THE ROBUST NONLINEAR SERVOMECHANISM PROBLEM

by

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Memorandum No. UCB/ERL M78/11

8 March 1978

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March 12, 1978

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ABSTRACT

We study the asymptotic tracking and disturbance rejection property of a general nonlinear multi-input multi-output distributed servomechanism which consists of input as well as output channel nonlinearity. We also explore the robustness of this property of such nonlinear servomechanism. Our result shows that the design principle of the robust linear servomechanism (i.e. replicating the dynamics of the reference and disturbance signals) works well for a large class of nonlinear servos provided that certain stability conditions are satisfied.

Research sponsored by the National Science Foundation Grant ENG76-84522 and the Joint Services Electronics Program Contract F44620-76-C-0100.

I. INTRODUCTION

One of the most important applications of feedback is to achieve servo-action, that is, to obtain a system that tracks a given class of input signals and rejects a given class of external disturbances with zero asymptotic error. This problem has been well understood for many years for the single-input, single-output, linear, time-invariant, lumped case (see e.g. Brown and Campbell 1948, James et al., 1965). In the last decade, this understanding has been extended to the multi-input multi-output case (see e.g. Bengtsson 1977, Davison 1976, Desoer and Wang 1977, Ferreira 1976, Francis 1975, Johnson 1976, Staats and Person 1977, Wolfe and Meditch 1977). Furthermore, the robustness of these properties with respect to linear perturbations of the plant has been settled. A self-contained exposition of these facts, including some distributed results, is to be found in Desoer and Wang (1977). For an abstract point of view, see Wonham (1976). Of course, any realistic model of a physical system can be linear only as a result of some approximations: realistic modeling of pneumatic valves, relays, servomotors, etc. require nonlinear models. So it is important to investigate the asymptotic tracking and the disturbance rejection properties of nonlinear servos, as well as the robustness of these properties under not necessarily linear perturbations of the linear and nonlinear subsystems. Note that the perturbations may originate from physical perturbation due to say, varying loads, changing operating conditions, ageing, etc. or they may be a conceptual tool to deal with the uncertainty of the plant due to, for example, manufacturing variations.

In this paper, we study the asymptotic tracking and disturbance rejection property of a general <u>nonlinear multi-input multi-output distributed</u> servomechanism which may consist of input channel nonlinearity (e.g. sensor nonlinearity, relay, pneumatic valve, etc.) as well as output channel nonlinearity (e.g. servomotor, etc.). We also explore the robustness of the asymptotic tracking and disturbance rejection of such nonlinear servomechanism. Our nonlinear results are extensions of the robust linear servomechanism theory: roughly speaking, our main result shows that the design principle of the robust linear servomechanism (i.e. replicating the dynamics of the reference and disturbance signals) works well for a large class of <u>nonlinear</u> servos provided that certain stability conditions are satisfied.

The organization of this paper is as follows. In section II, some notation and definitions are introduced, then the problem is precisely stated. In section III, the special case where the output channel nonlinearity is absent is treated. In section IV, the general nonlinear servomechanism problem for a class of reference and disturbance signals is solved by first considering the special case where the nonlinear servomechanism is free of input channel nonlinerity. The realism of some assumptions used in section IV is then discussed in section V. Finally, a simple example is given in section VI.

II. PROBLEM FORMULATION

A. NOTATION AND PRELIMINARIES

Let $\mathbb{R}$ ($\mathbb{C}$) denote the field of real (complex, respectively) numbers. Let $\mathbb{R}_+$ denote the positive real line $[0,\infty)$. Let θ_n denote the zero vector of $\mathbb{R}^n$. Let $\overset{\circ}{\mathbb{C}}_-$ ($\overset{\circ}{\mathbb{C}}_+$) denote the open left (right, respectively) half complex plane. Let $\sigma(A)$ denote the spectrum of $A \in \mathbb{R}^{nxn}$. Let

$\mathbb{R}[s]$ ($\mathbb{R}(s)$) be the set of all polynomials (rational functions, respectively) in s with real coefficients. Let $\mathbb{R}[s]^{pxq}$ ($\mathbb{R}(s)^{pxq}$) denote the set of all pxq matrices with elements in $\mathbb{R}[s]$ ($\mathbb{R}(s)$, respectively). Let $\partial p(s)$ denote the degree of $p(s) \in \mathbb{R}[s]$. Let $\mathcal{Z}[p(s)]$ denote the set of zeros of $p(s) \in \mathbb{R}[s]$. Let $g(s) \in \mathbb{R}(s)$, g(s) is said to be <u>exponentially stable</u> iff 1) g(s) is proper (i.e. g is bounded at ∞); 2) {poles of g(s)} $\subset \mathring{\mathbb{C}}_-$. Let $G(s) \in \mathbb{R}(s)^{pxq}$, G(s) is said to be <u>exponentially stable</u> iff every element of G(s) is exponentially stable. Let $N_\ell(s) \in \mathbb{R}[s]^{pxq}$, $D_\ell(s) \in \mathbb{R}[s]^{pxp}$; $M(s) \in \mathbb{R}[s]^{pxp}$ is said to be a <u>common left divisor</u> of $N_\ell(s)$ and $D_\ell(s)$ iff there exist $N_1(s) \in \mathbb{R}[s]^{pxq}$, $D_1(s) \in \mathbb{R}[s]^{pxp}$ such that $N_\ell(s) = M(s)N_1(s)$, and $D_\ell(s) = M(s)D_1(s)$; both N_ℓ and D_ℓ are said to be <u>right multiples</u> of M; $L(s) \in \mathbb{R}[s]^{pxp}$ is said to be a <u>greatest common left divisor</u> of N_ℓ and D_ℓ iff 1) it is a common left divisor of N_ℓ and D_ℓ, and 2) it is a right multiple of every common left divisor of N_ℓ and D_ℓ. When a greatest common left divisor L is unimodular (i.e. det L(s) = constant $\neq 0$), then N_ℓ and D_ℓ are said to be <u>left coprime</u>. $D_\ell^{-1}N_\ell$ is said to be a <u>left coprime factorization</u> of $G(s) \in \mathbb{R}(s)^{pxq}$ iff $D_\ell(s) \in \mathbb{R}[s]^{pxp}$, $N_\ell(s) \in \mathbb{R}[s]^{pxq}$ with D_ℓ, N_ℓ are left coprime and $G(s) = D_\ell(s)^{-1}N_\ell(s)$. The definitions of <u>right coprimeness</u> and <u>right coprime factorization</u> are similar. Let $\mathcal{A}$ be the following <u>convolution algebra</u> (Desoer and Vidyasagar 1975, Appendix D): f belongs to $\mathcal{A}$ iff for $t < 0$, $f(t) = 0$ and, for $t \geq 0$, $f(t) = f_a(t) + \sum_{i=0}^{\infty} f_i\delta(t-t_i)$, where $f_a(\cdot) \in L_1[0,\infty)$; $f_i \in \mathbb{R}$, $\forall i$; $t_i \geq 0$, $\forall i$ and $\sum_{i=0}^{\infty}|f_i| < \infty$. A pxq matrix $\mathcal{D} \in \mathcal{A}^{pxq}$ iff every element of $\mathcal{D}$ belongs to $\mathcal{A}$. Let $\hat{\mathcal{A}} \triangleq \mathcal{L}(\mathcal{A})$ ($\hat{\mathcal{A}}^{pxq} \triangleq \mathcal{L}(\mathcal{A}^{pxq})$), the Laplace transform of $\mathcal{A}$; hence $f \in \mathcal{A}$ iff the Laplace transform of f (denoted by $\hat{f}$) belongs to $\hat{\mathcal{A}}$. f belongs to $\mathcal{A}(-\alpha)$ for some α iff $t \to \exp(\ t)f(t) \in \mathcal{A}$. Let $|\cdot|$ denote some vector norm. Let $\|\cdot\|_p$ denote the standard L_p-norm. Let $f : \mathbb{R}_+ \to \mathcal{V}$, some normed space, the truncated function f_T is defined by $f_T(t) = \begin{cases} f(t), & t \leq T \\ 0, & t > T. \end{cases}$ For $p \in [1,\infty]$, let $L_{pe}(\mathbb{R}_+)$ (abreviated L_{pe}) denote the <u>extended space</u> associated with L_p and be defined by $L_{pe} = \{f : \mathbb{R}_+ \to L_p \mid \forall T \in \mathbb{R}_+, \|f_T\|_p < \infty\}$. For $i = 1,2$, $p \in [1,\infty]$, let $u_i : \mathbb{R}_+ \to \mathbb{R}^{n_i}$, $y_i : \mathbb{R}_+ \to \mathbb{R}^{m_i}$; a map $(u_1,u_2) \mapsto (y_1,y_2)$ is said to be (finite gain) L_p-<u>stable</u> iff 1) $u_i \in L_p^{n_i} \Rightarrow y_i \in L_p^{m_i}$; 2) $\exists\ k_i > 0$ such that $\|y_i\|_p \leq k_i\ (\|u_1\|_p + \|u_2\|_p)$, $\forall u_i \in L_p^{n_i}$. Operators mapping L_{pe}^n to L_{pe}^n are labelled by boldface symbols (e.g. $\boldsymbol{\Phi}$, $\mathbf{H}$). U.t.c. means "under these conditions." ":=" means "is defined by."

B. STATEMENT OF PROBLEM

Consider a nonlinear multi-input multi-output servomechanism S of the type shown on Fig. II.1, where

$$r(t),\ e(t),\ u(t),\ v(t),\ w(t),\ z(t),\ y(t) \in \mathbb{R}^n, \tag{II.1}$$

$r(\cdot)$ is the <u>reference signal</u> to be tracked, $w(\cdot)$ is the <u>disturbance signal</u> to be rejected,(1) $y(\cdot)$ is the <u>plant output</u> to follow the reference signal $r(\cdot)$, $e(\cdot)$ is the <u>tracking error</u>.

$$v(t) = (\mathbf{H}u)(t) = \int_0^t H(t-\tau)u(\tau)d\tau \tag{II.2}$$

$$u = \boldsymbol{\Phi}^I e, \quad y = \boldsymbol{\Phi}^o z, \tag{II.3}$$

where $\boldsymbol{\Phi}^I$, $\boldsymbol{\Phi}^o$ are some nonlinear causal operators, and $\mathbf{H}$ is a linear time-invariant convolution operator.

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(1) Note that the zero-input response of $\mathbf{H}$ can be included in $w(\cdot)$.

For a given class of reference signals $r(\cdot)$ and a given class of disturbance signals $w(\cdot)$, we wish to find sufficient conditions on Φ^I, Φ^O and H such that asymptotic tracking and disturbance rejection occurs in the nonlinear servomechanism S, i.e. for all such $r(\cdot)$'s and $w(\cdot)$'s, $e(t) \to \theta_n$, as $t \to \infty$.

III. ASYMPTOTIC TRACKING WITH INPUT CHANNEL NONLINEARITY: $\Phi^O = I$.

In this section, we examine a special case where $\Phi^O = I$, $w(\cdot) \equiv \theta_n$.[2] Thus the nonlinear servomechanism S is reduced to the system S^I shown on Fig. III.1.

We quickly derive a generalized version of Bergen and Iwens (1966). Note that in S^I,

$$e = r - H\Phi^I e \tag{III.1}$$

Assume that K is a linear time-invariant convolution operator, and add HKe to both sides of (III.1), then by linearity of H, obtain

$$(I+HK)e = r - H(\Phi^I - K)e \tag{III.2}$$

If

$$(I+HK)^{-1} \text{ is a well-defined causal operator} \tag{III.3}$$

then

$$e = (I+HK)^{-1}r - (I+HK)^{-1}H(\Phi^I - K)e \tag{III.4}$$

which characterizes the feedback system S_K^I shown on Fig. III.2, where $r_K := (I+HK)^{-1}r$, $\Phi_K^I := \Phi^I - K$ and, since H is linear, $H_K := H(I+KH)^{-1} = (I+HK)^{-1}H$. Now we can state the following theorem

(2) When $\Phi^O = I$, the effect of $w(\cdot)$ can be included in $r(\cdot)$.

Theorem III.1 (Asymptotic tracking with input channel nonlinearity)

Consider the nonlinear systems S^I and S_K^I described by (II.1) ~ (II.3) and (III.1) ~ (III.4). Suppose that there exists some $\underset{\sim}{K}$ such that

(a1) for a given class of reference signals $r(\cdot)$,

$$r_K := (\underset{\sim}{I}+\underset{\sim}{H}\underset{\sim}{K})^{-1}r \in L_2^n \text{ and } r_K(t) \to \theta_n \text{ as } t \to \infty;$$

(a2) $\underset{\sim}{\Phi}_K^I := \underset{\sim}{\Phi}^I - \underset{\sim}{K} \quad : \quad L_2^n \to L_2^n;$

(a3) $H_K(\cdot) \in L_2^{nxn}$, where $H_K(\cdot)$ is the impulse response of $\underset{\sim}{H}_K$.

