FREQUENCY DOMAIN ANALYSIS OF NONLINEAR SYSTEMS:

GENERAL THEORY

by

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## ABSTRACT

A unified study of the applications of Volterra functional series to nonlinear system analysis is presented with special emphasis on frequency-domain results which either have not been published before, or where rigor had been lacking. In particular, an in-depth analysis of the harmonic and intermodulation frequency components due to each Volterra kernel of a given order will be presented and explicit formulas which complement those already published will be given. The effects of manipulating various kernels, which need not be symmetric, on the overall response, as well as on the composite kernels will be investigated. These results are then applied to the analysis of dynamical systems described by a system of nonlinear state equations and a read-out map.

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## 1. Introduction

The Volterra functional series [1] has been used extensively in the analysis of nonlinear circuits and systems characterized by weak nonlinearities [2-11]. Our objective in this paper is to present a unified study of this subject with special emphasis on frequency-domain results which either have not been published before, or where rigor had been lacking. In particular, new explicit formulas will be presented which complement those already published [2-3]. These formulas are derived in Section 3 and are of particular interest to the frequency-domain analysis of nonlinear circuits. In particular, they show precisely which harmonic or intermodulation frequency components are generated by a Volterra kernel of a given order. Among other things, these explicit formulas show that the frequency components generated by an $\underline{n}$th order Volterra kernel are disjoint from those generated by the $\underline{(n+1)}$th order kernel. They also reveal a widely held assumption concerning the incommensurability of input frequencies as incorrect and the precise condition where this assumption holds is given.

Another widely used assumption in this area is that all kernels are symmetric so that one can freely manipulate them without questioning the validity of such mathematical operations as addition, multiplication, differentiation, etc., as well as other more complex system operations such as cascading one system into another etc. We show in Section 4 the precise conditions under which the above cited operations are valid even if the kernels are $\underline{not}$ symmetric. Moreover, we show why the usual symmetry assumption is undesirable for many applications.

The symmetrization results in Section 4 are then applied to develop a Volterra functional series associated with dynamical systems described by a nonlinear system of state equations and a read-out map in Section 5. Again, explicit formulas for deriving the Volterra kernels of any order are given. Finally, the generalization of the above cited results to multi-input systems are summarized briefly in Section 6.

Since one of the objectives of this paper is tutorial in nature, some well-known results are included in order to render this paper somewhat self-contained. Indeed, much of the materials presented in this paper will form the foundation for a subsequent paper on the applications of Volterra functional series to nonlinear circuits.

## 2. Functional Representation of Analytic Systems

For single input analytic systems [4-7], the output $y(t)$ can be expressed as a <u>Volterra functional series</u> of the input $u(t)$. Thus

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \qquad (2.1)$$

where

$$y_n(t) = \begin{cases} h_o & , n = 0 \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \ldots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i, & n > 0 \end{cases} \qquad (2.2)$$

is called the <u>nth order output</u> of the system, and $h_n(\tau_1, \tau_2, \ldots, \tau_n)$, a real valued function of $\tau_1, \tau_2, \ldots, \tau_n$, is called the <u>nth order kernel</u> or <u>nth order impulse response</u> of the system. The multiple Laplace Transform of the nth order impulse response

$$H_n(s_1, s_2, \ldots, s_n) = \begin{cases} H_o = h_o & , n = 0 \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) e^{-s_1\tau_1 - s_2\tau_2 - \cdots - s_n\tau_n} d\tau_1 \cdots d\tau_n, & n > 0 \end{cases} \qquad (2.3)$$

is called the <u>nth order transfer function</u> of the system. Observe that the nth order kernel, and hence also its transform is not unique in the sense that several distinct nth order kernels may give the same nth order output for the same input[1]. However, the <u>symmetrized kernel</u> and its associated <u>symmetrized transform</u> defined by

$$\overline{h}_n(\tau_1, \tau_2, \ldots, \tau_n) \triangleq \frac{1}{n!} \sum_{\substack{\text{all permutations} \\ \text{of } \tau_1, \ldots, \tau_n}} h_n(\tau_1, \tau_2, \ldots, \tau_n) \qquad (2.4)$$

and

$$\overline{H}_n(s_1, s_2, \ldots, s_n) \triangleq \frac{1}{n!} \sum_{\substack{\text{all permutations} \\ \text{of } s_1, s_2, \ldots, s_n}} H_n(s_1, s_2, \ldots, s_n) \qquad (2.5)$$

respectively are unique [4,7].

