GO is P space Hard

by

D. Lichtenstein and M. Sipser

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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David Lichtenstein*
Michael Sipser **

Computer Science Division
Department of Electrical Engineering and Computer Science
University of California
Berkeley, California 94720

1. Introduction

A great deal of effort has been spent in the search for optimal and computationally feasible game strategies. In some cases (e.g. Bridge-it, Nim) such strategies have been found, while others have been more resistant. Recently, it has become possible to provide compelling evidence that such strategies may not always exist. Even and Tarjan [1] and Schaefer [2] have shown that determining which player has a winning strategy in certain combinatorial games is a polynomial space complete problem [3]. (See also [4,5].)

We show that GO, a popular Oriental game with a long history, has a similar property. That is, given an arbitrary GO position on an n X n board, the problem of determining the winner is pspace-hard. To our knowledge, this is the first such result for a board game. Board games are, by their nature, planar -- a property which frequently complicates completeness proofs. We exploit a new technique developed in [6] to overcome this difficulty.

In practice, GO is played on a 19 X 19 board. As such it is a finite game for which a large table containing a winning

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strategy could, in principle, be given. Our generalization to an n X n board prevents such a solution while preserving the spirit of the game. We make no further modifications to the rules.

We prove that GO is pspace-hard rather than pspace-complete because GO is not known to be in pspace. If there were a polynomial bound on the length of GO games, then the completeness would follow trivially. While it happens that actual games seem never to approach 19^2 moves, we are unable to argue this in general. Finally, we acknowledge that our result has no *a priori* relevance to the problem of determining an optimal strategy when play begins on an empty board.

2. Preliminaries

Definition: The Quantified Boolean Formula Problem (QBF) = $\{Q_1 v_1 Q_2 v_2, \ldots, Q_n v_n F(v_1, v_2, \ldots, v_n) \mid Q_i \in \{\forall, \exists\}$, where the v_i are boolean variables, F is a boolean formula in conjunctive normal form with at most three variables in each of m clauses (3CNF), and the quantified formula is true$\}$. Wlog we demand that $Q_1 = \exists$, $Q_n = \forall$, and that $Q_i \neq Q_{i+1}$, for $1 \leq i < n$. In addition, m_i is the number of (possibly negated) occurrences of the variable v_i in F.

Definition: Q3CNF is the set of well formed formulae, as above, but which may be true or false.

Theorem 1: QBF is logspace complete for pspace. [1]

Given a boolean formula $B \in$ Q3CNF, we use theorem 3 to reduce it to a planar formula PB, theorems 2 and 4 to reduce PB to a planar geography game, PGG, and finally show how PGG can be transformed into an equivalent GO game. For the sake of motivating the definition of planar formulae, however, we first outline a simple proof of the pspace completeness of generalized geography.

3. Generalized Geography

Definition: Generalized Geography (GG) is a game played by two players on the nodes of a directed graph. Play begins when the first player puts a marker on a distinguished node. In subsequent turns, players alternately place a marker on any unmarked node q, such that there is a directed arc from the last node played to q. The first player who cannot move loses.

This is a generalization of a commonly played game in which players must name a place not yet mentioned in the game, and whose first letter is the same as the last letter of the last place named. The first player to be stumped loses. This instance of geography would be modeled by a graph with as many nodes as there are places. Directed arcs would go from a node, u, to all those nodes whose first letters are the same as u`s last letter.

Theorem 2: GG is logspace complete for pspace. [2]

Proof: Given a formula B $\in$ Q3CNF, $B=Q_1v_1,Q_2v_2,\ldots,Q_nv_nF(v_1,v_2,\ldots,v_n)$, we construct the following graph, GG(B):

Each variable, v_i, is represented by a diamond structure,

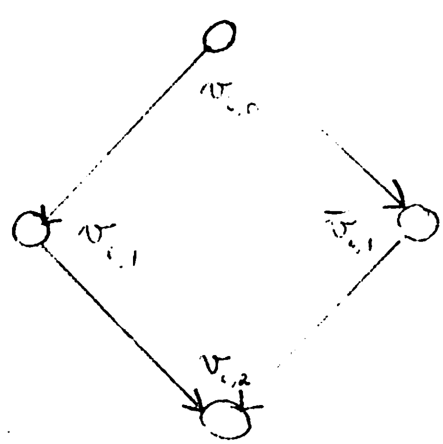


Fig. 1

and each clause, c_j, is represented by a single node. In addition, we have arcs $(v_{i,2}, v_{i+1,0})$ for $1 \leq i < n$, $(c_j, v_{i,1})$ for $v_i \in c_j$, and $(c_j, \bar{v}_{i,1})$ for $\bar{v}_i \in c_j$, and $(v_{n,2}, c_j)$ for $1 \leq j < m$. $v_{1,0}$ is the distinguished node.

Example:

$B = (\exists a)(\forall b)(\exists c)(\forall d)$
$[(a+\bar{b}+c)(b+\bar{d})]$

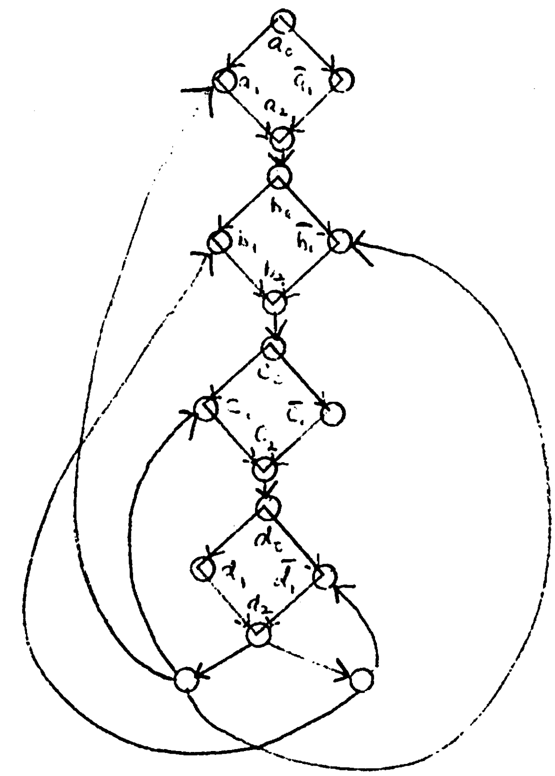


Fig 2

Play proceeds rather simply. One player chooses which path to take through $\forall$-diamonds (i.e. diamonds representing universally quantified variables), and the other player choses which path to take through $\exists$-diamonds. After all diamonds have been traversed, the $\forall$-player choses a clause, and the $\exists$-player then choses a variable from that clause. $\exists$ then wins immediately if the chosen variable satisfies the clause; otherwise, $\forall$ wins on the next move. It follows easily that $\exists$ wins iff B is true, and we leave the details to the reader.