U.t.c. if the map $r_K \mapsto e$, restricted to those r_K resulting from the r's under consideration, is L_2-stable, then

$$e(\cdot) \in L_2^n \text{ and } e(t) \to \theta_n \text{ as } t \to \infty.$$

Proof: see Appendix

Comments a) Typical reference signals $r(\cdot)$ and disturbances $w(\cdot)$ of interest are steps, ramps, parabolas, etc. (for curve following applications), sinusoids and linear combinations of these signals, (for vibration isolation, and/or rf pickup isolation).

b) Let $\lambda \in \mathbb{C}$ with Re $\lambda \geq 0$ and $\hat{H}(s)$, $\hat{K}(s)$ be rational; if $\hat{r}(s) = \bar{r}/(s-\lambda)^p$ then, $\forall \bar{r} \in \mathbb{C}^n$, $r_K(t) \to \theta_n$ as $t \to \infty$ exponentially (thus $r_K(\cdot) \in L_2^n$) provided that (i) $(\underset{\sim}{I}+\underset{\sim}{H}\underset{\sim}{K})^{-1}$ is exp. stable; (ii) $(\underset{\sim}{I}+\underset{\sim}{H}\underset{\sim}{K})^{-1}$ has a blocking zero of order p at λ (Ferreira and Bhattacharyya, 1977). Notice that in this case, the nonlinear servo shown on Fig. III.1 will perform asymptotic tracking (i.e. $e(t) \to \theta_n$ as $t \to \infty$) even when $\underset{\sim}{H}$ is subject to some (not necessarily small) linear perturbation provided that conditions (i), (ii) stated above are maintained for the perturbed $\underset{\sim}{H}$. For a detailed discussion of this robustness property, see Desoer and Wang (1977).

c) Consider the system shown on Fig. III.1, but assume $\underset{\sim}{\Phi}^I$ replaced by the linear map $\underset{\sim}{K}$ of Theorem III.1. Assumption (a1) of the Theorem asserts that, in the resulting linear servo, the error signal is in L_2^n and tends to zero as $t \to \infty$. Thus Theorem III.1 can be viewed as asserting that if $\underset{\sim}{K}$ is perturbed into $\underset{\sim}{\Phi}^I = \underset{\sim}{\Phi}_K^I + \underset{\sim}{K}$, then the tracking property is preserved provided some stability conditions are satisfied. So Theorem III.1 asserts the robustness of the tracking property under nonlinear perturbation of $\underset{\sim}{K}$.

d) Special case: if $\underset{\sim}{\Phi}_K^I$ is a time-varying memoryless nonlinearity, i.e. $u_K(t) = (\underset{\sim}{\Phi}_K^I e)(t) = \Phi_K^I(e(t),t)$ with $\Phi_K^I : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ (this will be the case if for example, $\underset{\sim}{\Phi}^I$ is a time-varying, memoryless, nonlinearity and $K \in \mathbb{R}^{n\times n}$), and if there exists some $\beta > 0$ such that $\forall x \in \mathbb{R}^n$, $\forall t \geq 0$ $|\Phi_K^I(x,t)| \leq \beta|x|$, then assumption (a2) is satisfied. Note that, in this case, a sufficient condition for the restricted map $r_K \mapsto e$ to be L_2-stable is that the map $r_K \mapsto e$ is L_2-stable which can be established by some well-known L_2-stability criterion e.g. circle criterion (see e.g. Desoer and Vidyasagar 1975, Vidyasagar 1978).

IV ASYMPTOTIC TRACKING AND DISTURBANCE REJECTION OF A NONLINEAR SERVOMECHANISM

We have noted the robustness of the tracking property under nonlinear perturbations of the input channel: $\underset{\sim}{K} \leftarrow \underset{\sim}{K} + \underset{\sim}{\Phi}_K^I$. Intuitively, one might ascribe this robustness to the fact that, as long as $e(\cdot) \in L_2^n$, for t large, $y(t)$ will be close to $r(t)$, hence only the behavior of $\underset{\sim}{\Phi}^I$ in the neighborhood of the origin is important. Suppose now that we have a nonlinearity in the output (i.e. $\underset{\sim}{\Phi}^o \neq \underset{\sim}{I}$), the situation is drastically different. Roughly speaking, if asymptotic tracking and disturbance rejection does occur in the nonlinear servomechanism S shown on Fig. II.1 for a class of reference signals $r(\cdot)$ and a class of disturbance signals $w(\cdot)$, then for large t, $y(\cdot)$ must be asymptotically close to $r(\cdot)$. This requires that $z(\cdot)$ tend to a preimage of $r(\cdot)$ with respect to $\underset{\sim}{\Phi}^{o}$. It seems that this will be very difficult to achieve except for the regulator case where $r(\cdot)$ tends to a constant $\mathbb{R}^n$-vector and $\underset{\sim}{\Phi}^{o}$ is time-invariant.

In part A of this section, we present a L_∞-stability theorem which forms the basis of our asymptotic tracking and disturbance rejection theorem for the nonlinear servomechanism S. In part B, we first develop a theorem which treats the special case that $\underset{\sim}{\Phi}^{I} = \underset{\sim}{I}$ (Theorem IV.2). Then it is seen that the general case where $\underset{\sim}{\Phi}^{I} \neq \underset{\sim}{I}$ is no more difficult than the special case (Theorem IV.3). Some discussion follows in part C.

A. L_∞-STABILITY RESULT

We state below a L_∞-stability theorem for multi-input, multi-output, nonlinear, time-varying, distributed systems. This theorem is a generalization of a result known in the literature (Desoer and Vidyasagar 1975, pp. 143).

Consider the feedback system shown in Fig. IV. 1a, where

$u_i(t),\ e_i(t),\ y_i(t) \in \mathbb{R}^n,\ i = 1,2$

$$y_1(t) = (\underset{\sim}{M}e_1)(t) = \int_0^t M(t-\tau)e_1(\tau)d\tau \qquad \text{(IV.1)}$$

$$y_2(t) = (\underset{\sim}{\Phi}e_2)(t) = \Phi(e_2(t),t) \qquad \text{(IV.2)}$$

with $\Phi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ being continuous in its 1st argument and piecewise continuous in its 2nd argument.

Applying the well-known exponential weighting technique (see e.g. Desoer and Vidyasagar 1975, pp. 143; Sandberg 1965, Zames 1965) to the system shown on Fig. IV.1a, we obtain its $\exp(\alpha t)$-weighted companion system which is shown on Fig. IV.1b, where

$$y_{1\alpha}(t) = (\underset{\sim}{M}_\alpha e_1)(t) = \int_0^t M_\alpha(t-\tau)e_{1\alpha}(\tau)d\tau \text{ with } M_\alpha(t) := \exp(\alpha t)M(t), \tag{IV.3}$$

$$y_{2\alpha}(t) = (\underset{\sim}{\Phi}_\alpha e_{2\alpha})(t) := \exp(\alpha t)\ \Phi[\exp(-\alpha t)e_{2\alpha}(t),t] \tag{IV.4}$$

and the subscript α in the symbols $u_{i\alpha}$, $e_{i\alpha}$, $y_{i\alpha}$, $i = 1,2$, has the following meaning:

$$f_{i\alpha}(t) := \exp(\alpha t)\ f_i(t) \tag{IV.5}$$

<u>Theorem IV.1</u> (<u>L_∞-stability theorem</u>)

Consider the feedback system described by (IV. 1,2) and its $\exp(\alpha t)$-weighted companion system described by (IV.3) ~ (IV.5). Suppose that

(i) $M(\cdot) \in L_1^{n\times n}$;

(ii) for some $\alpha > 0$, $M_\alpha(\cdot) \in L_2^{n\times n}$;

(iii) $\exists\ \beta > 0$ such that $\forall z \in \mathbb{R}^n$ and $\forall t \geq 0, |\Phi(z,t)| \leq \beta|z|$.

U.t.c. <u>if</u> the map $(u_{1\alpha},u_{2\alpha}) \mapsto (e_{1\alpha},e_{2\alpha})$ is L_2-stable (see Fig. IV.1b), <u>then</u> the maps $(u_1,u_2) \mapsto (e_1,e_2)$ and $(u_1,u_2) \mapsto (y_1,y_2)$ are L_∞-stable.

<u>Corollary IV.1</u>

Under the conditions of theorem IV.1, <u>if</u>, in addition, $\exists\ \mu_i > 0$ and $\gamma_i > 0$ such that

$$|u_i(t)| \leq \mu_i \exp(-\gamma_i t), \ \forall t \geq 0, \ i = 1,2 \tag{IV.6}$$

and

$$M_\alpha(\cdot) \in L_1^{nxn}, \tag{IV.7}$$

then, for $i = 1,2$, $\exists \ \mu_{ei} > 0$ and $\mu_{yi} > 0$ such that $\forall t \geq 0$,

$$|e_i(t)| \leq \mu_{ei} \exp(-\lambda t) \tag{IV.8}$$

$$|y_i(t)| \leq \mu_{yi} \exp(-\lambda t) \tag{IV.9}$$

where $\lambda := \min \{\gamma_1, \gamma_2, \alpha-\delta\}$ with $\delta > 0$ arbitrarily small.

Proof of Theorem IV.1 and Corollary IV.1: See Appendix

B. ASYMPTOTIC TRACKING AND DISTURBANCE REJECTION THEOREMS

We derive now sufficient conditions for a multi-input multi-output nonlinear time-invariant distributed servomechanism to track asymptotically a class of reference signals $r(\cdot)$ while the system is subject to a class of external disturbances $w(\cdot)$. Due to the enormous flexibility introduced by the output nonlinearity, we are obliged to make additional assumptions:

The class of reference signals $r(\cdot)$ and disturbance signals $w(\cdot)$ satisfy

(S1) $r(\cdot), w(\cdot) \in C^1$ and for $x = r,w$, $\exists \ \mu_x > 0$, $\gamma_x > 0$ such that

$$|\dot{x}(t)| \leq \mu_x \exp(-\gamma_x t), \quad \forall t \geq 0$$

For the class of reference and disturbance signals described in (S1), we shall require that

(S2) $\hat{H}(s) = \frac{1}{s}\hat{G}(s)$, where $\hat{G}(s) \in \mathcal{A}^{n\times n}$ and $\hat{G}(0) \neq \theta_{n\times n}$;

(S3) $u(t) = (\underset{\sim}{\Phi}^I e)(t) = \Phi^I(e(t))$

$y(t) = (\underset{\sim}{\Phi}^O z)(t) = \Phi^O(z(t))$

with $\Phi^I, \Phi^O : \mathbb{R}^n \to \mathbb{R}^n$ being C^1 functions.

Finally we assume that

(S4) $e, u, v, z, y \in L^n_{\infty e}$.

From assumptions (S1) ~ (S4) and Fig. II.1, it is easy to see that the functions e, u, v, z, y are differentiable on $\mathbb{R}_+$. Thus we can take time derivative of the equation

$$e = r - y = r - \underset{\sim}{\Phi}^O z \tag{IV.10}$$

and obtain

$$\dot{e} = \dot{r} - (D\underset{\sim}{\Phi}^O)z \tag{IV.11}$$

where the <u>linear</u> map $D\underset{\sim}{\Phi}^O : z \mapsto \frac{d}{dt}[(\underset{\sim}{\Phi}^O z)(t)]$ is the Fréchet derivative of $\underset{\sim}{\Phi}^O$ and $(D\underset{\sim}{\Phi}^O z)(t) = D\Phi^O(z(t))\dot{z}(t)$ (Dieudonné 1969, pp. 155; Martin 1976, pp. 33). Since

$$\dot{z} = \dot{w} + \dot{v} \tag{IV.12}$$

$$\dot{v} = G*u \quad \text{(by (S2))} \tag{IV.13}$$

and

$$\dot{u} = (D\underset{\sim}{\Phi}^I)\dot{e} \tag{IV.14}$$

where the <u>linear</u> map $D\underset{\sim}{\Phi}^I : e \mapsto \frac{d}{dt}[(\underset{\sim}{\Phi}^I e)(t)]$ is the Fréchet derivative of $\underset{\sim}{\Phi}^I$, eqn. (IV.11) becomes

$$\dot{e}(t) = \dot{r}(t) - D\Phi^{O}(z(t))\,[\dot{w}(t)+(H*\dot{u})(t)] \tag{IV.15}$$

Equations (IV.14) and (IV.15) characterize the <u>linear time-varying</u> feedback system $\dot{S}$ shown on Fig. IV.2

<u>Case I</u>: $\underset{\sim}{\Phi}^I = \underset{\sim}{I}$.

For simplicity, we first consider the special case where $\underset{\sim}{\Phi}^I = \underset{\sim}{I}$ (thus $D\Phi^I(e) = I$, $\forall e \in \mathbb{R}^n$ and $\dot{e} = \dot{u}$). Therefore $\dot{S}$ shown in Fig. IV.2 is reduced to the system $\dot{S}^o$ shown on Fig. IV.3a. For future reference, we shall define the K-shifted system $\dot{S}^o_K$ of $\dot{S}^o$ by applying the loop transformation technique with $K \in \mathbb{R}^{nxn}$ to $\dot{S}^o$ (see e.g. Desoer and Vidyasagar 1975, pp. 50). The system $\dot{S}^o_K$ is shown on Fig. IV.3b, where $D\Phi^o_K(z) = D\Phi^o(z) - K$. Furthermore, we define the exp(αt)-weighted system $\dot{S}^o_{K,\alpha}$ of $\dot{S}^o_K$. The system $\dot{S}^o_{K,\alpha}$ is shown on Fig. IV.3c, where $\dot{f}_\alpha(t) := \exp(\alpha t)\dot{f}(t)$ and $\hat{H}_{K,\alpha}(s) = \hat{H}_K(s-\alpha)$. (or equivalently, $H_{K,\alpha}(t) = \exp(\alpha t)H_K(t)$).

Now we can state the result.

<u>Theorem IV.2</u> <u>(Asymptotic tracking and disturbance rejection with output channel nonlinearity)</u>

Consider the system S described by (II.1) ~ (II.3) with $\underset{\sim}{\Phi}^I = \underset{\sim}{I}$ and the systems $\dot{S}^o$, $\dot{S}^o_K$, $\dot{S}^o_{K,\alpha}$ shown on Fig. IV.3a ~ 3c. Assume that (S1)~(S4) hold. Suppose that for some $K \in \mathbb{R}^{nxn}$

(a1) $\exists\ \alpha > 0$ such that $t \mapsto \exp(\alpha t)H_K(t) \in L_1^{nxn} \cap L_2^{nxn}$;

(a2) $\exists\ \beta > 0$ such that $\forall x,z \in \mathbb{R}^n$,

$$|D\Phi^o_K(z)\,x| \leq \beta|x|$$

U.t.c. if the map $(\dot{r}_{K,\alpha},\dot{w}_\alpha) \mapsto (\dot{e}_{K,\alpha},\dot{z}_\alpha)$ is L_2-stable (see Fig. IV.3c), then

(i) for x = e,z,v,y, $\exists\ \mu_x > 0$ such that

$$|\dot{x}(t)| \le \mu_x \exp(-\lambda t), \quad \forall t \ge 0 \tag{IV.16}$$

where $\lambda := \min\{\gamma_r, \gamma_w, \alpha-\delta\}$ with $\delta > 0$ arbitrarily small.

If, in addition,

(a3) $\hat{G}(0)$ is nonsingular,

then

(ii) $e(t) \to \theta_n$ as $t \to \infty$.