To emphasize the correspondence between the time-domain and frequency-domain quantities, let us replace the physical time variable $t$ in (2.2) by $n$ associated time variables $t_1, t_2, \ldots, t_n$:

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[1] Observe from (2.2) that an interchange of the argument of $h_n(\tau_1, \tau_2, \ldots, \tau_n)$ does not affect the input-output relationship. Hence there may be up to $n!$ distinct nth order kernels corresponding to the n! permutation of n variables.

$$y_n(t_1, t_2, \ldots, t_n) \triangleq \begin{cases} h_o & , n = 0 \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \ldots, \tau_n) \prod_{i=1}^{n} u(t_i - T_i) d\tau_i, & n > 0 \end{cases} \quad (2.6)$$

The multiple Laplace transform[2]

$$Y_n(s_1, s_2, \ldots, s_n) = \begin{cases} H_o & , n = 0 \\ H_n(s_1, s_2, \ldots, s_n) \prod_{i=1}^{n} U(s_i), & n > 0 \end{cases} \quad (2.7)$$

of (2.6) is called the _nth order output transform_, where $U(s)$ is the usual 1st order Laplace transform of the input.

We will define two systems as __equivalent__ if their respective __symmetrized__ transfer functions are identical. In particular, if $g_n(\tau_1, \tau_2, \ldots, \tau_n)$ denotes the nth order kernel of an equivalent system with output $z(t) = \sum_{n=0}^{\infty} z_n(t)$, then[3] for each n

$$h_n(\tau_1, \tau_2, \ldots, \tau_n) \doteq g_n(\tau_1, \tau_2, \ldots, \tau_n) \quad (2.8)$$

$$H_n(s_1, s_2, \ldots, s_n) \doteq G_n(s_1, s_2, \ldots, s_n) \quad (2.9)$$

$$Y_n(s_1, s_2, \ldots, s_n) \doteq Z_n(s_1, s_2, \ldots, s_n) \quad (2.10)$$

for all $u(t)$ in the input ensemble, where the symbol "$\doteq$" in (2.8)-(2.10) is used to mean the symmetrization of both sides of each equation are equal.

The following are some operations corresponding to various combinat-ons of analytic systems [4-7] which we will find useful in this paper. The quantities $a(t) \triangleq \sum_{n=0}^{\infty} a_n(t)$ and $b(t) = \sum_{n=0}^{\infty} b_n(t)$ denote the output of the two systems being combined, while $v_n(\tau_1, \tau_2, \ldots, \tau_n)$ and $w_n(\tau_1, \tau_2, \ldots, \tau_n)$ denote their corresponding _nth_ order kernels. The output of the composite system is denoted by $e(t) = \sum_{n=0}^{\infty} e_n(t)$, and its associated nth order kernel is denoted by $k_n(\tau_1, \tau_2, \ldots, \tau_n)$.

1. __Addition.__ For the system combination shown in Fig. 1, we have for each n and for all inputs $u(t)$,

$$e_n(t) = a_n(t) + b_n(t) \quad (2.11a)$$

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[2] We shall use capital letters to denote Laplace Transform variables.

[3] The symmetrized $h_n(\cdot)$ may differ from the symmetrized $g_n(\cdot)$ trivially.

$$E_n(s_1, s_2, \ldots, s_n) \doteq A_n(s_1, s_2, \ldots, s_n) + B_n(s_1, s_2, \ldots, s_n) \tag{2.11b}$$

$$K_n(s_1, s_2, \ldots, s_n) \doteq V_n(s_1, s_2, \ldots, s_n) + W_n(s_1, s_2, \ldots, s_n) \tag{2.11c}$$

Notice that (2.11b) is the multiple output transform of (2.11a).

