A THERMODYNAMIC APPROACH TO DISSIPATIVE DRIFT INSTABILITIES

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ABSTRACT

The transport of electrons and energy across a plasma slab associated with various dissipative drift instabilities (e.g., the dissipative trapped electron instability) is examined. A single electrostatic wave is considered. Equations describing the evolution of the plasma number density, momentum density, energy density, and entropy density are derived from the drift kinetic equation, while an equation for the evolution of the wave amplitude is obtained using the local approximation. The (possibly nonlinear) growth rate of the wave is found to be directly related to the electron flux. This connection between the wave growth and the electron flux is interpreted as a consequence of momentum conservation; this allows a simple and direct estimate of the growth rates of the various dissipative drift instabilities. In addition, it is shown that, when the thermodynamic forces are properly identified, the drift wave transport coefficients derived by previous authors do indeed satisfy the Onsager relations, and that the resulting plasma transport produces a net increase in the plasma entropy.

I. INTRODUCTION

The purpose of this paper is to develop a framework for the analysis of the cross-field transport associated with dissipative drift instabilities, such as the dissipative trapped electron instability (Kadomtsev and Pogutse, 1969), the dissipative trapped ion instability (Kadomtsev and Pogutse, 1971), and the dissipative drift instability (Hendel, et al., 1970). We find that even the collisionless drift instability (Galeev, et al., 1963) may be understood within this framework. Although we begin by focusing on the transport of electrons and energy, we find that a simple expression that allows approximate calculations of the growth rates of dissipative drift instabilities comes naturally out of this analysis. In addition, the plasma entropy is considered. Previous authors (e.g., Horton, 1976) have been unable to demonstrate that dissipative drift instabilities produce a net increase in the plasma entropy. We show that this problem is resolved when the thermodynamic forces are properly identified.

We adopt as our model a plasma slab in a uniform magnetic field (see Fig. 1). The magnetic field is taken parallel to the z-axis, while gradients in the plasma temperature and density are parallel to the x-axis. We consider the electron flux and energy flux driven by a single electrostatic wave. The wave potential is assumed to be of the form

$$\Phi(\theta,x,t) = \Phi_o(x,t)\, h(\theta) \tag{1.1}$$

where the wave phase θ is given by

$$\theta = k_y y + k_z z - \omega t \ .$$

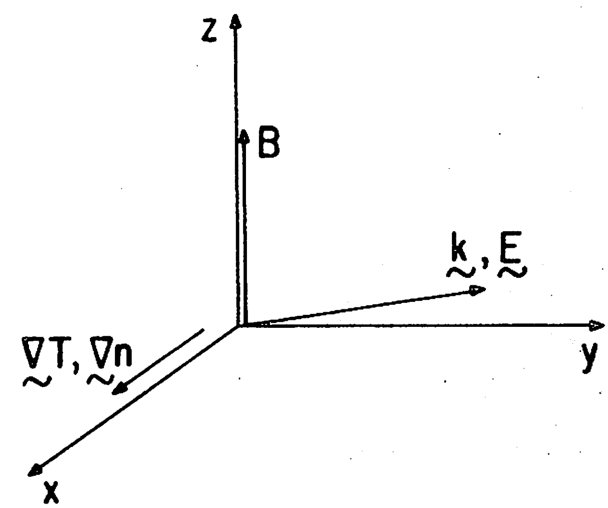

FIGURE 1. Directions of the Fields and Gradients

$\Phi_o(x,t)$ is the slowly varying wave amplitude. The wave form, $h(\theta)$, is assumed to be periodic in θ (with a period of 2π) and to be normalized such that

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \; h^2(\theta) = \tfrac{1}{2} \quad . \tag{1.3}$$

The phase velocity of the wave, $v_\phi = \omega/k_z$, is assumed to be much smaller than the electron thermal velocity, $v_{te} = (T_e/m_e)^{\frac{1}{2}}$. We will focus on the interaction between the wave and the <u>electron distribution</u>, and assume that $k\rho_e$ and ω/Ω_e are small enough that the electron dynamics are adequately described by the guiding center equations of motion. $\rho_e = v_{te}/\Omega_e$ and $\Omega_e = eB/m_e$ are the electron gyroradius and gyrofrequency, respectively.

We note that this single wave picture is sufficient to understand the transport associated with the linear development of dissipative drift instabilities, as each Fourier mode is treated independently in linear theory. In addition, certain nonlinear problems, such as the trapping of resonant particles by the wave, may be examined within this framework (Nevins, 1977a,b; Nevins, et al., 1977).

It is helpful to view the transport of electrons and energy driven by a dissipative drift instability as resulting from a random walk in which the $E \times B$ drift of particles in the field of the wave provides the "step" and collisions between particles yield a finite correlation time. The fundamental parameter for this transport process is $(\Delta x_T/L)$, where Δx_T is the characteristic step size and L is the scale length for variations in the plasma temperature and density. It is convenient to define a frequency, ω_T, such that

$$\Delta x_T \omega_T \equiv \frac{k_y \Phi_o}{B} . \tag{1.4}$$

ω_T is the characteristic frequency of the cross-field motion that produces the step, Δx_T . The manner in which $k_y \Phi_o / B$, the characteristic guiding center drift velocity, is factored to yield Δx_T and ω_T depends on the particular transport mechanism that is invoked. We will find that each transport mechanism is associated with a different instability mechanism. We will not consider particular transport mechanisms until after we have developed a framework that may be employed in the analysis of any dissipative drift instability.

Following the usual procedure in transport calculations we consider two time scales; the microscopic time scale, and the macroscopic or transport time scale. The division between these two time scales is made possible by ordering in the small parameter $(\Delta x_T/L)$. The assumption of local thermal equilibrium provides us with the condition that $\partial/\partial t$ cannot exceed $(\Delta x_T/L)\omega_T$. Within this constraint, the microscopic time scale is defined by $\partial/\partial t = \mathcal{O}[(\Delta x_T/L)\omega_T]$, while the transport time scale is defined by $\partial/\partial t = \mathcal{O}[(\Delta x_T/L)^2 \omega_T]$. We make this expansion about an equilibrium that includes the electrostatic wave propagating in the y-z plane. The phase velocity of this wave is assumed to be small. $(v_\phi/v_{te})^2 = \mathcal{O}(\Delta x_T/L)$. The amplitude of the wave is allowed to vary slowly with both x

$$\frac{\Delta x_T}{\Phi_o} \frac{\partial \Phi_o}{\partial x} = \mathcal{O}\left(\frac{\Delta x_T}{L}\right)$$

and t

$$\frac{1}{\Phi_o} \frac{\partial \Phi_o}{\partial t} = \mathcal{O}(\Delta x_T/L)^2 \omega_T .$$

Within this ordering in $(\Delta x_T/L)$, we will also make use of the small parameter $(e\Phi_o/T)$, keeping terms through second order in this parameter. In Sects. 2 and 3 we derive equations describing the evolution, on the macroscopic time scale, of the number density, momentum density, energy density, and entropy density by taking the appropriate moments of the drift kinetic equation.

In Sect. 4 we linearize the drift kinetic equation to obtain an equation describing the evolution of the electron distribution function on the microscopic time scale.

In Sect. 5 we investigate the thermodynamic properties of the plasma-wave system, identify the thermodynamic forces, and derive a wave phase averaged version of the Thermodynamic Identity.

In Sect. 6 we show that the growth rate of a low frequency drift wave is simply related to the particle flux driven by the wave.

In Sect. 7 we demonstrate that the equations for the evolution of the electron plasma and the low frequency drift wave derived here conserve both energy and momentum.

In Sect. 8 we show that a consideration of momentum conservation leads to a simple physical picture that allows "back of the envelope" calculations of the growth rates of the dissipative drift instabilities.

Finally, in Sect. 9 we demonstrate that, when the thermodynamic forces are properly identified, the anomalous transport coefficients derived by Horton (1976) obey Onsager's relations, and that the resulting macroscopic fluxes produce a net increase in the entropy, as is required physically.

2. FLUID EQUATIONS

In this section we derive equations describing the evolution of the plasma number density, momentum density, and entropy density by taking the appropriate moments of the drift kinetic equation. In deriving these equations we assume that the electrostatic wave is described by a potential that may be written as in Eq. (1.1), and that the electron distribution function depends on y and z only through the wave phase, θ. The distribution function is assumed to be periodic in θ, having the same period as the wave. In addition, we assume that $k\rho$, ρ/L, ω/Ω, $\omega_T/\Omega \ll 1$, so that the drift kinetic equation may be used to describe the evolution of the distribution function. The subscript e has been dropped. m and T are understood to refer to the electron mass and temperature.

We do not make use of the small parameters $(\Delta x_T/L)$ and $(e\Phi_o/T)$ in this section. Hence, these equations may be generally applied to describe the evolution of a plasma slab in the presence of a single traveling wave. We will focus on the evolution of the electron component. Similar equations for the evolution of the ion component may be obtained by replacing the electron charge, $-e$, and mass, m, by the ion charge and mass where ever they appear.

We find it convenient to use the variable set $(\theta,x,\mu,E,\sigma,t)$; where θ is the wave phase given by Eq. (1.2); x is the x-component of the guiding center position; μ is the magnetic moment; E is the particle energy in the wave frame,

$$E = \tfrac{1}{2} m q'^2 + \mu B - e\Phi \tag{2.1}$$

where q' is the velocity slip, defined by

$$q' \equiv v_z - v_\phi \ , \tag{2.2}$$

and σ is the sign of q' .

In this set of variables the drift kinetic equation is given by

$$\underset{(1)}{\frac{\partial f}{\partial t}} - \underset{(2)}{e \frac{\partial \Phi}{\partial t} \frac{\partial f}{\partial E}} + \underset{(3)}{k_z q' \frac{\partial f}{\partial \theta}} + \underset{(4)}{\frac{k_y}{B} \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial \theta}} - \underset{(5)}{\frac{k_y}{B} \frac{\partial \Phi}{\partial \theta} \frac{\partial f}{\partial x}} = \underset{(6)}{C_e(f)} \ . \tag{2.3}$$

Since a nonstandard set of variables is being employed, an effort will be made to describe the reduction of each term in the macroscopic equations. To this end the terms of Eq. (2.3) have been numbered for future reference.

2.1 THE NUMBER DENSITY

We begin by deriving an equation for the evolution of the wave phase averaged number density. Although this equation is easily derived using the variables set $(\theta,x,\underline{v},t)$, we will use the variables $(\theta,x,\mu,E,\sigma,t)$ in order to illustrate some useful techniques for reducing terms when this variables set is employed.

An equation for the number density is obtained by applying the phase space averaging operator,

$$\int \frac{d\theta}{2\pi} \int d^3\underline{v} = \sum_{\sigma} \frac{2B}{m^2} \int \frac{d\theta}{2\pi} \int \frac{d\mu dE}{|q'|} \tag{2.4}$$

to Eq. (2.3). The phase space average of terms (1) and (2) may be combined to obtain

$$(1) + (2) = \sum_{\sigma} \frac{2B}{m^2} \int \frac{d\theta}{2\pi} \int d\mu \; dE \left[\frac{\partial}{\partial t} \left(\frac{f}{|q'|} \right) - f \frac{\partial}{\partial t} \left(\frac{1}{|q'|} \right) - e \frac{\partial \Phi}{\partial t} \frac{\partial f}{\partial E} \right] \tag{2.5}$$

q' depends explicitly on time only through the wave potential, $\Phi(\theta,x,t)$. Hence, it follows from Eq. (2.1) that

$$\frac{\partial}{\partial t} \left(\frac{1}{|q'|} \right) = e \frac{\partial \Phi}{\partial t} \frac{\partial}{\partial E} \left(\frac{1}{|q'|} \right) . \tag{2.6}$$

Upon substituting this expression into Eq. (2.5), and integrating by parts in E, we find that the second and third terms inside the square brackets cancel, leaving

$$(1) + (2) = \frac{\partial n}{\partial t} \tag{2.7}$$

where the wave phase averaged number density, $n(x,t)$, is defined by

$$n(x,t) \equiv \int \frac{d\theta}{2\pi} \int d^3\underline{v} f \equiv \sum_{\sigma} \frac{2B}{m^2} \int \frac{d\theta}{2\pi} \int \frac{d\mu dE}{|q'|} f \; . \tag{2.8}$$

Term (3) must vanish upon performing the θ-integral because the distribution function, f , is continuous along particle orbits.