4. Planar Formulae

Definition: Let $B \in Q3CNF$. We call $G(B) = (N,A)$ the graph of B, where $N=\{c_j \mid 1 \leq j \leq m\} \cup \{v_i \mid 1 \leq i \leq n\}$

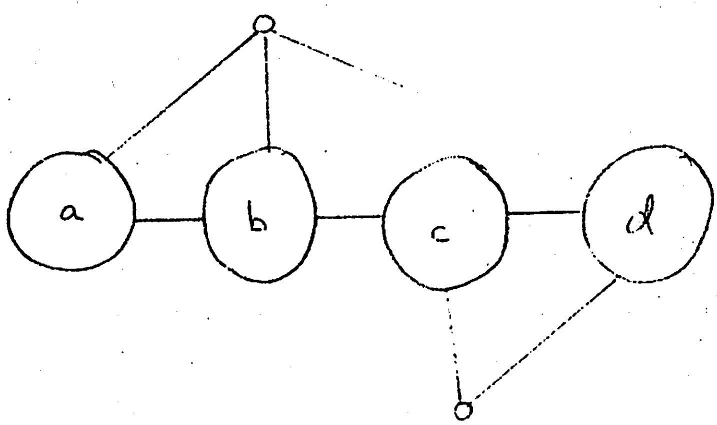
$A=A_1 \cup A_2$

where

$A_1=\{ \{c_i, v_j\} \mid v_j \in c_i \text{ or } \bar{v}_j \in c_i\}$

and

$A_2=\{ \{ v_j, v_{j+1}\} \mid 1<j<n\}$



Example.

$B=(\exists a)(\forall b)(\exists c)(\forall d)\left[(a+\bar{b}+d)\,(c+\bar{d})\right]$

Note that $(a+b+d)$ would give the same arcs as $(a+\bar{b}+d)$.

fig 3

Definition: The Planar Quantified Boolean Formula Problem (PQBF) is QBF restricted to formulae B such that G(B) is planar.

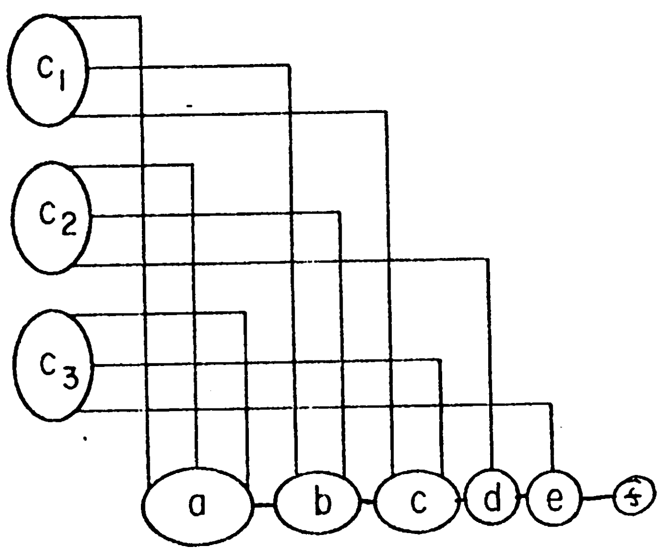
Theorem 3: PQBF is pspace complete.

Proof: We give a polynomial time algorithm that converts a formula B in Q3CNF into a formula PB such that:

i) G(PB) is planar

ii) PB <==> B

The algorithm proceeds as follows: Draw G(B) on a grid. The grid is 3m x 3m, with nodes arranged on the left and bottom borders. The set of clauses $\{c_i\}$ is along the left border, with each node 3 grid lines high. The set of variables $\{v_j\}$ is along the bottom border, with the width of a node v_i equal to m_i. Grid lines are then darkened in the obvious manner, so that each arc in A_1 consists of a horizontal segment and a vertical segment. A_2 is obtained simply by joining adjacent variables with an arc.



Example: $B = (\exists a)(\forall b)(\exists c)(\forall d)(\exists e)(\forall f)$
$[(a+\bar{b}+c)(a+\bar{b}+\bar{d})(\bar{a}+\bar{c}+e)]$ Fig 4

We now modify the formula so that non-planarity is eliminated in A_1 and then further modify the formula so that A_2 can be drawn without introducing non-planarity.

Pick a point in the graph where two arcs cross, involving, for instance, the variables a and b.

