ON OPTIMUM SINGLE-ROW ROUTING

by

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Abstract- The problem of single-row routing represents the backbone of the problem of general routing of multilayer printed circuit boards. In this paper, the necessary and sufficient condition for optimum single-row routing is obtained. By optimum routing we mean minimum street congestion. A novel formulation is introduced. Examples are given to illustrate how optimum routings are derived. A graph theory interpretation of the condition is also given.

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I. INTRODUCTION

Recent advances in microelectronics have drastically changed the tasks and design philosophy of circuit designers. One of the primary concerns nowadays is the efficient layout of chips and circuit modules which may contain thousands of interconnected devices and units. While CAD packages for layout are frequently used for various purposes in industry, the general problem of circuit layout is far from solved. As a matter of fact, basic study in the field from a theoretical point of view is lacking. One main problem seems to be the difficulty in formulating explicitly stated problems which are relevant to practical circuit layout.

In this paper we deal with a crucial problem, the problem of single-row routing.* It arises in the layout design of multilayer printed circuit boards and backplanes. It is a simple problem, it can be unambiguously stated, and it represents the backbone of the general routing problem. The problem was first introduced by H. So of the Bell Labs. [1]. Subsequently, algorithms and sufficient conditions for routing to minimize the tracks needed have been proposed [2]. In the present paper, we introduce a novel formulation of the same problem. With the new formulation, it becomes possible to understand the intricacies of the problem, thus we have been able to obtain a complete set of necessary and sufficient conditions for optimum routing. A graph theory interpretation is also given. Although an efficient algorithm has yet to be worked out to employ these conditions for general routing, examples are given to illustrate how the conditions are used to obtain optimum routing.

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*We sometimes use the term routing to mean realization.

II. FORMULATION OF THE PROBLEM

Given a set of n nodes evenly spaced on a row which is located on the real line R as shown in Fig.1. A net* list $L = \{N_1, N_2, \cdots, N_m\}$ is given which prescribes the connection pattern of the m nets to the n nodes. The specification can be expressed in terms of an $m \times n$ 0-1 matrix $\underset{\sim}{A}=[a_{ij}]$, where $a_{ij}= 1$ if net N_i is to connect node v_j, and $a_{ij}= 0$, otherwise. It should be stressed that a node is to be connected to one and only one net, thus there exists one and only one 1's in each column of the matrix.

A net list is to be realized with a set of m non-intersecting nets which consist of only horizontal and vertical paths connecting the nodes according to specification. An example depicting a realization of a given net list together with the matrix specification and some pertinent terminology is shown in Fig.2. The space above the real line R is referred to as the upper street and the space below R is referred to as the lower street. The number of horizontal tracks needed in the realization in the upper street, is called the upper-street congestion. Similarly, we define the lower-street congestion. In the realization, we allow a net to switch from upper street to lower street, and coversely, as shown by the net N_4 in Fig.2.

Previously, it has been shown that given a net list, a realization always exists [2]. An optimum realization is one which minimizes the street congestion in both streets. Sufficient conditions for realization to minimize street congestion together with a routing algorithm were proposed in [2]. Unfortunately, the algorithm has been found to be incomplete and it could fail. A counter examples is given in the Appendix. In this paper we give the necessary and sufficient condition for optimum realization. We introduce a new formulation of the problem as follows.

Consider the set of m nets given by the net list in terms of the matrix $\underset{\sim}{A}$. Let us draw a set of m horizontal intervals representing the m nets in the order given by the matrix from top down. This is illustrated in Fig.3

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*A net N is said to be the minimal net among a set of nets if there is no net N' in the set of nets with relation N' < N.

for the problem given by the matrix $\underset{\sim}{A}$ in Fig.2. Note that each horizontal line corresponds to the interval specified by a row of $\underset{\sim}{A}$ between the extreme left and right 1's. Each node is appropriately marked on the lines as shown. We call this the interval graphical representation of the matrix $\underset{\sim}{A}$. Since there are m! row permutations, there exist a total of m! interval graphical representations for a given net list. In the following we will first demonstrate that, for each representation, there corresponds a unique realization.

The interval graphical representation of the example in Fig.3 is redrawn in Fig.4a together with a set of line segments in broken lines connecting the nodes. Let us define a reference line as the continuous line segments which connect the nodes in succession from left to right. The m interval lines together with the reference line form a graph. The crux of our proposed realization lies in a topological mapping of the graph so constructed. Let us stretch out the reference line and set it on top of the real line R. The m horizontal interval lines are mapped topologically into vertical and horizontal paths. Nets and portions of nets which lie above the reference line are mapped into paths in the upper street. Similarly, nets and portions of nets below the reference line are mapped into paths in the lower street. This prosess defines a unique realization. For the example in Fig.4a, This mapping results in the realization as shown in Fig.4b. It becomes obvious that the problem of finding an optimum realization is reduced to that of finding a matrix $\underset{\sim}{A}$ which represents an optimum ordering of the m nets in the form of horizontal intervals. In order to pursue further we need to understand the property of street congestions in terms of the new formulation.

First, let us define the cut number of a node v_i, denoted by c_i, as in [2]. Let us draw a vertical line at v_i superimposed on the interval graphical representation as shown in Fig.3 for example by the line at v_6. The cut number c_i is defined as the number of nets cut by the vertical line, disregarding the net to which v_i belongs. The nets cut by the vertical line are called the nets which cover the node v_i. Thus at v_6 in Fig.3 $c_6=2$, and nets N_2 and

N_4 are said to cover v_6. We next introduce the upper cut number at v_i, c_{iu}, as the number of nets cut by the vertical line above v_i. Similarly, we define the lower cut number at v_i, c_{iw}, as the number of nets cut by the vertical line below v_i. Obviously, for all i, $c_i = c_{iu} + c_{iw}$. These cut numbers of all the nodes for the example are listed in the table in Fig.3. We similarly, define a net covering the node v_i from the above as the net which intersects with the vertical line above v_i, and a net covering the node v_i from below as the net which intersects with the vertical line below v_i. Thus in Fig.3, N_2 covers the node v_6 from the above and N_4 covers the node v_6 from below.

Let $$C_u = \max_i c_{iu}$$
and $$C_w = \max_i c_{iw} \qquad (1)$$

From the topological mapping just introduced, it becomes clear that C_u gives the track number in the upper street and is equal to the upper street congestion for the realization. Similarly, C_w is equal to the lower street congestion. Therefore, our problem of finding an optimum realization or routing* amounts to finding an ordering of the m nets among the m! permutations for which the max $\{C_u, C_w\}$ is a minimum.

It would be hopeless to generate all m! realizations in order to obtain an optimum one. In the next section, we will first study the characterestics of an optimum realization to gain some insight. The necessary and sufficient conditions for an optimum realization will be given in section IV.

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* We sometimes use the term routing to mean realization.

III. OPTIMUM ROUTING, A PREAMBLE

Before we discuss optimum routing, it is necessary to introduce the term, cut number of a net N_j, denoted by q_j as was done in [2]. We difine q_j as the maximum of the cut number of the node which belongs to the net N_j. For the example in Fig.3, we have $q_1 = 3$, $q_2 = 1$, $q_3 = 2$ and $q_4 = 3$. It is clear that the cut number of a net is an important property in determining the net ordering for an optimum routing. For example, if the first net from the top has a cut number q then C_w is at least q, because at one of the nodes which belong to the first net, the lower cut number is q. Similarly, if the last net chosen has a cut number q, then C_u is at least q, because at one of the nodes which belong to the last net, the upper cut number is q. Thus, it makes sense to select those nets with least cut numbers as the outer nets for optimum routing.

In an optimum realization, let Q_o be the street congestion, thus $Q_o = \max\{C_u, C_w\}$. Let us further denote by

$$q_m = \min_j q_j \qquad\qquad (2)$$
$$q_M = \max_j q_j$$

then we can state the following:

Propbsition 1 $\qquad Q_o \geqq \max\{q_m, q_t\}$ where $q_t = \lceil q_M/2 \rceil$ and $\lceil x \rceil$ is the smallest integer not smaller than x.

Poof $\qquad Q_o \geqq q_m$ has already been shown by the argument above. $Q_o \geqq q_t = \lceil q_M/2 \rceil$ is proven by first assuming that we assign the net with q_M in one of the middle rows. For example, if q_M is even, at the node v_i where $c_i = q_M$, the best we could do is to choose $q_M/2$ nets covering the node from the above and $q_M/2$ nets covering the node from below. Thus $c_{iu} = c_{iw} = q_M/2$. If q_M is odd, at the node where $c_i = q_M$, the best we could do is to choose $(q_M + 1)/2$ nets on one side of v_i and $(q_M - 1)/2$ nets on the other. Thus either $c_{iu} = (q_M + 1)/2$ and $c_{iw} = (q_M - 1)/2$, or $c_{iu} = (q_M - 1)/2$ and $c_{iw} = (q_M + 1)/2$. It is also clear that with any other ordering we cannot do better than this. Therefore, we have shown that $Q_o = \max\{C_u, C_w\} \geq \lceil q_M/2 \rceil$.

