PSEUDOCLASSICAL TRANSPORT II:

A NONLINEAR THEORY OF THE "COLLISIONLESS" DRIFT INSTABILITY

by

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ABSTRACT

A self-consistent theory of the evolution of a plasma slab supporting a finite amplitude drift wave that has trapped the resonant electrons is presented. In Part I it was shown how the trapping of resonant particles affects the particle and energy flux driven by the wave. Here in Part II this previous work is extended to include the effect of particle trapping on the evolution of the finite amplitude wave. The connection between this theory and other work, using quasilinear theory, on the anomalous transport associated with low frequency drift waves is considered. It is shown that, for parameters typical of tokamak plasmas, particle trapping may result in the nonlinear stabilization of the wave at amplitudes, $(e\Phi_o/T) \simeq 10^{-2}$, that are of the same order as those observed in experiments. The application of this work to experiments is discussed, and it is found to be potentially useful in understanding the drift wave spectrum anu transport rates observed in computer simulations, stellerators, and future tokamak experiments.

1. INTRODUCTION

There has been considerable recent interest in low frequency drift wave instabilities such as the collisionless drift instability (Galeev et al., 1963) and the dissipative trapped electron instability (Kadomtsev and Pogutse, 1969). These instabilities are thought to be responsible for the anomalous transport observed in tokamaks (Dean et al., 1974).

In many investigations of the anomalous transport associated with these low frequency drift wave instabilities (e.g., Liu et al., 1976; Horton, 1976), the "quasilinear" transport coefficients have been employed to relate the drift wave fluctuation spectrum to the anomalous transport rates. Other authors (Pogutse; 1972, Brambilla and Lichtenberg, 1973; Gell et al., 1975; Gell and Nevins, 1975) have focused on the phenomena of trapping of resonant particles by a finite amplitude low frequency drift wave, and have derived "pseudoclassical" transport coefficients. In these previous investigations of pseudoclassical transport theory the finite amplitude drift wave has not been treated self-consistently, and the relation between pseudoclassical theory and other work on the anomalous transport associated with low frequency drift wave instabilities has not been investigated.

In this paper we unite these two methods of investigating the anomalous transport associated with low frequency drift waves. We begin by extending previous work on pseudoclassical transport by presenting a self-consistent theory of the evolution of a plasma slab in which a finite amplitude, low frequency drift wave has trapped the resonant electrons. A complete set of equations for the evolution of both the finite amplitude wave and the background plasma is developed. In deriving these equations, we use both the results and the notation from a companion paper (Nevins, 1977a), hereafter refered to as I.

In I we derived the pseudoclassical fluxes of particles and energy across the magnetic field, and found that these pseudoclassical fluxes are smaller than the corresponding quasilinear fluxes by the factor $(\nu_{eff}/\omega_{BOUNCE})$. This factor is the ratio of the effective collision frequency for scattering electrons out of resonance with the finite amplitude wave to the bounce frequency of electrons trapped in that wave. The pseudoclassical theory considered here is valid in the limit

$$(\nu_{eff}/\omega_{BOUNCE}) < 1 \ . \tag{1.1}$$

This condition may be written in terms of the parameters of the plasma-wave system as

$$\left(\frac{e\Phi_o}{T}\right) > \left(\frac{\nu_e}{k_z v_{te}}\right)^{2/3} \ . \tag{1.2}$$

We find that the pseudoclassical transport theory considered here is associated with the nonlinear development of the collisionless drift instability (Galeev et al., 1963). Hence, this pseudoclassical transport theory is a nonlinear theory of the "collisionless" drift instability. The nonlinear effect considered is the trapping of resonant electrons by the wave. This nonlinear theory should be used <u>in place of</u> the usual linear theory when particle trapping occurs to determine both the anomalous transport coefficients and the evolution of the finite amplitude wave.

Having developed a satisfactory pseudoclassical transport theory, we proceed to consider the relationship between this theory and other work on drift wave instabilities and anomalous transport (e.g., Liu et. al., 1976;

Horton, 1976). We show how the effect of particle trapping by the wave can be combined with other effects (e.g., the trapping of electrons by the magnetic field) to determine both the nonlinear evolution of the finite amplitude drift wave and the anomalous transport driven by that wave.

In Sec. 2 the time evolution of the finite amplitude wave is examined. It is shown that, when inequality (1.2) is satisfied, the susceptibility of the resonant electrons, a destabilizing term in the dispersion relation, is reduced by the factor ($\nu_{eff}/\omega_{BOUNCE}$).

In Sect. 3 we complete our pseudoclassical theory by presenting a closed set of equations describing the evolution of both the plasma slab and the finite amplitude wave. These equations are shown to conserve both energy and momentum.

In Sect. 4 the relation between this pseudoclassical transport theory and other work on low frequency drift waves is examined. It is shown that the trapping of resonant electrons can lead to the stabilization of the low frequency drift wave at small, but finite amplitudes.

Finally, in Sect. 5, we conclude by examining the conditions for the validity of the pseudoclassical transport theory presented here. We find it potentially useful in understanding the drift wave spectrum and anomalous tranport observed in computer simulations, stellerators, and future tokamak experiments.

2. THE DISPERSION RELATION AND A NONLINEAR DISSIPATIVE INSTABILITY

In this section we find the dispersion relation of the finite amplitude drift wave and obtain equations describing the evolution of the wave amplitude, Φ_o. The wave amplitude evolves in time due to a nonlinear dissipative instability similar to the one discovered by Kadomtsev and Pogutse (1970), and later examined by Ott and Manheimer (1976). The instability considered here differs from the one described by these authors in that we are considering the effect of collisions on electrons trapped by a traveling wave, while the previous work dealt with the effect of collisions on electrons trapped between the maxima of a standing wave. The orbits of particles trapped by a standing wave are quite different from those of particles trapped by a traveling wave. We show elsewhere (Nevins, 1977b) that this difference in the character of the trapped particle orbits is responsible for the substantial differences between the nonlinear growth rates obtained here and those obtained previously.

