AN INTRINSIC CALCULUS FOR WEAK MARTINGALES IN THE PLANE

by

E. Wong and M. Zakai

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Eugene Wong and Moshe Zakai

University of California, Berkeley
and
Technion, Haifa, Israel

0. Introduction

Let $W(s,t)$, $0 \leq s$, $T < \infty$, be a two-parameter Wiener process. Stochastic integrals of four types:

$$\int \psi_z dW_z \quad , \int \psi_{z,z'} dW_z \, dW_{z'},$$
$$\int \psi_{z,z'} dW_z \, dz', \int \psi_{z,z'} dz \, dW_{z'}$$

have been defined [1,4,5,6] and form the basis for a theory of continuous-parameter martingales in the plane and for their associated stochastic calculus. Our earlier efforts in deriving a differentiation formula for this stochastic calculus were only partially successful in that the resulting generalization of the Ito lemma is too complicated to be truly useful. [9]

In part, the complexity of the differentiation formula reflects an inherently complex situation, but an important additional factor is that in [9] the differentiation formula is treated as a transformation rule for stochastic integrals (as is in the case of the Ito lemma) rather than a transformation rule for martingales. In this paper we shall derive an intrinsic form for the transformation of weak semi-martingales in the plane which is free of their stochastic-integral representation. The result provides not only a simpler formula, but also a greater elucidation of the underlying calculus.

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1. Preliminaries and Notation

The definitions and notation of this paper will follow those of [ ] and can be summarized as follows: Let $R_+^2 = [0,\infty) \times [0,\infty)$ denote the positive quadrant of the plane. For two points $z = (s,t)$ and $z' = (s',t')$ in R_+^2

(a) $z \succ z'$ will denote the condition $s \geq s'$ and $t \geq t'$, and $z \succ\succ z'$ the condition $s > s'$ and $t > t'$,

(b) $z \curlywedge z'$ will denote the condition $s \leq s'$ and $t \geq t'$ and $z \hat{\curlywedge} z'$ the condition $s < s'$ and $t > t'$,

(c) the function $h(z,z')$ will denote the indicator function on $R_+^2 \times R_+^2$ of the condition $z \curlywedge z'$,

(d) $z \times z'$ will denote the point (s,t') and $z \vee z'$ the point $(\max(s,s'), \max(t,t'))$.

(e) if $z \prec\prec z'$, $(z,z']$ will denote the rectangle $(s,s'] \times (t,t']$.

(f) 0 will denote the origin and R_z the rectangle $\{0 \prec \zeta \prec z\}$.

Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space and let $\{\mathcal{F}_z, z \in R_+^2\}$ be a family of σ-subfields satisfying the following conditions:

(F_1) $z \prec z'$ implies $\mathcal{F}_z \subset \mathcal{F}_{z'}$,

(F_2) $\mathcal{F}_0$ contains all the null sets of

(F_3) for every z $\mathcal{F}_z = \bigcap_{z' \succ\succ z} \mathcal{F}_{z'}$

(F_4) for every z $\mathcal{F}_z^1$ and $\mathcal{F}_z^2$ are conditionally independent given $\mathcal{F}_z$, where $\mathcal{F}_z^1 = \mathcal{F}_{s,\infty}$ and $\mathcal{F}_z^2 = \mathcal{F}_{\infty,t}$.

For a stochastic process $\{X_z, z \in R_+^2\}$, $X(z,z']$ will denote $X_{s',t'} + X_{s,t} - X_{s,t'} - X_{s',t}$. A process X is said to be $\mathcal{F}_z$-adapted if for each z X_z is $\mathcal{F}_z$-measurable. In the definitions that follow the process X is assumed to be $\mathcal{F}_z$-adapted and for each z X_z is integrable.

Definitions

(M_1) X_z is a martingale if $z' > z$ implies $E(X_{z'} | \mathcal{F}_z) = X_z$ a.s.

(M_2) X_z is an adapted 1-martingale (2-martingale) if $\{X_{s,t}, \mathcal{F}_{s,t}\}$ is a martingale in s for each fixed t (in t for each fixed s).

(M_3) X_z is a weak martingale if $z' \gg z$ implies $E\{X(z,z'] | \mathcal{F}_z\} = 0$

(M_4) X_z is a strong martingale if X vanishes at the axes and $E\{X(z,z'] | \mathcal{F}^1_z \vee \mathcal{F}^2_z\} = 0$ whenever $z' \gg z$.