Q.E.D.

From the above, we see that the strategy to obtain an optimum routing is to choose those nets with the lowest cut number as the outer rows and to divide up c_i properly between c_{iu} and c_{iw} at those nodes where the cut number is larger than q_m. Although detailed specifications are to be worked out for an optimum routing, it seems that an increasing ordering of nets based on the cut number of a net from the outer rows to the center rows is perhaps the right strategy for an optimum routing. Before we give the necessary and sufficient conditions for optimum realization, let us consider an example.

Example 1. Given the net list as represented by the interval graphical representation of Fig.5. The net list consists of 16 nodes and 8 nets. The cut numbers of the nodes and nets are marked on the figure. From that, we see $q_m = 2$ and $q_M = 3$. thus $Q_o \geq 2$. Our question is whether $Q_o = 2$ can be realized. For comparison, the problem does not satisfy the sufficient condition given in [2]. From our discussion so far, it is clear that we should select N_1 and N_7 as the outer rows because they have the lowest cut number; but how about the rest?

First we consider all the nodes with cut number less than 3. These are v_1, v_2, v_3, v_6, v_7, v_{10}, v_{11}, v_{14}, v_{15} and v_{16}. Clearly, at any of these nodes, since the cut number is less than 3, c_{iu} and c_{iw} at these nodes will not cause trouble. This means that we only need to concern ourselves with the remaining six nodes with cut number equal to three: v_4, v_5, v_8, v_9, v_{12} and v_{13}. At these nodes, we must make sure that in assigning the nets, the cut number is divided up between the lower street and the upper street. A 2-1 division or a 1-2 division of the cut number is fine. But a 3-0 or a 0-3 division is not. In Table 1 we list all necessary information at these nodes:

node	associated net	nets cut
v_4	N_4	N_1 N_2 N_3
v_5	N_3	N_1 N_2 N_4
v_8	N_6	N_1 N_2 N_5
v_9	N_5	N_1 N_2 N_6
v_{12}	N_8	N_2 N_6 N_7
v_{13}	N_2	N_6 N_7 N_8

Table 1. Pertinent information for optimum routing in Example 1

The first column gives all the nodes with cut number larger than 2. The second column indicates the net to which the node belongs. The third column gives the nets which cover the node or nets cut by a vertical line drawn at the node. From the table, we can decide an ordering of the eight nets such that nets in the second column will have at least one net above it and one net below it among the nets specified in the third column. For this example, there exist many such orderings, for example

$$N_1 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_2 \quad N_8 \quad N_7$$

The interval graphical representation and its corresponding realization are shown in Fig.6. Thus $Q_o = 2$ has been realized.

IV. THE NECESSARY AND SUFFICIENT CONDITIONS

In general, the task of obtaining an optimum routing is more involved. However, the concept is the same. We must check those nodes at which the cut number is large. A table similar to that of Example 1 needs to be constructed. We must test whether an optimum division of the cut numbers between the lower street and the upper street is possible. Although the determination of a feasible order may not be simple, a set of necessary and sufficient conditions can always be stated.

First we need to define some useful terms.

Let

$$\ell \triangleq q_M - q_t \qquad (3)$$

A net x is said to cover a net y at a node v which belongs to y if x covers v. Similarly, net x is said to cover net y from the above if net x covers node v from the above, and net x is said to cover net y from below if x covers v from below.

Theorem 1 There exists an optimum realization with street congestion $Q_o = q_t$ if and only if there exists an ordering such that for each v_i with $c_i = q_t + k$ $(k=1,\cdots,\ell)$ the net associated with v_i is covered from the above and below by at least k nets.

Proof Since at each v_i with $c_i \leq q_t$, c_{iu} and $c_{iw} \leq q_t$, we only need to concern ourselves with those v_i where $c_i > q_t$. At these nodes, with cut number $q_t + k$, $k > 0$, if there are at least k nets above the node and k nets below the node, the maximum cut number, C_w and C_u are at most q_t. Since from Proposition 1 $Q_o \geq q_t$ thus the optimum $Q_o = q_t$ is realized. This proves the sufficiency. To prove the necessity, we assume that at those nodes v_i with cut number $q_t + k$, there are less than k nets covering v_i from the above or below. Then, since $c_{ui} + c_{wi} = c_i = q_t + k$, either c_{ui} or c_{wi} must be greater than q_t. Therefore the street congestion is larger than q_t. Q.E.D.

To deal with the general situation we need to introduce the definition of p-excess property.

By <u>p-excess property</u> we mean that there exists an ordering such that for each v_i with $c_i = q_t + k$ ($k=p+1,\cdots,\ell$) the net associated with v_i is covered by at least k-p nets from the above and from the below.

<u>Theorem 2</u> There exists an optimum realization with street congestion $Q_o = q_t + p$, if and only if p is the least non-negative integer for which the p-excess property holds.

The proof of this theorem is exactly the same as that of Theorem 1 and is therefore omitted.

<u>Remark</u> Theorem 1 is a special case of Theorem 2 when p=0.

<u>Example 2</u> In this example there are 30 nodes and 15 nets. An interval graphical representation is shown in Fig.7 together with the cut number of the nodes and of the nets. For convenience we use alphabets in capital letters to designate nets. It is seen that $q_m = 4$, $q_M = 7$, $q_t = 4$ and $\ell = 3$. In Table 2, we give those nodes with cut number larger than 4, the associated nets and the nets that cover the pertinent nodes. The nodes are grouped into three parts according to their cut numbers. First, we must check whether the conditions in Theorem 1 are satisfied. To determine the net ordering, it is useful to note that there are two nets which have cut numbers less than 5. They are net D and net G, and they are assigned right away to the outer rows. As to the others, we will start from inside out by considering the first part of Table 2. Both nets K and L have cut number 7, we need to assign them in the middle. We next consider nets in the second part, namely: C, H and O. Since they have cut number 6, and we will temporarily assign them next to nets K and L. In checking with the nets which cover the nodes with cut number 7, a tentative ordering of

F C H L K J I G

will satisfy the conditions that both L and K have three nets above and below. Similarly, for the second part in Table 2 a tentative assignment of

M N F O C H L K J I G

is made. This will satisfy the conditions that O, C and H have two nets above them and two nets below them among those which cover the pertinent nodes.

Nodes	Cut number	Associated nets	Nets covering nodes
17	7	K	F C H L J I G
16	7	L	F C H K J I G
24	6	C	F O N M I G
18	6	H	F C L J I G
23	6	O	F C N M I G
15	6	K	F C H J I G
7	5	A	D B E F C
11	5	B	E F C H G
13	5	E	F C H I G
6	5	F	D A B E C
25	5	F	O N M I G
12	5	I	E F C H G
14	5	J	F C H I G
21	5	J	F C M I G
20	5	M	F C J I G
22	5	N	F C M I G
19	5	L	F C J I G
10	5	H	B E F C G

Table 2 Example 2:
Nodes with cut number larger than q_t, the associated nets and the nets which cover the nodes.

The remaining part of Table 2 contains nodes with cut number 5. An ordering must be obtained to satisfy the remaining conditions with the clue that net D and net G are assigned in the outer rows. We discover that there are various constraints among the nets with cut number 5. An ordering as

D M N F A B E O C H L K J I G

satisfies all conditions except at node 20 where net M should have a net from the above among the pertinent ones. After much cut and try, we are convinced that not all the conditions can be simultaneously satisfied. This means that the conditions in Theorem 1 cannot be met, thus $Q_o > q_t$. By applying Theorem 2, it becomes clear that Theorem 2 is satisfied with $p = 1$. Therefore, we conclude that $Q_o = q_t + 1 = 5$, and the ordering given above gives one such solution. The interval graphical representation is drawn in Fig.8 together with the reference line. The realization is shown in Fig.9. It is seen that at node v_{20}, $C_{20w} = 5$, which is the maximum C_w among the nodes. Thus an optimum $Q_o = 5$ has been realized.

To gain further insight of the conditions given in Theorem 1 and Theorem 2, we will classify the nodes into different sets according to their cut number. Let $\mathcal{S}_k$, $k=1,2,\cdots,\ell$ be the set of nodes with cut number equal to $q_t + k$. Then the test to be made can be stated in terms of net covering to satisfy a specific table generated as in Example 2. The necessary and sufficient conditions for an optimum realization can be summarized by the flow chart as given in Table 3. In the flow chart we use the terminology "cover $\mathcal{S}_k(j)$" to mean nets covering nodes in the set with cut number $q_t + k$ from the above and below j times.