This dissipative instability is nonlinear because trapped particle effects are considered and the growth rate is found to depend on the wave amplitude. Coupling between Fourier modes is not considered. The wave amplitude has been assumed to be a slowly varying function of x. Hence, we may use the local dispersion relation to determine the evolution of the wave amplitude (Krall and Rosenbluth, 1965; Mikhailovskii, 1967). This dispersion relation is obtained by ignoring the x dependence of Φ_o. Hence, the wave potential may be written as

$$\Phi(y,z,t) = \Phi_o(t)\ h(\theta) \tag{2.1}$$

where

$$\theta = k_y y + k_z z - \int_0^t \omega(\tau)\, d\tau \tag{2.2}$$

and

$$\Phi_o(t) = \Phi_o(t{=}0)\ \exp[\int_0^t \gamma(\tau)\, d\tau] \ . \tag{2.3}$$

We again choose the waveform to be

$$h(\theta) = \cos\theta \tag{2.4}$$

as we did in evaluating the particle flux in I.

Our immediate goal is to calculate the electron susceptibility, $\chi^{(e)}$, which is defined by

$$\mathrm{Re}(\chi^{(e)}\tilde{\phi}) \equiv \frac{4\pi e}{k^2}\delta n \ , \tag{2.5}$$

where $\tilde{\phi}$ is the complex potential,

$$\begin{aligned} \tilde{\phi} &= \Phi_o\, e^{i\theta} \\ &= \Phi_o \cos\theta + i\Phi_o \sin\theta, \end{aligned} \tag{2.6}$$

and δn is the perturbation in the electron density due to the wave. When coupling between Fourier modes is ignored, δn may be written as

$$\delta n(\theta) = a \cos\theta + b \sin\theta \tag{2.7}$$

where

$$a = \int \frac{d\theta}{\pi} n \cos\theta \tag{2.8}$$

$$b = \int \frac{d\theta}{\pi} n \sin\theta \ . \tag{2.9}$$

Using Eqs. (2.4) through (2.9) we find

$$\mathrm{Re}\,\chi^{(e)} = \frac{4\pi e}{k^2\Phi_o}\int \frac{d\theta}{\pi}\, n\, h(\theta) \tag{2.10}$$

$$\mathrm{Im}\,\chi^{(e)} = \frac{4\pi e}{k^2\Phi_o}\int \frac{d\theta}{\pi}\, n\, h'(\theta) \tag{2.11}$$

The electron density is given by

$$n = \int d^3v\, f(\theta,v) \tag{2.12}$$

We have numerically integrated Eqs. (2.10) through (2.12) using the electron distribution function described in I. We found that, through zero order in $(\nu_{eff}/\omega_{BOUNCE})$, the real part of the electron susceptibility is well approximated by the linear result,

$$\mathrm{Re}\,\chi^{(e)} = \frac{1}{k^2\lambda_d^2} \tag{2.13}$$

where

$$\lambda_d^2 = \frac{T}{4\pi ne^2} \tag{2.14}$$

and T is the electron temperature.

This linear susceptibility describes the adiabatic response of the electrons to the wave. It evolves slowly due to the evolution of the electron temperature and density. The real part of the electron susceptibility together with the ion susceptibility determines the real part of the wave frequency. Hence, the frequency of the finite amplitude wave will change slowly along with the electron temperature and density.

We must introduce an ion susceptibility to obtain a dispersion relation for our finite amplitude wave. In many experiments, the ion temperature is small compared to the electron temperature. In this limit the ion susceptibility is given by (Mikhailovskii, 1974)

$$\chi^{(i)} = \frac{\omega_{pi}^2}{\Omega_i \omega} \frac{k_y}{k^2} \frac{1}{n} \frac{dn}{dx} + \frac{\omega_{pi}^2}{\Omega_i^2} \frac{k_y^2}{k^2} \tag{2.15}$$

where ω_{pi} is the ion plasma frequency, and Ω_i is the ion gyro frequency.

The real part of the dielectric function is then

$$\varepsilon_r(\underline{k},\omega) = 1 + \frac{1}{k^2 \lambda_d^2} + \frac{\omega_{pi}^2}{\Omega_i \omega} \frac{k_y}{k^2} \frac{1}{n} \frac{dn}{dx} + \frac{\omega_{pi}^2}{\Omega_i^2} \frac{k_y^2}{k^2} \tag{2.16}$$

Hence, the real part of the wave frequency is given by solving $\varepsilon_r = 0$,

$$\omega = \frac{\omega_{ne}}{1 + \frac{k_y^2 T}{m_i \Omega_i^2}} \tag{2.17}$$

where

$$\omega_{ne} = - \frac{k_y T}{eB} \frac{1}{n} \frac{\partial n}{\partial x} \tag{2.18}$$

and we have assumed that $1/k^2\lambda_d^2$, $\omega_{pi}^2/\Omega_i^2 \gg 1$.