Comments:

(a) Theorem IV.2 gives a sufficient condition, for a nonlinear, time-invariant, multi-input, multi-output, distributed servo to track asymptotically any bounded reference signal r which goes to a constant exponentially and to reject asymptotically any bounded additive disturbance signal w which goes to a constant exponentially. Note that the conclusions (i) and (ii) provide a lower bound on the exponential rate at which the tracking error $e(\cdot)$ goes to θ_n; more precisely, the tracking error $e(t) \to \theta_n$ as $t \to \infty$ at an exponential rate no less than λ, where $\lambda = \min\{\gamma_r, \gamma_w, \alpha-\delta\}$ with $\delta > 0$ arbitrarily small.

(b) Stability is always a main concern for a servomechanism. Conclusion (i) of Theorem IV.2 guarantees that for the given class of reference and disturbance signals described by (S1), the functions $e, z, v, y \in L^n_\infty$.

(c) From a design point of view, one may be given a fixed nonlinear plant modeled by a linear plant $\hat{P}(s)$ followed by a nonlinearity $\underset{\sim}{\Phi}^o$. It is desired to have the plant output y track a given class of reference signals r which go to constants at most at an exponential rate γ_r and reject a given class of disturbance signals w which go to constants at most at an exponential rate γ_w; furthermore, the tracking error $e(\cdot)$

is required to go to zero at least at certain rate. Hence if one can design some compensator around $\hat{P}(s)$ thus forming a $\hat{G}(s)$, then precede it with the integrators to form $\hat{H}(s)$ such that the conditions of Theorem IV.2 are satisfied for some $\alpha > \min\{\gamma_r,\gamma_w\}$, then the plant output y will track the reference signal r and reject the disturbance signal w, and the tracking error $e(\cdot)$ will go to θ_n at an exponential rate no less than $\min\{\gamma_r,\gamma_w\}$.

(d) Note that in the single-input single-output lumped case, the knowledge of the root locus method shows us that a smaller time constant of $\hat{G}(s)$ does not necessarily guarantee the existence of a larger α such that (a1) is satisfied.

(e) A careful reader might worry that it is not explicitly assumed that the range of the nonlinear time-invariant operator $\underset{\sim}{\Phi}^o$ is the whole $\mathbb{R}^n$; if it is not, the theorem cannot possibly be true for any bounded reference signal which tends to any constant vector of $\mathbb{R}^n$. The fact is that the condition that the map $(\dot{r}_{K,\alpha}, \dot{w}_\alpha) \mapsto (\dot{e}_{K,\alpha}, \dot{z}_\alpha)$ is L_2-stable and the presence of the integrators implicitly force the range of $\underset{\sim}{\Phi}^o$ to be $\mathbb{R}^n$.

(f) The L_2-stability condition of the map $(\dot{r}_{K,\alpha},\dot{w}_\alpha) \mapsto (\dot{e}_{K,\alpha},\dot{z}_\alpha)$ in $\dot{S}_{K,\alpha}$ can be tested by some well-known L_2-stability criterion, e.g. the circle criterion.

(g) By assumption (S2), $s = 0$ is not a pole of $\hat{G}(s)$. Hence, for the lumped case, assumption (a3) is equivalent to that the transfer function $\hat{G}(s)$ has no transmission zero (Desoer and Schulman 1974, Pugh 1977, MacFarlane and Karcanias 1976) at $s = 0$ as we expected from the robust linear servomechanism theory (Davison 1976, Desoer and Wang 1977).

Proof of Theorem IV.2

(i) With the assumptions (S1), (a1) and (a2), we apply Corollary IV.1 to the system $\dot{S}_K$ shown on Fig. IV.3a and thus conclude that for $x_K = e_K, z, v, y_K$, $\exists\ \mu_{x_K} > 0$ such that

$$|\dot{x}_K(t)| \leq \mu_{x_K} \exp(-\lambda t) \qquad \forall t \geq 0 \tag{IV.17}$$

where λ is defined in (IV.16).

Since $\dot{e} = \dot{e}_K - K\dot{v}$ and $\dot{y} = \dot{y}_K + K\dot{z}$, we also conclude that there exist $\mu_e > 0$, $\mu_y > 0$ such that

$$\left.\begin{aligned} |\dot{e}(t)| &\leq \mu_e \exp(-\lambda t) \\ |\dot{y}(t)| &\leq \mu_y \exp(-\lambda t) \end{aligned}\right\} , \quad \forall t \geq 0$$

(ii) From (i), $\dot{z}(\cdot) \in L_1^{n\times n}$ and $z(t) \to z_\infty \in \mathbb{R}^n$ at an exponential rate λ. Thus $e(t) = r(t) - \Phi^o(z(t)) \to e_\infty := r_\infty - \Phi^o(z_\infty)$ at an exponential rate λ, where $r_\infty := \lim_{t\to\infty} r(t)$. We now prove $e_\infty = \theta_n$ by contradiction. Suppose $e_\infty \neq \theta_n$. Write

$$e(t) = e_\infty + e_o(t) \tag{IV.18}$$

where $e_o(t) \in L_1^n \cap L_\infty^n$ and $e_o(t) \to \theta_n$ exponentially. Consider the nonlinear servomechanism S (with $\underset{\sim}{\Phi}^I = \underset{\sim}{I}$) and calculate $p(\cdot)$, the output of the integrators:

$$p(t) = e_\infty t + p_o(t) \tag{IV.19}$$

where $p_o(t) := \int_0^t e_o(\tau)d\tau \in L_\infty^n$ and goes to a constant as $t \to \infty$. Now, for $t > 0$,

$$
\begin{aligned}
v(t) &= (H*e)(t) = (G*p)(t) \\
&= \int_0^t G(t-\tau)e_\infty \tau d\tau + (G*p_o)(t) \\
&= t\int_0^t G(t-\tau)d\tau\ e_\infty - \int_0^t (t-\tau)G(t-\tau)d\tau\ e_\infty + (G*p_o)(t) \\
&= t\left[\int_0^t G(\tau)d\tau\ e_\infty - \int_0^t \frac{\tau}{t} G(\tau)d\tau\ e_\infty + \frac{(G*p_o)(t)}{t}\right] \\
&= t\left[\int_0^\infty G(\tau)d\tau\ e_\infty - \int_t^\infty G(\tau)d\tau\ e_\infty - \int_0^t \frac{\tau}{t} G(\tau)d\tau\ e_\infty + \frac{(G*p_o)(t)}{t}\right]
\end{aligned}
\tag{IV.20}
$$

By assumption (S2), we have that $\forall\varepsilon > 0$, $\exists\ t_o$ such that

$$
\int_{t_o}^\infty |G(t)|dt < \varepsilon \tag{IV.21}
$$

Let $t = \beta t_o$ with $\beta > 1$, the second integral in (IV.20) is bounded by $\varepsilon|e_\infty|$, the third integral in (IV.20) is such that

$$
\begin{aligned}
\left|\int_0^{\beta t_o} \frac{\tau}{\beta t_o} G(\tau)d\tau\right| &\leq \frac{1}{\beta}\int_0^{t_o} |G(\tau)|d\tau + \int_{t_o}^{\beta t_o} |G(\tau)|d\tau \\
&\leq \frac{\|G\|_{\mathcal{A}}}{\beta} + \varepsilon
\end{aligned}
\tag{IV.22}
$$

Note that $p_o \in L_\infty^n$, thus by assumption (S2),

$$
G*p_o \in L_\infty^n \tag{IV.23}
$$

Hence asymptotically as $\beta \to \infty$ (equivalently as $t = \beta t_o \to \infty$),

$$
v(t) = t[\hat{G}(0)e_\infty + m(t)]
$$

where $|m(t)| \leq 2\varepsilon|e_\infty| + \|G\|_{\mathcal{a}}|e_\infty|/\beta + \|G*p_o\|_\infty/t$.

Therefore, by assumption (a3) and the assumption $e_\infty \neq \theta_n$, we have, for large $t > 0$, the asymptotic relation

$$v(t) \sim t\,\hat{G}(0)e_\infty \; . \tag{IV-24}$$

This contradicts the conclusion of (i) that $v(t) \to v_\infty \in \mathbb{R}^n$ at an exponential rate λ. Therefore, $e_\infty = \theta_n$ i.e. $e(t) \to \theta_n$ as $t \to \infty$.

Q.E.D.

Case II: $\underset{\sim}{\Phi}^I \neq \underset{\sim}{I}$

Let us now extend Theorem IV.2 to the general case where $\underset{\sim}{\Phi}^I \neq \underset{\sim}{I}$. Observe that the system $\dot{S}$ shown on Fig. IV.2 can be further transformed into the system shown on Fig. IV.4, where $\dot{\tilde{r}}(t) = D\Phi^I(e(t))\dot{r}(t)$. Note that if we assume that for some $\beta^I > 0$,

$$|D\Phi^I(e)x| \leq \beta^I|x| \;, \quad \forall x,e \in \mathbb{R}^n \tag{IV.25}$$

then

$$\begin{aligned} |\dot{\tilde{r}}(t)| &= |D\Phi^I(e)\dot{r}(t)| \\ &\leq \beta^I|\dot{r}(t)| \\ &\leq \beta^I\mu_r \exp(-\gamma_r t), \quad \forall t \geq 0 \quad \text{(by (S1))} \end{aligned} \tag{IV.26}$$

Comparing Fig. IV.4 with Fig. IV.3a, we see that we are right back to the previous case. Note that $u(t) \to \theta_n$ as $t \to \infty$ implies that $e(t) \to \theta_n$ as $t \to \infty$ provided that

$$\Phi^I(e) = \theta_n \quad \text{iff} \quad e = \theta_n \tag{IV.27}$$

To summarize, we state the following theorem.

Theorem IV.3 (Asymptotic tracking and disturbance rejection)

Consider the nonlinear time-invariant servomechanism S described by (II.1) ~ (II.3). Assume (S1) ~ (S4). Suppose that for some $K \in \mathbb{R}^{nxn}$

(a1) $\exists\ \alpha > 0$ such that $t \mapsto \exp(\alpha t)\ H_K(t) \in L_1^{nxn} \cap L_2^{nxn}$;

(a2) $\exists\ \beta > 0$ such that $\forall x \in \mathbb{R}^n$,

$$|D\Phi^I(e)\ D\Phi^o(z)x - Kx| \leq \beta|x|, \quad \forall\ e,z \in \mathbb{R}^n,$$

and

(a3) $\exists\ \beta^I > 0$ such that $\forall x \in \mathbb{R}^n$,

$$|D\Phi^I(e)x| \leq \beta^I|x| \ , \quad \forall e \in \mathbb{R}^n;$$

(a4) $\Phi^I(e) = \theta_n$ iff $e = \theta_n$.

U.t.c. if the map $(\ddot{r}_{K,\alpha},\dot{w}_\alpha) \mapsto (\dot{e}_{K,\alpha},\dot{z}_\alpha)$ is L_2-stable , then

(i) for x = e, u, z, v, y, $\exists\ \mu_x > 0$ such that

$$|\dot{x}(t)| \leq \mu_x \exp(-\lambda t), \qquad \forall t \geq 0,$$

where $\lambda := \min\{\gamma_r,\gamma_w,\alpha-\delta\}$ with $\delta > 0$ arbitrarily small.

If, in addition,

(a5) $\hat{G}(0)$ is nonsingular,

then

(ii) $e(t) \to \theta_n$ as $t \to \infty$.

C. DISCUSSION

For simplicity, we shall only discuss the case where $\underset{\sim}{\Phi}^I = \underset{\sim}{I}$.

(i) Robustness of the nonlinear servomechanism

As expected intuitively and as seen from the proof of Theorem IV.2, the crucial components of the nonlinear servo S are the integrators. If some of the integrators are subject to perturbation, then the whole argument will fail. However, there is quite noticeable margin in the nonlinearity $\underset{\sim}{\Phi}^o$ as we have seen from the assumption (a2) of Theorem IV.2. Furthermore for sufficiently small linear perturbation on $\hat{G}(s)$, say, from $\hat{G}(s) \in \hat{\mathcal{A}}^{nxn}$ to $\hat{G}(s) + \delta\hat{G}(s) \in \hat{\mathcal{A}}^{nxn}$, assumptions (S2), (a1), (a3) will remain valid. Thus if the map $(\dot{r}_{K,\alpha}, \dot{w}_\alpha) \mapsto (e_{K,\alpha}, z_\alpha)$ remains L_2-stable for the perturbed system, then asymptotic tracking and disturbance rejection still holds. Note that in contrast to the robust linear servomechanism theory (Ferreira 1976, Desoer and Wang 1977), we have not been able to assert that small perturbations on integrators will result in small asymptotic tracking error.

(ii) Asymptotic tracking $(w(\cdot) \equiv \theta_n)$ when $r(t) = \sum_{i=0}^{m} p_i t^i, p_i \in \mathbb{R}^n, m \geq 1$

We have discussed above the difficulties associated with guaranteeing robust asymptotic tracking for this case. However, under some additional assumptions, we are able to give a bound on the tracking error $e(t)$ as t becomes large. For the purpose of deriving this bound, we consider the lumped linear time-invariant servo shown on Fig. IV.5, where the closed-loop system is exp. stable and $e_L(t) \to \theta_n$ as $t \to \infty$ for any of the r's under consideration. Now let $\underset{\sim}{K}$ be subject to some nonlinear memoryless perturbation and let it thus become $\underset{\sim}{\Phi}^o = \underset{\sim}{K} + \underset{\sim}{\Phi}^o_K$. The perturbed system is the nonlinear servo S shown on Fig. II.1 with $\underset{\sim}{\Phi}^I = \underset{\sim}{I}$. If we apply the loop transformation with $\underset{\sim}{K}$ to S, then we obtain the system shown on Fig. IV.6.