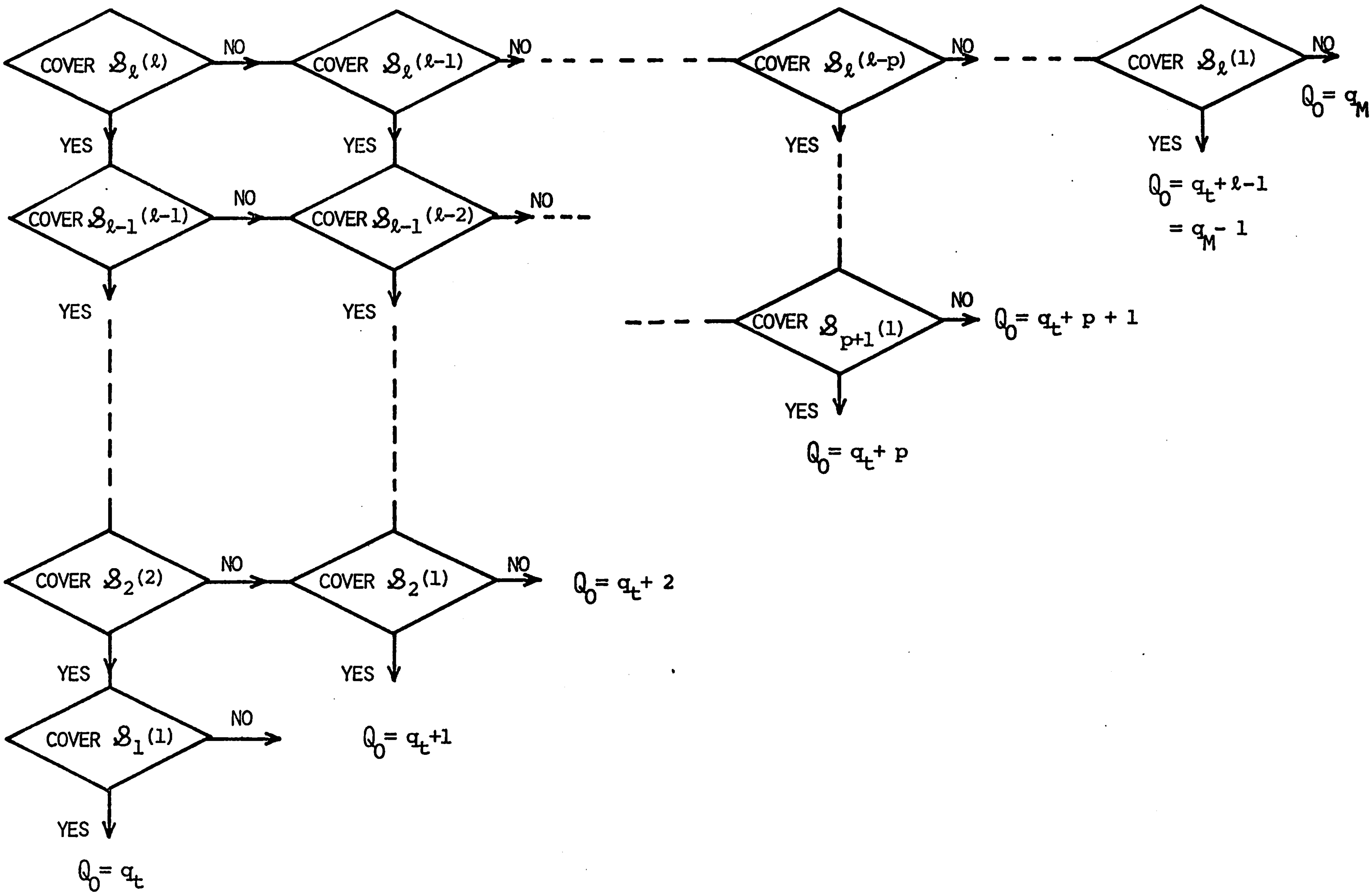

COVER $\mathcal{S}_\ell(\ell)$
NO
COVER $\mathcal{S}_\ell(\ell-1)$
NO
COVER $\mathcal{S}_\ell(\ell-p)$
NO
COVER $\mathcal{S}_\ell(1)$
NO
$Q_0 = q_M$
YES
$Q_0 = q_t + \ell - 1$
$= q_M - 1$
COVER $\mathcal{S}_{\ell-1}(\ell-1)$
COVER $\mathcal{S}_{\ell-1}(\ell-2)$
COVER $\mathcal{S}_{p+1}(1)$
$Q_0 = q_t + p + 1$
$Q_0 = q_t + p$
COVER $\mathcal{S}_2(2)$
COVER $\mathcal{S}_2(1)$
$Q_0 = q_t + 2$
COVER $\mathcal{S}_1(1)$
$Q_0 = q_t + 1$
$Q_0 = q_t$

table 3

V. GRAPH THEORY INTERPRETATIONS

The necessary and sufficient conditions for optimum routing can be interpreted in terms of graphs. This may lead to an efficient algorithm which is to be determined. For a given net list and a specified value of p we first construct a bipartite graph G as follows:

We assign a node v_i in G to every node v_i with cut number $c_i > q_t + p$. This forms the first set of nodes of the bipartite graph G. The second set of nodes consists of node N_j in G representing all the nets. The edges are defined according to the covering relation. Thus, there exists an edge between v_i and N_j if, in the net list, net N_j covers node v_i. Next we choose the orientation of the edges to satisfy the covering property at each node v_i. An edge enters v_i from node N_j if net N_j covers v_i from the above; and an edge leaves from v_i to node N_j if net N_j covers v_i from below. Thus to satisfy the p-excess property, for each node v_i with cut number $c_i = q_t + k$ $(k=p+1,\cdots,\ell)$, both the number of edges entering v_i and leaving from v_i must be at least k-p. For Example 1, with p=0, the bipartite graph is shown in Fig.10a with an edge orientation selected according to the ordering in Fig.6.

Next, we introduce a reduced digraph G' from the directed bipartite graph G by the following operations. First identify node v_i in G to its associated net, node N_j by shorting v_i and N_j. Parallel edges with same directions are then merged. The reduced digraph G' thus consists of m nodes representing all the nets. For the present example, G' is shown in Fig.10b. We note that G' defines a precedence relation among the nets. A proper ordering can be obtained by observing the precedence relation and tracing through the digraph G'. First, it is important to note that the reduced digraph G' has no cycle. This is almost self-evident because G' gives the precedence relation among nets. We are now in a position to state the following theorem.

Theorem 3 The necessary and sufficient condition for a net list to have the p-excess property is that there exists an orientation for the edges in the bipartite graph G with the following properties:

(i) For each node v_i, the number of edges entering v_i and leaving from v_i must be at least k-p, and (ii) the reduced digraph G' is acyclic.

Poof (i) is obvious and follows directly from the definitions of G and the p-excess property. To prove necessity for (ii), we assume that there is a directed cycle in G'

$$[N_1, N_2], [N_2, N_3], \cdots, [N_{k-1}, N_k], [N_k, N_1]$$

Then, reading the list from left to right, we see that N_1 is above N_2, N_2 above $N_3, \cdots, N_{k-1}$ above N_k, and N_k above N_1. Therefore, we conclude that N_1 is above N_1, which is a contradiction. To prove sufficiency for (ii), we introduce an ordering among nets based on the orientation of the edges in G'. For two distinct nets N and N' we say $N < N'$ if and only if there exists a directed path

$$[N_1, N_2], \cdots, [N_{k-1}, N_k]$$

where $N = N_1$ and $N' = N_k$**. Since there exists no directed cycles, one of the following cases must hold for any two distinct nets N and N'.

(1) $N < N'$

(2) $N' < N$

(3) There exists no order relation between N and N'.

Because of the transitive relation that $N < N'$ and $N' < N''$ imply $N < N''$, there exists a partial order < among the nets in G'. Therefore a proper ordering among the nets can be obtained.

Q.E.D.

In our example in Fig.10b, N_1 is the minimal net. By starting with N_1 and tracing through G', we obtain a proper ordering

$$N_1 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_2 \quad N_8 \quad N_7$$

** A net N is said to be the minimal net among a set of nets if there is no net N' in the set of nets with relation $N' < N$.