In I, we noted that the imaginary part of the electron susceptibility vanishes through zero order in $(\nu_{eff}/\omega_{BOUNCE})$. Hence, it is necessary to evaluate Eq. (2.11) through first order in $(\nu_{eff}/\omega_{BOUNCE})$ in order to obtain a nonvanishing contribution to Im $\chi^{(e)}$. In evaluating this integral, it is helpful to note that $h'(\theta)$ may be written as

$$h'(\theta) = - \frac{1}{\Phi_o} \frac{B}{k_y} v_{dr} \tag{2.19}$$

where v_{dr} is the x component of the guiding center drift velocity. Combining Eqs. (2.11), (2.12), and (2.19) the imaginary part of the electron susceptibility becomes

$$\mathrm{Im}\ \chi^{(e)} = -\left(\frac{8\pi}{k^2\Phi_o^2}\right)\frac{eB}{k_y}\int\frac{d\theta}{2\pi}\int d^3\underline{v}\ v_{dr}\ f \quad . \tag{2.20}$$

Comparing Eq. (2.20) with the definition of the particle flux, Γ_e (Nevins, 1977b), we find that the imaginary part of the electron susceptibility may be written as

$$\mathrm{Im}\ \chi^{(e)} = -\left(\frac{8\pi}{k^2\Phi_o^2}\right)\frac{eB}{k_y}\Gamma_e \tag{2.21}$$

This relation between Im $\chi^{(e)}$ and the electron flux holds whenever the magnetic field is uniform and the evolution of the electron distribution function is properly described by the drift kinetic equation, i.e., when $k_y\ \rho_e \ll 1$, and $\omega \ll \Omega_e$. We use this relation here to evaluate the nonlinear growth rate of the finite amplitude drift wave. In Sect. 3 we will use Eq. (2.21) to show that our pseudoclassical transport theory conserves both energy and momentum.

The enhanced electron flux due to the trapping of the resonant particles by the wave was evaluated in I. Using this result, we obtain the contribution of these trapped electrons to the imaginary part of the electron susceptibility,

$$\mathrm{Im}\ \chi_{NL}^{(e)} = -\ 3.59\ \chi_o\ (\Delta - 0.84\ \eta_e)\left[\frac{\nu_e}{k_z v_{te}}\left(\frac{e\Phi_o}{T}\right)^{-3/2}\right] \tag{2.22}$$

where

$$\chi_o \equiv \left(\frac{\pi}{2}\right)^{1/2} \frac{1}{k^2\lambda_d^2} \frac{\omega_{ne}}{k_z v_{te}} \qquad (2.23)$$

$$\Delta \equiv (1 - \omega/\omega_{ne}) \qquad (2.24)$$

and

$$\eta_e \equiv \frac{d(\ell n\ T)}{d(\ell n\ n)} \ . \qquad (2.25)$$

This nonlinear susceptibility describes the response of the resonant electrons to the wave in the limit $\nu_{eff}/\omega_{BOUNCE} < 1$. We use the subscript "NL" to distinguish it from the linear susceptibility of the resonant electrons (Mikhailovskii, 1974),

$$\mathrm{Im}\ \chi_L^{(e)} = - \chi_o(\Delta - 0.5\ \eta_e) \ . \qquad (2.26)$$

The imaginary part of the electron susceptibility is small, being of order $(\Delta x/L)$, while the real part of the electron susceptibility is of order 1. Hence the growth rate of the finite amplitude wave is given by

$$\gamma_{NL} = - \frac{\mathrm{Im}\ \chi_{NL}^{(e)}}{\frac{\partial \varepsilon_r}{\partial \omega}}$$

$$= 3.59 \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega^2}{k_z v_{te}} (\Delta - 0.84\ \eta_e)\ [(\nu_e/k_z v_{te})(e\Phi_o/T)^{-3/2}] \ . \qquad (2.27)$$

It is instructive to compare this nonlinear growth rate to the corresponding linear growth rate (Mikhailovskii, 1974). Using Eq. (2.26) we find

$$\gamma_L = \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega^2}{k_z v_{te}} (\Delta - 0.5\ \eta_e) \ . \qquad (2.28)$$

We see that, apart from differences in the numerical coefficients, γ_{NL} differs from γ_L in that γ_{NL} is smaller by the factor

$$\frac{\nu_e}{k_z v_{te}} \left(\frac{e\Phi_o}{T}\right)^{-3/2} = \nu_{eff}/\omega_{BOUNCE} \qquad (2.29)$$

This result is quite different from that of Kadomtsev and Pogutse (1970), and Ott and Manheimer (1976). These authors study a finite amplitude standing wave, and find that the nonlinear growth rate is larger than the corresponding linear growth rate. Our result, Eq. (2.27), applies to the nonlinear regime of traveling waves. It is remarkable that the nonlinear behavior of traveling and standing waves should be so different. We show elsewhere (Nevins, 1977b) that this difference between the nonlinear growth rates results from the increased width of the orbits of particles trapped by standing waves compared as to those of particles trapped by a traveling wave.

The dependence of γ_{NL} on $(\nu_{eff}/\omega_{BOUNCE})$ recalls work dealing with the effect of collisions on the damping of Langmuir waves. The growth of the low frequency drift waves and the damping of Langmuir waves are quite similar problems in that they are both a consequence of the interaction between the wave and the particles resonant with the wave. In Vlasov theory, the wave growth (or damping) results from the Landau prescription for choosing the integration contour when calculating the linear susceptibility (Landau, 1946). The effect of collisions on the Landau damping of Langmuir waves had been studied extensively. It has been found that when $\nu_{eff}/\omega_{BOUNCE} > 1$ the dominant effect of collisions is to prevent the trapping of the resonant electrons and maintain the Maxwellian character of the distribution function (Johnston, 1971; Auerbach, 1977). Hence, the damping of Langmuir waves is properly described by linear theory when $\nu_{eff}/\omega_{BOUNCE} > 1$. Zakharov and Karpman (1963) have studied the damping of Langmuir waves of larger amplitude, such that $\nu_{eff}/\omega_{BOUNCE} < 1$. These authors obtained a nonlinear damping rate for finite amplitude Langmuir waves quite similar to our nonlinear growth rate for finite amplitude drift waves. The nonlinear damping (or growth) rate is found to be smaller than the corresponding linear rate by the same factor, $\nu_{eff}/\omega_{BOUNCE}$.