(M_5) X_z is a Wiener process if X is a Gausian process satisfying $EX_z = 0$ for all z and $EX(z,z'] \, X(\zeta,\zeta'] = \text{Area } ((z,z'] \cap (\zeta,\zeta'])$.

A strong martingale is also a martingale which in turn is both an adapted 1-martingale and an adapted 2-martingale, either of which is also a weka martingale. A Wiener process is a strong martingale. If we assume that $\{\mathcal{F}_z\}$ is generated by a Wiener process W, i.e., $\mathcal{F}_z = \sigma(\{W_\zeta, \zeta \prec z\})$, then $\mathcal{F}_z$-martingales are representable as stochastic integrals in terms of W. A more general representation result will be stated below. Henceforth, we shall assume that $\mathcal{F}_z$ is generated by a Wiener process W.

Definition X_z is said to be square-integrable semimartingale if $X_z = M_z + M_{1z} + M_{2z} + B_z$ where M is a square-integrable martingale, M_{1z} (M_{2z}) is a sample-continuous square-integrable process which is an adapted 1-martingale (2-martingale) and mean-square differentiable in the 2-direction (1-direction), and $B_z = \int_{R_z} b_\zeta \, d\zeta$ where b is an $\mathcal{F}_z$-predictable process with $\int_{R_+^2} Eb_\zeta^2 \, d\zeta < \infty$.

It follows from the results of [4,5] that every square-integrable semimartingale has a unique representation of the form

$$(1.1) \quad X_z = \int_{R_z} \psi_\zeta \, dW_\zeta + \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \, dW_\zeta \, dW_{\zeta'} + \int_{R_z \times R_z} \alpha_{\zeta,\zeta'} \, d\zeta \, dW_{\zeta'} + \int_{R_z \times R_z} \beta_{\zeta,\zeta'} \, dW_\zeta \, d\zeta' + \int_{R_z} b_\zeta \, d\zeta$$

where θ and b are $\mathcal{F}_z$-predictable and square-integrable ($d\mathcal{P}$ dz measure) processes, ψ, α and β are $\mathcal{F}_{z \vee z'}$-predictable and square-integrable ($d\mathcal{P}$ dz dz' measure) processes.

Now, suppose that a process X is of the form (1.1) where the integrands satisfy the same predictability conditions as before, but instead of being $d\mathcal{P}$ dz or $d\mathcal{P}$ dz dz' square-integrable are now merely almost surely dz or dz dz' square-integrable. We shall call such a process a <u>locally square-integrable semimartingle</u> or a <u>local semimartingale</u> for short.

<u>Remark</u> In the one parameter case, local martingales and semimartingales are defined by a stopping argument. In the two parameter case the situation is different since stopped versions of processes can be defined only via integration ([5], see also [3]). It is still true in the two parameter case that if $M(\zeta)$ is a local semimartingale then there exists a sequence of square integrable semimartingales $M^n(\zeta)$ such that $M^n(\zeta) \xrightarrow{a.s.} M(\zeta)$ parameter stopping (this will follow directly from the results of the next section and the stopping line constructed in [7]).

2. A Calculus of Semimartingales

Let X be a local semimartingale and let ψ be a predictable process such that

$$(2.1) \qquad \text{Prob}\left(\sup_z |\psi_z| < \infty\right) = 1$$

Then the stochastic integral

$$(2.2) \qquad (\psi \circ X)_z = \int_{R_z} \psi_\zeta \, dX_\zeta$$

is defined by

$$(2.3) \qquad (\psi \circ X)_z = \int_{R_z} \psi_\zeta \, \theta_\zeta \, dW_\zeta + \int_{R_z \times R_z} \psi_{\zeta \vee \zeta'} \, \psi_{\zeta,\zeta'} \, dW_\zeta \, dW_{\zeta'} + \int_{R_z} \int_{R_z} \psi_{\zeta \vee \zeta'} \alpha_{\zeta,\zeta'} \, d\zeta \, dW_{\zeta'} + \int_{R_z \times R_z} \psi_{\zeta \vee \zeta'} \, \beta_{\zeta,\zeta'} \, dW_\zeta \, d\zeta' + \int_{R_z} \psi_\zeta \, b_\zeta \, d\zeta$$

Hence, for each ψ, ψ o X defines a mapping of local semimartingales into local semimartingales.