2. __Multiplication.__ For the system combination shown in Fig. 2, it can be shown using (2.2) that for each n, and for all inputs u(t),

$$e_n(t) = \sum_{i=0}^{n} a_i(t) b_{n-i}(t) \tag{2.12a}$$

$$E_n(s_1, s_2, \ldots, s_n) \doteq \sum_{i=0}^{n} A_i(s_1, s_2, \ldots, s_i) B_{n-i}(s_{i+1}, s_{i+2}, \ldots, s_n) \tag{2.12b}$$

$$K_n(s_1, s_2, \ldots, s_n) \doteq \sum_{i=0}^{n} V_i(s_1, s_2, \ldots, s_i) W_{n-i}(s_{i+1}, s_{i+2}, \ldots, s_n) \tag{2.12c}$$

3. __Differentiation.__ For the system combination shown in Fig. 3 in which the output a(t) is differentiated once, we have for each n and for all inputs u(t),

$$e_n(t) = \dot{a}_n(t) \tag{2.13a}$$

$$E_n(s_1, s_2, \ldots, s_n) \doteq (s_1 + s_2 + \ldots + s_n) A_n(s_1, s_2, \ldots, s_n) \tag{2.13b}$$

$$K_n(s_1, s_2, \ldots, s_n) \doteq (s_1 + s_2 + \ldots + s_n) V_n(s_1, s_2, \ldots, s_n) \tag{2.13c}$$

The Volterra Series representation (2.1) can be easily generalized to allow multiple inputs [7]. Thus, for a multi-input system with output y(t) and inputs $u_1(t), \ldots, u_N(t)$, we have

$$y(t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} y_{n_1 n_2 \ldots n_N}(t) \tag{2.14}$$

where

$$y_{n_1 n_2 \ldots n_N}(t) = \begin{cases} h_{00\ldots0} & , n_i = 0 \quad \forall \ i = 1, \ldots, N \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{n_1 n_2 \ldots n_N}\left(\tau_{11}, \tau_{12}, \ldots, \tau_{1n_1}; \tau_{21}, \tau_{22}, \ldots, \tau_{2n_2}; \ldots; \ldots; \tau_{N1}, \tau_{N2}, \ldots, \tau_{Nn_N}\right) \cdot \\ \qquad \prod_{i=1}^{N} \prod_{j=1}^{n_i} u_i(t-\tau_{ij}) d\tau_{ij}, \text{ not all } n_i = 0 \quad i = 1, \ldots, N \end{cases} \tag{2.15}$$

A similar expression for the transfer function is given by

$$H_{n_1 n_2 \ldots n_N}\left(s_{11}, s_{12}, \ldots, s_{1n_1}; s_{21}, s_{22}, \ldots, s_{2n_2}; \ldots; \ldots; s_{N1}, s_{N2}, \ldots, s_{Nn_N}\right)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{n_1 n_2 \ldots n_N}(\cdot) \prod_{i=1}^{N} \prod_{j=1}^{n_i} e^{-s\tau_{ij}} d\tau_{ij} \tag{2.16}$$

## 3. Steady State Analysis

In this section the steady state output of analytic systems with several sinusoidal inputs will be considered. Let the input $u(t)$ be represented by

$$u(t) = \sum_{k=1}^{K} A_k e^{j\omega_k t} \tag{3.1}$$

where $A_k$ may be complex, and $\omega_k$ may be any positive or negative real number. Let $y(t) = \sum_{n=0}^{\infty} y_n(t)$ denote the output. Since the zeroth order output $y_0$ does not depend on the input, when we refer to the order $n$ in the rest of this section we mean $n > 0$, unless otherwise specified. Thus from (2.2), the $\underline{n}$th order output is given by[4]

$$
\begin{aligned}
y_n(t) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n) \sum_{i=1}^{n} \sum_{k=1}^{K} A_k e^{j\omega_k(t-\tau_i)} \, d\tau_i \\
&= \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \cdots \sum_{k_n=1}^{K} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n) \prod_{i=1}^{n} A_{k_i} e^{j\omega_{k_i}(t-\tau_i)} \, d\tau_i \\
&= \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \cdots \sum_{k_n=1}^{K} \prod_{i=1}^{n} A_{k_i} e^{j\omega_{k_i} t} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n) \prod_{i=1}^{n} e^{-j\omega_{k_i}\tau_i} \, d\tau_i
\end{aligned}
$$

It follows from (2.3) that the $\underline{n}$th order output due to the input $u(t)$ in (3.1) is given by[5]

$$
\begin{aligned}
y_n(t) &= \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \cdots \sum_{k_n=1}^{K} \left( \prod_{i=1}^{n} A_{k_i} e^{j\omega_{k_i} t} \right) H_n\left( j\omega_{k_1}, j\omega_{k_2}, \ldots, j\omega_{k_n} \right) \\
&= \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \cdots \sum_{k_n=1}^{K} \left[ A_{k_1} A_{k_2} \cdots A_{k_n} H_n\left( j\omega_{k_1}, j\omega_{k_2}, \ldots, j\omega_{k_n} \right) \right] e^{j(\omega_{k_1} + \omega_{k_2} + \ldots + \omega_{k_n})t}
\end{aligned} \tag{3.2}
$$