We are not aware of any previous work dealing with the effect of collisions on the interaction between a low frequency drift wave and electrons resonant with this wave. Nevertheless, the similarity of this problem to the work on Langmuir waves described above, both in the physical mechanism for the growth (or damping) and in the mathematical formulation of the problem, leads us to the conclusion that in the limit $\nu_{eff}/\omega_{BOUNCE} > 1$ the dominant effect of collisions will be to prevent the trapping of resonant particles by the low frequency drift wave, and to maintain the Maxwellian character of the distribution function near the phase velocity of this wave. Hence, in the limit $\nu_{eff}/\omega_{BOUNCE} > 1$, the collisionless linear theory will correctly describe the response of the electrons to the low frequency drift wave.

We conclude that the pseudoclassical theory presented here is a nonlinear theory of the interaction of the low frequency drift wave with the resonant electrons. This nonlinear theory replaces the linear theory of this interaction when the wave amplitude satisfies

$$(e\Phi_o/T) > (\nu_e/k_z v_{te})^{2/3} \quad . \tag{2.30}$$

3. MACROSCOPIC EQUATIONS AND CONSERVATION LAWS

In this section we examine the macroscopic equations describing the evolution of the plasma-wave system, and show that the total momentum and energy of the system are conserved.

The equations describing the evolution of the temperature and wave phase averaged number density are obtained in Nevins (1977b) by taking the appropriate moments of the drift kinetic equation. It is found that

$$\frac{\partial n}{\partial t} = - \frac{\partial}{\partial x} \Gamma_e \tag{3.1}$$

$$\frac{\partial w}{\partial t} = - \frac{\partial}{\partial x} Q_e - \frac{\omega}{k_y} eB \, \Gamma_e \tag{3.2}$$

Equations (3.1) and (3.2) together with the relation

$$w = 3/2 \; n \; T \tag{3.3}$$

determine the evolution of the plasma temperature and density.

We find it convenient to describe the evolution of the finite amplitude drift wave with an equation for the evolution of the wave energy density, $\mathcal{W}$, where

$$\mathcal{W} = \omega \frac{\partial \varepsilon_r}{\partial \omega} (k^2 \, \Phi_o^{\,2}/16 \, \pi) \tag{3.4}$$

Taking the time derivative of Eq. (3.4) and employing Eqs. (3.3), (2.21), and (2.27) we find

$$\frac{\partial \mathcal{W}}{\partial t} = \frac{\omega}{k_y} eB \, \Gamma_e \tag{3.5}$$

From Eqs. (2.18) and (3.5) we see that the instability condition for waves propagating in the electron diamagnetic drift direction (i.e. waves satisfying $\frac{\omega}{\omega_{ne}} > 0$) is

$$\frac{\partial n}{\partial x} \Gamma_e < 0 \tag{3.6}$$

In I we noted that the anomalous flux due to particles trapped in the finite amplitude wave may be directed either up or down the density gradient. This instability condition states that those waves which drive an electron flux down the density gradient are unstable, while those waves that drive an electron flux up the density gradient will be damped. We show elsewhere that this behavior may be understood by considering the conservation of canonical momentum (Nevins, 1977b).

Using Eqs. (2.16) and (3.4) we find that $\left(\frac{e\Phi_o}{T}\right)$ is related to the wave energy density by

$$\left(\frac{e\Phi_o}{T}\right)^2 = 4 \frac{\omega}{\omega_{ne}} \left(\frac{\mathcal{W}}{nT}\right) \tag{3.7}$$

Equations (3.1), (3.2), (3,3), (3.5), and (3.7), together with the pseudoclassical expressions for the particle and energy flux derived in I, form a closed set of equations describing the macroscopic evolution of the plasma slab and finite amplitude wave due to the collisional transport of the electrostatically trapped particles. Hence, we may use these equations to check some basic conservation laws.

The total energy density, W, may be written as

$$W = w + \mathcal{W} \tag{3.8}$$

Taking the derivative of Eq. (3.8) with respect to time and using Eqs. (3.2) and (3.4) we find that the energy source terms cancel, leaving

$$\frac{\partial W}{\partial t} = - \frac{\partial}{\partial x} Q_e \tag{3.9}$$

Equation (3.9) describes a system in which the total energy is conserved. Hence, the energy source terms in Eqs. (3.2) and (3.5) represent a transfer of energy between the electrons and the finite amplitude wave.

We may derive similar equations describing the conservation of canonical momentum. In Nevins (1977b) it is shown that the y and z components of the electron momentum density satisfy the equations

$$\frac{\partial p_y}{\partial t} = - eB \Gamma_e \tag{3.10}$$

$$\frac{\partial p_z}{\partial t} = - \frac{k_z}{k_y} eB \Gamma_e - \frac{\partial}{\partial x} \Pi_{xz} \tag{3.11}$$

where Π is the stress tensor. The wave momentum density is taken to be

$$\underline{P} = \underline{k} \frac{\partial \varepsilon_r}{\partial \omega} \left(\frac{k^2 \phi_o^2}{16\pi} \right) \tag{3.12}$$

The rate of change of the wave momentum, $\underline{P}$ is found by taking the derivative of Eq. (3.12) with respect to time and using Eqs. (2.3), (2.21) and (2.27). This gives

$$\frac{\partial \underline{P}}{\partial t} = \frac{\underline{k}}{k_y} eB \Gamma_e \tag{3.12}$$

Defining the total momentum density, $\underline{\mathrm{P}}$, as

$$\underline{\mathrm{P}} = \underline{p} + \underline{P} \tag{3.14}$$

we find

$$\frac{\partial P_y}{\partial t} = 0 \tag{3.15}$$

$$\frac{\partial P_z}{\partial t} = - \frac{\partial}{\partial x} \Pi_{xz} \tag{3.16}$$

It is clear from Eqs. (3.15) and (3.16) that the total canonical momentum is conserved. Hence, the momentum source terms in Eqs. (3.10), (3.11) and (3.12) describe the transfer of momentum between the particles and the wave.