Note that, with z_L defined on Fig. IV.5, and $\underset{\sim}{H}$, $\underset{\sim}{K}$ both linear

$$z = \underset{\sim}{H}(\underset{\sim}{I}+\underset{\sim}{K}\underset{\sim}{H})^{-1}[r-\underset{\sim}{\Phi}{}_K^o z]$$

$$= z_L - \underset{\sim}{H}(\underset{\sim}{I}+\underset{\sim}{K}\underset{\sim}{H})^{-1}\underset{\sim}{\Phi}{}_K^o z \qquad \text{(IV.28)}$$

Thus $e = r - (\underset{\sim}{K}+\underset{\sim}{\Phi}{}_K^o)z$

$$= r - \underset{\sim}{K}z - \underset{\sim}{\Phi}{}_K^o z$$

$$= r - \underset{\sim}{K}z_L + \underset{\sim}{K}\underset{\sim}{H}(\underset{\sim}{I}+\underset{\sim}{K}\underset{\sim}{H})^{-1}\underset{\sim}{\Phi}{}_K^o z - \underset{\sim}{\Phi}{}_K^o z \qquad \text{(by IV.28)}$$

$$= e_L - [\underset{\sim}{I}-\underset{\sim}{K}\underset{\sim}{H}(\underset{\sim}{I}+\underset{\sim}{K}\underset{\sim}{H})^{-1}]\underset{\sim}{\Phi}{}_K^o z$$

$$= e_L - (\underset{\sim}{I}+\underset{\sim}{K}\underset{\sim}{H})^{-1}\underset{\sim}{\Phi}{}_K^o z \qquad \text{(IV.29)}$$

By assumption on the linear servo, $e_L(t)$ will tend exponentially to zero at a rate controlled by adjusting the closed-loop poles of the <u>linear</u> servo. Thus if $\underset{\sim}{\Phi}{}_K^o$ is memoryless with characteristic $\Phi_K^o : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ such that $\exists\ \xi > 0$ such that

$$|\Phi_K^o(z,t)| \leq \xi \quad \forall z \in \mathbb{R}^n, \quad \forall t \geq 0 \qquad \text{(IV.30)}$$

then, by (IV.29) and the closed-loop exp. stability of the linear servo, we conclude that for large t, $|e(t)|$ will be bounded by $\|(\underset{\sim}{I}+\underset{\sim}{K}\underset{\sim}{H})^{-1}\|_\infty \xi + |e_L(t)|$, where $\|\cdot\|_\infty$ denotes the L_∞-<u>induced</u> norm of $(\underset{\sim}{I}+\underset{\sim}{K}\underset{\sim}{H})^{-1}$.

(iii) <u>Asymptotic tracking $(w(\cdot) \equiv \theta_n)$ when $r(t) = \bar{r} \sin \omega t$</u>

Referring to Fig. II.1 with $w(\cdot) \equiv \theta_n$ and $\underset{\sim}{\Phi}{}^I = \underset{\sim}{I}$, if asymptotic tracking does occur in S, then z(t) must be, asymptotically, the preimage of $\bar{r} \sin \omega t$ with respect to the <u>nonlinear</u> operator $\underset{\sim}{\Phi}{}^o$. Hence, in general, z(t) will consist of an infinite number of harmonics and, possibly, subharmonics of $r(\cdot)$. To generate these <u>exactly</u> would require $\hat{H}(s)$ to have an infinite number of poles on the $j\omega$-axis at those frequencies

Therefore, we can not expect <u>perfect</u> tracking for this class of reference signals. Of course, we could bound the tracking error by using arguments similar to the one used in (ii).

V. APPLICABILITY OF LOOP TRANSFORMATION WITH $K \in \mathbb{R}^{nxn}$

The asymptotic tracking and disturbance rejection Theorem IV.2 does not give condition on $\hat{G}(s)$ under which some $K \in \mathbb{R}^{nxn}$ can be found such that the map $t \mapsto \exp(\alpha t)H_K(t) \in L_1^{nxn} \cap L_2^{nxn}$ for some $\alpha > 0$. We now investigate this question.

Assume that the system S described by (II.1) ~ (II.3) with assumptions (S1)~(S4) consists of a linear time-invariant convolution operator $\underset{\sim}{G}$ such that for some $\nu > 0$, $\hat{G}(s) \in \hat{\mathcal{A}}(-\nu)^{nxn}$ and $\hat{G}(0) \neq \theta_{nxn}$. It is obvious that such $\hat{G}(s)$ satisfies the assumption (S2). Now the following propositions show that there exists $K \in \mathbb{R}^{nxn}$ such that (a1) and (a3) of Theorem IV.2 are satisfied under one additional assumption on $\hat{G}(s)$.

<u>Proposition V.1</u>.

Let $\hat{H}(s) = \frac{I}{s}\hat{G}(s)$, where $\hat{G}(s) \in \hat{\mathcal{A}}(-\nu)^{nxn}$, $\nu > 0$. If, in addition, $\sigma[\hat{G}(0)] \subset \mathring{\mathbb{C}}_+$, then for $K = \varepsilon I$, with $\varepsilon > 0$ sufficiently small, there exists some $\alpha > 0$ such that

$$\hat{H}_K := \hat{H}[I+\varepsilon\hat{H}]^{-1} \in \hat{\mathcal{A}}(-\alpha)^{nxn} \tag{V.1}$$

<u>Corollary V.1</u>.

Let $\hat{H}(s) = \frac{I}{s}\hat{G}(s)$, where $\hat{G}(s) \in \mathbb{R}(s)^{nxn}$ is <u>exponentially stable</u> with $\sigma[\hat{G}(0)] \subset \mathring{\mathbb{C}}_+$, then for $K = \varepsilon I$, with $\varepsilon > 0$ sufficiently small, the transfer function matrix

$$\hat{H}_K := \hat{H}[I+\varepsilon\hat{H}]^{-1} \text{ is } \underline{\text{exp. stable}} . \tag{V.2}$$

Proof of Proposition V.1 and Corollary V.1: see Appendix

Remarks (a) Note that the condition $\sigma[\hat{G}(0)] \subset \mathring{\mathbb{C}}_+$ implies that $\hat{G}(0)$ is nonsingular, i.e. assumption (a3) of Theorem IV.2 is satisfied.

(b) Corollary V.1 concludes that $\hat{H}_K(s)$ is exp. stable which implies that for some $\alpha > 0$, $t \mapsto \exp(\alpha t)\, H_K(t) \in L_1^{nxn} \cap L_\infty^{nxn}$ and in particular, $t \mapsto \exp(\alpha t)\, H_K(t) \in L_1^{nxn} \cap L_2^{nxn}$, i.e. assumption (a1) of Theorem IV.2 is satisfied.

The following proposition shows that $\hat{G}(s) \in \hat{\mathcal{A}}(-\nu)^{nxn}$, for some $\nu > 0$ together with (V.1) is enough to guarantee that $t \mapsto \exp(\alpha t)\, H_K(t) \in L_1^{nxn} \cap L_2^{nxn}$.

Proposition V.2.

Let $\hat{H}(s) = \frac{I}{s}\hat{G}(s)$, where $\hat{G}(s) \in \hat{\mathcal{A}}(-\nu)^{nxn}$, $\nu > 0$. If there exists $K \in \mathbb{R}^{nxn}$ such that for some $\alpha \in (0,\nu)$

$$\hat{H}_K := \hat{H}[I+K\hat{H}]^{-1} \in \hat{\mathcal{A}}(-\alpha)^{nxn}$$

then

$$t \mapsto \exp(\alpha t) H_K(t) \in L_1^{nxn} \cap L_2^{nxn} \tag{V.3}$$

Corollary V.2.

Let $H(s) = \frac{I}{s}\hat{G}(s)$, where $\exp(\nu t)G(t) \in L_1^{nxn}$, $\nu > 0$. If there exists $K \in \mathbb{R}^{nxn}$ such that for some $\alpha \in (0,\nu)$

$$\hat{H}_K(s) = \hat{H}[I+K\hat{H}]^{-1} \in \hat{\mathcal{A}}(-\alpha)^{nxn}$$

then

$$t \mapsto \exp(\alpha t)\, H_K(t) \in L_1^{nxn} \cap L_\infty^{nxn}, \text{ in particular}$$

$$t \mapsto \exp(\alpha t)\, H_K(t) \in L_1^{nxn} \cap L_2^{nxn}$$

Proof of Proposition V.2 and Corollary V.2: see Appendix

Remarks: (a) The existence of $K \in \mathbb{R}^{nxn}$ such that (V.3) holds can be guaranteed under a very mild condition (see Proposition V.1).

(b) When $\hat{G}(s)$ does not satisfy $\sigma[\hat{G}(0)] \subset \mathring{C}_+$, the existence of $K \in \mathbb{R}^{nxn}$ such that (V.3) is satisfied could still be proved by other means, e.g. the root locus method for the case where $\hat{G}(s) \in \mathbb{R}(s)$.

VI. EXAMPLE

When the system S described by (II.1) ~ (II.3) with assumption (S1)~(S4) is single-input single-output, some well-known graphical tests are readily available. Note that in this case if $D\Phi_K^o$ belongs to the sector $[\tilde{\beta},\beta]$ with $\tilde{\beta} \geq -\beta$, $\beta > 0$, then assumption (a2) of Theorem IV.2 is satisfied. Now we can give the following algorithm:

Algorithm:

Data: $\hat{G}(s) \in \mathbb{R}(s)$, with $\hat{G}(s)$ exp. stable, $\underset{\sim}{\Phi}^o$.

Step 1: Use, say, the root locus method to find the (not necessarily connected) set $J \subset \mathbb{R}$ such that $\forall k \in J$, there exists some $\alpha > 0$ such that $\hat{H}_k(s-\alpha)$ is exp. stable (If J is an empty set, then Theorem IV.2 cannot be applied).

Step 2: Choose some $k \in J$ and choose some corresponding α to plot the α-shifted Nyquist diagram $\omega \mapsto \hat{H}_k(-\alpha+j\omega)$, then choose a critical disk $D[-\frac{1}{\tilde{\beta}}, -\frac{1}{\beta}]$ subject to the condition that $\tilde{\beta} \geq -\beta$, $\beta > 0$ (note that after choosing k and α, there is still some freedom in choosing the critical disk).

Step 3: If $D\Phi^o$ belongs to the sector $[\tilde{\beta}+k,\beta+k]$, then by the circle criterion, $(\dot{r}_{k,\alpha},\dot{w}_\alpha) \mapsto (\dot{e}_{k,\alpha},\dot{z}_\alpha)$ is L_2-stable and the conclusions (i) and

(ii) of the Theorem IV.2 follow. If not, go to step 2 and choose a new k, then repeat the process.

A simple example using this algorithm to predict asymptotic tracking is shown as follows

Consider the nonlinear servomechanism shown on Fig. VI.1

where $$\Phi^o(z) = \begin{cases} z^3 + 0.5z & , |z| \leq (\frac{1}{6})^{0.4} \\ (\operatorname{sgn} z)[\sqrt{|z|} - (1/6)^{0.2} + (1/6)^{1.2}] + 0.5z, & |z| \geq (\frac{1}{6})^{0.4} \end{cases} \tag{VI.1}$$

(see Fig. VI.2)

The root locus method shows that for any $k > 0, \hat{H}_k(s) = \hat{H}(s)[1+k\hat{H}(s)]^{-1}$ is exp. stable (see Fig. VI.3). Let

$$k = 0.5 \tag{VI.2}$$

we have

$$\hat{H}_k(s) = \frac{s^2+12s+32}{s^3+8.5s^2+21s+16} \tag{VI.3}$$

and

$$\Phi_k^o(z) = \begin{cases} z^3 & , |z| \leq (1/6)^{0.4} \\ (\operatorname{sgn} z)[\sqrt{|z|} - (1/6)^{0.2} + (1/6)^{1.2}], & |z| \geq (1/6)^{0.4} \end{cases} \tag{VI.4}$$

Note that $D\Phi_k^o$ belongs to the sector $[-0.089, 0.833]$ (See Fig. VI.4). Thus the circle criterion shows that the $\exp(\alpha t)$-weighted system $\dot{S}_{k,\alpha}$ is L_2-stable, since the α-shifted Nyquist diagram $\omega \mapsto \hat{H}_k(-\alpha+j\omega)$ with $\alpha = 1$ lies in the critical disk $[\frac{1}{0.089}, -\frac{1}{0.833}]$. (See Fig. VI.5). Therefore, by Theorem IV.2, the nonlinear servo shown on Fig. VI.1 will track any bounded reference signal $r(\cdot)$ which go to constant at some exp. rate. Figure VI.6 shows the tracking errors $e(\cdot)$ for three different reference signals.

VII. CONCLUSION

In our study of the robust nonlinear, multi-input, multi-output, distributed servomechanism problem, we found that the design principle prescribed by the robust linear servomechanism theory — i.e. replicating the dynamics of the reference and the disturbance signals — is still valid. Our work shows that the robustness of this design extends to (not necessarily small) nonlinear perturbations of the plant. When the output nonlinearity is absent, Theorem III.1 shows that some suitable stability conditions guarantee robust asymptotic tracking for a large class of reference signals, e.g. steps, ramps, parabolas,...,sinusoids, while the plant is subject to nonlinear time-varying input channel perturbations. When the output nonlinearity is present, robust asymptotic tracking and disturbance rejection can be achieved for reference signals and disturbance signals that tend to constants: for this class of reference and disturbance signals, Theorem IV.2 and IV.3 give sufficient conditions for achieving this. Note that in this case, although we know that asymptotic tracking of ramps, parabolas,...and sinusoids is robust with respect to linear perturbations of the linear plant, it may not be robust with respect to nonlinear output channel perturbations; with some additional assumption, we show how to bound the tracking error (sec. IV.C). In section V, we show how some of the conditions required by the previous theorems can be achieved.

APPENDIX

Proof of Theorem III.1:

By assumption (a1), for a given class of $r(\cdot)$, $r_K(\cdot) \in L_2^n$, thus $e(\cdot) \in L_2^n$ since the map $r_K \mapsto e$ restricted to those r_K resulting from the r's under consideration is L_2-stable. Furthermore, $u_K = \underset{\sim}{\Phi}_K^I e \in L_2^n$ since $\underset{\sim}{\Phi}_K^I : L_2^n \to L_2^n$ by (a2). Now $y_K = \underset{\sim}{H}_K u_K = H_K * u_K$, where $u_K \in L_2^n$ and $H_K \in L_2^{nxn}$ by (a3). Thus the Fourier transform of $y_K(\cdot)$, say $\hat{y}_K(j\omega)$, belongs to L_1^n (by Schwartz inequality). Finally, $y_K(t) \to \theta_n$ as $t \to \infty$ by Riemann-Lebesgue lemma. Hence $e(t) = r_K(t) - y_K(t) \to \theta_n$ as $t \to \infty$ since $r_K(t) \to \theta_n$ as $t \to \infty$ by assumption (a1).