A local semimartingale is a one-parameter local semimartingale in each direction and a rerepresentation of (1.1) makes this clear. Let X_{W1} and $X_{\mu 1}$ be defined as follows:

$$(2.4) \qquad X_{W1}(z,\zeta') = \theta_{\zeta'} + \int_{R_z} h(\zeta,\zeta')[\psi_{\zeta,\zeta'}\, dW_\zeta + \alpha_{\zeta,\zeta'}\, d\zeta]$$

$$(2.5) \qquad X_{\mu 1}(z,\zeta') = b_{\zeta'} + \int_{R_z} h(\zeta,\zeta')\beta_{\zeta,\zeta'}\, dW_\zeta$$

Then, (1.1) can be written as

$$(2.6) \qquad X_z = \int_{R_z} X_{W1}(z,\zeta')\, dW_{\zeta'} + \int_{R_z} X_{\mu 1}(z,\zeta')\, d\zeta'$$

Because $X_{W1}(z,\zeta')$ and $X_{\mu 1}(z,\zeta')$ are both $\mathcal{F}_{\zeta'\otimes z}$ measurable (2.6) is a local 1-semimartingale representation for X. It is convenient to denote the Lebesgue measure by μ and write (2.6) in a compact form as

$$(2.7) \qquad X = X_{W1} \text{ o } W + X_{\mu 1} \text{ o } \mu$$

Similarly, we can define X_{W2} and $X_{\mu 2}$ by

$$(2.4') \qquad X_{W2}(z,\zeta) = \theta_\zeta + \int_{R_z} h(\zeta,\zeta')[\psi_{\zeta,\zeta'} dW_{\zeta'} + \beta_{\zeta,\zeta'}\, d\zeta']$$

$$(2.5') \qquad X_{\mu 2}(z,\zeta) = b_\zeta + \int_{R_z} h(\zeta,\zeta')\, \alpha_{\zeta,\zeta'}\, d\zeta$$

and rewrite (1.1) to yield a local 2-semimartingale representation

$$(2.6') \qquad X_z = \int_{R_z} X_{W2}(z,\zeta)\, dW_\zeta + \int_{R_z} X_{\mu 2}(z,\zeta)\, d\zeta$$

which will be expressed in a more compact form as

$$(2.7') \qquad X = W \text{ o } X_{W2} + \mu \text{ o } X_{\mu 2}$$

Equations (2.7) and (2.7') can be thought of as partial differential formulas: $\partial_i X = X_{Wi}\, \partial_i W + X_{\mu i}\, \partial_i \mu$ so that the function X_{Wi} and $X_{\mu i}$ can be thought of as partial derivatives.[2] Intuitively, we can construct a local semimartingale Z from two existing ones X and Y by identifying $\partial_1\partial_2 Z =$

$\partial_2 Y\ \partial_1 X$. We shall call the operation composition and denote it by $Y * X$. The precise definition for $Y * X$ is given by

$$(2.8)\qquad (Y * X)_z = \int_{R_z \times R_z} Y_{W2}(\zeta \vee \zeta', \zeta)\, X_{W1}(\zeta \vee \zeta', \zeta')\, dW_\zeta\, dW_{\zeta'}$$

$$+ \int_{R_z \times R_z} Y_{\mu 2}(\zeta \vee \zeta', \zeta)\, X_{W1}(\zeta \vee \zeta', \zeta')\, d\zeta\, dW_{\zeta'}$$

$$+ \int_{R_z \times R_z} Y_{W2}(\zeta \vee \zeta', \zeta)\, X_{\mu 1}(\zeta \vee \zeta', \zeta')\, dW_\zeta\, d\zeta'$$

$$+ \int_{R_z \times R_z} Y_{\mu 2}(\zeta \vee \zeta', \zeta)\, X_{\mu 1}(\zeta \vee \zeta', \zeta')\, d\zeta\, d\zeta'$$

In an abbreviated but more suggestive form, (2.8) can be expressed as

$$(2.9)\qquad Y * X = W \circ Y_{W2}\, X_{W1} \circ W + \mu \circ Y_{\mu 2}\, X_{W1} \circ W + W \circ Y_{W2}\, X_{\mu 1} \circ \mu + \mu \circ Y_{\mu 2}\, X_{\mu 1} \circ \mu$$

The representation (2.8) shows that $Y * X$ is again a local semimartingale provided that the integrands are locally square-integrable. If X and Y are such that the integrands in the representation (1.1) are locally fourth-power integrable, then the integrands in (2.8) are indeed locally square-integrable. However, this seems to be an artificial and undesirable condition. Local semimartingales should be defined so that closure under composition comes naturally and without additional restrictions.