In Nevins (1977b) we note that the y component of the canonical momentum of a particle is proportional to the x guiding center position. Hence, the transport of particles across the magnetic field implies a change in the canonical momentum of the species being transported. As the total canonical momentum of the system is conserved, the flux of particles is determined by the rate at which momentum is transfered from one element of the system to another. In the pseudoclassical transport theory presented here, momentum balance is achieved by transfering momentum between the electrons and the low frequency drift wave. As a result, the pseudoclassical particle flux is accompanied by an instability of the low frequency drift wave.

This pseudoclassical transport is but one example of the link between particle transport and drift wave instabilities. We consider this link in more detail elsewhere (Nevins, 1977b) and show that the connection between drift wave instability and particle transport is useful in understanding many of the instability mechanisms of the low frequency drift wave.

4. NONLINEAR STABILIZATION AND ANOMALOUS TRANSPORT

In the preceeding sections we have developed a set of equations describing the evolution of a plasma slab in which the resonant electrons have been trapped by a finite amplitude low frequency drift wave. In this section we show how our work is related to previous work on low frequency drift wave instabilities. We then proceed to investigate the nonlinear saturation of the low frequency drift waves due to the trapping of the resonant electrons, and to make estimates of the saturation amplitude in various parameter ranges.

The linear stability of the low frequency drift wave has been studied extensively. It is now generally understood that several mechanisms can be important simultaneously in determining the growth rate of this wave [Hinton and Ross, 1976; Tang et al., 1976; Catto et al., 1976; Horton, 1976]. Two of the most important mechanisms affecting drift wave stability are the dissipative (magnetically) trapped electron instability mechanism, and the Landau resonance with the (magnetically) passing electrons.

When the local approximation is employed, the stability of the low frequency drift wave is determined by the imaginary part of the dielectric function. Im ε determines the rate at which energy is transfered between the background plasma and the wave. This energy transfer rate is given by

$$\dot{W} = -\omega \text{ Im } \varepsilon \ (\omega,k_y)(k^2\Phi_o{}^2/8\pi) \tag{4.1}$$

Low frequency drift waves have positive energy (i.e., $\omega \frac{\partial \varepsilon_r}{\partial \omega} > 0$). Hence, the instability condition is

$$\omega \text{ Im } \varepsilon(\omega,k_y) < 0 \tag{4.2}$$

Following Horton (1976)* we separate Im $\varepsilon(\omega,k_y)$ into a sum over the mechanisms by which energy is transfered between the wave and the background plasma.

$$\begin{aligned} \text{Im}\ \varepsilon(\omega,k_y) &= \text{Im}\ \chi_R^{(e)} + \sum_m{}' \ \text{Im}\ \chi_m^{(e)} + \sum_m \text{Im}\ \chi_m^{(i)} \\ &= \text{Im}\ \chi_R^{(e)} + \sum_m{}' \ \text{Im}\ \chi_m \end{aligned} \tag{4.3}$$

Im $\chi_R^{(e)}$ is the susceptibility of the resonant electrons. Im $\chi_m^{(e)}$ represent the susceptibilities associated with other mechanisms for the transfer of energy between the wave and the electron distribution. Im $\chi_m^{(i)}$ represent the susceptibilities associated with mechanisms for the transfer of energy between the wave and the ion distribution. The prime on the sum indicates that resonant electron effects are to be excluded.

The flux of particles and energy associated with the low frequency drift wave can be written in a similar fashion. The electron flux is given by

$$\Gamma_e = -\frac{k_y}{eB}(k^2\phi_o{}^2/8\pi)(\text{Im}\ \chi_R^{(e)} + \sum_m{}' \ \text{Im}\ \chi_m^{(e)}) \tag{4.4}$$

The energy flux may be decomposed in a similar manner (see,for example, Horton, 1976).

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* Horton introduces the function G_m^i in his discussion of drift wave stability. These functions are related to the susceptibilities considered here by:

$$\chi_m^{(e)} = -\frac{1}{k^2\lambda_d{}^2}\left[\Delta\ G_m^0 + \eta_e G_m^1\right]$$

Equation (4.4) follows directly from Eq. (2.21) above. It also follows from the expression for the particle flux derived by Horton (1976). This expression recalls the particle flux obtained in the quasilinear treatment of this problem (Krall and McBride, 1977). We note here that the derivation of Eq. (2.21) did not require the presence of many waves as in quasilinear theory. Similarly the derivations of drift wave transport presented by Horton (1976), and by Liu, Rosenbluth, and Tang (1976) do not require the presence of a spectrum of waves. Hence, Eq. (4.4) differs from the quasilinear flux in that it may be used to calculate the flux driven by a single wave. We show elsewhere (Nevins et. al., 1977b) that these fluxes result from a combination of the particle motion in the field of individual waves and Coulomb collisions.

When many waves are present, Eq. (4.4) gives the contribution of each wave to the total particle flux. There is a similar expression for the contribution of each wave to the energy flux. The total flux is obtained by summing the contributions from each wave.