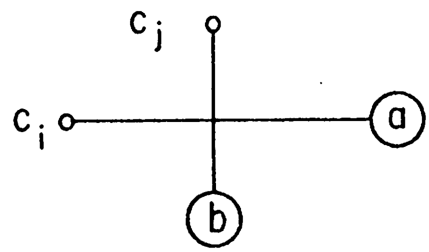


fig. 5

Replace that section of the graph by the following subgraph, G(X),

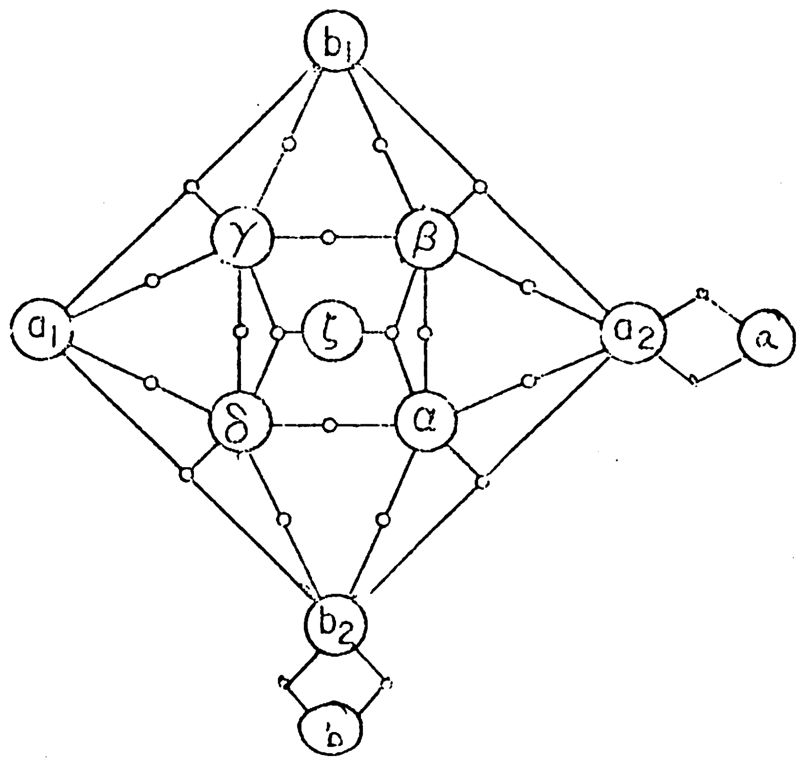


fig 6

where the small unlabeled nodes in the picture represent clauses of X, which is appended to B. X is comprised of:

$$(\bar{a}_2+\bar{b}_2+\alpha)(a_2+\bar{\alpha})(b_2+\bar{\alpha})$$

i.e. $a_2b_2 \iff \alpha$

$$(\bar{a}_2+b_1+\beta)(a_2+\bar{\beta})(\bar{b}_1+\bar{\beta})$$

i.e. $a_2\bar{b}_1 \iff \beta$

$$(a_1+b_1+\gamma)(\bar{a}_1+\bar{\gamma})(\bar{b}_1+\bar{\gamma})$$

i.e. $\bar{a}_1\bar{b}_1 \iff \gamma$

$$(a_1+\bar{b}_2+\delta)(\bar{a}_1+\bar{\delta})(b_2+\bar{\delta})$$

i.e. $\bar{a}_1b_2 \iff \delta$

$(\alpha+\beta+\xi)(\bar{\xi}+\gamma+\delta)$ i.e $(\alpha+\beta+\gamma+\delta)$

$$(\bar{\alpha}+\bar{\beta})(\bar{\beta}+\bar{\gamma})(\bar{\gamma}+\bar{\delta})(\bar{\delta}+\bar{\alpha})$$

and

$$(b_2+\bar{b})(b+\bar{b}_2)(a_2+\bar{a})(a+\bar{a}_2)$$

fig. 7

i.e. $a \iff a2$, $b \iff b2$

At the same time a or $\bar{a}$ is replaced in c_i with a_1 or $\bar{a}_1$ and b or $\bar{b}$ is replaced in c_j with b_1 or $\bar{b}_1$

It is clear from the picture that the new graph has one less crossover point, and one can easily verify that X is satisfiable iff $[a_1 \iff a]$and$[b_1 \iff b]$.

The algorithm repeats the above replacement at each crossover point, starting at lower left and moving up and right, using new auxiliary variables each time, until the graph is finally planar.

At each stage of the algorithm, only a constant amount of work is done, and there are no more than $9m^2$ stages.

Now we draw in A_2 without disturbing the planarity of the graph. Since all of the new variables are in the same exist block, we are free to order them arbitrarily. Taking another look at our planar crossover box, we notice that there is a simple path linking all of its variables (i.e. the dashed lines in figure 8).

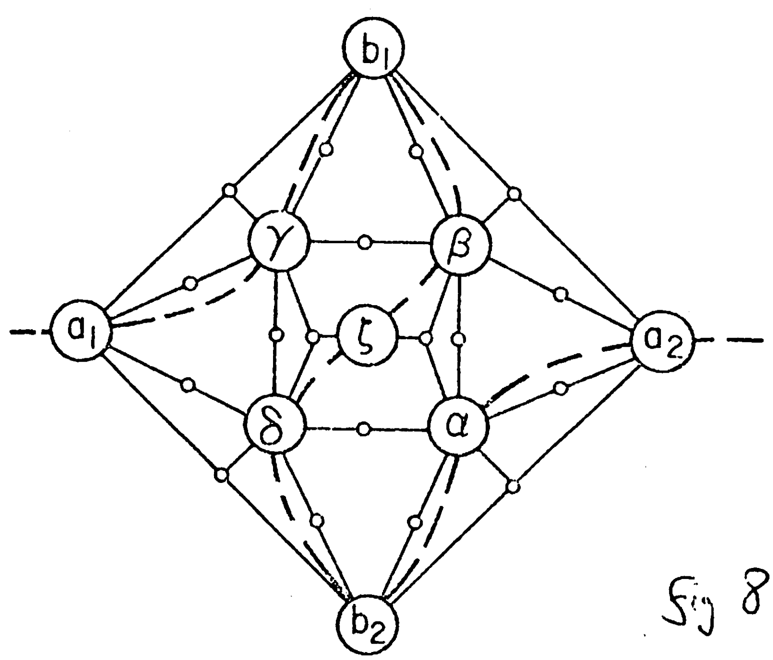


Fig 8

We use this fact to show how to connect all of the new variables together, as in figure 9.

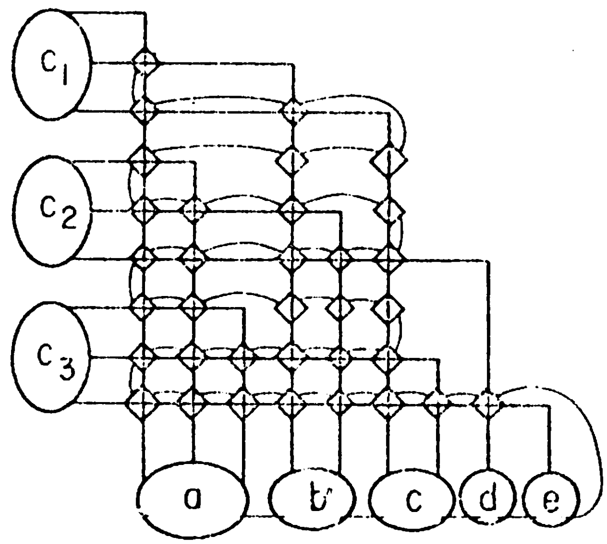


fig 9

Notice that we have used extra boxes to allow arcs in A_2 to cross A_1 arcs as necessary. This can add no more than $9m^2$ new boxes, and the algorithm is thus clearly polynomial. Q.E.D.