For one-parameter processes, if M is a sample-continuous local martingale, then there exists a sample-continuous increasing process $\langle M,M\rangle$ such that $M^2 - \langle M,M\rangle$ is a local martingale. If M is generated by a Wiener process W then M is necessarily of the form

$$M_t = M_0 + \int_0^t \psi_s\, dW_s$$

and $\langle M,M\rangle$ is given by

$$\langle M,M\rangle_t = \int_0^t \psi_s^2\, ds$$

The increasing process $\langle M,M\rangle$ has the interpretation of being the quadratic variation of M. Hence, if $X = M + B$ where B is both sample continuous and of bounded variation then the quadratic variation of X is just that of M, and it is consistent to define

$$\langle X,X\rangle = \langle M,M\rangle$$

Let M be a two-parameter square-integrable martingale generated by a Wiener process W. Then, M is necessarily of the form

$$M_z = M_0 + \int_{R_z} \theta_\zeta \, dW_\zeta + \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \, dW_\zeta \, dW_{\zeta'} \tag{2.10}$$

If we define the increasing process $[M,M]$ by

$$[M,M]_z = \int_{R_z} \theta_\zeta^2 \, d\zeta + \int_{R_z \times R_z} h(\zeta,\zeta') \, \psi_{\zeta,\zeta'} \, d\zeta \, d\zeta' \tag{2.11}$$

then $M^2-[M,M]$ is weak martingale. The process $[M,M]$ has the quadratic variation interpretation associated with $\langle M,M\rangle$ in one dimension.

For a two-parameter square-integrable semimartingale X as given by (1.1), the quadratic variation is equal to that of the martingale term. Hence, it is consistent to define

$$[X,X]_z = \int_{R_z} \theta_\zeta^2 \, d\zeta + \int_{R_z \times R_z} h(\zeta,\zeta') \, \psi_{\zeta,\zeta'} \, d\zeta \, d\zeta' \tag{2.12}$$

and to extend the same definition to local semimartingales.

Since a two-parameter local semimartingale X is a one-parameter local semimartingale in each direction, the quadratic variation process $\langle X,X\rangle_i$ is well defined for each direction i and is given by

$$\langle X,X\rangle_{iz} = \int_{R_z} X_{Wi}^2 (z,\zeta) \, d\zeta \tag{2.13}$$

From their definition (2.4 or 2.4') it can be seen that for each i direction X_{Wi} is a one-parameter local semimartingale in the other direction. Hence, the one-dimensional formula yields

$$X_{W1}^2(z,\zeta') = \theta^2_{\zeta'} + 2\int_{R_z} h(\zeta,\zeta')\, X_{W1}(\zeta\vee\zeta',\zeta')[\psi_{\zeta,\zeta'} dW_\zeta + \alpha_{\zeta,\zeta'}\, d\zeta]$$
$$+ \int_{R_z} h(\zeta,\zeta')\, \psi^2_{\zeta,\zeta'}\, d\zeta$$

which together with (2.13) yield

$$(2.14) \qquad \langle X,X\rangle_{1z} = [X,X]_z + 2\int_{R_z\times R_z} X_{W1}(\zeta\vee\zeta',\zeta')[\psi_{\zeta,\zeta'}\, dW_\zeta\, d\zeta' + \alpha_{\zeta,\zeta'}\, d\zeta\, d\zeta']$$

Similarly, $\langle X,X\rangle_{2z}$ can be written as

$$(2.14') \qquad \langle X,X\rangle_{2z} = [X,X]_z + 2\int_{R_z\times R_z} X_{W2}(\zeta\vee\zeta',\zeta)[\psi_{\zeta,\zeta'}\, d\zeta\, dW_{\zeta'} + \beta_{\zeta,\zeta'}\, d\zeta\, d\zeta']$$

Each of the processes $[X,X]$ and $\langle X,X\rangle_i$ is a local semimartingale, albeit rather special ones, so that stochastic integration with respect to each is well defined. Further, being a quadratic variation process in one form or another, each is also intrinsic to X and its existence is independent of the semimartingale representation (1.1).