When the magnetic field is inhomogeneous, the pseudoclassical fluxes and the nonlinear susceptibility must be modified to account for the trapping of particles by the magnetic field. The effect of magnetic trapping on the resonant interaction of drift waves with the electron distribution has been considered by many authors (see, e.g., Horton, 1976). The most important limit to our work is

$$\frac{\omega}{k_{\parallel} v_{te}} \gtrsim \delta^{1/2} \qquad (4.5)$$

where $k_{\parallel}$ is the component of the wave vector parallel to the magnetic field and $\delta^{1/2}$ is a measure of the magnetic field inhomogeniety. In tokamaks, δ is given by

$$\delta = (\text{minor radius}/\text{major radius})$$

Inequality (4.5) is most easily satisfied near the magnetic axis (where δ vanished) or near the mode rational surface (where $k_{\parallel}$ vanishes). In this limit the nonlinear susceptibility of the trapped electrons and the pseudoclassical fluxes are largely unaffected by the presence of the magnetic field inhomogeneity. Hence, the expressions that we have derived in I, and in Sect. 2, may be employed.

The nonlinear susceptibilities due to effects other than particle trapping (e.g. mode coupling or induced scattering) are smaller than the corresponding linear susceptibilities by the factor $(e\Phi_o/T)$. Particle trapping occurs when the wave amplitude satisfies inequality (1.2). In a nearly collisionless plasma $[(\nu_e/k_z v_{te}) \ll 1]$, this inequality can be satisfied while $(e\Phi_o/T)$ is still small. Hence, in nearly collisionless plasmas, at moderate wave amplitudes, given by

$$(\nu_e/k_z v_{te})^{2/3} < (e\Phi_o/T) \ll 1 , \qquad (4.6)$$

particle trapping will be the dominant nonlinear effect.

The nonlinear stabilization of the low frequency drift waves due to the trapping of the resonant electrons may be investigated by using the nonlinear susceptibility of Eq. (2.22) in evaluating Im ε. In a local treatment of drift wave stability, the rate at which energy is being transferred to the wave must vanish at saturation. This occurs when

$$\mathrm{Im}\ \varepsilon = 0 \tag{4.7}$$

Im ε is a function of the wave amplitude through Im $\chi^{(e)}_{NL}$. Hence Eq. (4.7) is an equation for the wave amplitude at saturation.

We first consider the nonlinear stabilization of the low frequency drift wave in a plasma with a moderate temperature gradient. Using Eq. (2.27) we find that when η_e satisfies

$$2.0\ \Delta > \eta_e > 1.2\ \Delta \tag{4.8}$$

the resonant electron susceptibility can change from a destabilizing term at low wave amplitudes, $(e\Phi_o/T) < (\nu_e/k_z v_{te})^{2/3}$, into a stabilizing term at larger wave amplitudes. If the energy transfer to the wave from the resonant electrons dominates other energy transfer mechanisms, then this change in the character of Im $\chi^{(e)}_R$ will cause the stabilization of the drift wave during the transition from the linear energy transfer rate described by Im $\chi^{(e)}_L$ to the nonlinear energy transfer rate described by Im $\chi^{(e)}_{NL}$. Hence, in the parameter range

$$2\left(\Delta - \frac{1}{\chi_o}\sum_m{}' \mathrm{Im}\ \chi_m\right) > \eta_e > 1.2\ \Delta - \frac{0.33}{\chi_0}\cdot\sum_m{}' \mathrm{Im}\ \chi_m \tag{4.9}$$

we may estimate the wave amplitude at saturation as

$$\left(\frac{e\Phi_o}{T}\right)_{\text{saturation}} \simeq \frac{\nu_e}{k_z v_{te}}^{2/3} \qquad (4.10)$$

Normalizing the electron temperature to 1 keV, the number density to $10^{13}/\text{cm}^3$, and the parallel wave length to 20π cm, we obtain

$$\left(\frac{e\Phi_o}{T}\right)_{\text{saturation}} \simeq 1.16\times10^{-2}\left\{\frac{\left(n/10^{14}\ \text{cm}^{-3}\right)}{\left(\frac{T}{1\ \text{kev}}\right)^2\left(\frac{k_z}{0.1\ \text{cm}^{-1}}\right)}\right\}^{\frac{2}{3}}$$

The trapping of resonant electrons can also lead to the nonlinear stabilization of low frequency drift waves in plasmas with small or negative temperature gradients. When $\eta_e < 1.2\ \Delta$ the resonant electrons continue to destabilize the drift wave even after they become trapped, but the destabilizing influence of the resonant electrons decreases with increasing wave amplitude. The nonlinear saturation of the wave due to particle trapping will occur when the rate at which the wave extracts energy from the resonant electrons is balanced by the rate at which the wave loses energy to the plasma by other mechanisms. Hence, in the parameter range

$$\sum_m{}' \operatorname{Im} \chi_m > 0$$

$$\eta_e < 1.2\ \Delta - \frac{0.33}{\chi_o}\sum_m{}' \chi_m \qquad (4.11)$$

the amplitude of the low frequency drift wave at saturation is somewhat larger, and is given by

$$\left(\frac{e\Phi_o}{T}\right) \simeq \left(\frac{\nu_e}{k_z v_{te}}\right)^{2/3} \left[\frac{3.59\ \chi_o\ (\Delta - 0.84\eta_e)}{\sum_m{}'\ \mathrm{Im}\ \chi_m}\right]^{2/3} \quad . \tag{4.12}$$

The regions of parameter space in which these stabilization mechanisms are operative are sketched in Fig. 1.

The estimates of the drift wave saturation level presented above are low enough to warrant the neglect of nonlinear effects other than particle trapping. These estimates are of the same order of magnitude as the drift wave fluctuation levels observed in tokamak experiments (Mazzucato, 1976; Surko and Slusher, 1976). Hence, we conclude that particle trapping is a possible saturation mechanism for low frequency drift waves when resonant electron effects are important in determining the linear stability of the wave.