Terms (4) and (5) may be combined to give

$$\frac{k_y}{B} \sum_{\sigma} \frac{2B}{m^2} \int \frac{d\theta}{2\pi} \int \frac{d\mu dE}{|q'|} \left(\frac{\partial\Phi}{\partial x} \frac{\partial f}{\partial\theta} - \frac{\partial\Phi}{\partial\theta} \frac{\partial f}{\partial x} \right) =$$
$$\frac{k_y}{B} \sum_{\sigma} \sigma \frac{2B}{m^2} \int \frac{d\theta}{2\pi} \int \frac{d\mu dE}{q'} \left[\frac{\partial}{\partial\theta} \left(\frac{\partial\Phi}{\partial x} f \right) - \frac{\partial}{\partial x} \left(\frac{\partial\Phi}{\partial\theta} f \right) \right] \; . \tag{2.9}$$

Referring to Eq. (2.1) we see that q' depends on x and θ only through $\Phi(\theta,x,t)$. Hence, (2.9) may be written as

$$(4) + (5) = \frac{k_y}{B} \sum_{\sigma} \sigma \frac{2B}{m^2} \int \frac{d\theta}{2\pi} \int d\mu dE \left[\frac{\partial}{\partial\theta} \left(\frac{1}{q'} \frac{\partial\Phi}{\partial x} f \right) - \frac{\partial}{\partial x} \left(\frac{1}{q'} \frac{\partial\Phi}{\partial\theta} f \right) \right] \; . \tag{2.10}$$

The first term within the square brackets is a perfect differential. Hence its θ-integral must vanish. The remaining term may be written as

$$(4) + (5) = \frac{\partial}{\partial x} \Gamma_e \tag{2.11}$$

where Γ_e is the wave phase averaged electron flux,

$$\Gamma_e[f] \equiv \int \frac{d\theta}{2\pi} \int d^3\underline{v}\, v_{dr}\, f \qquad (2.12)$$

and v_{dr}, the x component of the $\underline{E} \times \underline{B}$ drift velocity, is given by

$$v_{dr} = - \frac{k_y}{B} \frac{\partial \Phi}{\partial \theta} \quad . \qquad (2.13)$$

We use the notation $\Gamma_e[f]$ in Eq. (2.12) to emphasize that this equation defines the particle flux as a functional of the distribution function. The goal of a transport calculation is to obtain an algebraic expression for Γ_e involving the local values of the system parameters, n , T , B , and Φ_o , and their gradients.

Finally, term (6) vanishes due to the conservation of particles in collisions. Hence, the wave phase averaged number density obeys the equation

$$\frac{\partial n}{\partial t} = - \frac{\partial}{\partial x} \Gamma_e \quad . \qquad (2.14)$$

2.2 THE MOMENTUM DENSITY EQUATIONS

We find it most useful to derive equations for the canonical momentum density. The canonical momentum of a single particle is given by

$$\underline{p} = m\underline{v} - e\underline{A}$$

where $\underline{A}$ is the vector potential. The vector potential is not uniquely defined. Hence, the momentum density depends on the choice of gauge. In the slab model employed here the vector potential may be written as

$$\underline{A} = xB\hat{y} \quad .$$

With this choice of gauge, the x-component of the particle momentum density, mv_x , is a gyrophase dependent quantity. Thus, the x-component of the momentum density equation cannot be obtained from the drift kinetic equation, but must instead be obtained directly from the Boltzmann equation. We put off further discussion of this equation until Sect. 3.

An equation for the y-component of the momentum density may be obtained directly from the equation for the evolution of the wave phase averaged number density by applying the operator

$$\int_{-\infty}^{\infty} dx' \; p_y(x \mid x') \quad . \tag{2.15}$$

$p_y(x \mid x')$ is the momentum density at the field point x due to a particle at x' .

One natural choice of this function is to locate all of the particle momentum, $-eBx'$, at the particle position, i.e.,

$$p_y(x \mid x') = -eBx' \, \delta(x-x') \ . \tag{2.16}$$

We suggest an alternate expression,

$$p_y(x \mid x') = -eB \, [\, H(x'-x) - H(-x) \,] \tag{2.17}$$

where $H(x)$ is the Heavyside function (i.e., $H(x) = 1$ if $x > 0$, while $H(x) = 0$ if $x < 0$). This expression for the momentum density may be visualized as a tether connecting the particle to the flux surface on which A_y vanishes (with our choice of gauge this occurs at $x = 0$). This definition of $p_y(x \mid x')$ may be generalized to inhomogeneous magnetic fields (provided that $\underline{B}$ lies in the y-z plane) by simply replacing B in Eq. (2.17) with $B_z(x)$, giving

$$p_y(x \mid x') = -eB_z(x) \, [\, H(x'-x) - H(-x) \,] \ .$$

We note that both (2.16) and (2.17) give the correct total particle momentum,

$$\int_{-\infty}^{\infty} dx \; p_y(x \mid x') = -eBx' \ .$$

They differ only in how this momentum is distributed in space. Hence, either (2.16) or (2.17) may be employed in deriving a conservation law for the y-component of the momentum.

The application of a force, $F\hat{y}$, to the particle will cause the particle guiding center to move in x . If (2.16) is chosen as the particle momentum density, then this force will generate both a momentum source (as expected) and a momentum flux. When (2.17) is employed as the particle momentum density, the application of a force results <u>only</u> in a momentum source. Hence, the choice of (2.17) as the particle momentum density yields a simpler equation for the evolution of the total y-component of the canonical momentum density. For this reason we prefer (2.17) over (2.16). We note that the nonlocal character of the particle momentum results from the dependence of the particle y-momentum, $-eBx$, on position. Our choice of (2.17) as the particle momentum density merely reflects the nonlocal character of this canonical momentum.

Applying (2.15) to (2.14), and using (2.17) as the particle momentum density, we obtain

$$\frac{\partial p_y}{\partial t} = -eB\Gamma_e \tag{2.18}$$

where

$$p_y \equiv - \int_{-\infty}^{\infty} dx' \; eB \left[H(x'-x) - H(-x) \right] n(x') \; . \tag{2.19}$$

If (2.16) had been employed as the particle momentum density in place of (2.17), this equation would read

$$\frac{\partial p_y}{\partial t} = -eB\Gamma_e - \frac{\partial}{\partial x} \Pi_{xy}$$

where the x-y component of the stress tensor is given by

$$\Pi_{xy} = -eBx\Gamma_e \ .$$

Finally, we consider the equation of evolution for the z-component of the canonical momentum density. This equation is obtained by multiplying the drift kinetic equation through by

$$mv_z = m(q' + \frac{\omega}{k_z})$$

and then applying the phase space averaging operator, (2.4).

The reduction of terms (1) and (2) parallels the reduction of these terms in the equation for n , and yield $\partial p_z/\partial t$, where the z-component of the canonical momentum density, p_z , is defined by

$$p_z \equiv \int \frac{d\theta}{2\pi} \int d^3\underline{v}\ mv_z f \ . \tag{2.20}$$

After a little manipulation term (3) may be written as

$$(3) = m\ k_z \sum_\sigma \sigma \frac{2B}{m^2} \int d\mu\ dE \int \frac{d\theta}{2\pi}\ q' \frac{\partial f}{\partial \theta} \ . \tag{2.21}$$

This expression is integrated by parts over θ. Refering to Eqs. (2.1) and (2.13) it is found that

$$\frac{\partial q'}{\partial \theta} = -\frac{e\ B}{m\ k_y} \frac{1}{q'} v_{dr} \ . \tag{2.22}$$

Hence, term (3) may be expressed in terms of the particle flux as

$$(3) = \frac{k_z}{k_y} e\ B\ \Gamma_e \ . \tag{2.23}$$

Terms (4) and (5) are again treated together. We use the commutation relations

$$\frac{1}{q'}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{1}{q'} = \frac{1}{q'^3}\frac{e}{m}\frac{\partial\Phi}{\partial x}$$

and

$$\frac{1}{q'}\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\theta}\frac{1}{q'} = \frac{1}{q'^3}\frac{e}{m}\frac{\partial\Phi}{\partial\theta}$$

to write these terms in the form

$$\begin{aligned}(4) + (5) = {} & \frac{mk_y}{B}\sum_\sigma \frac{2B}{m^2}\int d\mu\ dE \int\frac{d\theta}{2\pi}\frac{\partial}{\partial\theta}\left(\frac{v_z}{|q'|} f \frac{\partial\Phi}{\partial x}\right) \\ & - \frac{\partial}{\partial x}\left(\frac{mk_y}{B}\sum_\sigma\frac{2B}{m^2}\int\frac{d\theta}{2\pi}\int\frac{d\mu\ dE}{|q'|} v_z\ f\ \frac{\partial\Phi}{\partial\theta}\right) .\end{aligned} \tag{2.24}$$

The integrand of the first term in (2.24) is again an exact differential. Hence, this term vanishes. We identify the remaining term in (2.24) as the divergence of the stress tensor,

$$(4) + (5) = \frac{\partial}{\partial x}\Pi_{xz} \tag{2.25}$$

where

$$\Pi_{xz} \equiv \int\frac{d\theta}{2\pi}\int d^3\underline{v}\ m\ v_z\ v_{dr}\ f\ . \tag{2.26}$$

Finally term (6) gives the collisional drag on the electrons

$$\left.\frac{\partial p_z}{\partial t}\right|_{coll} = \int\frac{d\theta}{2\pi}\int d^3\underline{v}\ m\ v_z\ C_{ei}(f)\ . \tag{2.27}$$

Hence, the z-component of the momentum balance equation is

$$\frac{\partial p_z}{\partial t} = -\frac{k_z}{k_y} eB\Gamma_e - \frac{\partial}{\partial x}\Pi_{xz} + \left.\frac{\partial p_z}{\partial t}\right|_{coll} . \tag{2.28}$$

The final term in Eq. (2.28) describes the collisional drag of the electrons on the ions. Many important low frequency instabilities, such as the dissipative trapped ion mode (Kadomstev and Pogutse, 1971), the dissipative trapped electron mode (Kadomtsev and Pogutse, 1969), and the collisionless drift mode (Galeev, et al., 1963) occur in nearly collisionless plasmas where this (essentially classical) term may be neglected when compared with the other terms on the right hand side of Eq. (2.28).

Dropping the collisional term, we obtain

$$\frac{\partial p_z}{\partial t} = -\frac{k_z}{k_y} eB\Gamma_e - \frac{\partial}{\partial x}\Pi_{xz} . \tag{2.29}$$

It is clear from Eq. (2.29) that the electrons may be expected to develope a drift parallel to $\underline{B}$ due to the momentum source term, $(-k_z/k_y)eB\Gamma_e$. In a more complete treatment of the anomalous transport driven by a single wave, this term would give rise to an anomalous "bootstrap" current. In this paper we confine our attention to the anomalous particle and energy flux. We do not investigate the effect of the wave on the plasma current.