(See Appendix 1 for a specific example with $K=2$ and $n=2$). Observe that $A_{k_1} A_{k_2} \cdots A_{k_n} H_n(j\omega_{k_1}, j\omega_{k_2}, \ldots, j\omega_{k_n})$ is the phasor associated with frequency $\omega_{k_1} + \omega_{k_2} + \ldots \omega_{k_n}$. Different terms in (3.2) may give rise to the same frequency, and each permutation of $(j\omega_{k_1}, j\omega_{k_2}, \ldots, j\omega_{k_n})$ in the argument of $H_n(\cdot)$ gives rise to a term in the $\underline{n}$th order output. Let us look at a specific example with $K = n$ and $A_k = 1 \ \forall k = 1,\ldots,n$, i.e., $u(t) = \sum_{k=1}^{n} e^{j\omega_k t}$. Then

---

[4] Observe that $\prod_{i=1}^{n} \left( \sum_{k=1}^{K} A_k e^{j\omega_k(t-\tau_i)} \right) = \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \cdots \sum_{k_n=1}^{K} \left( \prod_{i=1}^{n} A_{k_i} e^{j\omega_{k_i}(t-\tau_i)} \right)$

[5] Equation (3.2) is given in [2,4].

$$y_n(t) = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_n=1}^{n} H_n(j\omega_{k_1}, j\omega_{k_2}, \ldots, j\omega_{k_n}) e^{j(\omega_{k_1}+\ldots+\omega_{k_n})t} \qquad (3.3)$$

It follows from (3.3) that there are n! terms in $y_n(t)$ with frequency $\omega_1+\omega_2+\ldots+\omega_n$, each corresponding to a permutation of $(j\omega_1, j\omega_2, \ldots, j\omega_n)$ in the argument of $H_n(\cdot)$. Hence, if y(t) contains no component with frequency $\omega_1+\ldots+\omega_n$ other than those n! terms in $y_n(t)$, then it follows from (2.5) that

$$\overline{H}_n(j\omega_1, j\omega_2, \ldots, j\omega_n) = \frac{1}{n!} \left( \text{coefficient of } e^{j(\omega_1+\omega_2+\ldots+\omega_n)t} \text{ in } y(t) \right) \qquad (3.4)$$

This is one method usually used to derive symmetric transfer functions by exponential inputs. It has been suggested by several authors [2,3] that (3.4) holds since y(t) contains no components with the same frequency $\omega_1+\omega_2+\ldots+\omega_n$, other than those n! terms in $y_n(t)$, if the set of input frequencies $\{\omega_1, \omega_2, \ldots, \omega_n\}$ is incommensurable. Unfortunately, this need not be the case. Indeed, if we choose $\omega_1 = \sqrt{2}$, $\omega_2 = 2\sqrt{2} - 1$ and $\omega_3 = 1$, then $\{\omega_1, \omega_2, \omega_3\}$ are incommensurable. Yet we have $\omega_1+\omega_2+\omega_3 = \omega_1+\omega_1+\omega_1 = 3\sqrt{2}$. To eliminate this possibility, we must impose a slightly stronger condition as specified in our next definition.

Def. 3.1 A set of frequencies $\{\omega_1, \omega_2, \ldots, \omega_k\}$ is called a _frequency base_, if there does not exist a set of rational numbers $\{r_1, r_2, \ldots, r_K\}$ (not all zero) such that

$$r_1\omega_1 + r_2\omega_2 + \ldots + r_K\omega_K = 0$$

Since any output frequency $\omega_{k_1}+\omega_{k_2}+\ldots+\omega_{k_n}$ in (3.2) can be expressed as $m_1\omega_1+m_2\omega_2+\ldots+m_K\omega_K$, where $m_i$, $i = 1,2,\ldots,k$, are _nonnegative integers_, it is convenient to define a compact notation to denote the output frequencies.

Def. 3.2 A vector $\underset{\sim}{M} = (m_1, m_2, \ldots, m_K)$, where $m_i$, $i = 1, \ldots, K$, are nonnegative integers, is called a _module vector_ of the input frequencies $\{\omega_1, \omega_2, \ldots, \omega_K\}$. For each module vector M, we define the corresponding _output frequency_

$$\omega_{\underset{\sim}{M}} \triangleq m_1\omega_1 + m_2\omega_2 + \ldots + m_K\omega_K$$

Module vectors have the following properties.