5. Planar Generalized Geography

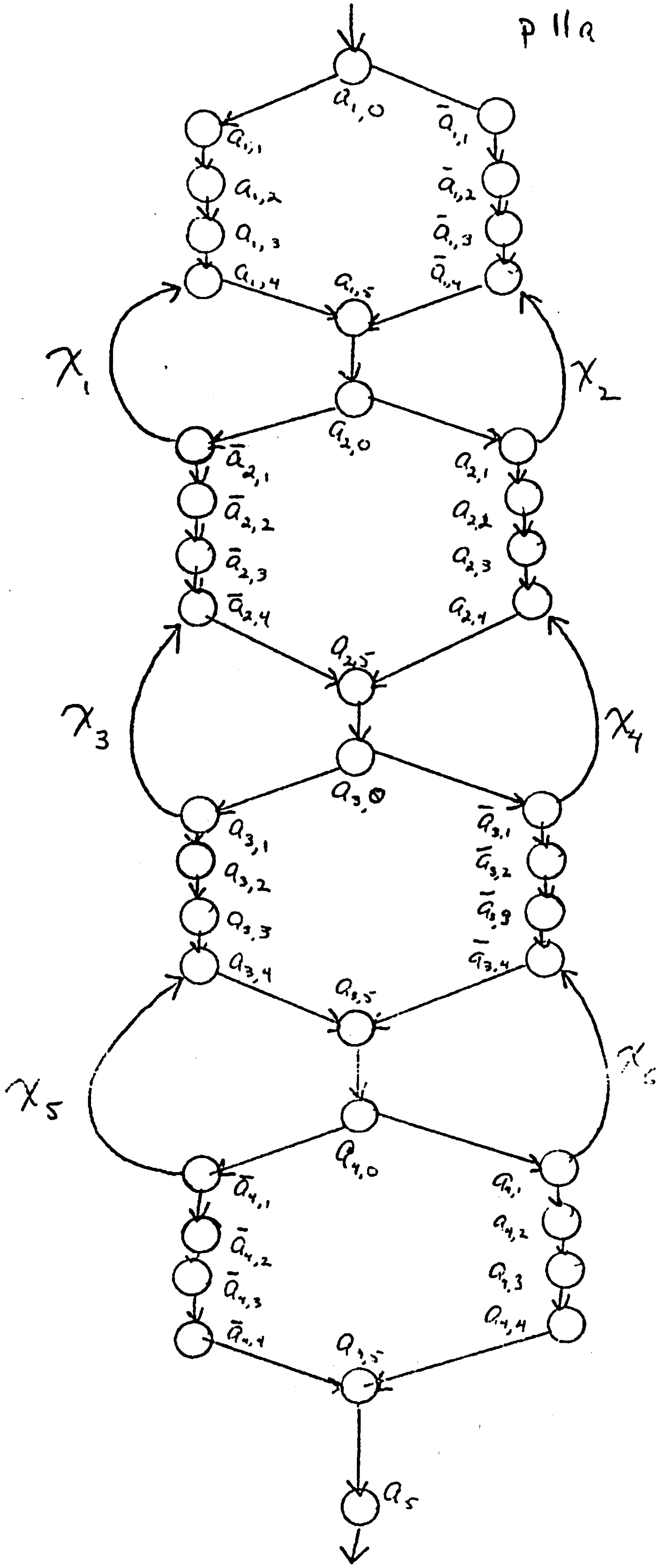
Theorem 4: Generalized Geography is pspace complete even when played only on planar graphs.

Proof: There are two problems which prevent us from merely invoking the previous theorem to give us the proof. The first is that arcs (c_j, v_i) must be drawn to one side of the diamond representing v_i, and arcs $(c_j,\bar{v}_i)$ must be drawn to the other side; in drawing the graph of a formula, we permitted ourselves any ordering of arcs around nodes v_i. The second problem is the set of arcs, $\{ (v_n,c_j) \mid 1\leq j\leq m\}$.

The first problem can be solved simply: The diamond representing v_i is replaced by a chain of $2m_i$ expanded diamonds, as in figure 10.