2.3 THE ENTROPY DENSITY

It is convenient to use the variable set $(\theta, x, \underline{v}, t)$ in deriving an equation for the wave phase averaged entropy density. Using this variable set the drift kinetic equation may be written as

$$\overset{(1)}{\frac{\partial f}{\partial t}} + \overset{(2)}{\frac{e}{m} k_z \frac{\partial \Phi}{\partial \theta} \frac{\partial f}{\partial v_z}} - \overset{(3)}{\frac{k_y}{B} \frac{\partial \Phi}{\partial \theta} \frac{\partial f}{\partial x}} + \overset{(4)}{\frac{k_y}{B} \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial \theta}} + \overset{(5)}{k_z v_z \frac{\partial f}{\partial \theta}} = \overset{(6)}{C_e(f)} \quad . \qquad (2.30)$$

Following Boltzmann (1896) the wave phase averaged entropy density is taken as

$$S_e = - \int \frac{d\theta}{2\pi} \int d^3\underline{v} \; f \; (\ln f) \; . \qquad (2.31)$$

Thus, an equation of evolution for the entropy density may be obtained by multiplying the drift kinetic equation, (2.30), by $-(1 + \ln f)$ and applying the phase space averaging operator, $\int (d\theta/2\pi) d^3\underline{v}$.

Term (1) immediately reduces to $\partial S_e/\partial t$.

Term (2) vanishes on integration over v_z .

Terms (3) and (4) may be combined to yield

$$(3) + (4) = \frac{\partial}{\partial x} J_S$$

where the entropy flux, J_S , is given by

$$J_S[f] \equiv - \int \frac{d\theta}{2\pi} \int d^3\underline{v} \; v_{dr} \; f \; (\ln f) \; . \qquad (2.33)$$

Term (5) vanishes upon integration over θ.

Term (6) gives rise to an entropy source,

$$\dot{S}_e[f] \equiv - \int \frac{d\theta}{2\pi} \int d^3\underline{v} \, (\ln f) \, C_e(f) \ . \tag{2.34}$$

Hence, the wave phase averaged entropy density satisfies the equation

$$\frac{\partial S_e}{\partial t} = - \frac{\partial}{\partial x} J_S + \dot{S}_e \ . \tag{2.35}$$

The first term on the right hand side of Eq. (2.35) describes the convection of entropy. This term can cause $\partial S_e/\partial t$ to become negative. The Second Law of Thermodynamics applies only to the entropy source, and states that

$$\dot{S}_e + \dot{S}_i \geq 0$$

where S_i is the entropy source of the ions.

In analyzing dissipative drift instabilities the ions are often taken to be massive, fixed, scattering centers so that the Lorentz operator may be employed to describe electron-ion collisions. We make this approximation here. It is then a simple matter to show

$$\dot{S}_e \geq 0 \tag{2.36}$$

directly from Eq. (2.34). The derivation of inequality (2.36) follows closely the proof of the Boltzmann H-theorem. This proof may be found in many textbooks (e.g., DeGroot and Mazur, 1962). We will discuss the entropy source further in Sect. 5.

3. THE ENERGY DENSITY

The derivation of an equation describing the evolution of the electron energy density has been postponed until this section because we find it helpful to introduce the small parameters $(\Delta x_T/L)$ and $(e\Phi_o/T)$ in reducing terms (1) and (2), as these small parameters allow us to separate the wave energy from the electron thermal energy. We adopt the ordering described in Sect. 1. The wave phase averaged energy density will evolve on the transport time scale. Hence, we must keep terms through second order in $(\Delta x_T/L)$ in the energy density equation.

Before considering the evolution of the energy density, we briefly review the x-component of the momentum equation. As we mentioned previously, this equation must be derived directly from the Boltzmann equation. We do not present this derivation here, but simply point out that the wave phase average of v_x is given by the diffusion velocity,

$$u_x \equiv \frac{\Gamma_e}{n} = \mathcal{O}\left(\frac{\Delta x_T}{L}\right) v_{dr} \ .$$

In equilibrium both n and Γ_e vary only on the transport time scale, so the inertial term in the x-momentum equation,

$$\frac{\partial u_x}{\partial t} \sim \left(\frac{\Delta x_T}{L}\right)^3 \omega_T \, v_{dr}$$

vanishes through second order in $(\Delta x_T/L)$. Hence, the leading terms in this equation describe the equilibrium balance of forces.

The finite amplitude wave introduces some ponderomotive-like forces that are smaller than the pressure gradient term by the factor $(e\Phi_o/T)^2$. We ignore these terms, and obtain the usual equilibrium condition,

$$\frac{\partial}{\partial x}(P + B^2/8\pi) = 0 \tag{3.1}$$

where the pressure is given by

$$P = nT . \tag{3.2}$$

The energy equation is obtained by multiplying the drift kinetic equation (2.3) through by the energy of a single particle,

$$\mathcal{E} \equiv \tfrac{1}{2} mv^2 - e\Phi = E + mv_\phi q' + \tfrac{1}{2} mv^2 \tag{3.3}$$

and applying the phase space averaging operator, (2.4).

Terms (1) and (2) are again treated together. Noting that

$$\begin{aligned}\frac{1}{q'}\,\mathcal{E}\frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\mathcal{E}f}{|q'|}\right) - f\frac{\partial}{\partial t}\left(\frac{\mathcal{E}}{|q'|}\right) \\ &= \frac{\partial}{\partial t}\left(\frac{\mathcal{E}f}{|q'|}\right) - e\frac{\partial\Phi}{\partial t}f\frac{\partial}{\partial E}\left(\frac{\mathcal{E}}{|q'|}\right) + e\frac{\partial\Phi}{\partial t}\frac{1}{|q'|}f\end{aligned}$$

we write terms (1) and (2) in the form

$$(1) + (2) = \sum_\sigma \frac{2B}{m^2}\int\frac{d\theta}{2\pi}\int d\mu dE\left\{\frac{\partial}{\partial t}\left(\frac{\mathcal{E}f}{|q'|}\right) - e\frac{\partial\Phi}{\partial t}f\frac{\partial}{\partial E}\left(\frac{\mathcal{E}}{|q'|}\right) + e\frac{\partial\Phi}{\partial t}\frac{f}{|q'|} + e\frac{\partial\Phi}{\partial t}\mathcal{E}\frac{\partial f}{\partial E}\right\} . \tag{3.4}$$

Integrating the second term in square brackets on the right hand side of Eq. (3.4) by parts, we find that it exactly cancels the fourth term in the brackets, leaving

$$(1) + (2) = \frac{\partial}{\partial t} \overline{w} - \dot{\mathcal{W}}^{(e)} \qquad (3.5)$$

where $\overline{w}$ is the wave phase average of the total electron energy density,

$$\overline{w} \equiv \int \frac{d\theta}{2\pi} \int d^3\underline{v} \, (\tfrac{1}{2} mv^2 - e\Phi) \, f \qquad (3.6)$$

and $\dot{\mathcal{W}}^{(e)}$ is defined by

$$\dot{\mathcal{W}}^{(e)} \equiv - \int \frac{d\theta}{2\pi} \int d^3v \, e \frac{\partial \Phi}{\partial t} f \; . \qquad (3.7)$$

The reduction of terms (3) thru (5) follows closely the reduction of the corresponding terms in the z-momentum equation. Hence, term (3) becomes

$$(3) = \frac{\omega}{k_y} \; eB\Gamma_e \qquad (3.8)$$

while terms (4) and (5) combine to yield

$$(4) + (5) = \frac{\partial}{\partial x} Q_e \qquad (3.9)$$

where

$$Q_e[f] \equiv \int \frac{d\theta}{2\pi} \int d^3\underline{v}\; v_{dr}\; (\tfrac{1}{2}\, mv^2 - e\Phi)\; f\;. \tag{3.10}$$

Term (6) describes the collisional transfer of energy between the electrons and the ions. This energy transfer rate is small in $(m_e/m_i)^{\frac{1}{2}}$. Hence, the leading terms in the energy density equation are

$$\frac{\partial \overline{w}}{\partial t} - \dot{\mathcal{W}}^{(e)} = -\frac{\partial}{\partial x} Q_e - \frac{\omega}{k_y} eB\Gamma_e\;. \tag{3.11}$$

To proceed further in the reduction of this equation, we assume that the wave amplitude varies only on the transport time scale,

$$\frac{1}{\Phi_o} \frac{\partial \Phi_o}{\partial t} \lesssim \left(\frac{\Delta x_T}{L}\right)^2 \omega_T\;. \tag{3.12}$$

When (3.12) is satisfied the perturbation in the electron distribution function [which is of order $(\Delta x_T/L)$] may be ignored in evaluating $\dot{\mathcal{W}}^{(e)}$. Hence,

$$\dot{\mathcal{W}}^{(e)} = -\int \frac{d\theta}{2\pi} \int d^3\underline{v}\; e\, \frac{\partial \Phi}{\partial t}\, f_o + \mathcal{O}(\Delta x_T/L)^3$$

where f_o is a local Boltzmann distribution,

$$f_o = n(m/2\pi T)^{3/2} \exp\left(-\frac{\mathcal{E}}{T}\right)\,. \tag{3.13}$$

A further expansion in $(e\Phi_o/T)$ yields

$$\dot{\mathcal{W}}^{(e)} = -\frac{1}{k^2\lambda_D^2} \frac{\partial}{\partial t} (k^2\Phi_o^2/16\pi)\;. \tag{3.14}$$

We show in Appendix A that (3.14) may be interpreted as the local time rate of change in the contribution of the electrons to the wave energy. Hence, the left hand side of Eq. (3.11) is the time rate of change in the thermal energy of the electrons,

$$w = \overline{w} - \mathcal{W}^{(e)} \tag{3.15}$$

where

$$\mathcal{W}^{(e)} = -\frac{1}{k^2\lambda_D^2}\,(k^2\phi_o^2/16\pi)\ . \tag{3.16}$$

Hence, Eq. (3.11) may be written as

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial x} Q_e - \frac{\omega}{k_y} eB\Gamma_e\ . \tag{3.17}$$

This equation is remarkable in that the energy source, $eB\Gamma_e(\omega/k_y)$, is simply related to the particle flux. We show in Sect. 7 that this term describes the heating (or cooling) of the electron distribution by the wave.

The evolution of the plasma energy is often described by providing an equation for the temperature. It follows from Eqs. (3.6), (3.15), and (3.16) that the electron temperature is related to the electron thermal energy by

$$w = \frac{3}{2}\,nT\ . \tag{3.18}$$

Equations (2.14), (3.1), (3.17), and (3.18) may be combined to obtain an equation for the evolution of the plasma temperature,

$$\frac{3}{2} n \frac{d}{dt} T + \frac{\partial}{\partial x} \tilde{Q}_e = - P \frac{\partial}{\partial x} u_x + u_x \frac{\partial}{\partial x} (B^2/8\pi) - \frac{\omega}{k_y} eB\Gamma_e \tag{3.19}$$

where $\frac{d}{dt}$ is the convective derivative,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x}$$

and $\tilde{Q}_e$ is the heat flux (Braginskii, 1965),

$$\tilde{Q}_e = Q_e - \frac{5}{2} T \Gamma_e . \tag{3.20}$$

We may interpret the terms on the right hand side of Eq. (3.19) as the work done on a fluid element by compression, the work done by the magnetic field, and the heating of the plasma by the wave.

4. THE MICROSCOPIC TIME SCALE

In this section we consider the evolution of the electron distribution on the microscopic time scale. We assume that the plasma-wave system is in a "steady state" in which the temporal variation on the microscopic time scale comes only through the dependence of the distribution function and the wave amplitude on θ. This assumption is consistent with the linear analysis of dissipative drift instabilities. For nonlinear problems, such as the trapping of particles by the wave, this assumption must be considered on a case by case basis.