Property a.

If $m_1 + m_2 + \ldots + m_K = n$, where $n \geq 0$, then $\omega_{\underset{\sim}{M}}$ is an <u>nth order output frequency</u>; i.e., $\omega_{\underset{\sim}{M}}$ is generated by an <u>n</u>th order kernel.

Proof: Obvious from (3.2)  ◻

Property b.

Let the input frequencies $\{\omega_1, \omega_2, \ldots, \omega_K\}$ constitute a frequency base and let $\underset{\sim}{M}$, $\underset{\sim}{M}'$ be any two module vectors. Under these conditions

$$\underset{\sim}{M} = \underset{\sim}{M}' \text{ iff } \omega_{\underset{\sim}{M}} = \omega_{\underset{\sim}{M}'}$$

Proof: Follows immediately from Definitions 3.1 and 3.2.  ◻

Property c.

If the <u>n</u>th order transfer function is <u>symmetric</u> and the input frequencies constitute a frequency base, then the sum of all terms with frequency $\omega_{\underset{\sim}{M}}$ in (3.2) for $y_n(t)$ is given by[6]

$$y_{\underset{\sim}{M}}(t) = n! \left( \prod_{k=1}^{K} \frac{A_k^{m_k}}{m_k!} \right) H_n \left( m_1\{j\omega_1\}, m_2\{j\omega_2\}, \ldots, m_K\{j\omega_K\} \right) e^{j\omega_{\underset{\sim}{M}} t} \qquad (3.5)$$

where $\underset{\sim}{M} = (m_1, m_2, \ldots, m_K)$, $\sum_{k=1}^{K} m_k = n$, and $m_k\{j\omega_k\}$ denotes "$m_k$" consecutive arguments in $H_n(\cdot)$ having the same frequency $j\omega_k$.

(See Appendix 2 for a specific example with $K = 5$, $\underset{\sim}{M} = (2,0,1,3,5)$).

Proof: Equation (3.5) is obtained by summing all terms in (3.2) with frequency $\omega_{\underset{\sim}{M}}$. This corresponds to terms in which the argument of $H_n(\cdot)$ contains "$m_i$" entries having the same frequency $\omega_i$, where $i = 1, 2, \ldots$  Since (3.2) contains all permutation terms of $m_1\{j\omega_1\}, m_2\{j\omega_2\}, \ldots, m_K\{j\omega_K\}$ in the argument of $H(\cdot)$ and $\sum_{i=1}^{K} m_i = n$, there are $\dfrac{n!}{m_1! m_2! \ldots m_K!}$ of them. Thus

$$y_{\underset{\sim}{M}}(t) = \frac{n!}{m_1! m_2! \ldots m_K!} A_1^{m_1} A_2^{m_2} \ldots A_K^{m_K} H_n \left( m_1\{j\omega_1\}, m_2\{j\omega_2\}, \ldots, m_K\{jw_K\} \right) e^{j\omega_{\underset{\sim}{M}} t} \qquad ◻$$

Remarks:

1. In view of the preceeding properties of $\underset{\sim}{M}$, we will call $\underset{\sim}{M}$ an <u>nth order module vector</u> if $\sum_{i=1}^{K} m_i = n$.

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[6] A similar expression has been given in [2].

2.  It follows from property b that if the input frequencies $\{\omega_1, \omega_2, \ldots, \omega_n\}$ in (3.2) constitute a frequency base, then y(t) does not contain output terms with frequency $\omega_1 + \omega_2 + \ldots + \omega_n$ other than those n! terms in $y_n(t)$. Thus (3.4) holds.

3.  It follows from property b that if the input frequencies constitute a frequency base, and if $\underline{M}$ is an $\underline{n}$th order module vector, then the output frequency $\omega_{\underline{M}}$ only appears in the $\underline{n}$th order output $y_n(t)$ and is unique.

4.  It follows from property b that if the input frequencies constitute a frequency base, and if two module vectors $\underline{M}$ and $\underline{M}'$ are not equal, then $y_{\underline{M}}(t)$ does not contain terms in $y_{\underline{M}'}(t)$, and vice-versa. Hence the terms in the total output

$$y(t) = \sum_{\substack{\text{all} \\ \text{possible} \\ \underline{M}}} y_{\underline{M}}(t) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_K=0}^{\infty} y_{\underline{M}}(t) \qquad (3.6)$$

do not have overlapping output frequency components. In any event (3.6) holds in general regardless of whether the input frequencies form a frequency base or not.