This is the order which we obtained in Fig.6. Obviously, there exist many proper orderings because, at each stage, we may pick up arbitrarily a minimal net among many. For example, N_5 may be picked ahead of N_3 to obtain the following proper orderings:

$$N_1 \quad N_5 \quad N_6 \quad N_3 \quad N_4 \quad N_2 \quad N_8 \quad N_7$$

and

$$N_1 \quad N_5 \quad N_3 \quad N_4 \quad N_6 \quad N_2 \quad N_8 \quad N_7 \quad .$$

VI. CONCLUSIONS

In this paper we have demonstrated that by the use of an interval graphical representation of the net list of the single row routing problem we obtain a set of necessary and sufficient conditions for optimum realization. The conditions are given in terms of feasible orderings of the nets. A graph theory interpretation is also given. So far, an efficient algorithm has not yet been worked out to implement a procedure for properly ordering the nets. However, cut-and-try has been rather easy for modestly complicated problems.

The special case with $Q_o \leq 2$ is of interest for practical printed-circuit-board routing. For this case, a simple algorithm can be derived by the use of directed graphs. As the general problem of multilayer printed-circuit-board routing depends in a crucial way on single-row routing, especially situation with maximum of two rows, the major problem remains to be solved is optimum layering. The problem is to decompose in an optimum way a single row, multilayer net list into a number of single-row, single-layer problems.[3]

In conclusion, as far as optimum single-row routing is concerned, the present paper gives a complete set of necessary and sufficient conditions. Work is still needed to derive an algorithm for the implementation of net ordering to satisfy these conditions.

APPENDIX

In [2], Theorem 2 stated that a net list L over R is realizable with $C_u = C_w = M \geq \rho' = \lceil \rho/2 \rceil$ if the following holds: for every unit interval (a,b) with density d(a,b)= I, $I \geq M + 1$, there exists at least 2(I-M) nets covering (a,b) such that each of them has cut number less than I. The theorem gives a sufficient condition for realization (not necessarily the optimum). An Assignment algorithm for realization was also given. In the Assignment algorithm, there usually exists leeway as to which nets among several eligible ones should be assigned at each step. No specific information on the ordering was given. In the following example, it is shown that an arbitrary ordering will fail in obtaining a realization.

Example 3 Consider the net list given in Fig.11 together with the density for all the intervals, the cut number of the nodes and of the nets. Since $\rho = 6$, $M \geq 3$, we only need to check first those intervals with density larger than or equal to 3. The conditions given in Theorem 2 in [2] are satisfied with M = 3, therefore a realization with street congestion equal to 3 should exist.

Let us test the Assignment algorithm. The first interval from the left is (4,5), and we have four nets, N_1 , N_2 , N_3 and N_4 to choose for U and W. Since the algorithm does not indicate priority, we choose N_4 in U and N_1 in W. The next interval (8,9) can be passed. The next one is (10,11), and we have N_2 and N_3 to choose for U. We assign N_3 in U. The next interval (12,13) can be passed. The next interval is (16,17), and we assign N_2 in U. The last one (18,19) can be passed. We thus completed all intervals with density 4. This is summarized in Fig.12.

The leftmost unit interval with density 5 is (5,6). There are 4 nets, N_1 , N_2 , N_3 and N_4 two of which must be in U and two remaining must be in W. Thus the algorithm fails.

The same problem can be solved easily with our present method. From the given information, we have $q_m = 0$ $q_M = 5$, $q_t = 3$ and $\ell = 2$. The information to satisfy Theorem 1 is summarized in Table 4. It is easily checked that an ordering as

$$N_1 , N_2 , N_5 , N_6 , N_7 , N_8 , N_{10} , N_{11} , N_9 , N_3 , N_4$$

satisfies all the conditions. Thus $Q_o = q_t = 3$, and an optimum ordering is shown in Fig.13.

It should be noted that the Assignment algorithm given in [2] always assigns nets from outside inward. This is because the way that routing is defined in the upper street and lower street. The interval graphical representation given in the present paper avoids completely the difficulty of sequential routing.

Nodes	Cut number	Net Asso.	Nets cut
6	5	N_6	N_1 N_2 N_3 N_4 N_5
7	5	N_5	N_1 N_2 N_3 N_4 N_6
5	4	N_5	N_1 N_2 N_3 N_4
8	4	N_6	N_1 N_2 N_3 N_4
11	4	N_8	N_1 N_2 N_3 N_7
12	4	N_7	N_1 N_2 N_3 N_8
16	4	N_{11}	N_1 N_2 N_9 N_{10}
17	4	N_{10}	N_1 N_2 N_9 N_{11}

Table 4 Example 3

References

1. H. So, "Some theoretical results on the routing of multilayer printed-wiring boards", Proc. 1974 IEEE International Symp. on Circuits and Systems. pp. 296-303.

2. B.S. Ting, E.S. Kuh and I. Shirakawa, "The multilayer routing problem: algorithms and necessary and sufficient conditions for the sigle-row, single-layer case", IEEE Trans. on Circuits and Systems, vol. CAS-23, no. 12, pp. 768-778, Dec. 1976.

3. B.S. Ting and E.S. Kuh "An approach to the routing of multilayer printed circuit boards", Proc. 1978 IEEE International Symp. on Circuits and Systems.

Fig. 1. A set of n nodes on a single-row.

Fig. 2. A realization of the net list $L = \{N_1, N_2, N_3, N_4\}$ where $N_1 = \{v_1, v_5\}$, $N_2 = \{v_2, v_8\}$, $N_3 = \{v_3, v_6\}$ & $N_4 = \{v_4, v_7, v_9\}$. Upper street congestion = 1 track, lower street congestion = 3 tracks in this realization.

Fig. 3. Interval graphical representation of the matrix $\underset{\sim}{A}$ in Fig.2 and the node cut numbers.

Fig. 4a. Interval graphical representation together with the reference line.

Fig. 4b. Net list realization which corresponds to the interval graphical representation of Fig.4a and the matrix $\underset{\sim}{A}$ of Fig.2.

Fig. 5. Example 1, an arbitrary ordering together with cut numbers.

Fig. 6. An optimum realization of the example given in Fig.5.

Fig. 7. Example 2 indicating an interval graphical representation with cut numbers of the nodes and nets.

Fig. 8. A specific ordering representing an optimum with $Q_o = q_t + 1 = 5$.

Fig. 9. An optimum realization with $Q_o = 5$.

Fig. 10a. A bipartite graph G representing net ordering of Example 1.

Fig. 10b. The reduced digraph G'.

Fig. 11. A counter example to the Algorithm in Ref.[2].

Fig. 12. An ordering which fails.

Fig. 13. An optimum ordering for Example 3.

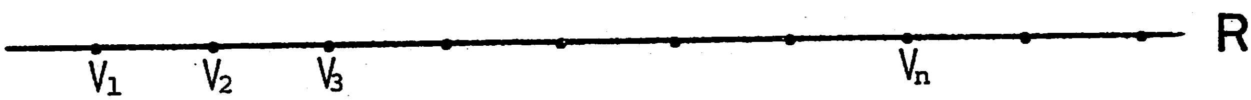
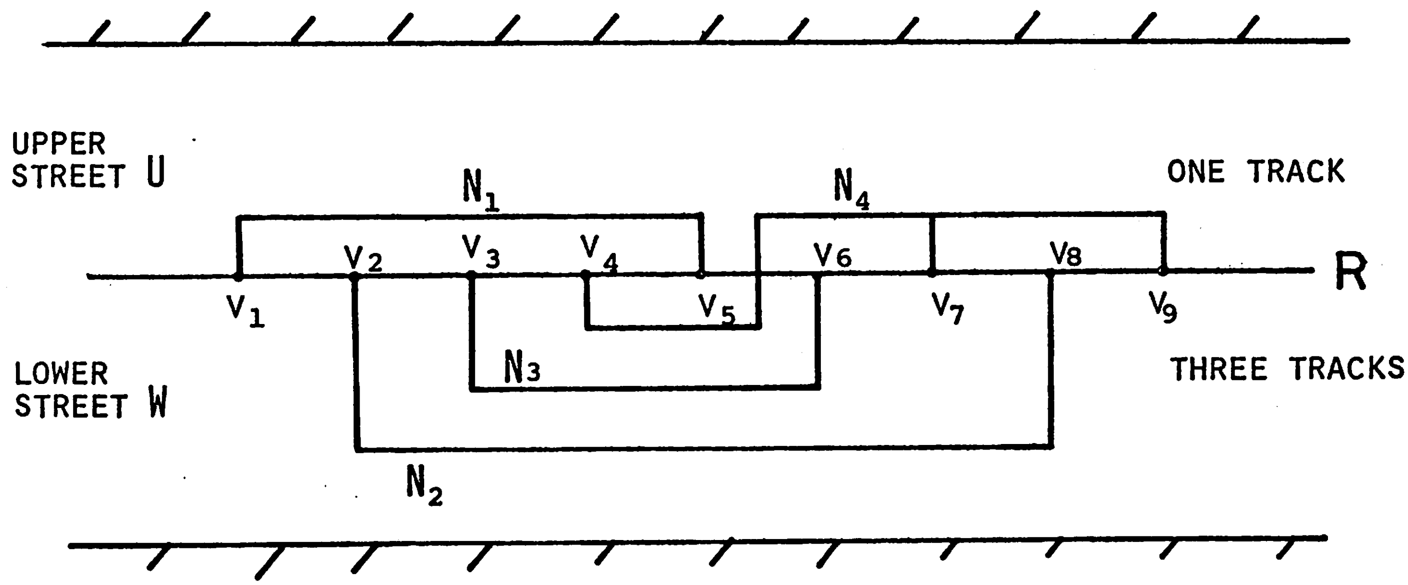
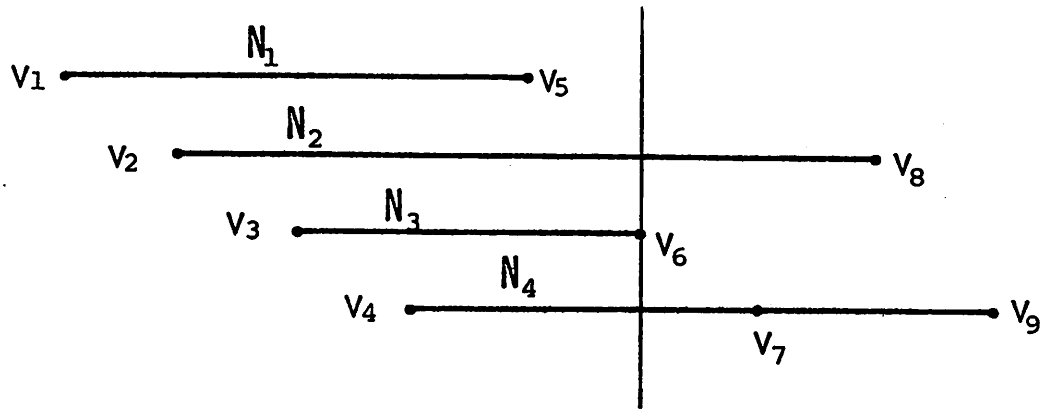