In this steady state the drift kinetic equation may be written, through first order in $(\Delta x_T/L)$ as

$$\left\{ q + h(\theta) \frac{\Delta x_T}{\Phi_o} \frac{\partial \Phi_o}{\partial x} \right\} \frac{\partial f}{\partial \theta} - h'(\theta) \, \Delta x_T \frac{\partial f}{\partial x} = C_e(f)/\omega_T \tag{4.1}$$

where

$$h'(\theta) = \frac{dh}{d\theta} ,$$

we have introduced the normalized velocity-slip,

$$q \equiv \frac{v_z - v_\phi}{v_T} \tag{4.2}$$

and a characteristic velocity,

$$v_T \equiv \frac{\omega_T}{k_z} . \tag{4.3}$$

We wish to expand Eq. (4.1) in powers of $(\Delta x_T/L)$ about an equilibrium that <u>includes</u> the low frequency drift wave. Low frequency drift waves arise from perturbations in the ion density caused by the self-consistent $E \times B$ convection of ions across the zero order density gradient. The role of the electron distribution is to provide Debye shielding of these ion perturbations (Mikhailovskii, 1974; Kadomtsev, 1965). The effect of the Debye shielding may be included at zero order in $(\Delta x_T/L)$ by expanding the electron distribution function about a local Boltzman distribution:

$$f_o = n(x) \left(\frac{m}{2\pi T(x)} \right)^{3/2} \exp \left(- \frac{1}{T} \left(\tfrac{1}{2} mv^2 - e\Phi \right) \right) \tag{4.4}$$

where n and T are functions of x only. This choice of the zero order distribution function is consistent with the electron distribution function obtained in the linear analysis of low frequency drift waves.

Written in terms of our adopted set of variables, $(\theta, x, \mu, E, \sigma)$, the Boltzman distribution becomes

$$f_o = n \left(\frac{m}{2\pi T} \right)^{3/2} \exp \left(- \frac{1}{T} \left(E + mqv_\phi v_T + \tfrac{1}{2} mv_\phi^2 \right) \right). \tag{4.5}$$

The electron distribution function may be written as

$$f = f_o (1 + \hat{f}) \tag{4.6}$$

where $\hat{f} = \mathcal{O}(\Delta x_T/L)$. This expansion of the electron distribution function is put into the kinetic equation (4.1) and terms in like powers of $(\Delta x_T/L)$ are equated. The terms involving $\partial f_o/\partial \theta$ require special attention. From Eq. (4.5) we see that f_o depends on θ only through q. Evaluating this factor we find

$$\frac{\partial f_o}{\partial \theta} = + \frac{h'(\theta)}{q} (a_0 \Delta x) \; f_o \tag{4.7}$$

where

$$a_0 \equiv \frac{eB}{k_y T} \omega . \tag{4.8}$$

Low frequency drift instabilities have frequencies of order

$$\omega_{ne} = - \left(\frac{k_y T}{eB} \right) \left(\frac{1}{n} \right) \left(\frac{\partial n}{\partial x} \right) .$$

Hence, $a_0 = \mathcal{O}(1/L)$; and $\partial f_o / \partial \theta$ is first order in $(\Delta x_T / L)$. The steady state kinetic equation is then satisfied at zero order in $(\Delta x_T / L)$ as

$$C_e(f_o) = 0 .$$

At first order in $(\Delta x_T / L)$ we obtain

$$q f_o \frac{\partial \hat{f}}{\partial \theta} - \Delta x_T \left(\frac{\partial f_o}{\partial x} - a_0 f_o \right) h'(\theta) = C_e(f)/\omega_T . \tag{4.9}$$

In Eq. (4.9) we require an expression for f_o valid only to zero order in $(\Delta x_T / L)$. To this order f_o is given by

$$f_o = n \left(\frac{m}{2\pi T} \right)^{3/2} e^{-E/T} + \mathcal{O}(\Delta x_T / L) \tag{4.10}$$

so $\partial f_o/\partial x$ may be written as

$$\frac{\partial f_o}{\partial x} = - (a_1 + A_2 E)\ \dot{f}_o \tag{4.11}$$

where we have defined

$$a_1 \equiv - \frac{\partial}{\partial x}\ \frac{M}{T} = - \frac{1}{n}\frac{\partial n}{\partial x} + \frac{3}{2}\frac{1}{T}\frac{\partial T}{\partial x} \tag{4.12}$$

$$A_2 \equiv \frac{\partial}{\partial x}\ \frac{1}{T} \tag{4.13}$$

and M, the chemical potential of the electrons, is given by:

$$M = T\ \ell n \left[n(2\pi\hbar^2/mT)^{3/2} \right] . \tag{4.14}$$

These three quantities, a_0, a_1, and A_2 are all of order $(1/L)$. a_0 is a measure of the departure of the system from thermal equilibrium because of the variations in the wave potential with time. a_1 and A_2 measure the departure from thermal equilibrium due to variations in the density and temperature with x. In Sect. 5 we find that the quantities

$$A_1 \equiv a_0 + a_1 \tag{4.15}$$

and A_2 are the thermodynamic forces acting on the plasma (DeGroot and Mazur, 1962).

Using the definitions (4.13) and (4.15) we may write the kinetic equation to first order in $(\Delta x_T/L)$ as

$$f_o \left[\Delta x_T (A_1 + A_2 E) h'(\theta) + q \frac{\partial \hat{f}}{\partial \theta} \right] = C_e(f)/\omega_T \ . \qquad (4.16)$$

Equation (4.16) describes the evolution of the electron distribution function on the microscopic time scale. It may be used to determine $\hat{f}$. This distribution function, together with Eqs. (2.12) and (3.10) then determine Γ_e and Q_e . Hence, the solution of equation (4.16) is the basic problem in determining the flux of particles and energy driven by a low frequency drift wave. We do not attempt to solve for $\hat{f}$ here, but only note that when Δx_T is proportional to the wave amplitude, Eq. (4.16) is equivalent to the linear (in wave amplitude) kinetic equation considered by many authors (e.g., Horton, 1976; Liu, et al., 1976).

5. THERMODYNAMIC PROPERTIES

We now proceed to investigate the evolution of the entropy density for plasma-wave systems that are "near thermal equilibrium". Operationally, what we mean by "near thermal equilibrium" is that

$$\frac{\Delta x_T}{L} \ll 1 \ . \tag{5.1}$$

When inequality (5.1) is satisfied, the entropy source, $\dot{S}_e$, and the entropy flux, J_S , may be expressed in terms of the particle and energy flux.

We begin by considering the entropy source,

$$\dot{S}_e = - \int \frac{d\theta}{2\pi} \int d^3\underline{v}\ \ell n\ f\ C_e(f)$$

the distribution function is written as it was in Eq. (4.6),

$$f = f_o(1 + \hat{f})$$

where $\hat{f} = \mathcal{O}(\Delta x_T/L)$. The local Boltzmann distribution, f_o , may be written in the variable set $(\theta,x,\underline{v},t)$ as

$$f_o = n(x) \left[\frac{m}{2\pi T(x)}\right]^{3/2} \exp\left[-\frac{\frac{1}{2}mv^2 + e\Phi}{T(x)}\right]. \tag{5.2}$$

The contribution of $\ell n\ f_o$ to $\dot{S}_e$ vanishes due to the conservation of particles and energy* in collisions. Hence, through second order in

*The collision operator, C_e , includes both electron-electron and

$(\Delta x_T/L)$, $\dot{S}_e$ may be written as

$$\dot{S}_e = -\int \frac{d\theta}{2\pi} \int d^3\underline{v}\ \hat{f}\ C_e(f) \ . \tag{5.3}$$

At this point it is helpful to change to the variable set (θ,μ,E,σ) . In this set of variables $\dot{S}_e$ may be written as

$$\dot{S}_e = -\sum_\sigma \frac{2B}{m^2 v_T} \int \frac{d\theta\ d\mu\ dE}{2\pi|q|}\ \hat{f}\ C_e(f) \ . \tag{5.4}$$

The first order in $(\Delta x_T/L)$ part of the drift kinetic equation, [i.e., Eq. (4.16)] may now be used to replace the collision operator with an expression involving f_o and $\hat{f}$,

$$C_e(f) = f_o \left[\Delta x(A_1 + A_2 E)h'(\theta) + q \frac{\partial f}{\partial \theta}\right] \omega_T \ .$$

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electron-ion collisions. In electron-ion collisions the electron energy is only conserved at zero order in the mass ratio, (m_e/m_i) . We neglect the higher order terms in (m_e/m_i) which describe the energy transfer between electrons and ions; i.e., we take the ions to be infinitely massive, fixed, scattering centers. This approximation is consistent with the use of the Lorentz collision operator to describe the action of electron-ion collisions on the electron distribution function.

This replacement does not yield a trivial identity (e.g.,

$$\dot{S}_e = \frac{\partial S}{\partial t} + \frac{\partial}{\partial x} J_s \;)$$

because Eq. (4.16) differs from the full drift kinetic equation in that only the first order terms have been included, and we have assumed that the system is in a steady state.

Inserting this expression into Eq. (5.4) we obtain

$$\dot{S}_e = - \sum_\sigma \frac{2B}{m^2 v_T} \int \frac{d\theta \; d\mu \; dE}{2\pi |q|} f_o \hat{f} \; (A_1 + A_2 E) \frac{k_y \Phi_o}{B} h'(\theta)$$

$$+ \sum_\sigma \sigma \frac{2B}{m^2} k_z \int \frac{d\mu \; dE}{\pi} f_o \int d\theta \; \frac{\partial}{\partial \theta} (\hat{f})^2 \qquad (5.5)$$

where we have used the fact that, to lowest order in $(\Delta x_T / L)$, f_o may be written as

$$f_o = n \left[\frac{m}{2\pi T} \right]^{3/2} e^{-E/T} + \mathcal{O}\left(\frac{\Delta x}{L}\right) \quad .$$

The θ-integral in the second term of Eq. (5.5) must vanish because f_o is a single-valued function of θ, with the same periodicity as the wave. Hence, $\dot{S}_e$ may written as

$$\dot{S}_e = A_1 \Gamma_e + A_2 Q_e \; , \qquad (5.6)$$

where the particle and energy fluxes are given by

$$\begin{pmatrix} \Gamma_e \\ Q_e \end{pmatrix} = \sum_\sigma \frac{2B}{m^2 v_T} \int \frac{d\theta \, d\mu \, dE}{2\pi |q|} f_o \hat{f} \begin{pmatrix} 1 \\ E \end{pmatrix} v_{dr} . \tag{5.7}$$

These fluxes can be put into a more recognizable form by returning to the variable set $(\theta, \underline{v})$ and noting that, to lowest order in $(\Delta x_T/L)$, E may again be replaced by $(\frac{1}{2} mv^2 - e\Phi)$. This procedure yields

$$\begin{pmatrix} \Gamma_e \\ Q_e \end{pmatrix} = \int \frac{d\theta}{2\pi} \int d^3\underline{v} \; f_o \hat{f} \begin{pmatrix} 1 \\ \frac{1}{2} mv^2 - e\Phi \end{pmatrix} v_{dr} . \tag{5.8}$$

Equation (5.6) plays a central role in the thermodynamics of our plasma slab, as it allows us to identify the thermodynamic forces acting on the plasma-wave system. It is well known in the study of non-equilibrium thermodynamics that the entropy source may be written as a sum of products between the macroscopic fluxes and the thermodynamic forces conjugate to those fluxes (DeGroot and Mazur, 1962). Hence, A_1 is the thermodynamic force conjugate to the particle flux while A_2 is the thermodynamic force conjugate to the energy flux. We note that A_1 may be written in the form

$$A_1 = -\frac{\Delta}{n}\frac{\partial n}{\partial x} + \frac{3}{2}\frac{1}{T}\frac{\partial T}{\partial x} \tag{5.9}$$

where

$$\Delta \equiv \left(1 - \frac{\omega}{\omega_{ne}}\right) . \tag{5.10}$$

This thermodynamic force differs from the expression employed by previous workers by the factor Δ multiplying $(1/n)(\partial n/\partial x)$. In Sect. 9 we will show that, when this factor is included, the transport coefficients associated with the various dissipative drift instabilities that have been derived by Horton (1976) and others (e.g., Liu, et al., 1976; Manheimer, 1977; Krall and McBride, 1977) satisfy the Onsager reciprocity relations (Onsager, 1931; Casimir, 1945) and that the resultant particle and energy flux cause an increase in the entropy of the plasma.