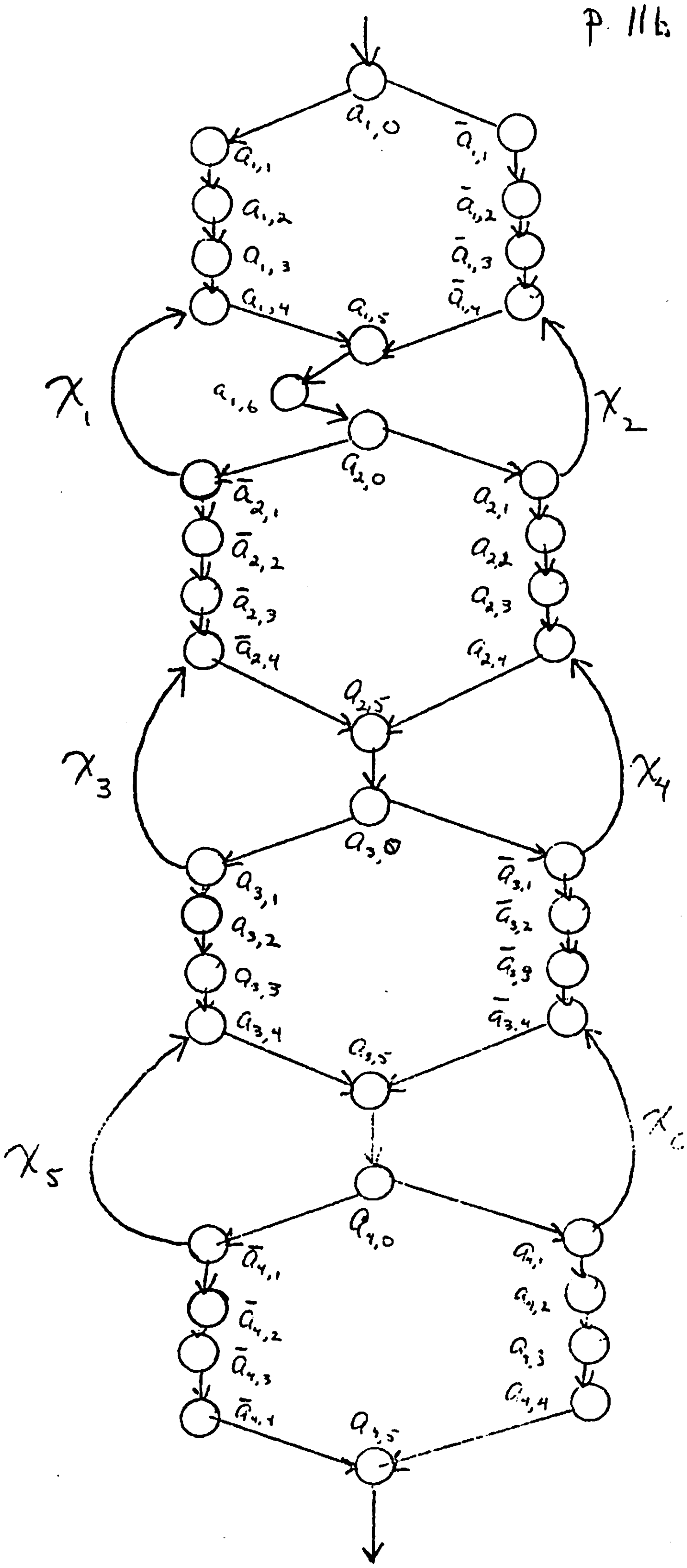
The function of arcs x_i is an XOR between successive diamonds, ensuring that the choice of path in adjacent diamonds will be opposite (Note that in figure 10, $a_{i,3}$ is on the opposite side from $a_{i+1,3}$). Arcs from clause nodes are then directed into the appropriate $a_{i,3}$ or $\bar{a}_{i,3}$, with no two clauses using the same i. The function of a_{2m_a+1} is to shift the choice of truth value of the next variable to the other player. (We can simplify the reduction of this problem to GO if we allow arcs x_i to be traversed only by $\exists$, and so an existentially quantified variable will be as in figure 10a.)

p 11a



Example: Here, $m_a = 2$, where a is universally quantified.

fig. 10



Example: Here, $m_a = 2$, where a is existentially quantified.

Note the presence of $a_{1,6}$ and the absence of a_5. The structure is otherwise identical to fig 10.

fig. 10a

To solve the second problem, we make the following observation: There is no need to wait until all variables have had their truth values chosen before allowing the $\forall$-player to test the truth of a clause; in fact, each clause can be tested as soon as its last variable has had its value fixed. Moreover, it is only necessary to allow testing of clauses not satisfied by their last chosen variable.

The clause construction therefore becomes:

Let $c_i=(a+\bar{b}+d)$, where d is the variable with the highest index of the three (i.e. is quantified last). The corresponding arcs in GG(B) are: $(c_i,a_{i,2})(c_i,\bar{b}_{j,2})$, and $(\bar{d}_{k,2},c_i)$ if d is a $\forall$-variable, else $(\bar{d}_{k,3},c_i)$.

Notice that if d is chosen true, there is no way for the $\forall$-player to test c_i. In fact, it would not be in $\forall$'s interest to do so.

At this point, we draw a dotted line around the chain of diamonds representing each variable, and notice that the resulting graph, ignoring direction of arcs, is exactly the graph of the formula we started with, and hence planar for planar formulae. Q.E.D.