Fig. 1.



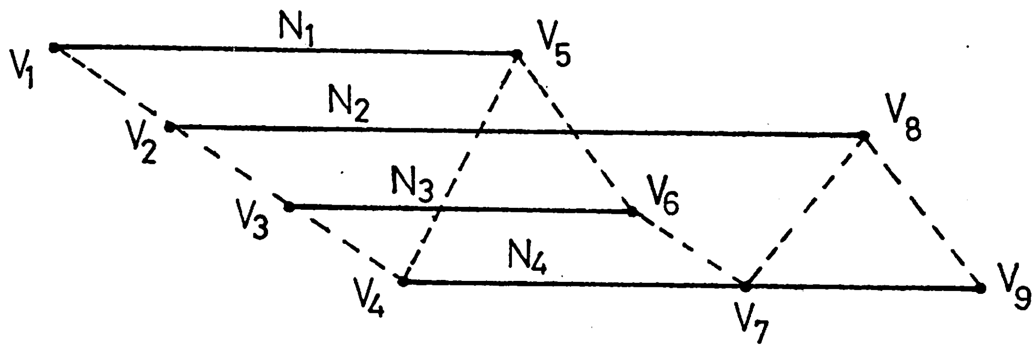
$$\underset{\sim}{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Fig. 2.



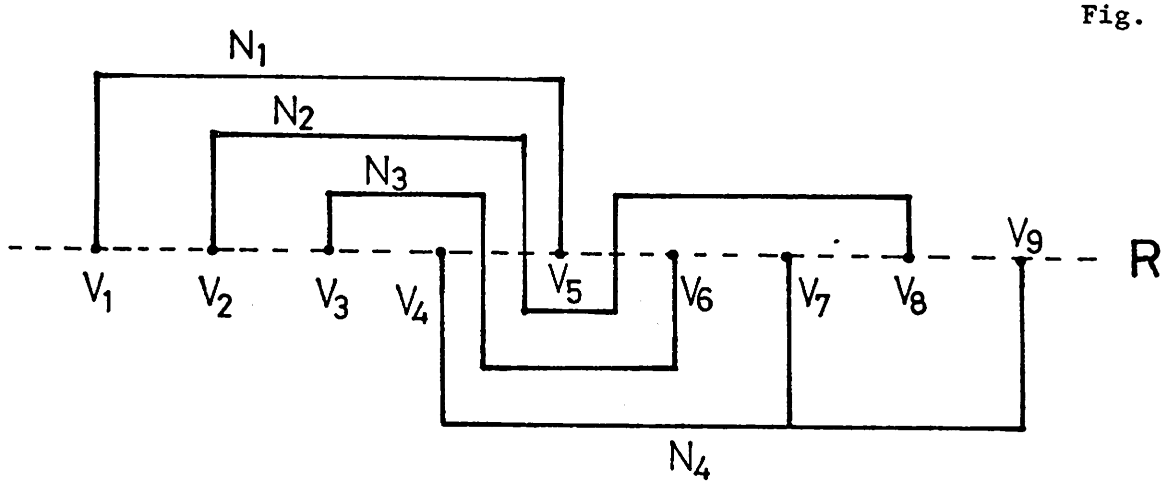
	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
C_{iu}	0	1	2	3	0	1	1	0	0
C_{iw}	0	0	0	0	3	1	0	1	0
C_i	0	1	2	3	3	2	1	1	0

Fig. 3.



N1
V5
V1
N2
V8
V2
N3
V6
V3
N4
V9
V4
V7

Fig. 4a.



N1
N2
N3
V9
R
V1
V2
V3
V4
V5
V6
V7
V8
N4

Fig. 4b.

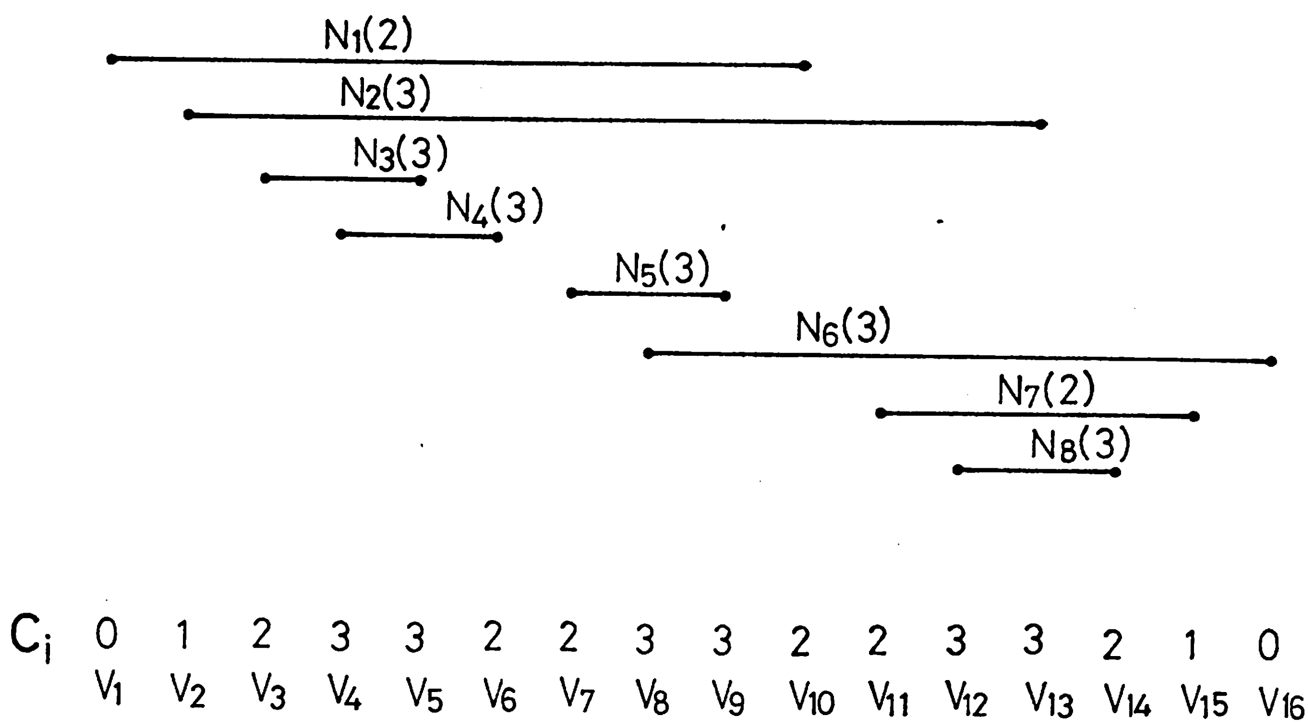

N1(2)
N2(3)
N3(3)
N4(3)
N5(3)
N6(3)
N7(2)
N8(3)
Ci 0 1 2 3 3 2 2 3 3 2 2 3 3 2 1 0
V1 V2 V3 V4 V5 V6 V7 V8 V9 V10 V11 V12 V13 V14 V15 V16

Fig. 5.