In systems near thermal equilibrium one expects that the fluxes may be written as a sum of products between the thermodynamic forces and the transport coefficients (DeGroot and Mazur, 1962). Hence, a further implication of our identification of A_1 as the thermodynamic force conjugate to the particle flux is that $(1/n)(\partial n/\partial x)$ must always appear multiplied by Δ in expressions for the particle or energy flux associated with the low frequency drift wave. This is indeed the case in every calculation that we are aware of (e.g., Horton, 1976; Liu, et al., 1976; Manheimer, 1977).

We now turn our attention to the entropy flux, J_S. Expanding the wave phase averaged entropy flux in powers of $(\Delta x_T/L)$, we find at zero order that

$$J_S^{(0)} = -\int \frac{d\theta}{2\pi} \int d^3\underline{v}\, v_{dr}\, f_o \ln f_o$$

$$= \frac{k_y}{B} \int d^3\underline{v} \int \frac{d\theta}{2\pi} \frac{\partial\Phi}{\partial\theta} f_o \ln f_o \ . \qquad (5.11)$$

The right-hand side of Eq. (5.11) must vanish because the zero-order distribution function, f_o , depends on the wave phase only through the wave potential, Φ . Thus, the entropy flux vanishes at zero order in $(\Delta x_T/L)$. Through first order in $(\Delta x_T/L)$, the wave phase averaged entropy flux is given by

$$J_S = -\int \frac{d\theta}{2\pi} \int d^3\underline{v}\ f_o\ \hat{f}\ (1 + \ln f_o)\ v_{dr} + \mathcal{O}\left(\frac{\Delta x_T}{L}\right)^2 . \tag{5.12}$$

Using Eqs. (2.12), (3.10), and (3.13) the entropy flux may be written as

$$J_S = -\frac{M}{T}\Gamma_e + \frac{Q_e}{T} \tag{5.13}$$

where M is the chemical potential,

$$M = T \ln\left[n\left(\frac{2\pi\hbar^2}{mT}\right)^{3/2}\right] . \tag{5.14}$$

The chemical potential enters this calculation in the form

$$\ln\left[n\left(\frac{m}{2\pi T}\right)^{3/2}\right] .$$

The argument of this log function is not dimensionless, but rather has the dimensions of 1/(unit phase space volume). There is no solution to this dilemma in classical physics, and, as a result, the chemical potential can only be defined in classical physics to within a constant.

We know from quantum mechanics that the natural unit of phase space volume is Planck's constant, h . Hence, the appropriate dimensionless argument for the log function is

$$\ell n \left[n \left(\frac{2\pi\hbar^2}{mT} \right)^{3/2} \right] = \frac{M}{T} .$$

This logarithm will be large compared to one provided that the quantum states are sparsely occupied, i.e., when the electron plasma may be treated with Boltzmann statistics, rather than Fermi-Dirac statistics.

Equation (5.13) may be used together with Eqs. (2.14) and (3.17) to rewrite the equation of evolution for the entropy density, (2.35), as

$$\frac{\partial S_e}{\partial t} = \frac{1}{T} \frac{\partial w}{\partial t} - \frac{M}{T} \frac{\partial n}{\partial t} . \tag{5.15}$$

We recognize Eq. (5.15) as a fluid version of the Thermodynamic Identity (Landau and Lifshitz, 1958),

$$dS = \frac{dQ}{T} - \frac{M}{T} dN . \tag{5.16}$$

Equation (5.15) is important for two reasons. First, the similarity between Eqs. (5.15) and (5.16) confirms our identification of w as the thermal energy of the electrons. In addition, Eq. (5.15) provides a starting point for thermodynamic discussions of dissipative drift instabilities. Other authors (e.g., Liu, et al., 1976) have employed such thermodynamic arguments in an effort to set upper bounds on the amplitude of various dissipative drift instabilities.

6. EVOLUTION OF THE WAVE AMPLITUDE

In this section we derive an equation describing the time evolution of the electrostatic wave. The wave amplitude has been assumed to be a slowly varying function of x . Hence, we may use the local dispersion relation to determine the evolution of the wave amplitude (Krall and Rosenbluth, 1965; Mikhailovskii, 1967). This dispersion relation is obtained by suppressing the x dependence of Φ_o , n , and T . The wave potential may then be written as

$$\Phi(y,z,t) = \Phi_o(t)\ h(\theta) \tag{6.1}$$

where

$$\Phi_o(t) = \Phi_o(t{=}0)\ \exp \int_0^t \gamma(\tau)\ d\tau \tag{6.2}$$

and we commit ourselves to the waveform

$$h(\theta) = \cos\theta \quad . \tag{6.3}$$

Our immediate goal is to calculate the electron susceptibility, $\chi^{(e)}$, which is defined by

$$\mathrm{Re}\ (\chi^{(e)}\tilde{\phi}) \equiv \frac{4\pi e}{k^2}\,\delta n(\theta) \quad , \tag{6.4}$$

where $\tilde{\phi}$ is the complex potential,

$$\tilde{\phi} = \Phi_o e^{i\theta}$$

$$= \Phi_o \cos\theta + i\Phi_o \sin\theta \ , \tag{6.5}$$

and δn is the perturbation in the electron density due to the wave. When coupling between Fourier modes is ignored, δn may be written as

$$\delta n(\theta) = a\cos\theta + b\sin\theta \tag{6.6}$$

where

$$a = \int \frac{d\theta}{\pi}\, n(\theta)\cos\theta \tag{6.7}$$

$$b = \int \frac{d\theta}{\pi}\, n(\theta)\sin\theta \tag{6.8}$$

and the local (in θ) electron density is given by

$$n(\theta) = \int d^3\underline{v}\ f(\theta,\underline{v}) \ . \tag{6.9}$$

Using Eqs. (6.3) through (6.9) we find

$$\mathrm{Re}\ \chi^{(e)} = \frac{8\pi e}{k^2\Phi_o}\int \frac{d\theta}{2\pi}\int d^3\underline{v}\ h(\theta)\ f(\theta,\underline{v}) \tag{6.10}$$

$$\mathrm{Im}\ \chi^{(e)} = \frac{8\pi e}{k^2\Phi_o}\int \frac{d\theta}{2\pi}\int d^3\underline{v}\ h'(\theta)\ f(\theta,\underline{v}) \ . \tag{6.11}$$

At zero order in $(\Delta x_T/L)$ and $(e\Phi_o/T)$, the real part of the electron susceptibility is easily evaluated, yielding

$$\mathrm{Re}\ \chi^{(e)} = \frac{1}{k^2\lambda_D^2} \tag{6.12}$$

where

$$\lambda_D^2 \equiv \frac{T}{4\pi ne^2} \quad . \tag{6.13}$$

This linear susceptibility describes the adiabatic response of the electrons to the wave. It evolves slowly due to the evolution of the electron tempertature and density. The real part of the electron susceptibility together with the ion susceptibility determines the real part of the wave frequency. In many experiments, the ion temperature is small compared to the electron temperature. In this limit the ion susceptibility is given by (Mikhailovskii, 1974)

$$\chi^{(i)} = \frac{\omega_{pi}^2}{\Omega_i \omega} \frac{k_y}{k^2} \frac{1}{n} \frac{dn}{dx} + \frac{\omega_{pi}}{\Omega_i^2} \frac{k_y^2}{k^2} \tag{6.14}$$

where ω_{pi} is the ion plasma frequency, and Ω_i is the ion gyro frequency.

The real part of the dielectric function is then

$$\varepsilon_r(\underline{k},\omega) = 1 + \frac{1}{k^2\lambda_d^2} + \frac{\omega_{pi}^2}{\Omega_i \omega} \frac{k_y}{k^2} \frac{1}{n} \frac{dn}{dx} + \frac{\omega_{pi}^2}{\Omega_i^2} \frac{k_y^2}{k^2} \quad . \tag{6.15}$$

Solving $\varepsilon_r = 0$ for ω , we find

$$\omega = \frac{\omega_{ne}}{1 + k_y^2 \rho_o^2} \tag{6.16}$$

where

$$\rho_o \equiv (m_i T_e / eB)^{\frac{1}{2}} \, . \tag{6.17}$$

This expression for the real part of the frequency is shared by several drift instabilities, including the dissipative trapped electron instability, the "collisionless" drift instability, and the collisional drift instability (Horton, 1976), as well as the nonlinear dissipative instabilities of Kadomtsev and Pogutse (1970), and Nevins (1977b). These dissipative drift instabilities are different instability mechanisms that effect the same branch of the dispersion relation. We will refer to this branch of the dispersion relation as the "low frequency drift wave". This branch goes over into the ion acoustic wave when $k_z/k_y \gtrsim \rho_o/L$ (Mikhailovskii, 1974). Hence, it has been called the "fast ion acoustic wave" by some authors.

The real part of the low frequency drift wave dispersion relation has been investigated by many authors (e.g., Horton, 1976; Liu, et al., 1976). These authors have considered the effects of finite ion temperature, ion temperature gradients, and the trapping of electrons in local magnetic wells, on the real part of the wave frequency.

We now turn our attention to the imaginary part of the dispersion relation. In evaluating $\mathrm{Im}\,\chi^{(e)}$ it is helpful to express $h'(\theta)$ in terms of v_{dr},

$$h'(\theta) = -\frac{1}{\Phi_o}\frac{B}{k_y} v_{dr} \tag{6.18}$$

where v_{dr}, the x-component of the $E \times B$ drift velocity, is given by Eq. (2.13). $\mathrm{Im}\,\chi^{(e)}$ may then be written as

$$\mathrm{Im}\,\chi^{(e)} = -\frac{8\pi}{k^2\Phi_o^2}\frac{eB}{k_y}\int\frac{d\theta}{2\pi}\int d^3\underline{v}\, v_{dr}\, f \tag{6.19}$$

or, using Eq. (2.12),

$$\mathrm{Im}\,\chi^{(e)} = -\left(\frac{8\pi}{k^2\phi_o^2}\right)\frac{eB}{k_y}\Gamma_e \,. \qquad (6.20)$$

The growth rate of the wave is then given by

$$\gamma(t) = -\frac{\mathrm{Im}\,\chi^{(e)}(t)}{\frac{\partial\varepsilon_r}{\partial\omega}} \,. \qquad (6.21)$$

In a dielectric medium the energy density of a wave is given by

$$\mathcal{W} = \frac{\partial}{\partial\omega}(\omega\varepsilon_r)\left(\frac{k^2\phi_o^2}{16\pi}\right). \qquad (6.22)$$

Hence, we may use Eqs. (6.2), (6.21), and (6.22) to write an equation describing the evolution of the wave energy density,

$$\frac{\partial\mathcal{W}}{\partial t} = \frac{\omega}{k_y}eB\Gamma_e \,. \qquad (6.23)$$

This relation between the time rate of change in the wave energy and the particle flux does not depend on the form of ε_r. Hence, Eq. (6.23) may be applied to determine the time evolution of the wave even when ε_r is not adequately approximated by Eq. (6.15). The wave amplitude may then be determined from the wave energy together with Eq. (6.22) and an appropriate expression for the real part of the dielectric function.

When Eq. (6.15) does adequately approximate the real part of the wave amplitude, the wave amplitude and wave energy are related by

$$\left(\frac{e\Phi_o}{T}\right)^2 = 4\,\frac{\omega}{\omega_{ne}}\left(\frac{\mathcal{W}}{nT}\right). \tag{6.24}$$

We note that Eq. (6.23) may be written as

$$\frac{\partial\mathcal{W}}{\partial t} = \left(\frac{\omega}{\omega_{ne}}\right) T \left(-\frac{1}{n}\frac{\partial n}{\partial x}\,\Gamma_e\right) \quad . \tag{6.25}$$

The low frequency drift wave has positive energy [i.e., $\frac{\partial}{\partial\omega}(\omega\varepsilon_r) > 0$] and propagates in the electron diamagnetic drift direction (i.e., $\omega/\omega_{ne} > 0$) . Hence, a low frequency drift wave which drives the electron flux down the density gradient (giving $-\frac{1}{n}\frac{\partial n}{\partial x}\,\Gamma_e > 0$) will be unstable, while a wave which drives the electron flux up the gradient will be damped. We will show in the next section that this behavior may be understood by considering the conservation of canonical momentum.