For two local semimartingales X and Y, X + Y is again a local semimartingale. Hence,

$$[X,Y] = \frac{1}{2}\ \{[X+Y,\ X+Y] - [X,X] - [Y,Y]\}$$
$$\langle X,Y\rangle_i = \frac{1}{2}\ \{\langle X+Y,\ X+Y\rangle_i - \langle X,X\rangle_i - \langle Y,Y\rangle_i\}$$

are well-defined bilinear operators on (X,Y). Together with Y * X, these comprise a set of binary operators on local semimartingales. The algebra of these operators may be interesting and has not yet been explored.

3. Differentiation Formula

In [9] a differentiation formula was derived which makes explicit use of the weak semimartingale representation (1.1). In the notation of this paper, that differentiation formula can be expressed as follows:

Let $F: R \longrightarrow R$ be a function with bounded continuous derivation through the fourth order, and let X be a process of the form (1.1). Then,

$$
\begin{aligned}
(3.1)\qquad F(X_z) = {} & F(X_0) + \int_{R_z} F_1(\theta_\zeta\, dW_\zeta + b_\zeta\, d\zeta) + \frac{1}{2}\int_{R_z} F_2\, \theta_\zeta^2\, d\zeta \\
& + \int_{R_z \times R_z} (F_1\, \psi + F_2\, X_{W2}\, X_{W1})\, dW_\zeta\, dW_{\zeta'} \\
& + \int_{R_z \times R_z} [F_1\, \alpha + F_2(X_{\mu 2}\, X_{W1} + \psi X_{W2}) + \frac{1}{2} F_3\, X_{W2}^2\, X_{W1}]\, d\zeta\, dW_{\zeta'} \\
& + \int_{R_z \times R_z} [F_1\, \beta + F_2(X_{W2} X_{\mu 1} + \psi X_{W1}) + \frac{1}{2} F_3\, X_{W2}\, X_{W1}^2]\, dW_\zeta\, d\zeta' \\
& + \int_{R_z \times R_z} [F_2(X_{\mu 2}\, X_{\mu 1} + \alpha X_{W1} + \beta X_{W2} + \frac{1}{2}\psi^2) \\
& \qquad + F_3\, (X_{W2}\, X_{W1}\, \psi + \frac{1}{2} X_{W2}^2\, X_{\mu 1} + \frac{1}{2} X_{\mu 2}\, X_{W1}^2) \\
& \qquad + \frac{1}{4} F_4\, X_{W2}^2\, X_{W1}^2]\, d\zeta\, d\zeta'
\end{aligned}
$$

where $F_k(x) = \dfrac{\partial^k}{\partial x^k} F(x)$, F_k have argument $X_{\zeta \vee \zeta'}$ or X_ζ, ψ, α and β have arguments (ζ,ζ'), X_{W1} and $X_{\mu 1}$ have arguments $(\zeta \vee \zeta', \zeta')$ and X_{W2} and $X_{\mu 2}$ have arguments $(\zeta \vee \zeta', \zeta)$. All integrals on $R_z \times R_z$ are restricted to the set $\zeta \wedge \zeta'$.

The same differentiation formula can be expressed in an intrinsic and representation-free form as follows:

$$
\begin{aligned}
(3.2)\qquad F(X) = {} & F(X_0) + F_1(X) \circ X + F_2(X) \circ (X * X) \\
& + \frac{1}{2} F_2(X) \circ (\langle X,X\rangle + \langle X,X\rangle_2 - [X,X]) \\
& + \frac{1}{2} F_3(X) \circ (X * \langle X,X\rangle_1 + \langle X,X\rangle_2 * X - 2\,[X, X * X]) \\
& + \frac{1}{4} F_4(X) \circ \langle X,X\rangle_2 * \langle X,X\rangle_1
\end{aligned}
$$

A verification of (3.2) is easily obtained by evaluating the terms $X * X$, $\langle X,X\rangle_i$, $[X,X]$, $\langle X, X * X\rangle_i$, $[X, X * X]$ and $[X * X, X * X]$ according to the formulas given in section 2. The details are too straightforward to be reproduced in full, but a term by term identification between (3.1) and (3.2) is useful and appears as follows:

$$
\begin{aligned}
F_1(X) \circ X = {} & \int_{R_z} F_1(\theta_\zeta dW_\zeta + b_\zeta\, d\zeta) \\
& + \int_{R_z \times R_z} F_1(\psi dW_\zeta dW_{\zeta'} + \alpha d\zeta dW_{\zeta'} + \beta\, dW_\zeta\, d\zeta')
\end{aligned}
$$