Q.E.D

Proof of Theorem IV.1

The feedback system described by (IV.1,2) is characterized by

$$e_2(t) = u_2(t) + \int_0^t M(t-\tau)u_1(\tau)d\tau - \int_0^t M(t-\tau)\Phi[e_2(\tau),\tau]d\tau \qquad \text{(A1)}$$

Premultiplying (A1) by $\exp(\alpha t)$ and using the notations of (IV.3) ~ (IV.5), we obtain

$$e_{2\alpha}(t) = u_{2\alpha}(t) + \int_0^t M_\alpha(t-\tau)u_{1\alpha}(\tau)d\tau - \int_0^t M_\alpha(t-\tau)\Phi_\alpha[e_{2\alpha}(\tau),\tau]d\tau \qquad \text{(A1}_\alpha\text{)}$$

which characterizes the $\exp(\alpha t)$-weighted companion system shown in Fig. IV.1b.

By assumption, the map $(u_{1\alpha},u_{2\alpha}) \mapsto (e_{1\alpha},e_{2\alpha})$ is L_2-stable, thus $\exists\ \mu_2 > 0$ such that for each $T \in \mathbb{R}_+$

$$\|e_{2\alpha,T}\|_2 \le \mu_2(\|u_{1\alpha,T}\|_2 + \|u_{2\alpha,T}\|_2) \qquad \text{(A2)}$$

where $u_{i\alpha,T}$ denotes the function $u_{i\alpha}$ truncated at time T, i = 1,2. Note that $\forall T \in \mathbb{R}_+$, i = 1,2,

$$\|u_{i\alpha,T}\|_2 := \left(\int_0^T |u_{i\alpha}(t)|^2 dt\right)^{1/2}$$

$$= \left(\int_0^T \exp(2\alpha t)|u_i(t)|^2 dt\right)^{1/2}$$

$$\leq \frac{\exp(\alpha T)}{\sqrt{2\alpha}} \|u_i\|_\infty \tag{A3}$$

Hence by (A2) and (A3)

$$\|e_{2\alpha,T}\|_2 \leq \frac{\mu_2 \exp(\alpha T)}{\sqrt{2\alpha}} (\|u_1\|_\infty + \|u_2\|_\infty) \tag{A4}$$

Further, on noting that $\forall t \geq 0$

$$|\Phi(z(t),t)| \leq \beta|z(t)| \Rightarrow |\Phi(z_\alpha(t),t)| \leq \beta|z_\alpha(t)| \tag{A5}$$

we have, $\forall T \in \mathbb{R}_+$

$$\|\Phi(e_{2\alpha}(t),t)_T\|_2 := \left(\int_0^T |\Phi_\alpha(e_{2\alpha}(t),t)|^2 dt\right)^{1/2}$$

$$\leq \beta\left(\int_0^T |e_{2\alpha}(t)|^2 dt\right)^{1/2} \qquad \text{(by assumption (iii))}$$

$$= \beta\|e_{2\alpha,T}\|_2$$

$$\leq \frac{\beta\mu_2 \exp(\alpha T)}{\sqrt{2\alpha}} (\|u_1\|_\infty + \|u_2\|_\infty) \quad \text{(by (A4))} \tag{A6}$$

Now we can prove that $y_1 \in L_\infty^n$: $\forall t \in \mathbb{R}_+$, we have

$$y_1(t) = (M*u_1)(t) - \int_0^t M(t-\tau)\Phi[e_2(\tau),\tau]d\tau \tag{A7}$$

The first term $(M*u_1)(\cdot) \in L_\infty^n$ and $\|M*u_1\|_\infty \leq \|M\|_1 \|u_1\|_\infty$, since $M(\cdot) \in L_1^{nxn}$ (by assumption (i)) and $u_1 \in L_\infty^n$. The second term is also in L_∞^n, this can be seen from the following: let

$$\tilde{y}_1(t) := \int_0^t M(t-\tau)\Phi[e_2(\tau),\tau]d\tau = \exp(-\sigma t)\int_0^t M_\alpha(t-\tau)\Phi_\alpha[e_{2\alpha}(\tau),\tau]d\tau \quad \text{(A8)}$$

then

$$|\tilde{y}_1(t)| \leq \exp(-\alpha t)\int_0^t |M_\alpha(t-\tau)||\Phi_\alpha[e_{2\alpha}(\tau),\tau]|d\tau$$

$$= \exp(-\alpha t)\int_0^\infty |M_\alpha(t-\tau)||\Phi_\alpha[e_{2\alpha}(\tau),\tau]_t|d\tau$$

$$\leq \exp(-\alpha t)\left(\int_0^\infty |M_\alpha(\tau)|^2 d\tau\right)^{1/2}\left(\int_0^\infty |\Phi_\alpha[e_{2\alpha}(\tau),\tau]_t|^2 d\tau\right)^{1/2}$$

(by Schwartz inequality)

$$= \exp(-\alpha t)\|M_\alpha\|_2 \|\Phi_\alpha[e_{2\alpha}(t),t]_t\|_2$$

$$\leq \exp(-\alpha t)\|M_\alpha\|_2 \frac{\beta\mu_2 \exp(\alpha t)}{\sqrt{2\alpha}}(\|u_1\|_\infty + \|u_2\|_\infty) \quad \text{(by (A6))}$$

$$= \frac{\beta\mu_2\|M_\alpha\|_2}{\sqrt{2\alpha}}(\|u_1\|_\infty + \|u_2\|_\infty) \quad \text{(A9)}$$

The right-hand side of (A9) is independent of t, thus

$$\|\tilde{y}_1\|_\infty \leq \tilde{\rho}_1(\|u_1\|_\infty + \|u_2\|_\infty) \quad \text{(A10)}$$

where $\tilde{\rho}_1 := \dfrac{\beta\mu_2}{\sqrt{2\alpha}} \|M_\alpha\|_2$

Return to (A7), we conclude that

$$\|y_1\|_\infty \leq \|M\|_1\|u_1\|_\infty + \tilde{\rho}_1(\|u_1\|_\infty + \|u_2\|_\infty)$$

$$\leq \rho_1(\|u_1\|_\infty + \|u_2\|_\infty) \quad \text{(A11)}$$

where $\rho_1 := \tilde{\rho}_1 + \|M\|_1$

Since $e_2 = y_1 + u_2$, we have

$$\|e_2\|_\infty \leq \|y_1\|_\infty + \|u_2\|_\infty$$

$$\leq \rho_1(\|u_1\|_\infty+\|u_2\|_\infty) + \|u_2\|_\infty \qquad \text{(by (A11))}$$

$$\leq (\rho_1+1)(\|u_1\|_\infty+\|u_2\|_\infty) \tag{A12}$$

Now $\|y_2\|_\infty = \|\Phi[e_2(t),t]\|_\infty$

$$= \operatorname*{ess\,sup}_{t\in\mathbb{R}_+} |\Phi[e_2(t),t]|$$

$$\leq \beta \operatorname*{ess\,sup}_{t\in\mathbb{R}_+} |e_2(t)|$$

$$= \beta\|e_2\|_\infty$$

$$\leq \beta(\rho_1+1)(\|u_1\|_\infty+\|u_2\|_\infty) \qquad \text{(by (A12))} \tag{A13}$$

Finally, $e_1 = u_1 - y_2$, thus

$$\|e_1\|_\infty \leq \|u_1\|_\infty + \|y_2\|_\infty$$

$$\leq \|u_1\|_\infty + \beta(\rho_1+1)[\|u_1\|_\infty+\|u_2\|_\infty] \quad \text{(by (A13))}$$

$$\leq [1+\beta(\rho_1+1)][\|u_1\|_\infty+\|u_2\|_\infty] \tag{A14}$$

From (A11)~(A14), we conclude that the maps $(u_1,u_2) \mapsto (e_1,e_2)$ and $(u_1,u_2) \mapsto (y_1,y_2)$ are L_∞-stable.

Q.E.D

<u>Proof of Corollary IV.1</u>

By assumption (IV.6), $\exists\ \mu_i > 0$ and $\alpha_i > 0$ such that

$$|u_i(t)| \leq \mu_i \exp(-\gamma_i t) \qquad \forall t \geq 0, \qquad i = 1,2.$$

Let $\lambda := \min\{\lambda_1,\lambda_2,\alpha-\delta\}$ where $\delta > 0$ is arbitrarily small. Premultiplying (A2) by $\exp(\lambda t)$, we obtain

$$e_{2\lambda}(t) = u_{2\lambda}(t) + \int_0^t M_\lambda(t-\tau)u_{1\lambda}(\tau)d\tau - \int_0^t M_\lambda(t-\tau)\Phi_\lambda[e_{2\lambda}(\tau),\tau]d\tau \tag{A15}$$

where the subscript λ has the same meaning as the α defined in (IV.5).

Note that

(a) the functions $t \mapsto u_{i\lambda}(t)$ is bounded on $\mathbb{R}_+$ since $\lambda = \min\{\gamma_1,\gamma_2,\alpha-\delta\}$;

(b) the map $z_\lambda(t) \mapsto \Phi_\lambda(z_\lambda(t),t)$ satisfies $|\Phi_\lambda(z_\lambda(t),t)| \leq \beta|z_\lambda(t)|$, $\forall t \geq 0$;

(c) $M_\lambda(\cdot) \in L_1^{nxn}$ by assumption (IV.7);

(d) $\exp[(\alpha-\lambda)t]\ M_\lambda(t) = M_\alpha(t) \in L_2^{nxn}$.

Thus by the Theorem IV.1 with α being substituted by $\alpha-\lambda$, we conclude that $\exists\ \mu_{ei} > 0$ and $\mu_{yi} > 0$ such that

$$|e_{i\lambda}(t)| = \exp(\lambda t)|e_i(t)| \leq \mu_{ei}$$

$$|y_{i\lambda}(t)| = \exp(\lambda t)|y_i(t)| \leq \mu_{yi}$$

Hence the conclusion of (IV.8) and (IV.9) follow. Q.E.D.

Proof of Proposition V.1

In the following we count zeros according to their multiplicities. We prove this proposition in three steps:

Step 1: We claim that $\exists\ \varepsilon_1 > 0$ such that $\forall \varepsilon \in (0,\varepsilon_1]$, $\det[sI+\varepsilon\hat{G}(s)]$ has n zeros close to $s = 0$; furthermore, they belong to $\mathring{\mathbb{C}}_-$. To see this, note that the Taylor series expansion of $\varepsilon\hat{G}(s)$ at $s = 0$ gives

$$\varepsilon\hat{G}(s) = \varepsilon\hat{G}(0) + s\varepsilon\ \hat{G}'(0) + \ldots \tag{A16}$$

Thus, $\exists\ \varepsilon_1 > 0$ such that $\forall \varepsilon \in (0,\varepsilon_1]$, the zeros of $\det[sI+\varepsilon\hat{G}(s)]$ around $s = 0$ are given by (within $O(\varepsilon^2)$) the n eigenvalues of $-\varepsilon\hat{G}(0)$. By assumption, $\sigma[\hat{G}(0)] \subset \mathring{\mathbb{C}}_+$, hence these n zeros are in $\mathring{\mathbb{C}}_-$.

Step 2: We claim that given any $\zeta \in (0,\nu)$, $\exists\ \varepsilon_2 > 0$ such that $\forall \varepsilon \in [-\varepsilon_2,\varepsilon_2]$, $\det[sI+\varepsilon\hat{G}(s)]$ has only n zeros in $\mathrm{Re}\ s \geq -\zeta$. To see this, let

$$A = \{s \in \mathbb{C} | \mathrm{Re}\ s > -\nu\} \subset \mathbb{C}$$

$$B = \{s \in \mathbb{C} | \mathrm{Re}\ s > -\zeta\ ,\ |s+\zeta| < \xi\} \subset A$$

where ξ is chosen such that $\det[sI+\varepsilon\hat{G}(s)]$ has no zeros in the set $\{s \in \mathbb{C} | \mathrm{Re}\ s \geq -\zeta,\ |s+\zeta| \geq \xi\}$ (this is possible since for $|s|$ sufficiently large, we have asymptotically $\det[sI+\varepsilon\hat{G}(s)] \sim \det[sI] = s^n$). Note that (i) for each $\varepsilon \in \mathbb{R}$, $\det[sI+\varepsilon\hat{G}(s)]$ is analytic in A; (ii) $\bar{B} \subset A$ is compact; (iii) when $\varepsilon = 0$, the n zeros of $\det[sI+0\cdot\hat{G}(s)] = s^n$ are at $0 \in \mathbb{C}$ which is in the interior of B. Hence, by the "Continuity of the roots of an equation as a function of parameters" (Dieudonné 1969, pp. 248) $\exists\ \varepsilon_2 > 0$ such that (a) $\forall \varepsilon \in [-\varepsilon_2,\varepsilon_2]$, $\det[sI+\varepsilon\hat{G}(s)]$ has no zeros on the boundary of B; (b) $\forall \varepsilon \in [-\varepsilon_2,\varepsilon_2]$, $\det[sI+\varepsilon\hat{G}(s)]$ has only n zeros in B.

Step 3: From step 1 and step 2, given any $\zeta \in (0,\nu)$, with $\varepsilon_3 := \min\{\varepsilon_1,\varepsilon_2\}$; we have that $\forall \varepsilon \in (0,\varepsilon_3]$, $\det[sI+\varepsilon\hat{G}(s)]$ has n zeros $\{z_1(\varepsilon),z_2(\varepsilon),\ldots,z_n(\varepsilon)\}$ such that $-\zeta < \mathrm{Re}\ z_i(\varepsilon) < 0$, for $i = 1,2,\ldots,n$. Fix $\varepsilon \in (0,-\varepsilon_3)$, choose $\alpha > 0$ such that $\max\limits_{1\leq i\leq n} \mathrm{Re}(z_i) < -\alpha < 0$, then we have $\det[sI+\varepsilon\hat{G}(s)] \neq 0$, $\forall s \in \mathbb{C}$ with $\mathrm{Re}\ s \geq -\alpha$, (note $\nu > \zeta > \alpha > 0$). Consequently, since $\frac{s}{s+1} \to 1$ as $|s| \to \infty$, we have

$$\inf_{\mathrm{Re}\ s \geq -\alpha} \left|\det\left[\frac{s}{s+1} I + \frac{\varepsilon}{s+1} \hat{G}(s)\right]\right| > 0 \tag{A17}$$

Now, since K is chosen to be εI

$$\hat{H}_K := \hat{H}(I+\varepsilon\hat{H})^{-1}$$

$$= \hat{G}(sI+\varepsilon\hat{G})^{-1}$$

$$= \frac{\hat{G}}{s+1}\left[\frac{s}{s+1} I + \frac{\varepsilon}{s+1} \hat{G}\right]^{-1} \tag{A18}$$

where $\frac{\hat{G}(s)}{s+1}$ and $\left[\frac{s}{s+1} I + \frac{\varepsilon}{s+1} \hat{G}(s)\right] \in \hat{\mathcal{A}}(-\alpha)^{n\times n} \subset \hat{\mathcal{A}}(-\nu)^{n\times n}$.