It is clear from (3.2) that the total phasor associated with the $\underline{n}$th order output frequency $\omega_{k_1} + \omega_{k_2} + \ldots + \omega_{k_n}$ is always a symmetric function of $j\omega_{k_1}$, $j\omega_{k_2}, \ldots, j\omega_{k_n}$. Therefore, unless otherwise specified, all nonlinear transfer functions are henceforth assumed to be symmetric in the rest of this section.

Now let us consider a K-tone real input given by

$$u(t) = \sum_{i=1}^{K} |A_i| \cos(\omega_i t + \angle A_i) = \sum_{i=1}^{K} \left( \frac{A_i}{2} e^{j\omega_i t} + \frac{A_i^*}{2} e^{-j\omega_i t} \right) \qquad (3.7)$$

where $A_i^*$ is the complex conjugate of $A_i$. By Definition $A_{-i} \triangleq A_i^*$ and $\omega_{-i} \triangleq -\omega_i$, (3.7) can be rewritten compactly as follows:

$$u(t) = \sum_{\substack{i=-K \\ i \neq 0}}^{K} \frac{A_i}{2} e^{j\omega_i t} \qquad (3.8)$$

The $\underline{n}$th order module vector of the input frequencies in this case assumes the form $\underline{M} \triangleq (m_{-K}, \ldots, m_{-1}, m_1, \ldots, m_K)$, where $m_i$, $i = \pm 1, \pm 2, \ldots, \pm K$ are

<u>nonnegative</u> integers with $\sum\limits_{\substack{i=-K \\ i\neq 0}}^{K} m_i = n$. Observe that the input frequencies

$\{\omega_{-K}, \ldots, \omega_{-1}, \omega_1, \ldots, \omega_K\}$ in this case do not form a frequency base because $\omega_i + \omega_{-i} = 0$. For each <u>n</u>th order module vector, we have

$$\omega_{\underset{\sim}{M}} = \sum_{\substack{i=-K \\ i\neq 0}}^{K} m_i \omega_i = \sum_{i=1}^{K} (m_i \omega_i + m_{-i} \omega_{-i}) = \sum_{i=1}^{K} (m_i - m_{-i}) \omega_i \qquad (3.9)$$

and from (3.5)

$$y_{\underset{\sim}{M}}(t) = \frac{n!}{2^n} \left( \prod_{\substack{i=-K \\ i\neq 0}}^{K} \frac{A_i^{m_i}}{m_i!} \right) H_n\left( m_{-K}\{j\omega_{-K}\}, \ldots, m_{-1}\{j\omega_{-1}\}, m_1\{j\omega_1\}, \ldots, m_K\{j\omega_K\} \right) e^{j\omega_{\underset{\sim}{M}} t} \qquad (3.10)$$

(See Appendix 3 for a specific example with $K = 2$, $\underset{\sim}{M} = (2,0,2,3)$).

To illustrate the significance of (3.10), we use it to derive Table 1 [2] which gives the output corresponding to each module vector $\underset{\sim}{M}$ from order 0 to 3 for the 2-tone input case, i.e.

$$u(t) = \sum_{i=1}^{2} |A_i| \cos(\omega_i t + \measuredangle_i) = \sum_{\substack{i=-2 \\ i\neq 0}}^{2} \frac{A_i}{2} e^{j\omega_i t}$$

Since the constraints for $\underset{\sim}{M} = (m_K, \ldots, m_{-1}, m_1, \ldots, m_K)$ to be an <u>n</u>th order vector are $\sum\limits_{\substack{i=-K \\ i\neq 0}}^{K} m_i = n$ and $m_i \geq 0 \; \forall i = \pm 1, \ldots, \pm K$, it can be shown easily

that the number of distinct <u>n</u>th order module vectors for a K-tone input is given by $\binom{2K+n-1}{n}$. Observe also that distinct module vectors of the same order may give rise to the same output frequency because $\{\omega_{-K}, \ldots, \omega_{-1}, \omega_1, \ldots, \omega_K\}$ is not a frequency base.