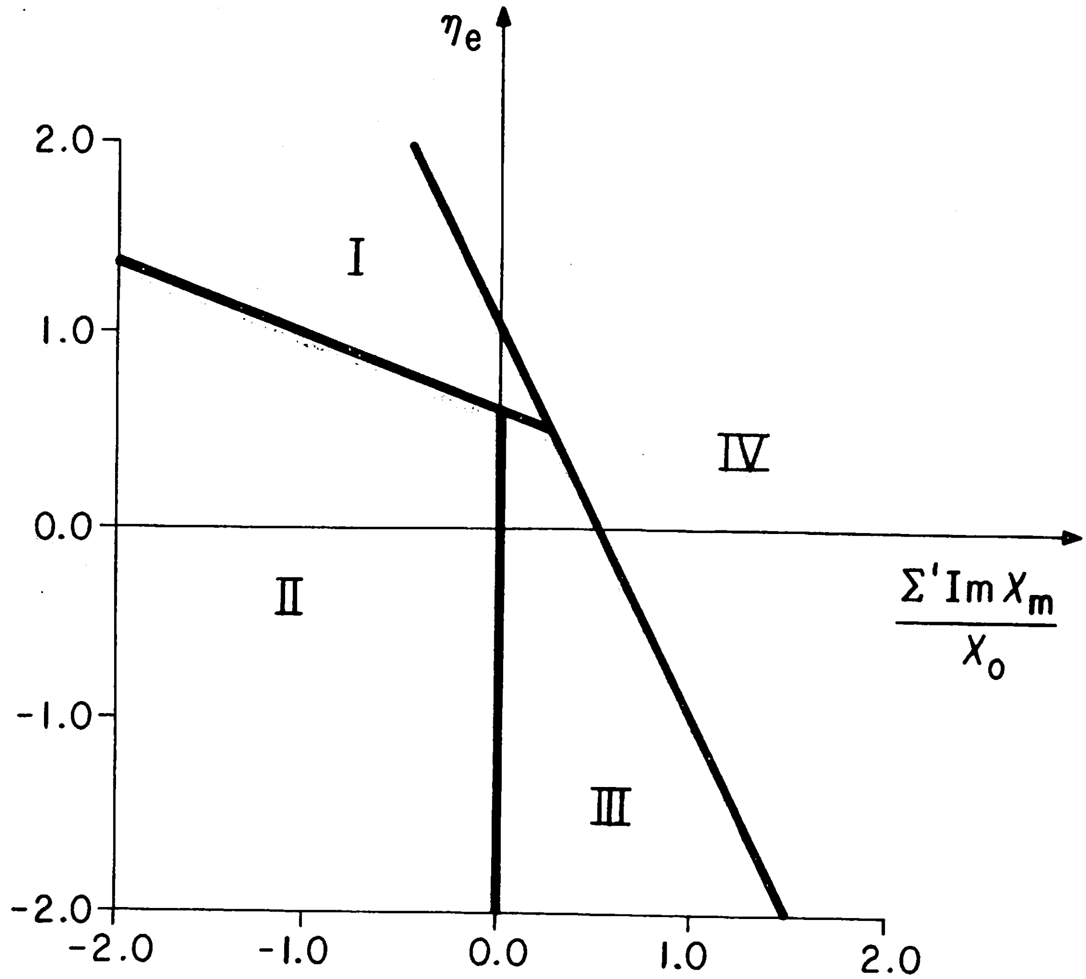

FIGURE I

This Figure shows a sketch of $(\eta_e, \sum_m{}' \mathrm{Im}\, \chi_m)$ space. The regions in which particle trapping leads to the nonlinear stabilization of low frequency drift waves are indicated. In region I the drift wave saturates at the level given by Eq. (4.10). In region II particle trapping does not lead to saturation. In region III the saturation level due to particle trapping is given by Eq. (4.12). egion IV is stable in both the linear and non-linear regime. We have taken $\Delta = (1-\omega/\omega_{ne}) = \frac{1}{2}$ making this sketch.

5. APPLICATIONS

In this section we will briefly review the assumptions made in this pseudoclassical transport theory and discuss possible applications of the pseudoclassical transport theory presented here.

We believe that the most restrictive assumption made in this work is the modeling of a spectrum of drift waves by a single, coherent traveling wave. Hence, we begin by examining this assumption.

In experimental plasmas a continuous fluctuation spectrum is observed, and the spectral density, S_Φ, may be measured directly (Mazzucato, 1976; Surko and Slusher, 1976). For our purposes it is convenient to consider the projection of the spectral density into the two dimensional space $(v_{\phi y}, v_{\phi z})$, where

$$\begin{aligned} v_{\phi y} &\equiv \omega/k_y \\ v_{\phi z} &\equiv \omega/k_z \end{aligned} \tag{5.1}$$

We identify each peak in this spectral density as a "mode". Each mode may be characterized by its width, $\delta \underline{v}_\phi$; and its amplitude, Φ_o. We define Φ_o in terms of the integral of S_Φ across the mode by

$$\Phi_o \equiv \left[\int_{\delta \underline{v}_\phi} S_\Phi(\underline{v}_\phi)\; d^2\underline{v}_\phi \right]^{1/2} \tag{5.2}$$

We will also be interested in the characteristic separation between modes in parallel phase velocity, $\Delta(\omega/k_z)$.

Particles may be trapped by a mode when the auto-correlation time of the mode is greater than the bounce period of a trapped particle. This condition may be written as

$$\delta(\omega/k_z) < v_{TRAP} \tag{5.3}$$

where v_{TRAP} is evaluated using the mode amplitude given by Eq. (5.2). Inequality (5.3) is a necessary, but not a sufficient condition for the application of our pseudoclassical transport theory. If the pseudoclassical transport coefficients and the nonlinear susceptibility are to be well defined, then we must require that the width of the spectral density in perpendicular phase velocity, ω/k_y, satisfies

$$\delta\left(\frac{\omega}{k_y}\right) << v_{te}\frac{\rho_e}{L_n} \tag{5.4}$$

and that the angular distribution of wave vectors satisfies

$$\delta\left(\frac{k_y}{k_z}\right) << \left|\frac{k_y}{k_z}\right| \quad .$$

k_y and k_z are related to $\underline{v}_\phi$ through Eqs. (5.1). Hence, this condition together with Eqs. (5.3) and (5.4) may be combined to yield

$$\delta\left(\frac{\omega}{k_z}\right) << \min\{\frac{\omega}{k_z}, \quad v_{TRAP}\} \tag{5.5}$$

A mode satisfying inequalities (5.4) and (5.5) may reasonably be approximated by a coherent traveling wave.