$$F_2(X) \circ (X * X) = \int_{R_z \times R_z} F_2(X_{W2}\, dW_\zeta + X_{\mu 2}\, d\zeta)(X_{W1} dW_{\zeta'} + X_{\mu 1}\, d\zeta')$$

$$\frac{1}{2} F_2(X) \circ (\langle X,X\rangle_1 + \langle X,X\rangle_2 - [X,X])$$
$$= \int_{R_z \times R_z} F_2\Big[X_{W2}\, d\zeta(\psi dW_{\zeta'} + \beta\, d\zeta') + (\psi dW_\zeta + \alpha\, d\zeta)\, X_{W1}\, d\zeta' + \frac{1}{2}\psi^2\, d\zeta\, d\zeta'\Big]$$

$$\frac{1}{2} F_3(X) \circ (X * \langle X,X\rangle_1 + \langle X,X\rangle_2 * X + 2[X, X * X])$$
$$= \int_{R_z \times R_z} F_3\Big[\frac{1}{2}(X_{W2}\, dW_\zeta + X_{\mu 2}\, d\zeta)\, X_{W1}^2\, d\zeta' + \frac{1}{2} X_{W2}^2\, d\zeta\, (X_{\mu 1}\, dW_{\zeta'} + X_{\mu 1}\, d\zeta') + \psi X_{W2} X_{W1} d\zeta\, d\zeta'\Big]$$

$$\frac{1}{4} F_4(X) \circ \langle X,X\rangle_2 * \langle X,X\rangle_1 = \frac{1}{4} F_4(X) \circ [X * X, X * X]$$
$$= \frac{1}{4} \int_{R_z \times R_z} F_4\, X_{W2}^2\, X_{W1}^2\, d\zeta\, d\zeta'$$

4. A Characterization of Positive Martingales

A problem related to the characterization of likelihood ratios [8] is the following. Let X be a local weak semimartingale such that $X_0 = 0$. What conditions must X satisfy in order for e^X to be a local martingale? A simple application of (3.2) gives us the answer.

Let X be written as

$$X = m + m_1 + m_2 + b \tag{4.1}$$

where m is a local martingale, m_i a proper local i-martingale and b a process of bounded variation. Let $M_i = m + m_i$. Since e^X is an i-martingale for $i = 1,2$, characterization of one-parameter continuous positive local martingales yields

$$\begin{aligned} X &= M_1 - \frac{1}{2}\langle X,X\rangle_1 \\ &= M_2 - \frac{1}{2}\langle X,X\rangle_2 \end{aligned} \tag{4.2}$$

Hence,

$$(4.3) \qquad X * X = M_2 * M_1 - \frac{1}{2} M_2 * \langle X,X\rangle_1 - \frac{1}{2} \langle X,X\rangle_2 * M_1$$
$$+ \frac{1}{4} \langle X,X\rangle_2 * \langle X,X\rangle_1$$
$$= M_2 * M_1 - \frac{1}{2}X * \langle X,X\rangle_1 - \frac{1}{2} \langle X,X\rangle_2 * X$$
$$- \frac{1}{4} \langle X,X\rangle_2 * \langle X,X\rangle_1$$

Letting $F(X) = e^X$ in (3.2) and making use of (4.3), we get

$$(4.4) \qquad e^X = 1 + e^X \circ (X + \langle X,X\rangle_1 + \langle X,X\rangle_2 - [X,X]$$
$$+ 2[X,X * X]) + e^X \circ M_2 * M_1$$

Since $M_2 * M_1$ is a local martingale, e^X is a local martingale if and only if

$$(4.5) \qquad X + \frac{1}{2} \{\langle X,X\rangle_1 + \langle X,X\rangle_2 - [X,X] + 2[X,X * X]\} = m$$

is a local martingale. This is essentially the likelihood ratio formula of [8] in intrinsic form.

We recall that in one dimension the condition equivalent to (4.5) is: $X + \frac{1}{2} \langle X,X\rangle = m$. Since in that case $\langle m,m\rangle = \langle X,X\rangle$, we can write a one parameter positive local martingale as

$$e^X = e^{m - \frac{1}{2} \langle m,m\rangle}$$

Although each of the terms $\langle X,X\rangle_i$, $[X,X]$ and $[X,X * X]$ appears to be uniquely determined by m, we have yet to find a way of expressing them explicitly in terms of m.

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