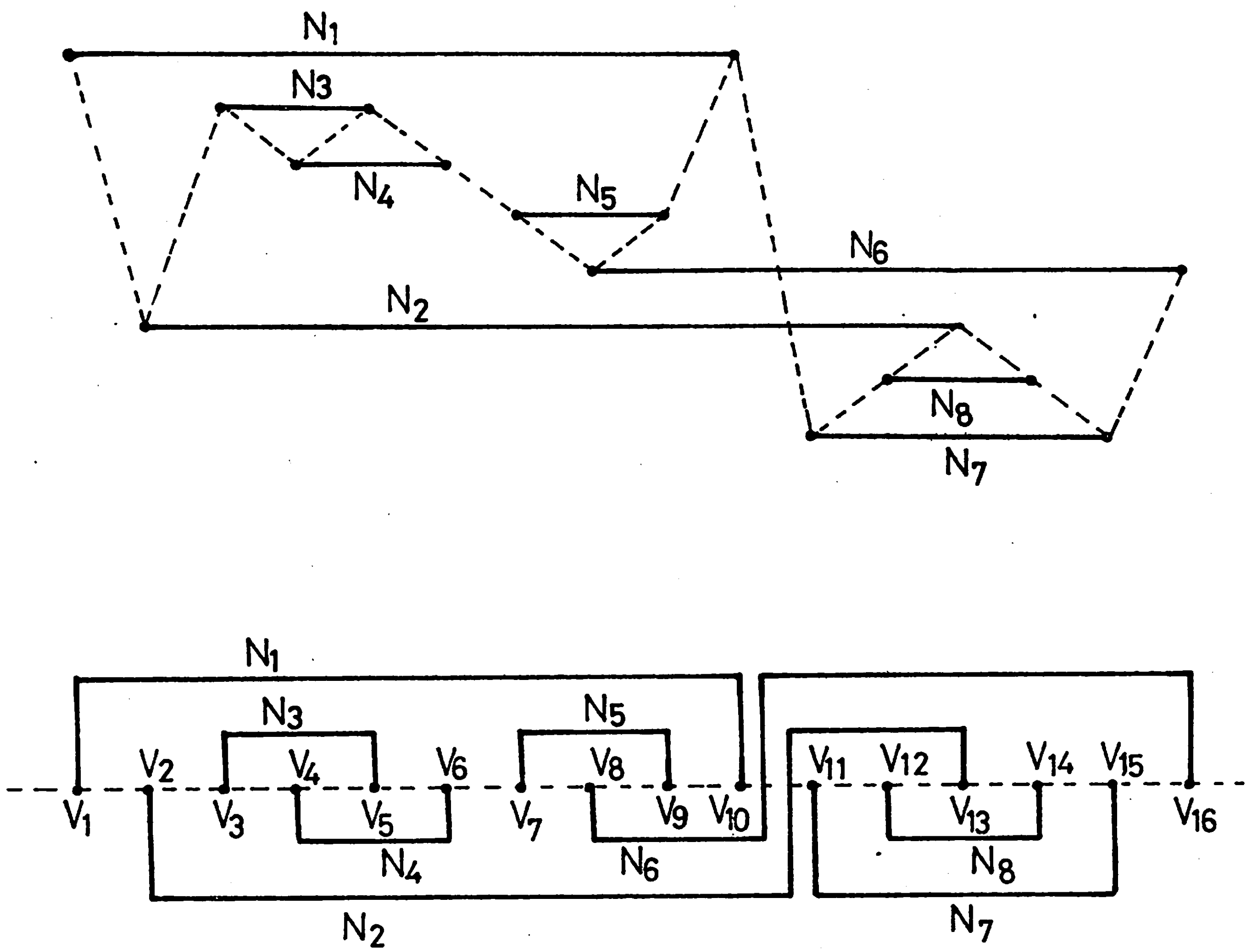
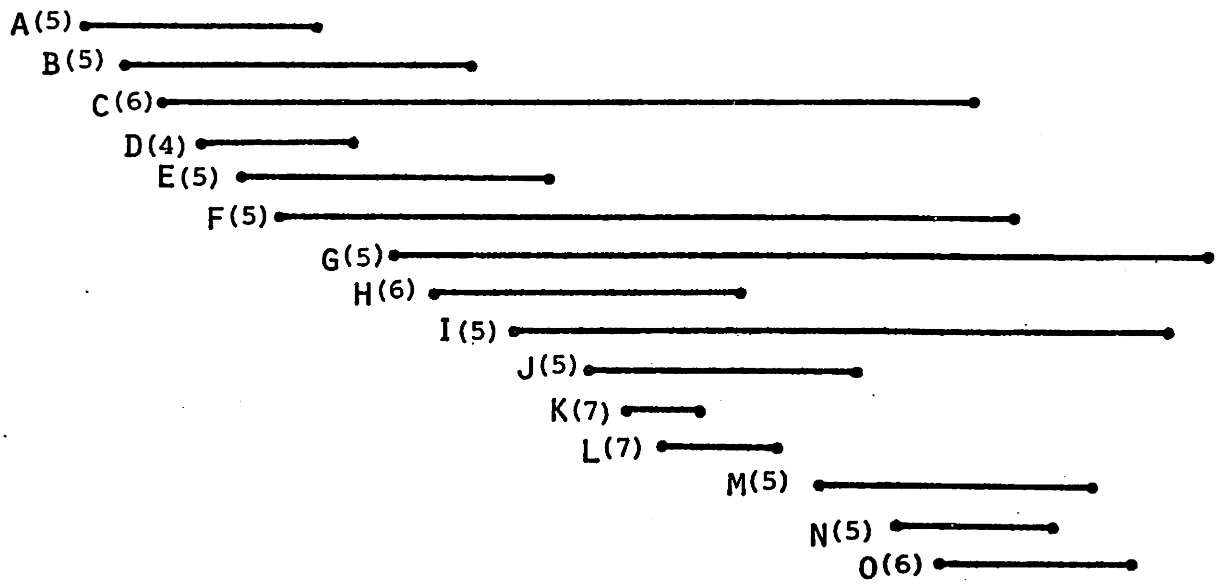

N1
N3
N4
N5
N6
N2
N8
N7
N1
N3
N5
V1
V2
V3
V4
V5
V6
V7
V8
V9
V10
V11
V12
V13
V14
V15
V16
N4
N6
N2
N8
N7

Fig. 6.



V_i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
C_i	0	1	2	3	4	5	5	4	4	5	5	5	5	5	6	7	7	6	5	5	5	5	6	6	5	4	3	2	1	0

Fig. 7.

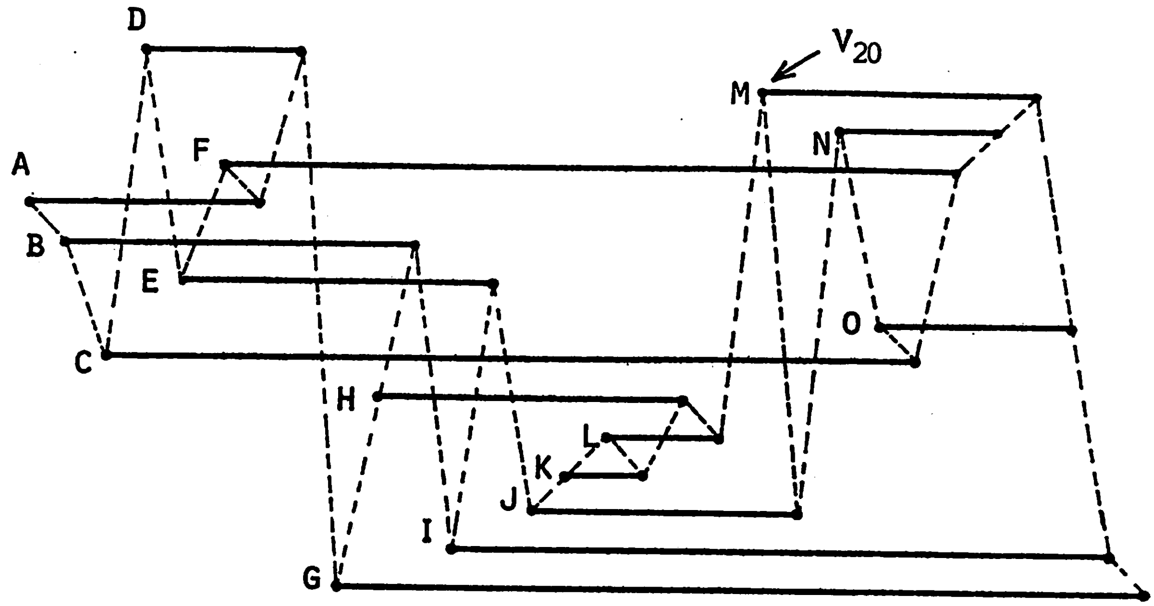


Fig. 8.