By (Desoer & Vidyasagar 1975, p. 250, Corollary 3) and (A17),

$$\left[\frac{s}{s+1} I + \frac{\varepsilon}{s+1} \hat{G}\right]^{-1} \in \hat{\mathcal{A}}(-\alpha)^{n\times n} \tag{A19}$$

Finally, by the closure of the algebra $\hat{\mathcal{A}}(-\alpha)^{n\times n}$, we conclude from (A18) and (A19), $\hat{H}_K \in \hat{\mathcal{A}}(-\alpha)^{n\times n}$. Q.E.D

<u>Proof of Corollary V.1</u>

Let $N(s)\ D(s)^{-1}$ be a right coprime factorization of $\hat{G}(s)$. Then $\hat{H}_K(s) = N(s)[sD(s)+\varepsilon N(s)]^{-1}$. Now $p(s,\varepsilon) := \det[sD(s)+\varepsilon N(s)]$ is a polynomial in s with coefficients which are <u>continuous</u> functions of ε. It is sufficient[(3)] to show that for sufficiently small $\varepsilon > 0$, $\mathcal{Z}[p(s,\varepsilon)] \subset \mathring{\mathbb{C}}_-$, then $\hat{H}_K(s)$ is exp. stable. Note that

$$p(s,\varepsilon) = \det D(s) \times \det[sI+\varepsilon\hat{G}(s)] \tag{A20}$$

As shown in step 1 of Proposition V.1, we know that for sufficiently

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[(3)] Actually, $\hat{H}_K$ is exp. stable $\Leftrightarrow$ $\mathcal{Z}[p(s,\varepsilon)] \subset \mathring{\mathbb{C}}_-$ since $N(s)$, $sD(s) + \varepsilon N(s)$ are right coprime if $(N(s), D(s))$ is right coprime and $\hat{G}(0)$ is nonsingular.

small $\varepsilon > 0$ the zeros of $\det[sI+\varepsilon\hat{G}(s)]$ around $s = 0$ are (within $O(\varepsilon^2)$) the eigenvalues of $-\varepsilon\hat{G}(0)$ and are in $\mathring{\mathbb{C}}_-$ (by assumption $\sigma[\hat{G}(0)] \subset \mathring{\mathbb{C}}_+$). Now $\mathcal{Z}[p(s,0)] = \mathcal{Z}[s^n \det D(s)] = \mathcal{Z}[\det D(s)] \cup \underbrace{\{0,0,\dots,0\}}_{\text{n-tuple}}$ and

$$\partial p(s,\varepsilon) = n + \partial[\det D(s)] \quad \text{(because } \hat{G}(s) \text{ is proper)} \tag{A21}$$

Therefore for $\varepsilon > 0$ sufficiently small, by continuity of the zeros of the polynomial $p(s,\varepsilon)$ as a function of ε, the $n + \partial[\det D(s)]$ zeros of $p(s,\varepsilon)$ are close to those of $p(s,0)$. We have seen that the n zeros of $p(s,\varepsilon)$ which are close to zero are in $\mathring{\mathbb{C}}_-$; the remaining $\partial[\det D(s)]$ zeros are close to the zeros of $\det D(s)$ which are in $\mathring{\mathbb{C}}_-$ by the assumption that $\hat{G}(s)$ is exp. stable. Hence, for sufficiently small $\varepsilon > 0$, $\mathcal{Z}[p(s,\varepsilon)] \subset \mathring{\mathbb{C}}_-$. This implies that $\hat{H}_K$ is exp. stable. Q.E.D

Proof of Proposition V.2

Note that

$$\hat{H}_K := \hat{H}(I+K\hat{H})^{-1}$$

$$= \hat{G}[sI+K\hat{G}]^{-1} \tag{A23}$$

Thus,

$$\hat{H}_K[sI+K\hat{G}] = \hat{G}, \quad \text{or}$$

$$s\hat{H}_K = \hat{G} - \hat{H}_K K\hat{G} \tag{A24}$$

Since, by assumption, $\hat{G} \in \hat{\mathcal{A}}(-\nu)^{n\times n} \subset \hat{\mathcal{A}}(-\alpha)^{n\times n}$ and $\hat{H}_K \in \hat{\mathcal{A}}(-\alpha)^{n\times n}$, we conclude that by the closure of the algegra $\hat{\mathcal{A}}(-\alpha)^{n\times n}$ and (A24), $s\hat{H}_K \in \hat{\mathcal{A}}(-\alpha)^{n\times n}$, i.e. $\dot{H}_K \in \mathcal{A}(-\alpha)^{n\times n}$. Therefore H_K contains no impulse functions and $\exp(\alpha t)H_K(t) \in L_1^{n\times n}$ (by assumption $\hat{H}_K \in \hat{\mathcal{A}}(-\alpha)^{n\times n}$). Now consider the Fourier transform of $\exp(\alpha t)H_K(t)$ which is given by $\hat{H}_K(-\alpha+j\omega) = \hat{G}(-\alpha+j\omega)[(-\alpha+j\omega)I + K\hat{G}(-\alpha+j\omega)]^{-1}$ (see (A23)). For $|\omega|$ large, since $\hat{G}(-\alpha+j\omega)$ is bounded, elements of $\hat{H}_K(-\alpha+j\omega)$ is of $O(\frac{1}{|\omega|})$. Thus $\hat{H}_K(-\alpha+j\omega) \in L_2^{nxn}$ and therefore $\exp(\alpha t)H_K(t) \in L_2^{nxn}$. Q.E.D.

Proof of Corollary V.2

Following from Proposition V.2, $\exp(\alpha t)H_K(t) \in L_1^{nxn}$. Now from (A24) we have

$$\dot{H}_K(t) = G(t) - (H_K*KG)(t) \tag{A25}$$

Since $\exp(\nu t)G(t)$, $\exp(\alpha t)H(t) \in L_1^{nxn}$, $(\nu>\alpha>0)$ we have

$$\exp(\alpha t)\dot{H}_K(t) \in L_1^{nxn} \tag{A26}$$

Now for each $t \in \mathbb{R}_+$,

$$\|\exp(\alpha\cdot)\dot{H}_K(\cdot)\|_1 = \int_0^\infty |\exp(\alpha t')\dot{H}_K(t')|dt'$$

$$\geq \left|\int_0^t \exp(\alpha t')\dot{H}_K(t')dt'\right|$$

$$= \left|\exp(\alpha t)H_K(t) - H_K(0_+) - \alpha\int_0^t \exp(\alpha t')H_K(t')dt'\right| \qquad \text{(integration by parts)}$$

$$\geq |\exp(\alpha t)H_K(t)| - |H_K(0_+)| - \alpha\left|\int_0^t \exp(\alpha t')H_K(t')dt'\right| \qquad \text{(triangular inequality)}$$

$$\geq |\exp(\alpha t)H_K(t)| - |H_K(0_+)| - \alpha\|\exp(\alpha\cdot)H_K(\cdot)\|_1 \qquad \text{(by Proposition V.2)}$$

Thus, for some m,

$$|\exp(\alpha t)H_K(t)| \leq m < \infty, \quad \forall t \in \mathbb{R}_+$$

which implies that $\exp(\alpha t)H_K(t) \in L_\infty^{nxn}$.

Therefore $\exp(\alpha t)H_K(t) \in L_1^{nxn} \cap L_\infty^{nxn}$ which implies that $\exp(\alpha t)H_K(t) \in L_p^{nxn}$, $\forall p \in [1,\infty]$. Q.E.D.

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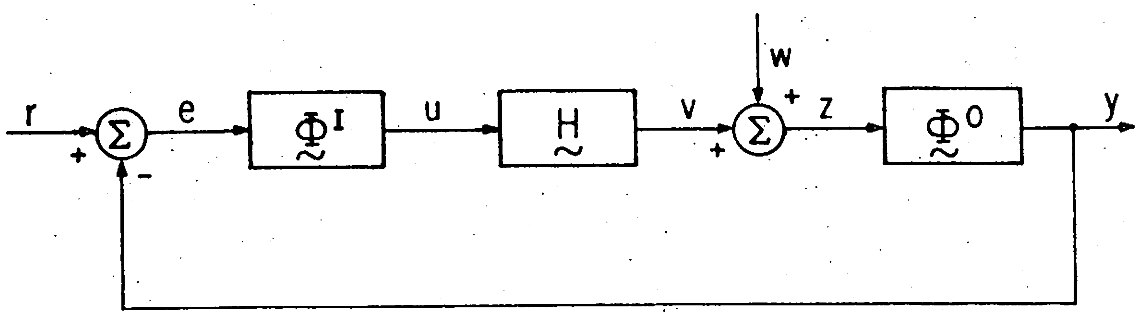

Fig. II.1. Nonlinear servomechanism S under study: $r(\cdot)$, $w(\cdot)$ are the reference and disturbance signals, respectively; $e(\cdot)$ is the tracking error.

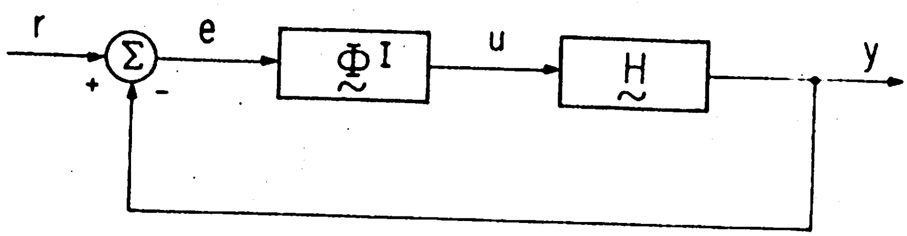

Fig. III.1. Nonlinear servomechanism S^I: a special case of S - the output nonlinearity is an identity, i.e. $\Phi^o = I$.

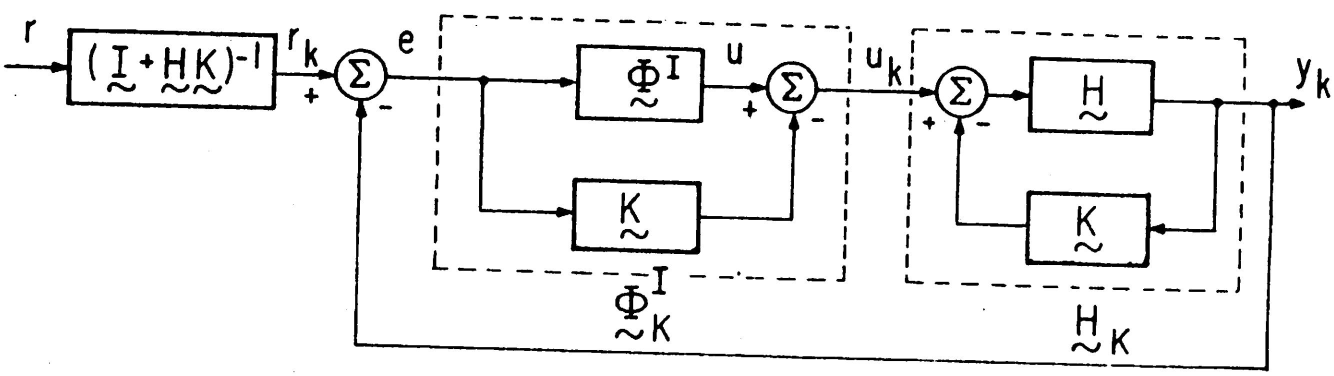

Fig. III.2. Nonlinear feedback system S^I_K: the loop-transformed system of S^I.

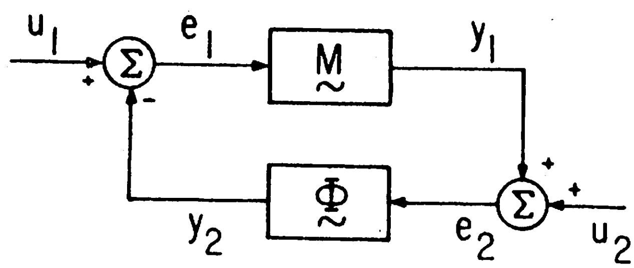

Fig. IV.1a. The feedback system under the consideration for L_∞-stability.

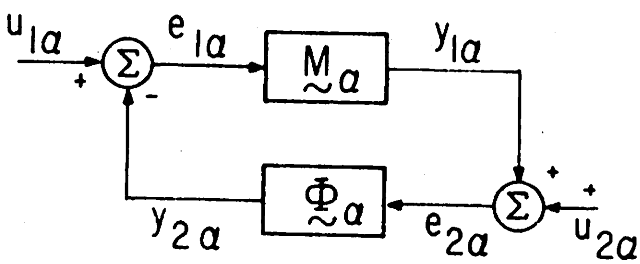

Fig. IV.1b. The exp(αt)-weighted feedback system of the system shown on Fig. IV.1a.

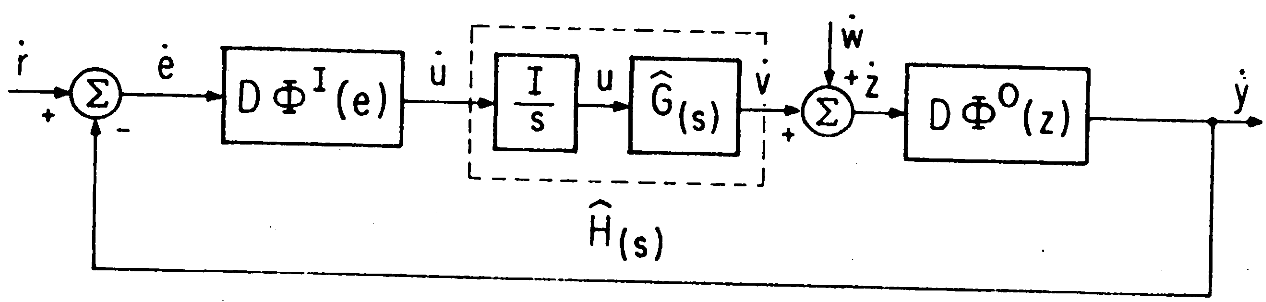

Fig. IV.2. The feedback system $\dot{S}$ which relates the derivatives of the signals in S.