p. 12a

Example

$(\exists a)(\forall b)(\exists d)(\forall e)\left[(a+\bar{b}+d)\right]$

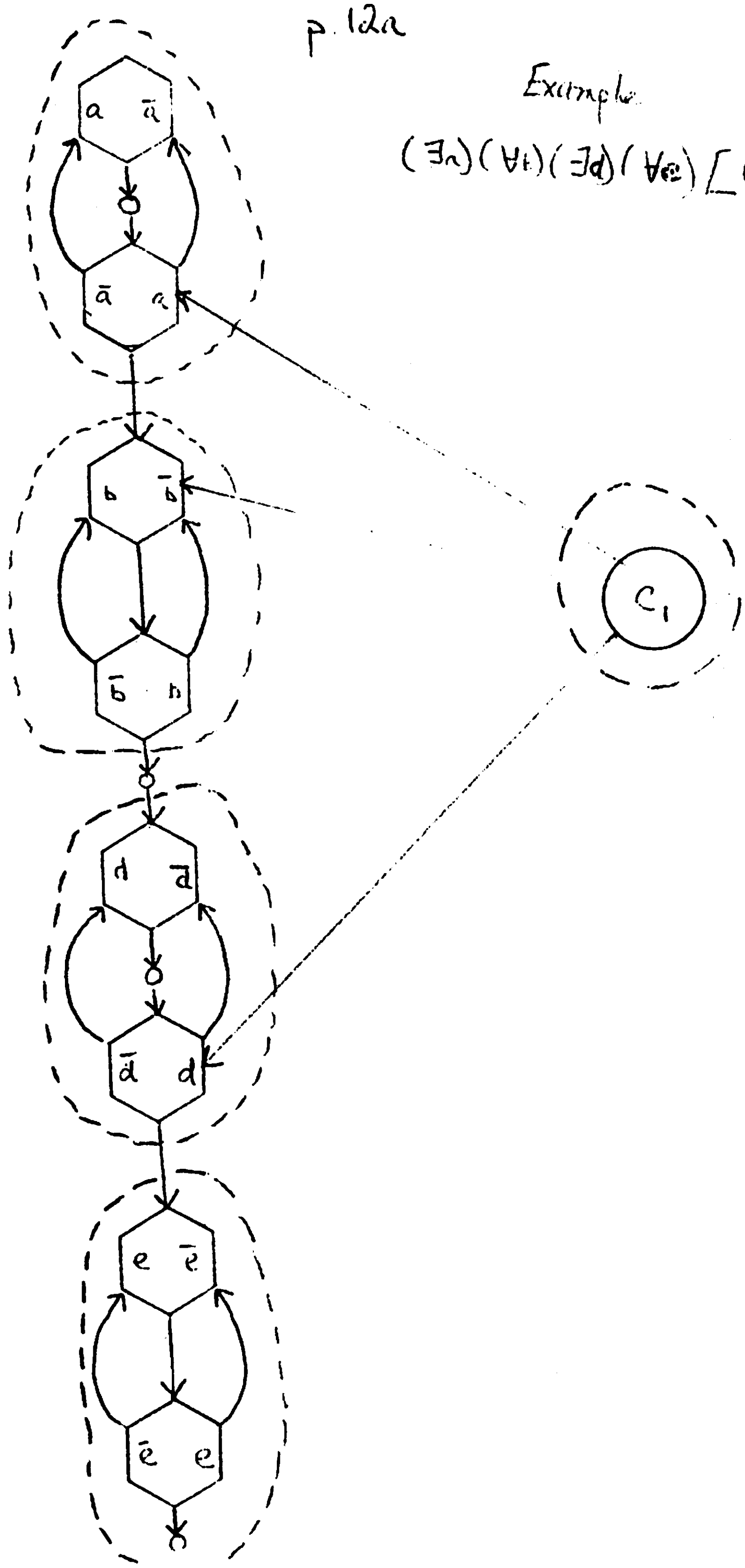


fig. 11

6. The Rules of GO

Go is played on a board which is a grid of 19 x 19 locations called points. There are two players, Black and White, for whom the rules are symmetric except that Black moves first. A player moves by placing a stone of his own color on a vacant point. The moves alternate between players, except that any player may pass at any time. The game terminates when both players pass.

As the game progresses, the stones form clusters called groups. A group is a maximal, uniformly colored set of stones which occupy a connected region of the board. A group of stones becomes surrounded if none of them is adjacent to a vacant point. After each black (white) move, all surrounded white (black) groups are removed, followed by all surrounded black (white) groups.

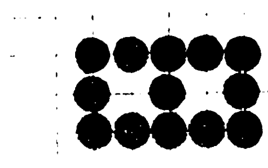
Scoring

At the end of the game all dead stones are removed from the board. A stone is dead if it ultimately can be surrounded, despite any attempts to save it. A vacant point is said to be white territory if it is surrounded on all sides by either white stones or the edge of the board. Black territory is similarly designated. The final score for White is the count of all the white territory minus the number of white stones which have been captured (removed at any time). The black score is similarly calculated and the highest scorer wins.

These rules are a subset of the actual rules of GO, though they are adequate for our purposes. The major omission concerns the situation of KO which has a special rule designed to prevent infinite repetitions of the same position. A complete, concise treatment is given in [8].

Eyes

An important consequence of the GO rules is that certain configurations of stones cannot be captured. If a configuration surrounds two separated, vacant points, it is said to have two eyes. It then cannot be surrounded because it is impossible for the opponent to fill both eyes simultaneously.



An uncapturable configuration.

Frequently, in the course of actual games, a player may have a nearly surrounded group of stones which he is desperately trying to connect to a group having two eyes. At the same time his opponent is trying to cut him off. We exploit such a situation later in our proof.

7. Construction of the GO Position

We now encode the constructed planar generalized geography game as a GO position. We refer to the GO players as Black and White and the geography players as the $\exists$-player and the $\forall$-player. The GO position to be constructed will have the property that Black has a winning strategy iff the $\exists$-player has a winning strategy.

The overall plan behind the construction is to have a very large white group of stones which is nearly surrounded. (fig. 13) It is so large that the outcome of the game hinges upon its survival, that is, Black will win iff he can capture it. White's only hope is first to escape through the small breach in the surrounding black stones, and then ultimately connect to a group with two eyes. This breach, however, leads to a structure which is patterned after the given geography graph. White and Black are then, in effect, forced to play the geography game with each other.