We will now derive some properties of the output frequency components when the input is a K-tone as defined by (3.7), where $\{-\omega_K, -\omega_{K-1}, \ldots, -\omega_1, \omega_1, \ldots, \omega_K\}$ are the input frequencies. We will make the <u>standing assumption</u> that the subset $\{\omega_1, \omega_2, \ldots, \omega_K\}$ of input frequencies form a frequency base. Under this assumption, a module vector $\underset{\sim}{M}$ that corresponds to a d.c. output component (i.e., $\omega_{\underset{\sim}{M}} = 0$) must have $m_i - m_{-i} = 0$ in view of (3.9); i.e., $m_i = m_{-i}$, $i = 1, \ldots, K$. Hence, the order

$$n = \sum_{\substack{i=-K \\ i\neq 0}}^{K} m_i = \sum_{i=1}^{K} (m_i + m_{-i}) = \sum_{i=1}^{K} 2m_i = \text{even integer.} \quad \text{Consequently,}$$

the order of a module vector that gives rise to a d.c. output component must be even.  Let us denote a 2rth order module vector which gives rise to a d.c. output by $\underset{\sim}{Z}_{2r} = (z_{-k},...,z_{-1},z_1,...,z_k)$, where $z_i = z_{-i}$, $i = 1,...,k$.

<u>Property 1</u>

For any output frequency $\omega_o = \sum_{i=1}^{K} p_i \omega_i$, where $p_i$ is an integer $\forall i = 1,...,K$, the lowest order module vector $\underset{\sim}{M} = (m_{-K},...,m_{-1},m_1,...,m_K)$ that gives rise to this frequency is at order $n = \sum_{i=1}^{K} |p_i|$.  This module vector is unique at order n and is given by

$$
\begin{cases}
m_i = p_i & \text{and} \quad m_{-i} & \text{if} \quad p_i \geq 0 \\
m_i = 0 & \text{and} \quad m_{-i} & \text{if} \quad p_i < 0
\end{cases}
\qquad \forall i = 1,...,k \qquad (3.11)
$$

(See Appendix 4 for a specific example with $K = 3$, $\omega_o = 4\omega_1 + 0\omega_2 + (-2)\omega_3$).

<u>Proof:</u>

By hypothesis, $\omega_{\underset{\sim}{M}} = \omega_o$ and $\underset{\sim}{M}$ is of order $n = \sum_{i=1}^{K} |p_i|$.  To prove that $\underset{\sim}{M}$ is of lowest order and unique, assume that there exists another module vector $\underset{\sim}{M}' = (m'_{-k},...,m'_{-1},m'_1,...,m'_k)$ of order n or lower such that $\omega_{\underset{\sim}{M}'} = \omega_{\underset{\sim}{M}}$ and $\underset{\sim}{M}' \neq \underset{\sim}{M}$.  It follows from (3.9) that

$\sum_{i=1}^{K} (m'_i - m'_{-i}) \omega_i = \sum_{i=1}^{K} (m_i - m_{-i}) \omega_i = \sum_{i=1}^{K} p_i \omega_i$.  Since $\{\omega_1,...,\omega_K\}$ forms a frequency base, $m'_i - m'_{-i} = m_i - m_{-i} = p_i$, $\forall i = 1,...,K$.  This implies that $m'_i - m_i = m'_{-i} - m_{-i}$.  As $m'_i$ and $m'_{-i}$ are nonnegative integers and either $m_i$ or $m_{-i}$ or both are equal to 0, $m'_i - m_i = m'_{-i} - m_{-i} \geq 0$.  Thus $m'_i + m'_{-i} \geq m_i + m_{-i}$.  Since $\underset{\sim}{M}' \neq \underset{\sim}{M}$, by assumption, there exist an "i" such that $m'_i - m_i = m'_{-i} - m_{-i} > 0$, i.e., $m'_i + m'_{-i} > m_i + m_{-i}$.  Therefore $\sum_{i=1}^{K} (m'_i + m'_{-i}) > \sum_{i=1}^{K} (m_i + m_{-i}) = n$.  This contradicts the assumption that $\underset{\sim}{M}'$ is of order n or lower.  ◻

<u>Corollaries</u>

1.  If an nth order module vector does not have both $m_i$ and $m_{-i}$ nonzero $\forall i = 1,2,...,K$, then no other module vector having the same or lower order can give rise to the same output frequency $\omega_{\underset{\sim}{M}}$.