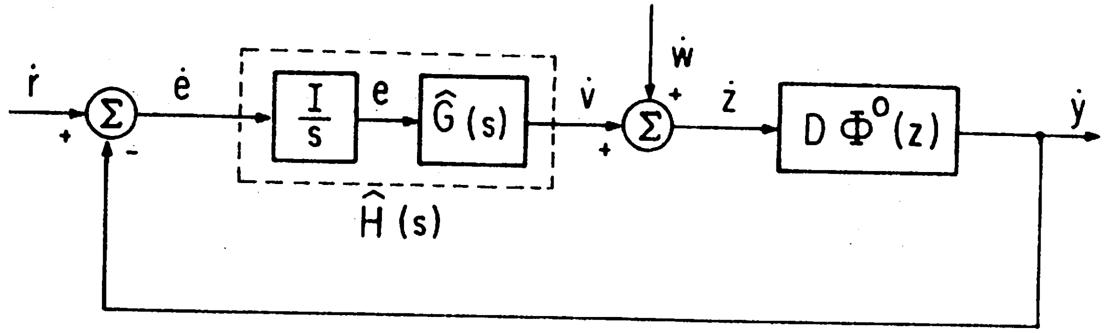

Fig. IV.3a. A special case of $\dot{S}$: $\dot{S}^o$ is obtained from $\dot{S}$ by letting $\underset{\sim}{\Phi}^I = \underset{\sim}{I}$.

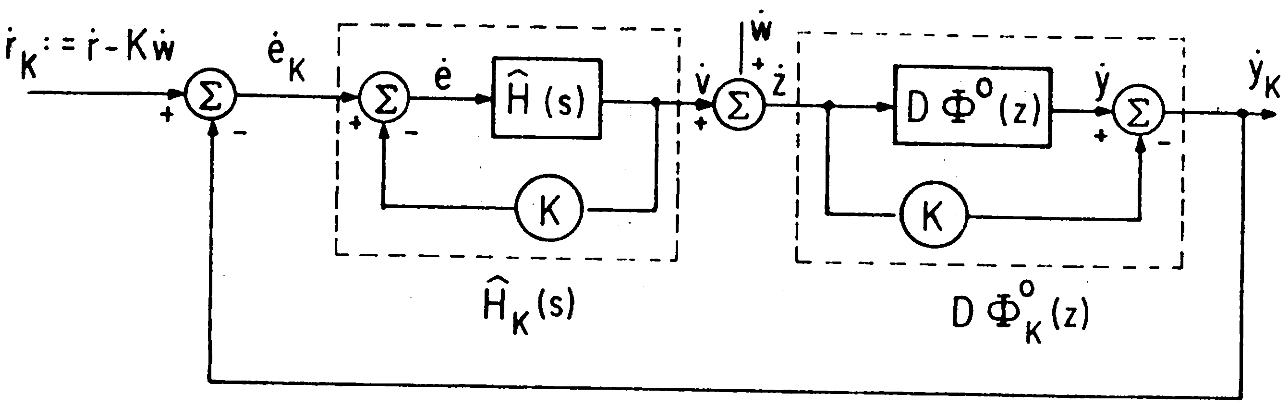

Fig. IV.3b. The K-shifted system $\dot{S}^o_K$ of the system $\dot{S}^o$ shown on Fig. IV.3a.

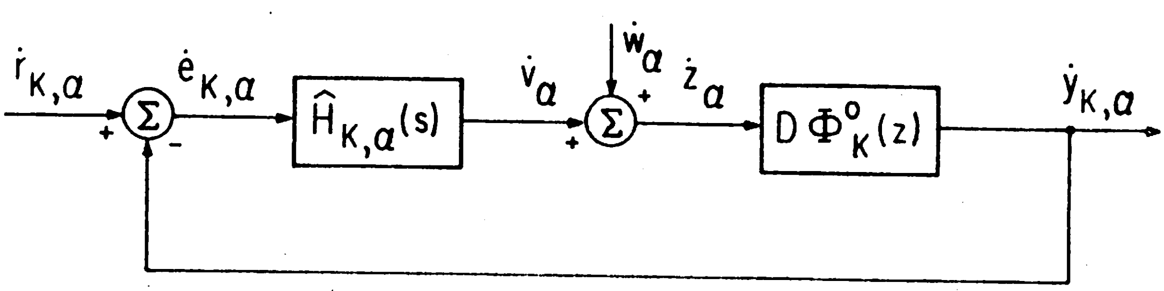

Fig. IV.3c. The $\exp(\alpha t)$-weighted system $\dot{S}^o_{K,\alpha}$ of the system $\dot{S}^o_K$ shown on Fig. IV.3b.

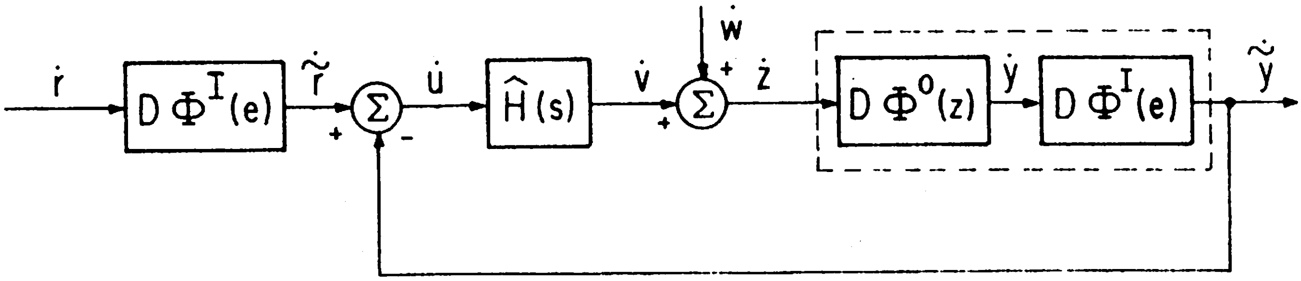

Fig. IV.4. The transformed feedback system of $\dot{S}$ which has the same structure as the system $\dot{S}^o$.

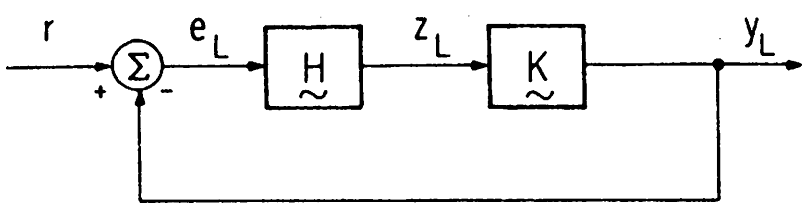

Fig. IV.5. The lumped linear time-invariant servo under consideration.

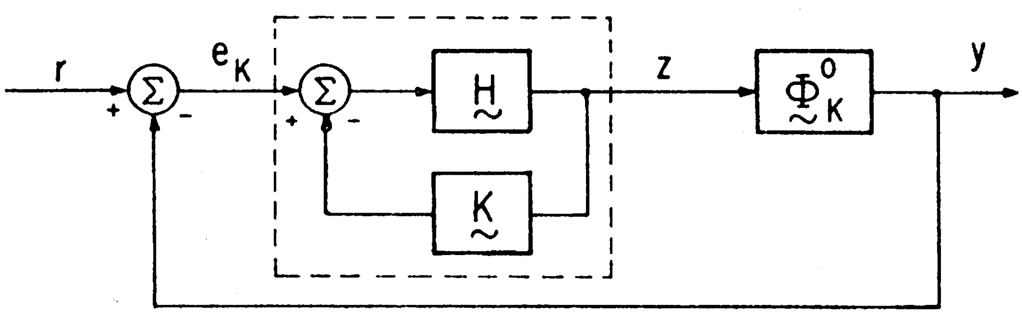

Fig. IV.6. The $\underset{\sim}{K}$-shifted system of the nonlinear system S with $\underset{\sim}{\Phi}^I = \underset{\sim}{I}$.

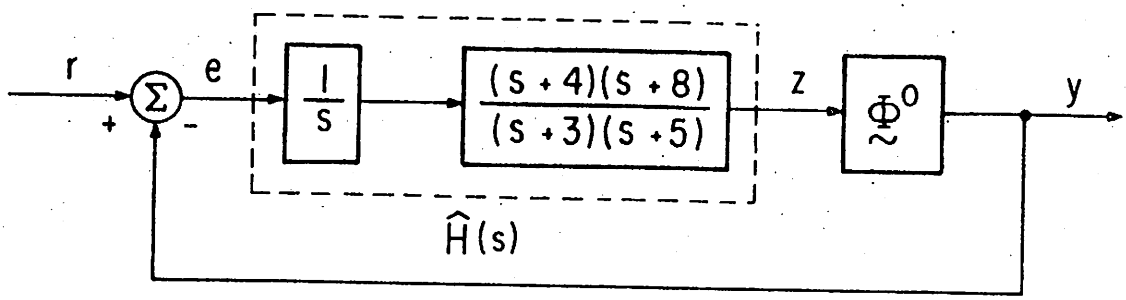

Fig. VI.1. An example of a single-input single-output nonlinear servo.

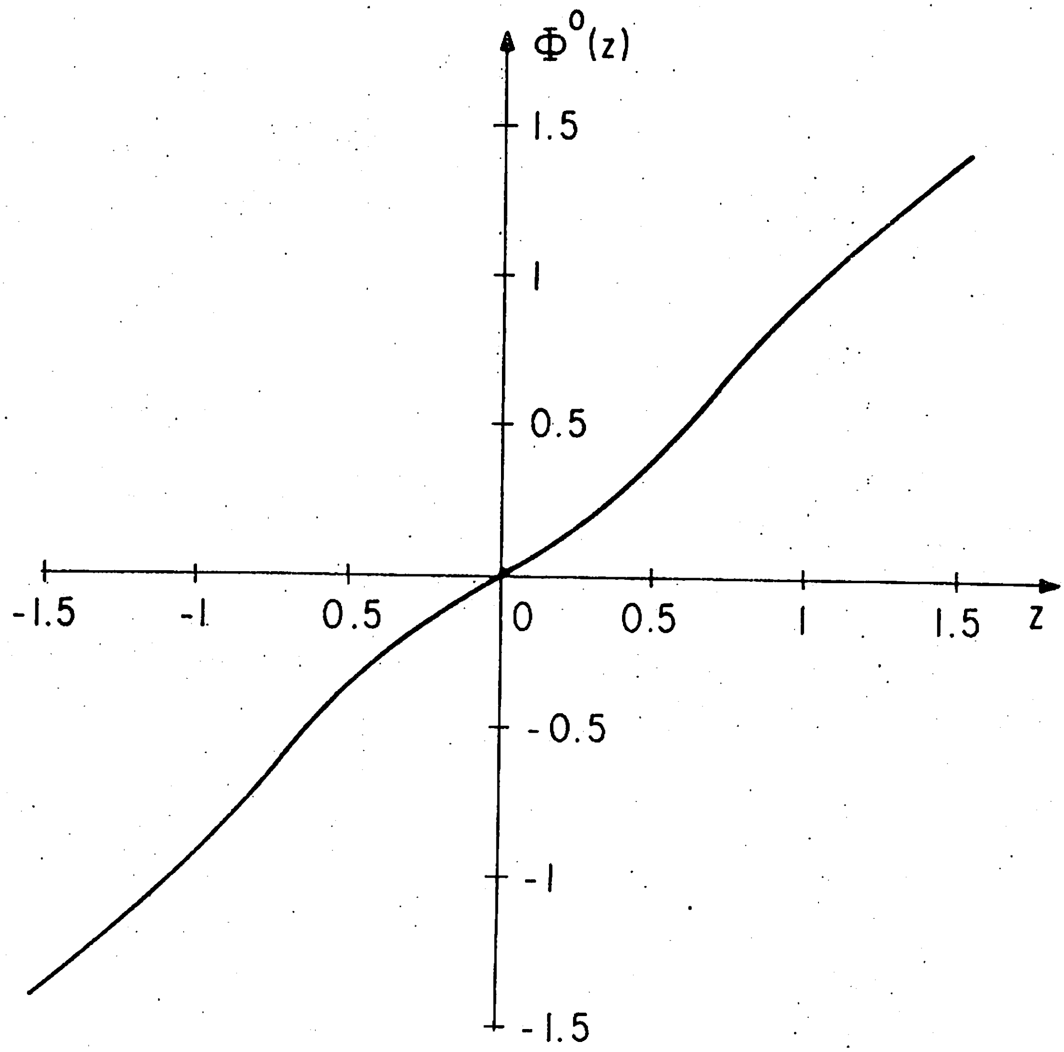

Fig. VI.2. The characteristic of $\underset{\sim}{\Phi}^{o}$ – the output nonlinearity in the nonlinear servo shown on Fig. VI.1.