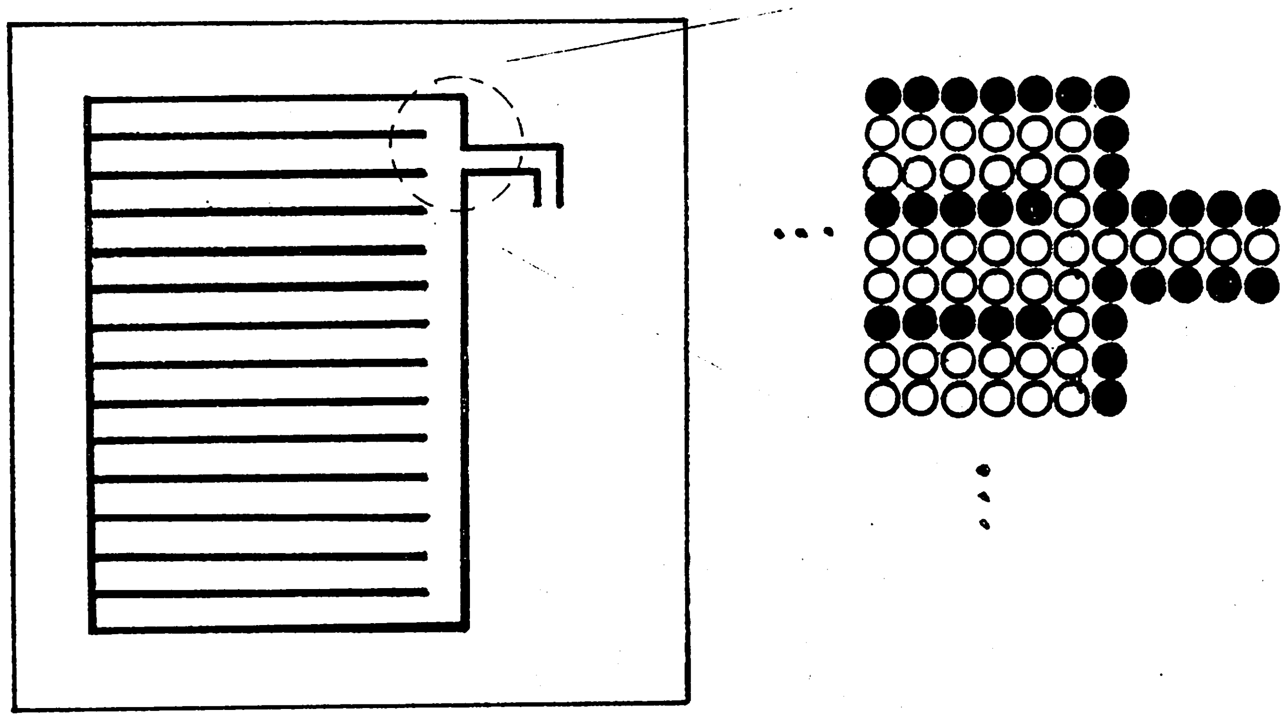
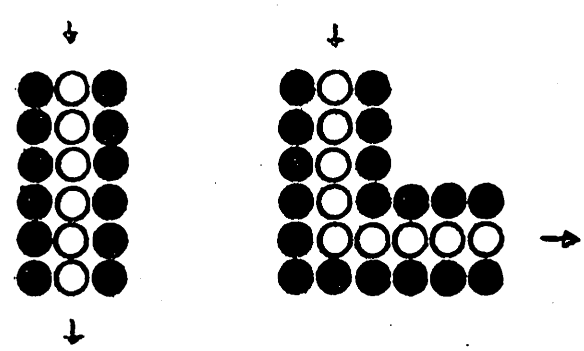
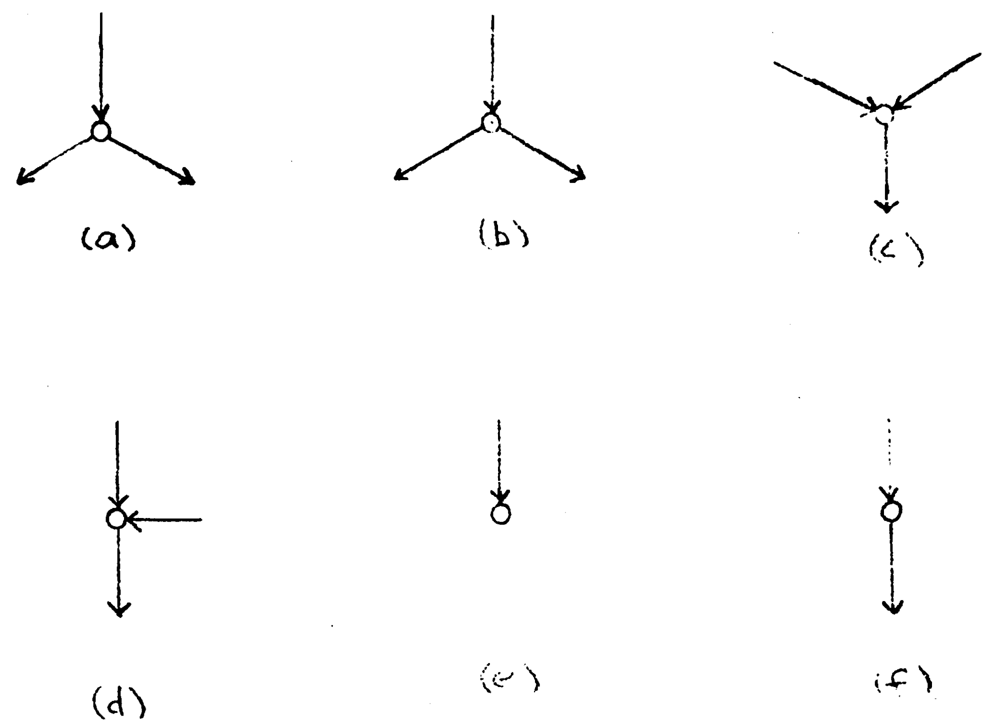


Figure 13

Each arc and vertex in the geography graph is represented by a corresponding <u>pipe</u> and <u>junction</u> in the GO position.

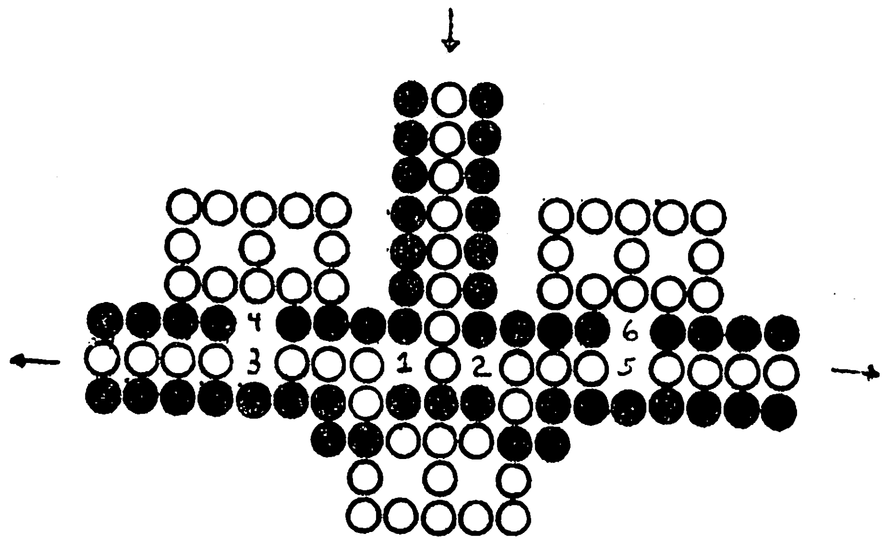


There are essentially six types of vertices which arise in our geography graphs.

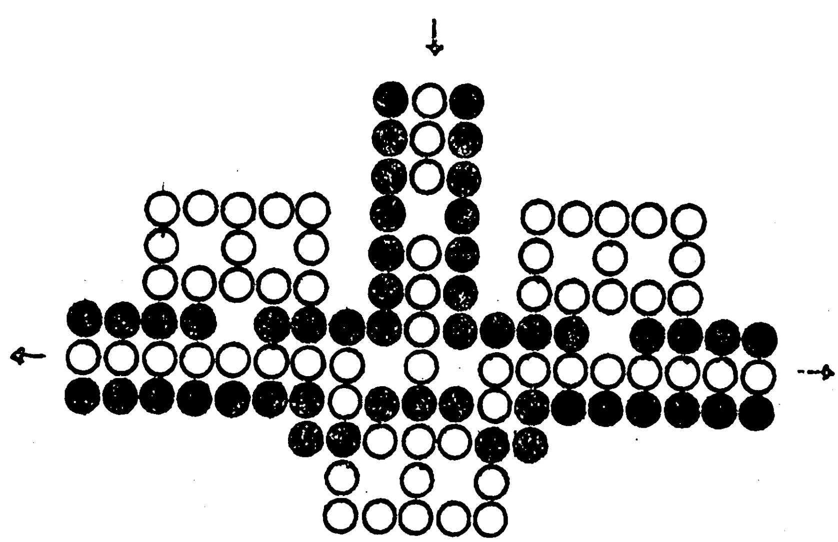


a) $\forall$-player choice b) $\exists$-player choice
c) join d) test e) dead end f) trivial