In general the fluctuation spectrum may contain several modes. Our pseudoclassical transport theory must be applied to each of these modes individually, determining the time evolution of the amplitude of each mode, as well as the transport associated with each mode. This transport theory is based on the phenomena of particle trapping by a wave. Hence, the procedure outlined here will be reasonable when the individual waves are able to trap the particles resonant with them. The motion of particles in the presence of several waves has been studied (Zaslavskii and Chirikov, 1972), and it has been determined that when

$$v_{TRAP} < \Delta\left(\frac{\omega}{k_z}\right) \qquad (5.6)$$

particles can become trapped by individual waves.

Armed with inequalities (5.4) thru (5.6) we proceed to review the experimental observations of drift waves. Low frequency drift waves have long been identified in Q-machines (some recent references are Politzer, 1971; Deschamps et al., 1973; Prager et al., 1975), in tokamaks (e.g., Mazzucato, 1976; Surko and Slusher, 1976), in stellerators (e.g., Okabayashi, Arunasalam, 1977), and in computer simulations (e.g., Lee and Okuda, 1976; Matsuda and Okuda, 1976; Cheng and Okuda, 1977). In Q-machines and in computer experiments the boundary conditions are important as they limit the number of unstable modes. Hence, in these systems, a sharply peaked fluctuation spectrum satisfying inequalities (5.4) thru (5.6) is observed.

Q-machine experiments have been performed over a wide range of collision frequencies. In these experiments, the parallel wavenumbers of the unstable modes, k_z, are determined by the length of the machine. Hence inequality (1.2),

$$\left(\frac{\nu_e}{k_z v_{te}}\right) < \left(\frac{e\Phi_0}{T}\right)^{3/2}$$

can only be satisfied at low collision frequencies, when the electron mean free path is greater than the length of the machine. In Q-machines electrons are emitted and absorbed by hot plates at each end of the machine. When the electron mean free path is greater than the machine length, this emission and absorption is important in determining the form of the electron distribution function. We have not considered this effect in our treatment of the electron distribution function. Hence, the theory presented here cannot as yet be said to apply to the Q-machine experiments.

Electron transport is of great interest in tokamak plasmas. Recently measurements have been made of the drift wave fluctuation spectrum in the ATC tokamak. The parallel wave numbers of the drift wave fluctuation spectrum were not measured, but an experimental upper limit on k_z of 0.6 cm^{-1} was reported (Surko and Slusher, 1976). At the fluctuation levels observed in these experiments, a value of k_z greater than 10^{-2} cm^{-1} is required to satisfy inequalities (1.2) and (5.3). Such values of k_z are consistent with the radial normal mode analysis of low frequency drift waves (Pearlstien and Berk, 1969), and are safely below the experimental upper limit on k_z. Hence, it appears likely that the resonant electrons can be trapped by the drift wave spectrum observed in these experiments.

In tokamaks, the observed fluctuation spectrum does not have a well defined frequency. Hence, it appears that both particle trapping and the finite width of the spectrum must be included in a satisfactory theory relating this fluctuation spectrum to the electron transport rates. The pseudoclassical transport theory presented above does not allow for this finite spectral width.

However, our pseudoclassical transport theory does provide a model problem in which the effects of particle trapping by drift waves has been properly dealt with. Hence, it should provide a qualitative indication of the role particle trapping plays both in stabilizing the drift wave fluctuation spectrum, and in determining the transport rates. A theory of drift wave turbulence that successfully incorporates particle trapping together with a finite spectral width should give results similar to these presented here, in the appropriate limit.

Experiments in stellerators (Okabayashi and Arunasalam, 1977) indicate that the drift wave fluctuation spectrum becomes more coherent as the shear in the magnetic field is increased. This increased coherence presumably results from a reduction in the number of linearly unstable modes as the magnetic shear is increased. This suggests that attempts to stabilize the low frequency drift wave by a combination of shear in the magnetic field and an inverted temperature gradient (Horton, 1976), will also yield a more coherent spectrum of low frequency drift waves. Hence, the pseudoclassical theory presented here may prove useful in understanding the fluctuation levels and transport rates in future tokamak experiments.

Finally, we consider the application of this theory to the understanding of computer simulations of low frequency drift wave instabilities (Lee and Okuda, 1976; Matsuda and Okuda, 1976; Cheng and Okuda, 1977). We expect that computer simulations will provide an excellent testing ground for this theory. The drift wave spectrum observed in these simulations often involves only a single Fourier mode. Hence, inequalities (5.4) thru (5.6) are well satisfied. Other plasma parameters such as $(\nu_e/k_z v_{te})$ are easily varied, and complete information is available about the state of the plasma wave system. Hence, our predictions on the form of the electron distribution function can be directly tested. In I we remarked on the similarity between the distribution function examined there, and the electron distribution function observed by Lee and Okuda (1976) in a computer simulation of the nonlinear behavior of a low frequency drift wave driven unstable by the interaction with the resonant electrons. In addition, we expect that the pseudoclassical theory presented here will be useful in understanding the particle and energy transport rates observed in computer simulations, and that the nonlinear dissipative instability discussed in Sec. 2 will be important in the energy balance of the drift wave at saturation.

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