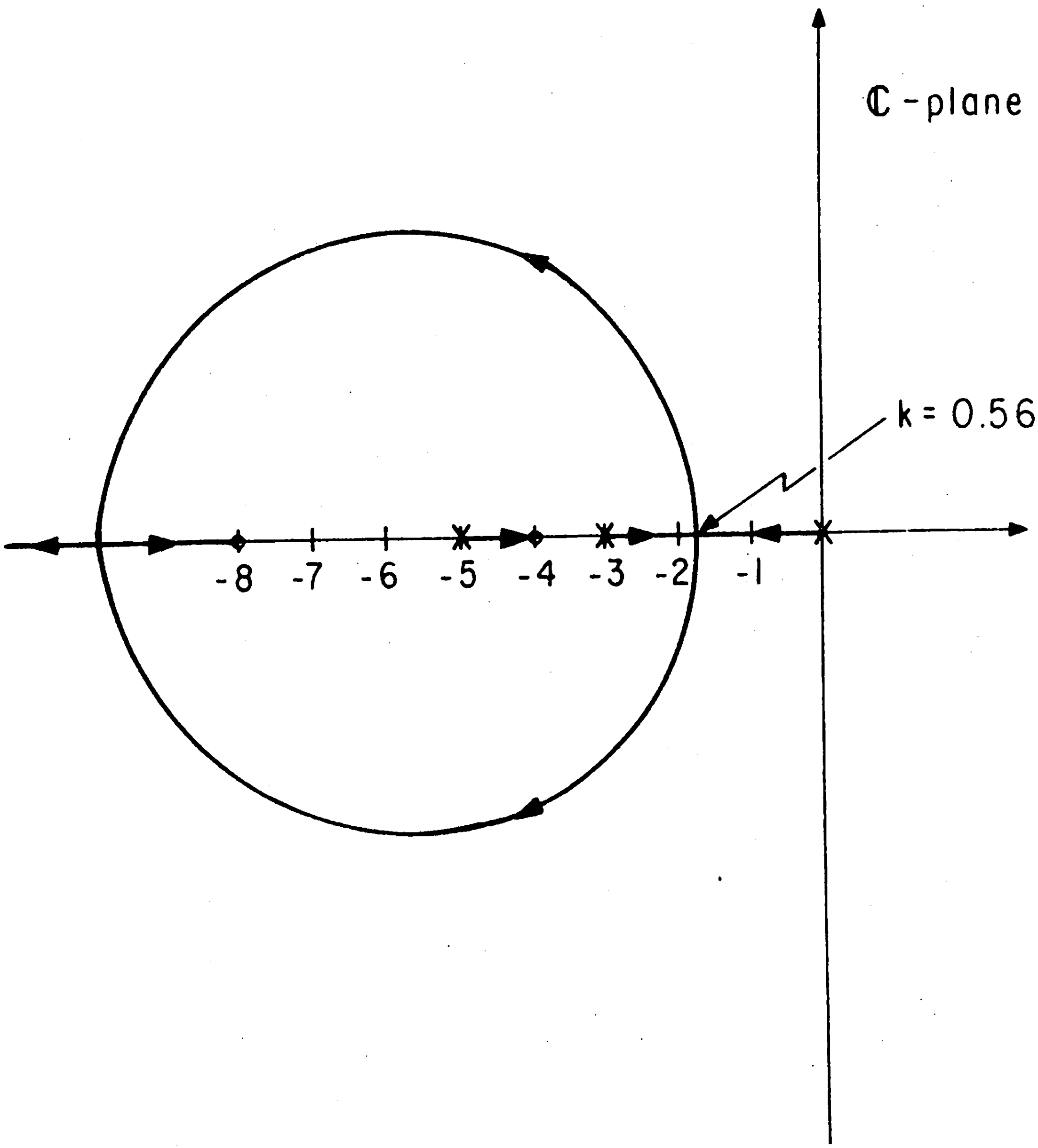

Fig. VI.3. Sketch of the root loci of $s^3 + (s+k)s^2 + (15+12k)s + 32k$.

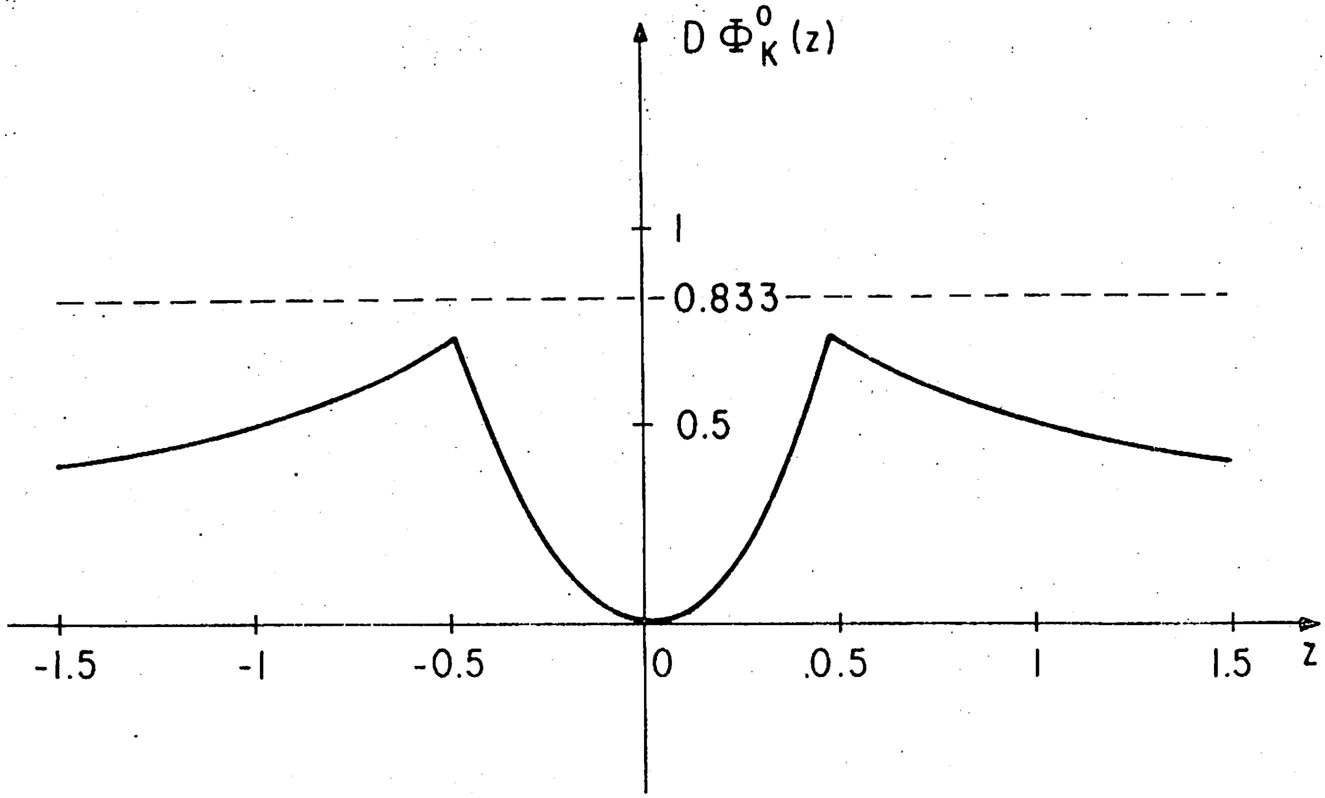

Fig. VI.4. The value of $D\Phi_K^o$ evaluated at z with k = 0.5. Note that the time-varying gain $-0.089 < D\Phi_k^o(z) < 0.833$, $\forall z \in \mathbb{R}$.

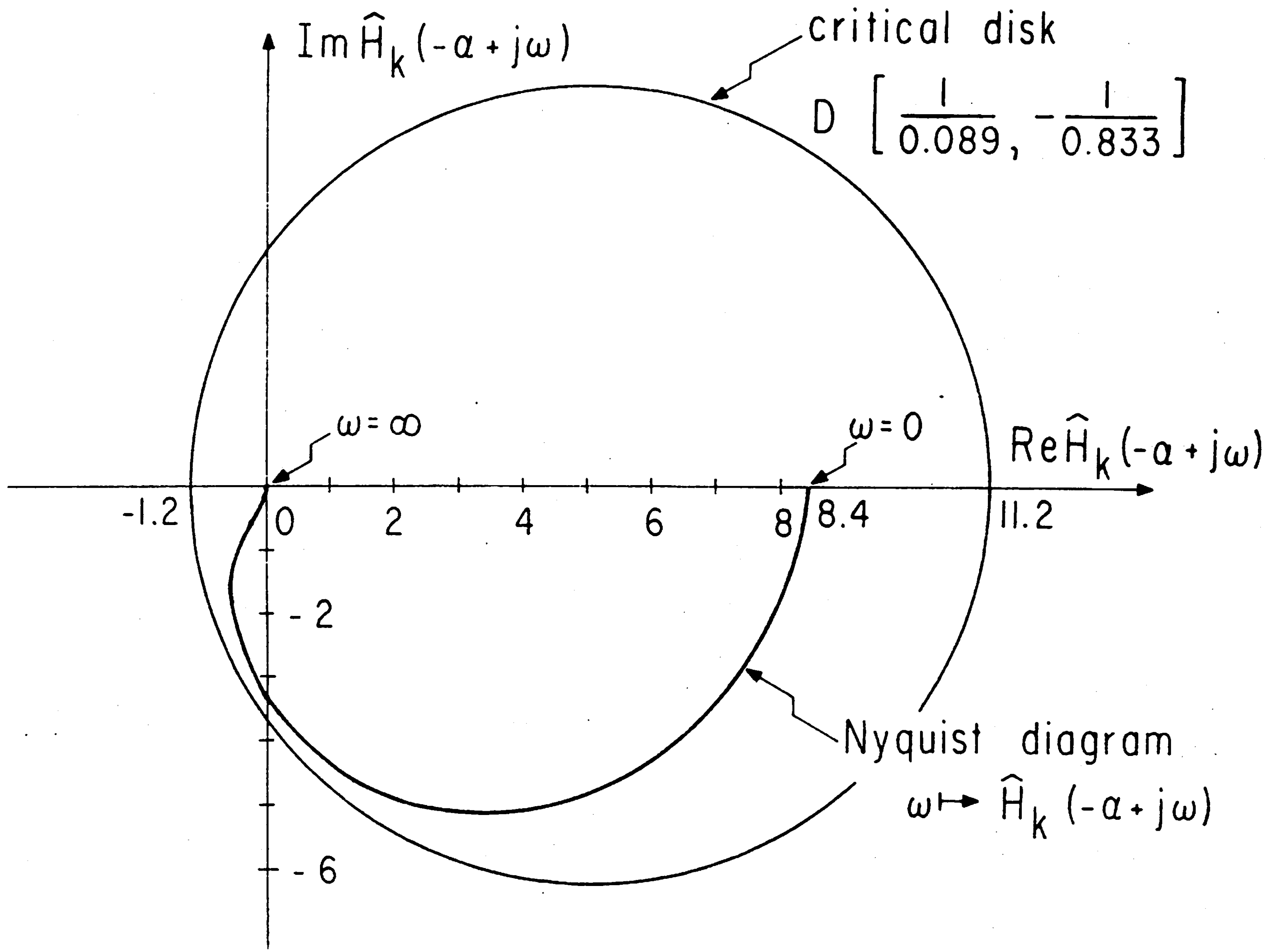

Fig. VI.5. The α-shifted Nyquist diagram $\omega \to \hat{H}_k(-\alpha+j\omega)$, with $k = 0.5$, $\alpha = 1$, $\hat{H}_k(s) = \dfrac{s^2+12s+32}{s^3+8.5s^2+21s+16}$, lies in the critical disk $D[\frac{1}{0.089}, -\frac{1}{0.833}]$ with center (5.0,0) and radius 6.2.

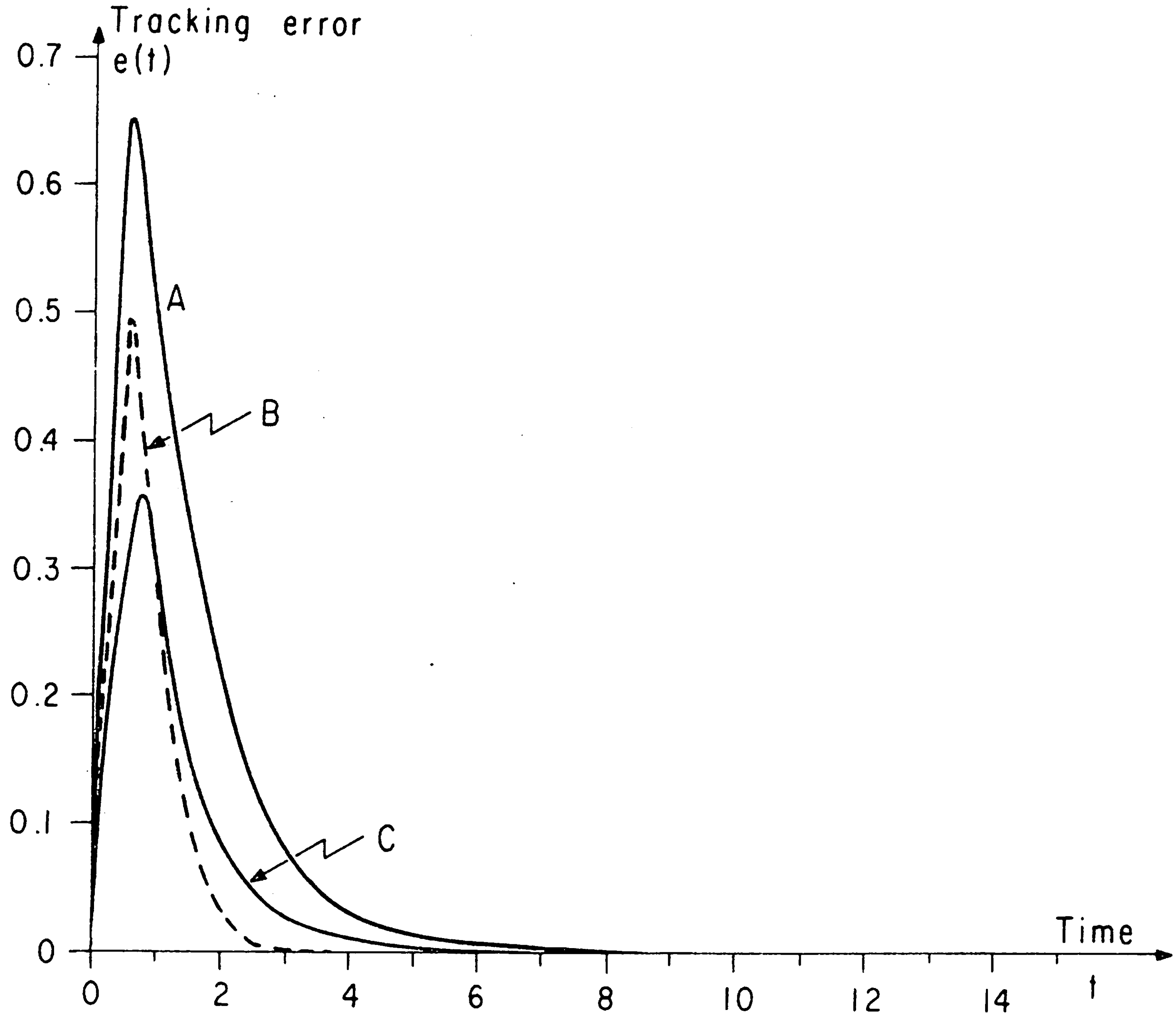

Fig. VI.6. Tracking error $e(\cdot)$ for three different reference signals

$$r(t) = \begin{cases} 2-2e^{-t} & \text{in A} \\ 1-e^{-2t} & \text{in B} \\ 1-e^{-t}. & \text{in C} \end{cases}$$