7. CONSERVATION LAWS

In this section we examine the equations describing the evolution of the plasma-wave system, and show that the total energy and momentum are conserved.

We first consider the total energy,

$$W = w + \mathcal{W} \quad . \tag{7.1}$$

Taking the time derivative of Eq. (7.1) and using the electron energy equation, (3.17), along with the wave energy equation, (6.23), we find that the energy source terms cancel, leaving

$$\frac{\partial \mathcal{W}}{\partial t} = -\frac{\partial}{\partial x} Q_e \quad . \tag{7.2}$$

Equation (7.2) describes a system in which the total energy is conserved. Hence, the energy source terms in Eqs. (3.17) and (6.23) describe the transfer of energy between the wave and the electron distribution.

Similarly, the total momentum density of the system is given by

$$\underline{P} = \underline{p} + \underline{\mathcal{P}} \tag{7.3}$$

where the wave momentum density is

$$\underline{\mathcal{P}} = \frac{\partial}{\partial \omega}\left[\underline{k}\ \epsilon_r(\omega,k)\right]\left(\frac{k^2\phi_o^2}{16\pi}\right) \quad . \tag{7.4}$$

It follows from Eqs. (6.2), (6.20), (6.21), and (7.4) that

$$\frac{\partial \mathcal{P}_y}{\partial t} = eB\Gamma_e \tag{7.5}$$

and

$$\frac{\partial \mathcal{P}_z}{\partial t} = \frac{k_z}{k_y} \, eB\Gamma_e \tag{7.6}$$

taking the time derivative of Eq. (7.3), and using the electron momentum equations, (2.18) and (2.19), together with the wave momentum equations, (7.5) and (7.6) we find that the source terms again cancel, giving

$$\frac{\partial P_y}{\partial t} = 0 \tag{7.7}$$

$$\frac{\partial P_z}{\partial t} = - \frac{\partial}{\partial x} \, \Pi_{xz} \ . \tag{7.8}$$

It is clear from Eqs. (7.7) and (7.8) that the total canonical momentum of the plasma-wave system is conserved. Hence, the source term in the electron momentum equations, (2.18) and (2.19), and the wave momentum equations. (7.5) and (7.6), describe the transfer of momentum between the wave and the electron distribution.

In Sect. 2 we noted that the y-component of an electron's canonical momentum is proportional to its x guiding center position. Hence, the transport of particles across the magnetic field implies a change in the canonical momentum of the species being transported. As the total canonical momentum of the system must be conserved, the particle flux is determined by the rate at which momentum is transferred from one element of the system to another.

It is evident from Eqs. (2.18), (7.5), and (7.7) that the momentum is being transferred between the electron distribution and the low frequency drift wave. This momentum transfer occurs because the guiding center motion of the electrons in the electric field of the wave combine with Coulomb collisions to produce an electron flux, i.e., a change in the electron momentum. We consider this process in more detail in Sect. 8, and show that the concept of momentum conservation may be employed together with random walk estimates of the electron flux in approximate calculations that yield the correct growth rates for the various dissipative instabilities of the low frequency drift wave.

8. A PHYSICAL INTERPRETATION OF DISSIPATIVE DRIFT INSTABILITIES

Conservation laws are useful in gaining a qualitative understanding of instability mechanisms. An example of this is the use of energy conservation in analyzing the Landau resonance. It has been shown that the wave damping or growth associated with a Landau resonance may be understood as resulting from the transfer of energy between the wave and the resonant particles (Jackson, 1960). We find that a simple and coherent picture emerges when momentum conservation is applied to the analysis of dissipative drift instabilities. Specifically, we consider the y-component of the canonical momentum. We have shown above [c.f., Eqs. (2.18) and (7.5)] that this momentum is transfered between the electron distribution and the wave at a rate

$$\dot{p}_y = eB\Gamma_e \ .$$

In this section we adopt the view that dissipative drift instabilities result from this momentum transfer. Our approach is then to use a random walk model to estimate the particle diffusion coefficient, D , and hence, the particle flux,

$$\Gamma_e \simeq nA_1 D \ . \tag{8.1}$$

In the absence of a temperature gradient, the particle flux may be written as

$$\Gamma_e = - D\left(1 - \frac{\omega}{\omega_{ne}}\right) \frac{\partial n}{\partial x} \ . \tag{8.2}$$

The factor $(1-\omega/\omega_{ne})$ enters Eq. (8.2) as part of the thermodynamic force (c.f., Sect. 5).

Combining Eq. (8.2) with Eqs. (7.4) and (7.5), we may estimate the wave growth that results from this transport by

$$\gamma \simeq -\frac{1}{2} \frac{eBD}{k_y(\partial\varepsilon_r/\partial\omega)(k^2\phi_o^2/16\pi)} (1 - \frac{\omega}{\omega_{ne}}) \frac{\partial n}{\partial x} . \tag{8.3}$$

The real part of the dielectric function is often adequately approximated by Eq. (6.15). Using this approximation we obtain

$$\gamma \simeq 2\left(\frac{k_y\Phi_o}{\omega B}\right)^{-2} D \; (1 - \frac{\omega}{\omega_{ne}}) . \tag{8.4}$$

A good test of any qualitative model is the ability to employ it in making direct, accurate estimates of physically interesting quantities. We show below that Eq. (8.4) together with estimates of D for various transport processes in which the "step" is provided by the coherent motion of particles in the electric field of the wave and the correlation time is provided by Coulomb collisions, yields accurate estimates of the growth rates of many dissipative drift instabilities.

For the model system considered in this paper, we consider the conservation of the y-component of the canonical momentum because this component of a particle momentum labels the particle magnetic flux surface. However, we wish to emphasize that similar results may be obtained for axisymmetric systems with a non-zero poloidal field (e.g., tokamaks and stellerators) by invoking the conservation of the toroidal component of the canonical momentum, and using the small parameter ρ_θ/L, where ρ_θ is the "poloidal" gyro radius. The toroidal momentum may then be taken as labeling the particle flux surface, and dissipative drift instabilities may be viewed as a result of the transfer of toroidal momentum between the electron distribution and a low frequency drift wave.

We also note that the y-component of the electron momentum equation was a consequence of the continuity equation,

$$\frac{\partial n}{\partial t} = - \frac{\partial}{\partial x} \Gamma_e \ .$$

Clearly, such an equation may be derived when the magnetic field is non-uniform. Similarly, the derivation of the wave momentum equation, (7.5), did not require a uniform magnetic field. Hence, the y-component of the momentum balance equation may be employed in the analysis of the dissipative trapped electron instability (for which the magnetic field inhomogeneities are important).

8.1 THE DISSIPATIVE TRAPPED ELECTRON INSTABILITY

With each dissipative drift instability mechanism there is an associated transport process. The transport process associated with the dissipative trapped electron instability involves the $E \times B$ drift of magnetically trapped electrons in the field of the wave. We consider the limit

$$\omega_{eff} < \omega < \omega_b \tag{8.5}$$

where ω_b is the bounce frequency of the magnetically trapped electrons,

$$\nu_{eff} = \frac{\nu_e}{\delta} \tag{8.6}$$

is the effective collision frequency for scattering electrons out of the local magnetic wells, and δ is a measure of the magnetic field inhomogeneity. In tokamaks, δ is given by the inverse aspect ratio.

The magnetically trapped particles oscillate coherently across the magnetic field due to the $E \times B$ drift in the wave electric field. A characteristic width of this oscillation is

$$\Delta x_{DTE} = \frac{k_y \Phi_o}{\omega B} . \tag{8.7}$$

The passing particles see the wave at the Doppler shifted frequency, and consequently have orbits with a width of order

$$\Delta x_p = \frac{k_y \Phi_o}{k_z v_{te} B} = \left(\frac{v_\phi}{v_{te}}\right) \Delta x_{DTE} . \tag{8.8}$$

Figure 2 illustrates the result of a collision between a trapped electron and another particle in which the electron is scattered into the loss cone of the local magnetic well. This collision results in a displacement of the mean particle position (i.e., the oscillation center position) by an amount Δx. We note that the coherent oscillations of the particles about their mean position are properly included as part of the wave momentum (Dewar, 1973). Hence, the collision has also resulted in a net transfer of momentum between the electron and the wave in the

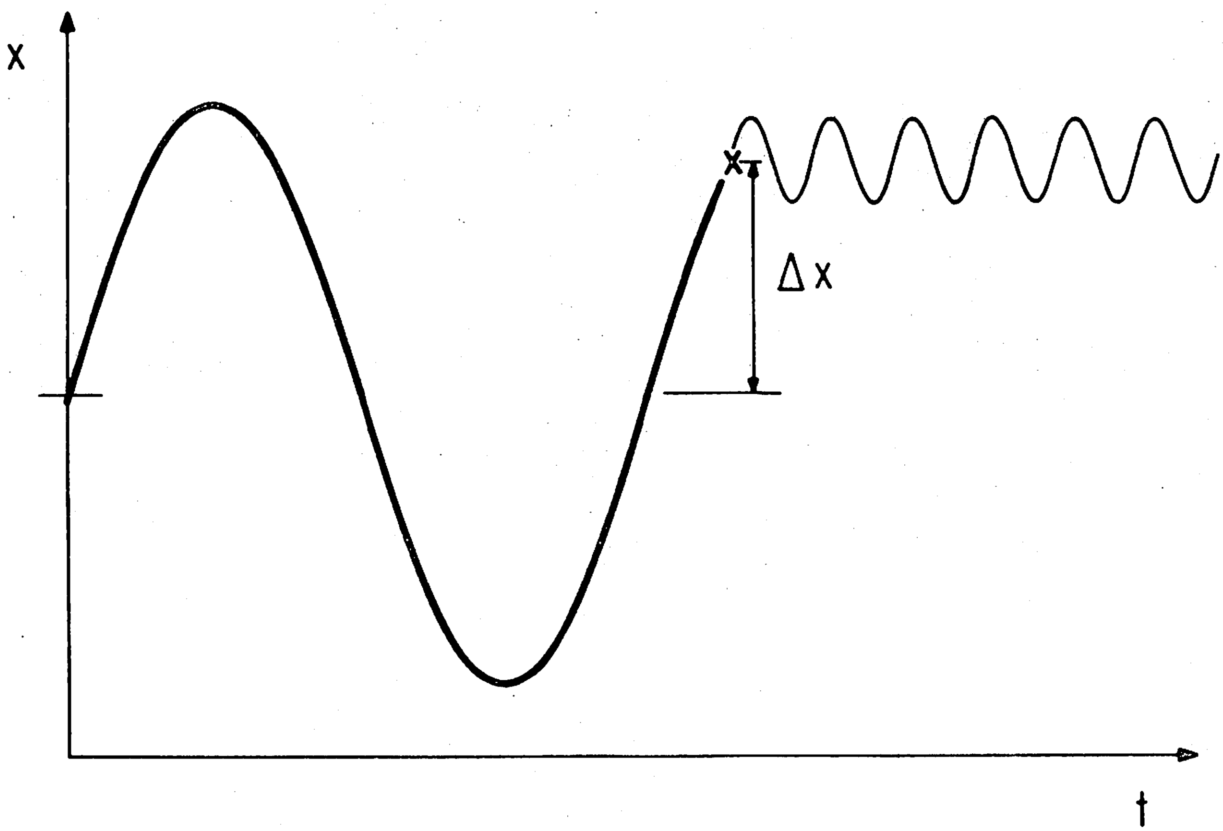

FIGURE 2. An illustration of the transport process responsible for the dissipative trapped electron instability. At X the trapped electron suffers a collision and scatters into a passing orbit. This results in a "step", Δx , and a momentum transfer between the electron distribution and the wave in the amount $\Delta p_y = eB\Delta x$.

amount $\Delta p_y = eB\Delta x$. Note that it does not matter whether the target particle is an electron or an ion — in either case the transfer of y-momentum occurs between the electron scattering from trapped to passing and the wave. Hence, both electron-ion and electron-electron collisions contribute to the anomalous particle flux associated with the dissipative instabilities of the low frequency drift wave.