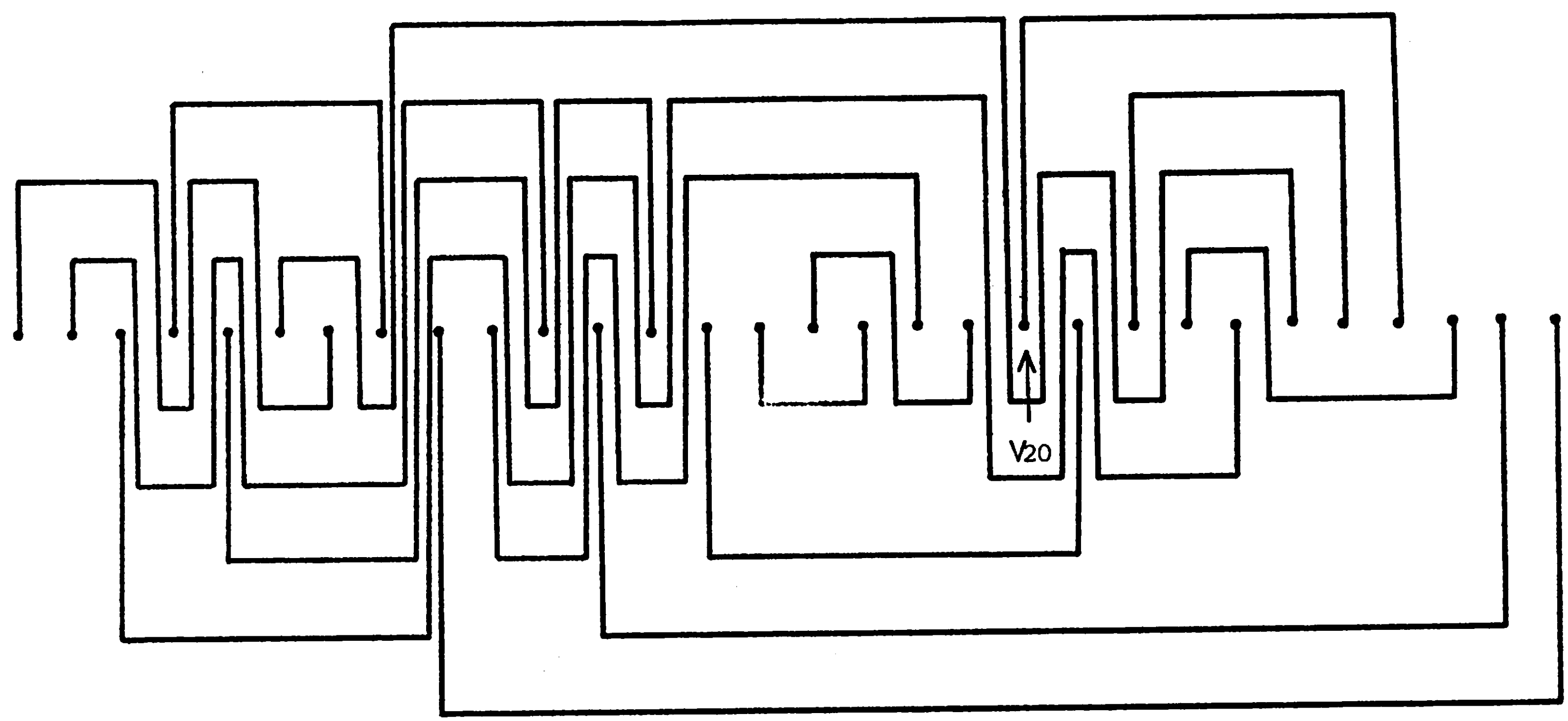


Fig. 9.

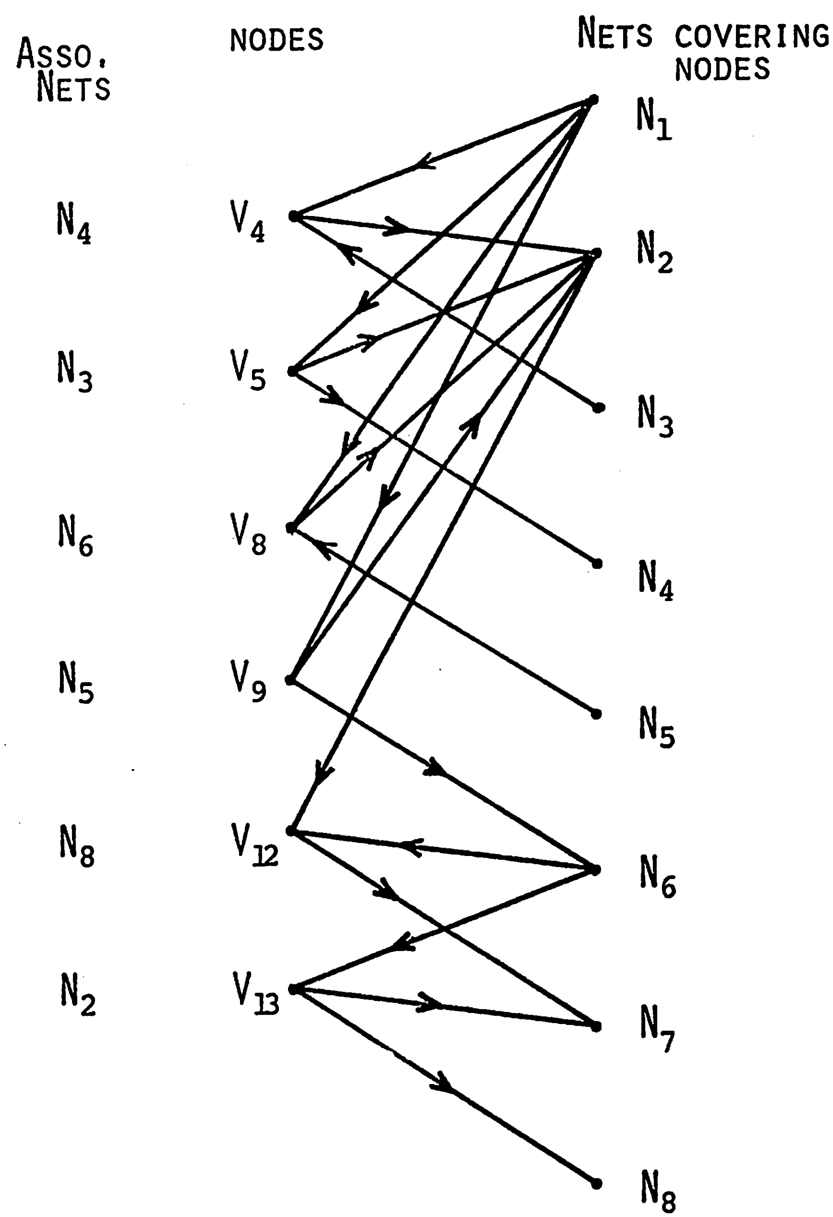


Fig. 10a.