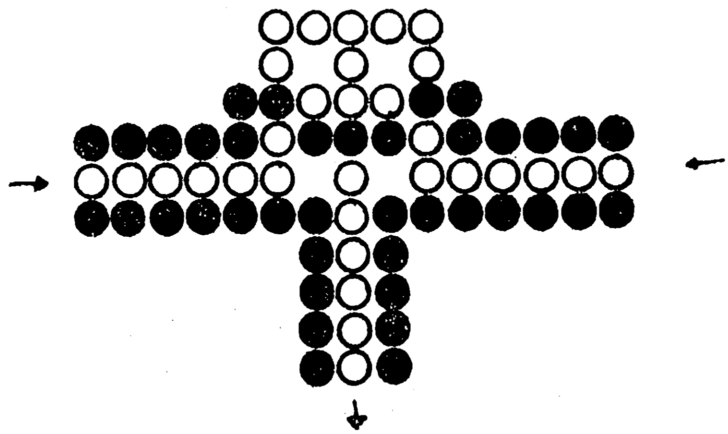
We give the corresponding GO junction for vertices (a) thru (d), and leave (e) for the reader. Note that in the generalized geography graphs which we construct, the position of a choice vertex (i.e. a vertex with outdegree > 1) determines which player makes the choice. This necessitates the occasional use of trivial vertices to switch the initiative. In the GO construction, the nature of the junction determines which player makes the choice. Thus the trivial vertices become unnecessary and are treated as arcs.



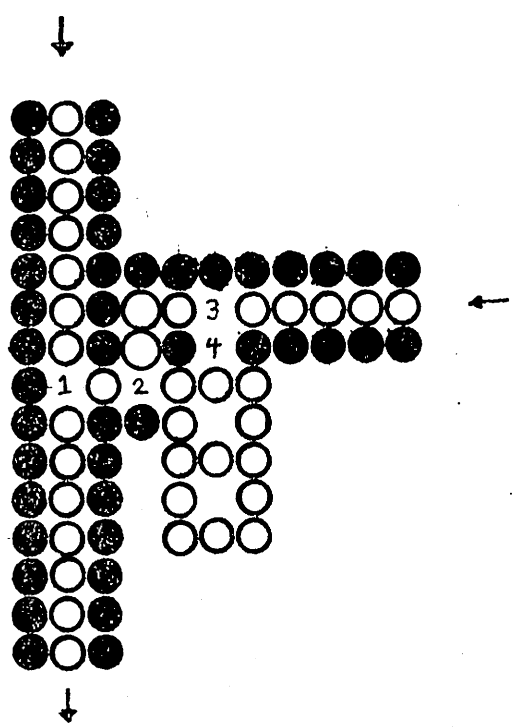
(a)



(b)



(c)



(d)

The desired GO position is obtained by joining the appropriate pipes and junctions in a way which embeds the geography graph. The pipe entering the first choice junction (the first diamond) is connected to the breach in Black's wall around the large white group. White moves first.

We now argue that if the players play "correctly" then the ensuing game will mimic a geography game, in that the course of play will travel through a sequence of pipes corresponding to a valid sequence of geography arcs. Furthermore, if any player does not play correctly his opponent will be able to win within a few moves.

Upon entering or leaving any junction it will be White's turn. Inductively, we assume that the large white group is completely surrounded except for the tip of the pipe entering the current junction. Let us consider the case where the play is about to enter an (a)-junction, corresponding to a choice by the $\forall$-player. We assume wlog that he wishes to go left.

Proposition: If White's first move is not at either point 1 or point 2 then Black can win in two moves.

Proof: Assume that White does not play at either 1 or 2. Further assume that White does not play at 3. In that case Black plays at 2 forcing White to respond at 1 whereby Black wins at 3. If White had played at 3 initially then we symmetrically reverse the argument and Black again has a win.

Proposition: If Black does not respond at point 2, then White can win in two moves.

Proof: Suppose White played at 1 and black failed to play at 2. Then White plays at 2, capturing three black stones. Black cannot now prevent White from connecting to the White group with two eyes, winning for White.

Proposition: White must now continue at point 3, or else lose immediately.

Proof: Clear.

Proposition: Black must respond at point 4 or lose immediately.

Proof: Clear, using the white group which has two eyes and which is directly above 4.

Thus if White chooses the left pipe and both players play correctly the sequence of moves would be: White - 1; Black - 2; White - 3; Black - 4. The play now continues as before, down the left pipe. The large white group is again completely surrounded, except for the tip of the left pipe and it is White's turn to move, fulfilling the induction assumptions. Both the (b) and the (c) junctions can be argued similarly. The (d) junction, corresponding to a selection of a variable to test by Black, is somewhat different and we analyze it here. We show that if the play enters through the right hand pipe then Black wins iff the play had previously passed down through the vertical pipe.

Proposition: If play first enters this junction at the top then it will leave at the bottom and there will be a white stone placed at point 1 and a black stone at point 2.

Proof: Clear, using the white group with two eyes to force Black's move.

Proposition: If play subsequently enters through the right hand pipe then Black wins.

Proof: White is forced to move at 3, followed by the winning black move at 4.

Proposition: If play enters through the right hand pipe prior to entering through the top pipe then White wins.

Proof: White moves at 3, and then has a win at either 2 or 4.

8. Conclusion

The concept of planar formulae has proved useful in demonstrating psapce hardness for GO and Checkers [7]. It would be interesting to know if similar techniques could be used to obtain pspace hardness proofs for games like Othello, Hex, and Chess (given some "natural" n X n generalization of the last). It would also be interesting to show that GO or Chess is in pspace, or that either is complete for exponential time.

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