Viewing the cross field motion as a random walk, the diffusion coefficient may be estimated as

$$D_{DTE} \simeq f \frac{\Delta x_{DTE}^2}{\tau} \tag{8.9}$$

where f , the fraction of electrons participating in this process at any given time, may be estimated as

$$f \simeq \delta^{\frac{1}{2}} \tag{8.1}$$

and the correlation time, τ , is the reciprocal of ν_{eff} . Hence,

$$D_{DTE} \simeq \delta^{\frac{1}{2}} \left(\frac{k_y \Phi_o}{\omega B} \right)^2 \nu_{eff} \; . \tag{8.11}$$

Note that (8.11) differs from the "quasilinear" diffusion coefficient associated with the dissipative trapped electron instability derived by other authors (e.g., Horton, 1976; Liu, et al., 1976; Manheimer, 1977) only by a factor of order unity.

Combining Eqs. (8.4) and (8.11) we find

$$\gamma_{DTE} \simeq 2 \; \delta^{-\frac{1}{2}} \; (1 - \frac{\omega}{\omega_{ne}}) \; \nu_e \; . \tag{8.12}$$

Liu, et al. (1976), have calculated the growth rate of the dissipative trapped electron instability in the absence of a temperature gradient. In the limit $\nu_{eff} < \omega < \omega_b$ and $T_i \ll T_e$ their result reduces to

$$\gamma_{DTE} = \delta^{-\frac{1}{2}} \left(1 - \frac{\omega}{\omega_{ne}}\right) \nu_e \left\{ \frac{\ell n \, (\omega\delta/\nu_e)}{1 - (2\delta)^{\frac{1}{2}}} \right\} \quad . \tag{8.13}$$

Comparing Eqs. (8.12) and (8.13) we see that our qualitative argument has yielded a very good estimate of the growth rate of the dissipative trapped electron instability. A similar argument yields the correct growth rate in the opposite limit, $\omega_b > \omega > \nu_{eff}$.

8.2 PSEUDOCLASSICAL TRANSPORT AND DISSIPATIVE DRIFT INSTABILITIES

In a uniform magnetic field the guiding center drifts of electrons in the electric field of a low frequency drift wave gives rise to a sequence of related transport mechanisms. We use the name "pseudoclassical transport" to describe these transport mechanisms, both for historical reasons, and because of the close analogy between pseudoclassical transport and neoclassical transport. We consider these pseudoclassical transport mechanisms elsewhere (Nevins, 1977a; Nevins, et al., 1977) and find that pseudoclassical transport has three regimes, determined by the parameter $\nu_* = \nu_e/k_z v_{te}$.

In the strongly collisional limit, $\nu_* > 1$, a random walk model yields (Nevins, et al., 1977)

$$D_{CDI} \simeq \left(\frac{\nu_e}{k_z^2 \, v_{te}^2}\right)\left(\frac{k_y \Phi_o}{B}\right)^2 \quad . \tag{8.14}$$

Combining this transport coefficient with Eq. (8.4), we find that the growth rate of the associated instability is

$$\gamma_{CDI} \simeq (1 - \frac{\omega}{\omega_{ne}})\left(\frac{\omega^2}{k_z^2\ v_{te}^2}\right)\nu_e \quad . \tag{8.15}$$

This growth rate should be compared to the growth rate of the collisional drift instability (Hendel, et al., 1970),

$$\gamma_{CDI} = (1 - \frac{\omega}{\omega_{ne}})\left(\frac{\omega^2}{k_z^2 v_{te}^2}\right)\nu_e \ . \tag{8.16}$$

At lower collision frequencies, such that $(e\Phi_o/T)^{3/2} < \nu_* < 1$, a random walk model, in which the correlation time is determined by Coulomb collisions, yields (Nevins, et al., 1977)

$$D_{CLDI} \simeq \left(\frac{1}{k_z v_{te}}\right)\left(\frac{k_y \Phi_o}{B}\right)^2 \quad . \tag{8.17}$$

We note that this diffusion coefficient differs from the "quasilinear" diffusion coefficient associated with the "collisionless" drift mode (Sagdeev and Galeev, 1969; Horton, 1976) only by the factor $(\pi/8)^{\frac{1}{2}}$. Using Eq. (8.4) together with Eq. (8.17) we find that the growth rate of the associated dissipative drift instability is

$$\gamma_{CLDI} \simeq \left(1 - \frac{\omega}{\omega_{ne}}\right)\left(\frac{\omega^2}{k_z v_{te}}\right). \tag{8.18}$$

This result should be compared with the growth rate of the "collisionless" drift mode (Sagdeev and Galeev, 1969),

$$\gamma_{CLDI} = \left(\frac{\pi}{8}\right)^{\frac{1}{2}} \left(\frac{\omega^2}{k_z v_{te}}\right)(1 - \frac{\omega}{\omega_{ne}}) \ . \tag{8.19}$$

Eq. (8.19) differs from Eq. (8.18) by this same factor, $\left(\frac{\pi}{8}\right)^{\frac{1}{2}}$. Hence, the so-called "collisionless" drift instability may be understood as a dissipative instability! The role of the collisions is to prevent particle trapping and maintain the Maxwellian character of the distribution function in the resonant region.

Finally, we consider the low collision frequency limit, $\nu_* < (e\Phi_o/T)^{3/2}$. This inequality may be viewed as a condition on the wave amplitude,

$$(e\Phi_o/T) > (\nu_e/k_z v_{te})^{2/3} \ . \tag{8.20}$$

In this limit the random walk model yields (Nevins, 1977a).

$$D_{NLDI} \simeq \left(\frac{1}{k_z v_{te}}\right)\left(\frac{k_y \Phi_o}{B}\right)^2 \left(\frac{\nu_e}{k_z v_{te}}\right) \left(\frac{e\Phi_o}{T}\right)^{-3/2} \ . \tag{8.21}$$

Hence, the growth rate associated with this transport mechanism is

$$\gamma_{NLDI} \simeq \left(\frac{1}{k_z v_{te}}\right)(1 - \frac{\omega}{\omega_{ne}}) \left(\frac{\nu_e}{k_z v_{te}}\right) \left(\frac{e\Phi_o}{T}\right)^{-3/2} \ . \tag{8.22}$$

We have investigated this nonlinear dissipative instability in detail elsewhere (Nevins, 1977b), and found that it is associated with the nonlinear development of a finite amplitude traveling wave that has trapped the resonant electrons.

8.3 THE NONLINEAR DISSIPATIVE INSTABILITY OF KADOMTSEV AND POGUTSE

It is evident from Eq. (8.22) that the qualitative picture outlined here may be used in the analysis of nonlinear as well as linear drift wave instabilities. In addition to the nonlinear dissipative instability of a traveling wave discussed above, Kadomtsev and Pogutse (1970) have found a nonlinear dissipative instability of a finite amplitude standing wave. This instability has been studied more recently by Ott and Manheimer (1976). We show here that this nonlinear dissipative instability may be understood by the method of momentum conservation outlined above.

Kadomtsev and Pogutse assume a wave of the form

$$\Phi = \Phi_o(t) \cos(k_y y - \omega t) \cos k_z z \tag{8.23}$$

and consider the limit $\omega \ll \omega_{BOUNCE}$, where $\omega_{BOUNCE} = k_z v_{te} (e\Phi_o/T)^{\frac{1}{2}}$ is the bounce frequency of a particle trapped by the wave.

In the traveling wave considered in Sect. 8.2, E_y and E_z were in phase. Hence, in traveling waves the electric fields felt by a trapped particle oscillate together at the bounce frequency. In contrast, the standing wave considered by Kadomtsev and Pogutse has E_y and E_z 90°

out of phase. The trapped particles, which oscillate about a zero of E_z remain near an extrema of E_y. Hence, the characteristic frequency of the cross-field motion is not ω_{BOUNCE}, but rather the (much smaller) wave frequency, ω. The width in x of these trapped particle orbits may be estimated as

$$\Delta x_{K-P} \simeq \frac{k_y \Phi_o}{\omega B} \ . \tag{8.24}$$

The effective collision frequency for scattering particles out of the local potential wells is

$$\nu_{eff} = \nu_e / \left(\frac{e\Phi_o}{T}\right) \tag{8.25}$$

while the fraction of electrons trapped by the wave may be estimated as

$$f \simeq \left(\frac{e\Phi_o}{T}\right)^{\frac{1}{2}} \ . \tag{8.26}$$

Hence, the diffusion coefficient may be estimated as

$$D_{K-P} \simeq \left(\frac{e\Phi_o}{T}\right)^{-\frac{1}{2}} \left(\frac{k_y \Phi_o}{\omega B}\right)^2 \nu_e \ . \tag{8.27}$$

It then follows from Eqs. (8.27) and (8.4) that

$$\gamma_{K-P} \simeq \left(\frac{e\Phi_o}{T}\right)^{-\frac{1}{2}} \left(1 - \frac{\omega}{\omega_{ne}}\right) \nu_e \ . \tag{8.28}$$

We compare this estimate with the growth rate for this nonlinear instability calculated by Ott and Manheimer (1976),

$$\gamma_{K-P} = 1.92 \left(\frac{e\Phi_o}{T}\right)^{-\frac{1}{2}} \left(1 - \frac{\omega}{\omega_{ne}}\right) \nu_e \; . \tag{8.29}$$

Again, we find good agreement between the estimate from our qualitative model and a kinetic calculation of the growth rate.

We conclude this section by pointing out that the method of analyzing dissipative drift instabilities presented in this section is approximate only through the random walk estimates of the electron flux. If this electron flux is obtained from kinetic theory, then Eqs. (6.20) and (6.21) (which are exact relations) may be employed to find the growth (or damping) rate of the wave driving the electron flux. In general, the calculation of the electron flux from kinetic theory is quite similar to the calculation of the imaginary part of the electron susceptibility, as both quantities involve the same moment of the electron distribution function. Hence, this method does not result in any mathematical simplification.

On the other hand, many powerful techniques have been developed for calculating transport coefficients using realistic collision operators. These techniques are at our disposal when calculating the electron flux driven by the wave. In another paper (Nevins, 1977a) we treat the nonlinear dissipative instability described in Section 8.2. Using a variational method due to Rosenbluth, et al. (1973), we are able to calculate the electron flux, and hence, the nonlinear growth rate using the full Fokker-Planck collision operator. This accurate treatment of the collisions is in contrast to other work on dissipative drift instabilities, in which Krook models are often employed (e.g., Kadomtsev and Pogutse, 1969,1970,1971; Ott and Manheimer, 1976; Liu, et al., 1976; Chen, et al., 1977), or more recent work in which a Lorentz model is used to accurately describe electron-ion collisions, while electron-electron collisions are evidently ignored (Hinton and Ross, 1976; Horton, 1976).

9. THE ONSAGER RELATIONS AND THE SECOND LAW OF THERMODYNAMICS

Previous workers (e.g., Horton, 1976; Manheimer, 1977) who have examined the anomalous transport associated with low frequency drift wave instabilities have been unable to demonstrate that the anomalous transport coefficients which they have derived satisfy the Onsager relations (Onsager, 1931) or that the resulting plasma transport results in a net increase in the entropy of the plasma. This is troubling, as the Onsager relations are a consequence of microscopic reversibility (which is certainly satisfied), while the Second Law of Thermodynamics follows from Boltzmann's H-theorem, which may be proven for the system in question. Hence, if the anomalous transport coefficients do not satisfy the Onsager relations and the Second Law of Thermodynamics, then these transport coefficients are open to question. In this section we show that the anomalous transport coefficients derived by Horton (1976) do indeed satisfy the Onsager relations, and that the resulting fluxes do produce an increase in the entropy of the plasma.

