eqs. (C.6), (C.9) and (C.10) we write I_k^{e-i} as

$$I_k^{e-i} = \int_0^\infty dx\ x^{2k}\ \mathrm{Erf}'(x) . \tag{C.27}$$

Thus we must evaluate the family of integrals

$$M_\ell \equiv \int_0^\infty dx\ x^{2\ell}\ \mathrm{Erf}'(x) . \tag{C.28}$$

Using (C.17) this can be written as

$$M_\ell = -\frac{1}{2}\int_0^\infty dx\ x^{2\ell-1}\ \mathrm{Erf}''(x) . \tag{C.29}$$

Integrating by parts and using (C.28) we find the recursion relation:

$$M_\ell = \frac{2\ell-1}{2}\ M_{\ell-1} . \tag{C.30}$$

M_0 may be evaluated directly from (C.28). We find

$$M_0 = 1 \tag{C.31}$$

The values of the integrals M_ℓ are given in Table C-1, while the values of I_k^{e-i} are given in Table C-2.

ℓ	0	1	2
J_ℓ		0.318	0.159
K_ℓ	0.5	0.409	0.693
M_ℓ	1.0	0.5	0.75

$L = -0.273$

Table C-1.

Values of various integrals needed in evaluating the energy integral I_k.

k	0	1	2
I_k^{e-e}	0.73	0.64	1.14
I_k^{e-i}	1.0	0.5	0.75
I_k	1.73	1.14	1.89

Table C-2.

Values of the energy integrals, I_k.

Appendix D - ISOLATING THE TRANSPORT DUE TO ELECTRON-ION COLLISIONS (Section 5)

In comparing the transport coefficients predicted by this theory with the results of numerical experiments (Nevins and Harte, 1977), it is necessary to include only the contribution of electron-ion collisions to the transport coefficients. We showed in Appendix A that the local entropy sources, K_{ee} and K_{ei}, are separately self-adjoint. Hence, the derivation of the pseudoclassical flux presented in Section 3-5 goes through as before when only the electron-ion collision term is retained.

In the algebraic expression for the entropy source, Eq. (5.25), only the contribution of the electron-ion collision operator should be included in calculating the energy integrals, I_k. The contribution of electron-ion collisions to these energy integrals, I_k^{e-i}, is calculated in Appendix C.

By comparing Eq. (5.25) with Eq. (4.11) we find that the coefficient of particle diffusion, $\mathcal{D}$, is given by

$$\mathcal{D} = 1.30\ D\ I_0 \tag{D.1}$$

Hence the contribution of electron-ion collisions to this diffusion coefficient, $\mathcal{D}^{e-i}$, is given by

$$\mathcal{D}^{e-i} = 1.30\ D, \tag{D.2}$$

where we have used the result from Appendix C that $I_0^{e-i} = 1.0$, and we recall from Section 5 that

$$D = \left(\frac{e\Phi_o}{T}\right)^{\frac{1}{2}} \Delta x^2\ \nu_{eff}\ .$$

Hence,

$$\mathcal{D}^{e-i} = 1.30 \left(\frac{k_y}{k_z}\right)^2 \sqrt{\frac{e\Phi_o}{T}}\ \rho^2 \nu_e . \tag{D.3}$$

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