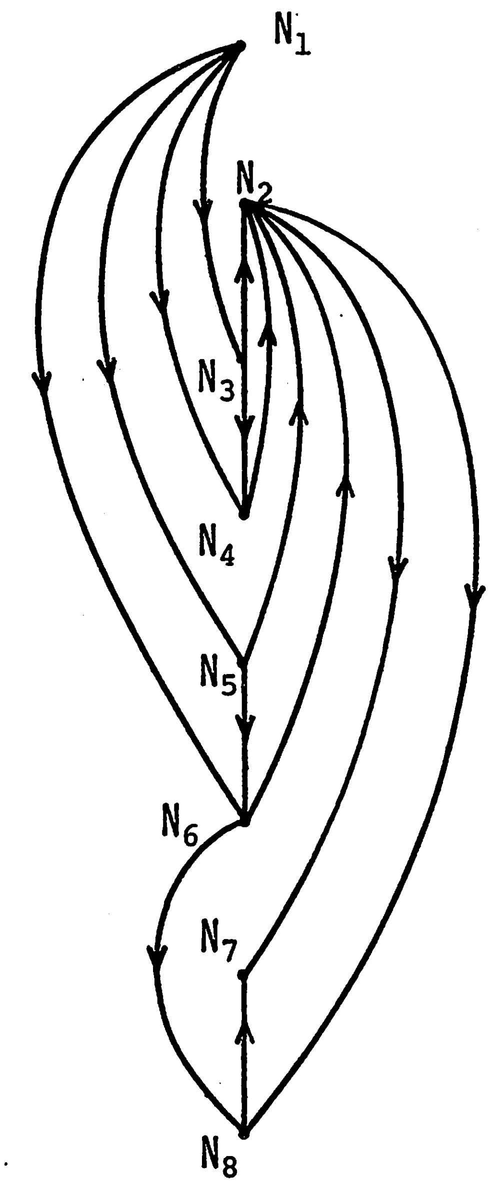
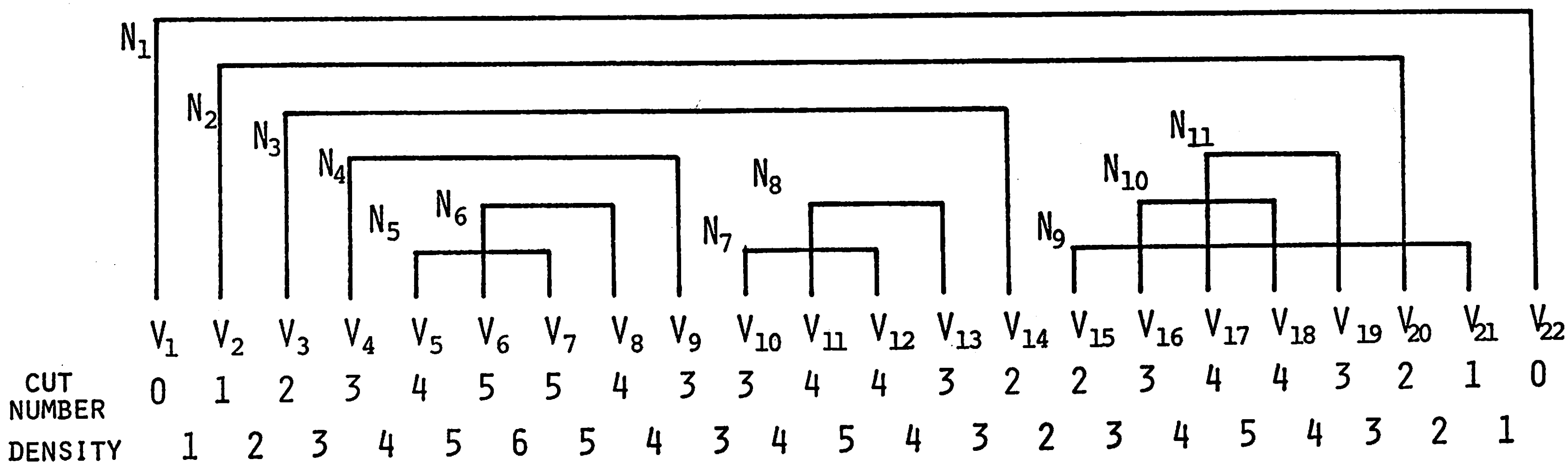


Fig. 10b.



NET	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}
CUT NUMBER	0	2	2	3	5	5	4	4	2	4	4

Fig. 11.

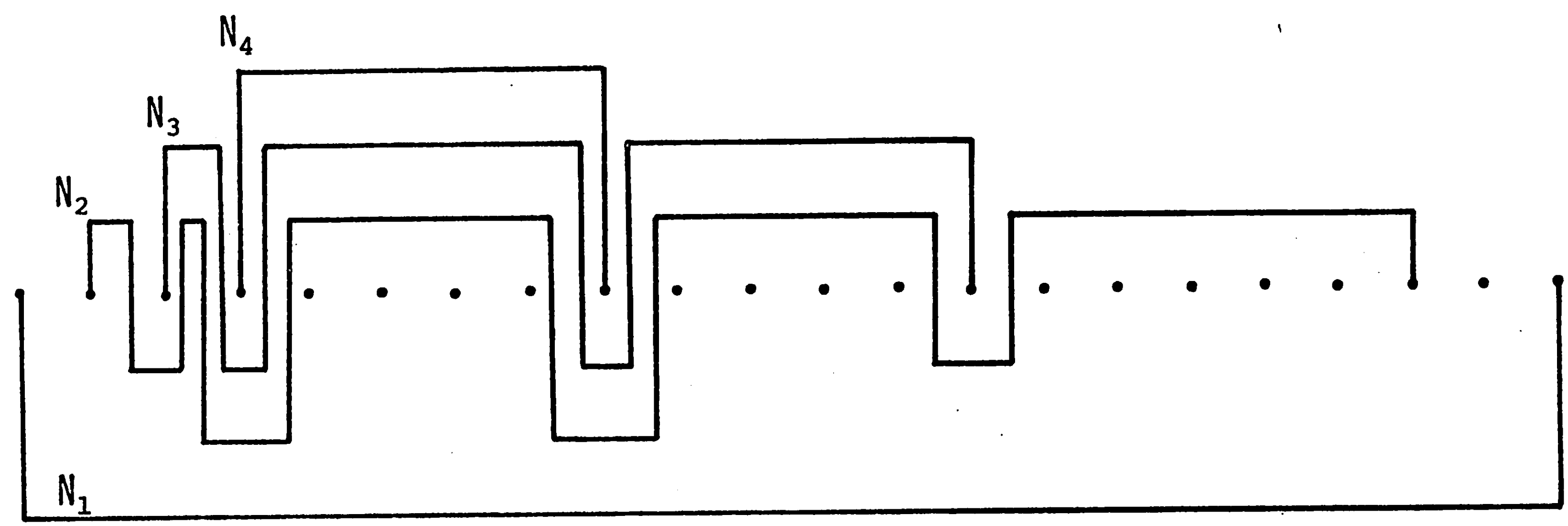


Fig. 12.

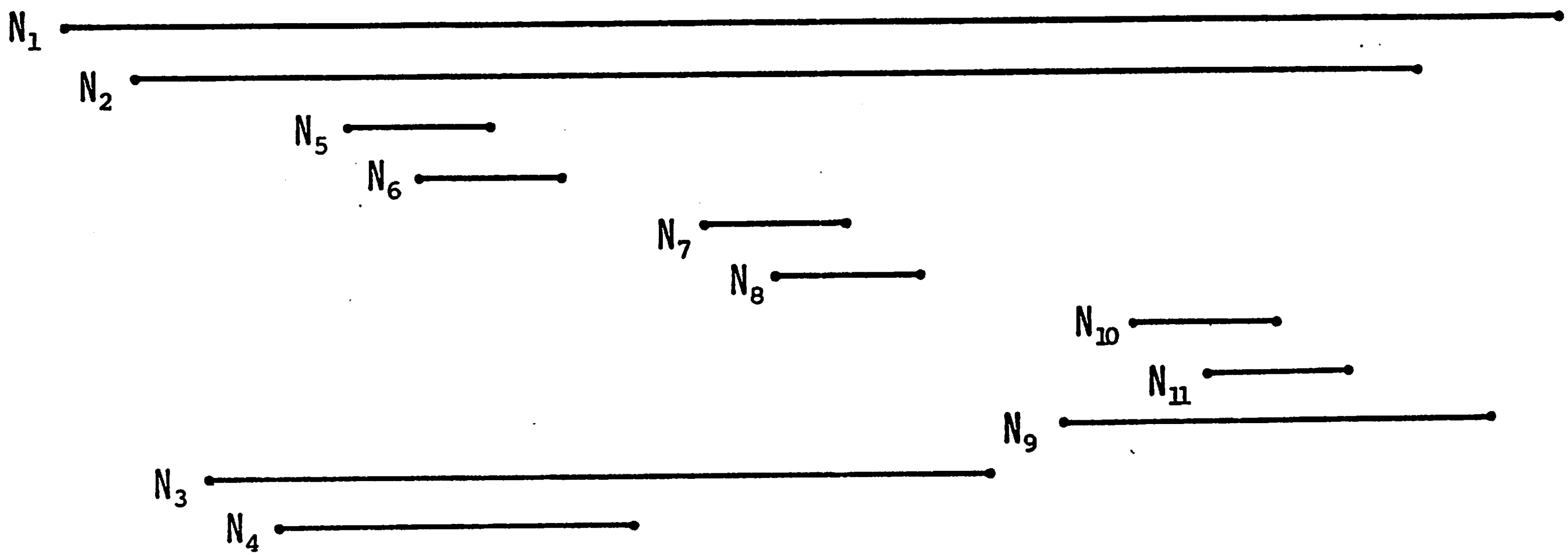


Fig. 13.