We begin by considering the transport due to a single wave. Horton writes the particle and energy fluxes as sums over the wave spectrum. We identify each term in these sums as the flux due to an individual wave. We are guided in this identification by the fact that Horton's derivation of the particle and energy flux does not require the presence of many waves, and by the results of the previous section, where we showed that the transport due to the various dissipative instability mechanisms considered by Horton may be understood as the result of a random walk in which the "step" is due to the coherent motion in the electric field of a single wave, combined with a coherence time that is determined by Coulomb collisions.

Horton choses to describe the transport in terms of the particle flux, Γ_e , and a "thermal" flux, K_e , given by

$$K_e = Q_e - \frac{3}{2}\Gamma_e \ . \tag{9.1}$$

We note that Horton's thermal flux is distinct from both our energy flux, Q_e , and the conventional heat flux (Braginskii, 1965), $\tilde{Q}_e = Q_e - \frac{5}{2} T \Gamma_e$.

Rewriting the entropy source, Eq. (5.6), using Horton's fluxes, we find

$$\dot{S}_e = (A_1 + \frac{3}{2} T A_2)\, \Gamma_e + A_2 K_e \ . \tag{9.2}$$

Hence, the thermodynamic forces, A_1' and A_2' , conjugate to these fluxes are

$$A_1' = A_1 + \frac{3}{2}\, T\, A_2 = -\frac{\Delta}{n}\frac{\partial n}{\partial x} \tag{9.3}$$

$$A_2' = A_2 \qquad\qquad = -\frac{1}{T^2}\frac{\partial T}{\partial x} \ . \tag{9.4}$$

Using Eqs. (9.3) and (9.4) together with the transport coefficients derived by Horton, the particle flux and thermal flux due to a single wave may be written as

$$\Gamma_e = \alpha\left(\sum_{i=1}^{3} \text{Im } G_i^0\right) A_1' + \alpha\left(\sum_{i=1}^{3} \text{Im } G_i^1\right) A_2' \tag{9.5}$$

$$K_e = \alpha\left(\sum_{i=1}^{3} \text{Im } G_i^1\right) A_1' + \alpha\left(\sum_{i=1}^{3} \text{Im } G_i^2\right) A_2' \tag{9.6}$$

where

$$\alpha = \tfrac{1}{2}\, n\, |\omega_{ne}| \left(\frac{e\Phi_o}{T}\right)^2 \tag{9.7}$$

and the functions G_m^n are given in Horton's paper. We note that these functions satisfy

$$G_m^0\ ,\ G_m^2 \geq 0 \qquad\qquad m = 1,2,3 \tag{9.8}$$

as well as

$$\left(\sum_{i=1}^{3} \text{Im } G_m^0\right)\left(\sum_{i=1}^{3} \text{Im } G_m^2\right) \geq \left(\sum_{i=1}^{3} \text{Im } G_m^1\right)^2 . \tag{9.9}$$

It is apparent from Eqs. (9.5) and (9.6) that the Onsager relations are indeed satisfied.

Using Eqs. (9.8) and (9.9) the entropy source may be written as

$$\dot{S}_e = \alpha\left[A_1'^2\left(\sum_{i=1}^{3} \text{Im } G_m^0\right) + 2A_1'A_2'\left(\sum_{i=1}^{3} \text{Im } G_m^1\right) + A_2'^2\left(\sum_{i=1}^{3} \text{Im } G_m^2\right)\right].$$

Multiplying and dividing by $(\sum \text{Im } G_m^0)$, and using Eq. (9.9) we find

$$\dot{S}_e = \frac{\alpha}{\sum_{i=1}^{3} \text{Im}\, G_m^0} \left[A_1' \left(\sum_{i=1}^{3} \text{Im}\, G_m^0 \right) + A_2' \left(\sum_{i=1}^{3} \text{Im}\, G_m^1 \right) \right]^2 . \qquad (9.11)$$

Each term on the right hand side of (9.11) is non-negative. Hence, the electron entropy source is non-decreasing.

The entropy source may also be written in terms of the plasma gradients,

$$\frac{1}{n}\frac{\partial n}{\partial x} \quad \text{and} \quad \frac{1}{T}\frac{\partial T}{\partial x} .$$

We find it instructive to do so:

$$\dot{S}_e = -\frac{1}{n}\frac{\partial n}{\partial x}\Gamma_e - \frac{1}{T^2}\frac{\partial T}{\partial x} K_e + \frac{1}{T}\left(-\frac{\omega}{k_y} eB\Gamma_e\right) . \qquad (9.12)$$

We recognize $\left(-\frac{\omega}{k_y} eB\Gamma_e\right)$ as the source term in the equation for the electron thermal energy (3.17). Hence, we may interpret the third term on the right hand side of Eq. (9.12) as the rate of entropy increase due to heating (or cooling) of the electron distribution by the wave. The neglect of this term by previous workers has prevented them from demonstrating that the anomalous transport associated with various dissipative drift instabilities produces a net increase in the plasma entropy.

When many waves are present, it is convenient to write the particle flux and the thermal flux as products between the gradients of n and T, and transport coefficients that involve sums over the wave spectrum. In such a formulation one cannot expect these transport coefficients to satisfy Onsager's relations because $-\frac{1}{n}\frac{\partial n}{\partial x}$ and $-\frac{1}{T^2}\frac{\partial T}{\partial x}$ are not the thermodynamic forces conjugate to the particle and thermal flux.

If interactions between waves are ignored, the total particle flux, total thermal flux, and total entropy source may be obtained by simply summing the contributions from each wave in the spectrum . The total entropy source is then non-negative because it is a sum of non-negative terms. We note that interactions between waves (e.g., induced scattering) may affect the particle flux, thermal flux, and anomalous heating rate. Hence, when wave-wave interactions are invoked to obtain the saturated wave spectrum, the effect of these wave-wave interactions on the anomalous transport should also be considered.

10. CONCLUSION

We presented a framework for the analysis of the various dissipative instability mechanisms of the low frequency drift waves. This framework differs from previous approaches to these instabilities in that it focuses directly on the transport of particles and energy associated with these instability mechanisms. The transfer of momentum and energy which accompanies this transport has been analyzed, and we have found that momentum and energy conservation is achieved by balancing the change in the momentum (or energy) of the electron distribution with a corresponding change in the momentum (or energy) of the low frequency drift wave. One consequence of these conservation laws is a simple relation between the electron flux and the growth rate of the low frequency drift wave, given by Eqs. (6.20) and (6.21). This relation may be employed in approximate calculations of the growth rates of the dissipative drift instabilities.

In addition, we have considered the thermodynamic properties of these dissipative drift instabilities. The thermodynamic forces conjugate to the particle and energy flux have been identified. We have shown that the anomalous transport coefficients derived by Horton (1976) obey Onsager's relations, and that the resulting transport must provide a net increase in the plasma entropy.

In another paper (Nevins, 1977b) we employ this framework in an investigation of the nonlinear behavior of the "collisionless" drift instability.

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Appendix - CONTRIBUTION OF THE ELECTRONS TO THE WAVE ENERGY

The energy density of an electrostatic wave may be written as

$$\mathcal{W} = \frac{\partial}{\partial\omega}\left(\omega\varepsilon(\omega,k)\right)\left(\frac{k^2\Phi_o^2}{16\pi}\right). \tag{A.1}$$

Decomposing the dielectric function into a sum over the susceptibilities of each species, Eq. (A.1) becomes

$$\mathcal{W} = \left(\frac{k^2\Phi_o^2}{16\pi}\right) + \sum_s \frac{\partial}{\partial\omega}\left[\omega\chi^{(s)}(\omega,k)\right]\left(\frac{k^2\Phi_o^2}{16\pi}\right). \tag{A.2}$$

We assume that

$$k \gg \frac{1}{\Phi_o}\frac{\partial\Phi_o}{\partial x} = \mathcal{O}\left(\frac{1}{L}\right) .$$

Then the first term on the right hand side of Eq. (A.2) may be recognized as the potential energy,

$$\frac{k^2\Phi_o^2}{16\pi} = \tfrac{1}{2}\int \frac{d\theta}{2\pi}\,\rho\,\Phi . \tag{A.3}$$

Hence, we are led to interpret the remaining terms as the kinetic energy associated with the coherent motion of particles in the electric field of the wave.

The electrons contribute to the wave energy through both the potential energy term, (A.3), and the kinetic energy term,

$$\frac{\partial}{\partial\omega}\left[\omega\chi^{(e)}(\omega,k)\right]\left(\frac{k^2\phi_o^2}{16\pi}\right)=\frac{1}{k^2\lambda_D^2}\left(\frac{k^2\phi_o^2}{16\pi}\right)+\mathcal{O}\left(\frac{\Delta x_T}{L}\right) \tag{A.4}$$

where we have used the fact that

$$\left(\frac{v_\phi}{v_{te}}\right)^2 = \mathcal{O}\left(\frac{\Delta x_T}{L}\right)$$

in evaluating $\chi^{(e)}$.

The electron charge density is given by

$$\rho^{(e)} = -e\, n\, e^{e\Phi/T} . \tag{A.5}$$

Hence, the contribution of the electrons to the potential energy density is

$$-e\, n\int \frac{d\theta}{2\pi}\,\Phi\, e^{e\Phi/T} = -\frac{2}{k^2\lambda_D^2}\left(\frac{k^2\phi_o^2}{16\pi}\right) + \mathcal{O}\left[\left(\frac{\Delta x_T}{L}\right),\left(\frac{e\Phi_o}{T}\right)^3\right] \tag{A.6}$$

combining Eqs. (A.4) and (A.6), the contribution of the electrons to the wave energy, $\mathcal{W}^{(e)}$, may be written as

$$\mathcal{W}^{(e)} = -\frac{1}{k^2\lambda_D^2}\left(\frac{k^2\phi_o^2}{16\pi}\right) + \mathcal{O}\left(\frac{\Delta x_T}{L}\right) \tag{A.7}$$

$$= -\tfrac{1}{4}\, nT\left(\frac{e\Phi_o}{T}\right)^2 + \mathcal{O}\left(\frac{\Delta x_T}{L}\right) . \tag{A.8}$$

Our purpose in this paper is to show that dissipative drift instabilities may be understood as the result of a transfer of energy (and momentum) between the plasma and the wave. Hence, we must assume that $\partial \mathcal{W}^{(e)}/\partial t$ and $\frac{\partial}{\partial t}(nT)$ are of the same order. Taking the time derivative of (A.8) and recalling that

$$\frac{\partial}{\partial t} = \mathcal{O}\left[\left(\frac{\Delta x_T}{L}\right)^2 \nu_e\right] ,$$

we find

$$\frac{\partial \mathcal{W}^{(e)}}{\partial t} = -\tfrac{1}{4}\, nT \frac{\partial}{\partial t}\left(\frac{e\Phi_o}{T}\right)^2 - \tfrac{1}{4}\left(\frac{e\Phi_o}{T}\right)^2 \frac{\partial}{\partial t}(nT) + \mathcal{O}\left(\frac{\Delta x}{L}\right)^3 . \tag{A.9}$$

Keeping only the leading terms in $(\Delta x_T/L)$ and $(e\Phi_o/T)^2$ yields

$$\frac{\partial \mathcal{W}^{(e)}}{\partial t} = -\tfrac{1}{4}\, nT \frac{\partial}{\partial t}\left(\frac{e\Phi_o}{T}\right)^2 \tag{A.10}$$

$$= -\frac{1}{k^2\lambda_D^2} \frac{\partial}{\partial t}\left(\frac{k^2\Phi_o^2}{16\pi}\right). \tag{A.11}$$

This expression, (A.11), represents the rate of change in the energy (both kinetic and potential) associated with the coherent motion of the electrons in the field of the wave. Hence, this term must be subtracted from the rate of change in the total electron energy density, $\partial \overline{w}/\partial t$, to obtain the time rate of change in the electron thermal energy, $\partial w/\partial t$.

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