2. By relabelling the index i of $\omega_i, \forall i = 1, 2, \ldots, K$, we can always write

$$\omega_0 = \sum_{\substack{i=-K \\ i \neq 0}}^{K} p_i \omega_i, \text{ where } p_i \text{ is a } \underline{\text{nonnegative}} \text{ integer } \forall i = \pm 1, \ldots, \pm K. \text{ In}$$

this case, $\underset{\sim}{M}$ defined in (3.11) is equal to $(p_{-K}, \ldots, p_{-1}, p_1, \ldots, p_K)$.

**Property 2.**

Given any output frequency $\omega_0 = \sum_{i=1}^{K} p_i \omega_i$, where $p_i$ is an integer

$\forall i = 1, 2, \ldots, K.$ Let $\underset{\sim}{M}$ be the unique module vector of the lowest order

$n \underset{\triangleq}{\sum_{i=1}^{K}} |p_i|$ (as defined in property 1) such that $\omega_{\underset{\sim}{M}} = \omega_0$. If

$\underset{\sim}{M}' = (m'_{-K}, \ldots, m'_{-1}, m'_1, \ldots, m'_K)$ is an $\underline{n}'$th order module vector, then

$$\omega_{\underset{\sim}{M}} = \omega_{\underset{\sim}{M}'} \text{ iff } \underset{\sim}{M}' = \underset{\sim}{M} + \underset{\sim}{Z}_{2r}$$

where $\underset{\sim}{Z}_{2r}$ is a module vector of order $2r = n'-n$ which gives rise to a d.c.
response.

**Proof:**

Suppose $\underset{\sim}{M}' = \underset{\sim}{M} + \underset{\sim}{Z}_{2r} = (m_{-K}+z_{-K}, \ldots, m_{-1}+z_{-1}, m_1+z_1, \ldots, m_K+z_K)$, where $z_i = z_{-i}$,
$\forall i = 1, 2, \ldots, K.$ Then it follows from (3.9) that

$$\omega_{\underset{\sim}{M}'} = \sum_{i=1}^{K} (m'_i - m'_{-i}) \omega_i = \sum_{i=1}^{K} (m_i + z_i - m_{-i} - z_{-i}) \omega_i = \sum_{i=1}^{K} (m_i - m_{-i}) \omega_i = \omega_{\underset{\sim}{M}}$$

Conversely, if $\omega'_{\underset{\sim}{M}} = \omega_{\underset{\sim}{M}}$, as in the proof of property 1, we have
$m'_i - m'_{-i} = m_i - m_{-i} = p_i$ so $m'_i - m_i = m'_{-i} - m_{-i} \geq 0$. By defining
$\underset{\sim}{Z}_{2r} = (z_{-K}, \ldots, z_{-1}, z_1, \ldots, z_K)$ such that $z_i = m'_i - m_i \geq 0 \ \forall i = \pm 1, \ldots, \pm K$,

and $r = \sum_{i=1}^{K} z_i$, it is clear that $\underset{\sim}{Z}_{2r}$ is a $\underline{2r}$th order module vector

which gives rise to a d.c. response. Moreover, $\underset{\sim}{M}' = \underset{\sim}{M} + \underset{\sim}{Z}_{2r}$ and $2r = n'-n$.

$\square$

**Property 3.**

All $\underline{n}$th order output frequencies are present in the $\underline{(n+2)}$th order
output.

**Proof:**

Follows immediately from property 2 with $r = 1$.

$\square$

## Property 4.

Any consecutive pair of $\underline{n}$th and $\underline{(n+1)}$th order outputs do not contain any identical output frequency components.

### Proof:

If a pair of outputs share a common frequency component, their orders must differ by an even integer in view of property 2.　　　　　□

### Remarks:

1. It follows from properties 3 and 4 that only even order outputs have d.c. components.

2. Even and odd order outputs do not contain any identical output frequency components.

3. All output frequencies from order 0 up to n can be generated from an $\underline{(n-1)}$th and $\underline{n}$th order output.

## Property 5.

The total d.c. output of order $2r$, $r > 0$ can be obtained from the expression[7]

$$\sum_{z_1=0}^{r} \sum_{z_2=0}^{r-z_1} \sum_{z_3=0}^{r-z_1-z_2} \cdots \sum_{z_{K-1}=0}^{r-z_1-z_2-\cdots-z_{K-2}} \frac{(2r)!}{2^{2r}} \left[ \prod_{i=1}^{K} \frac{|A_i|^{2z_i}}{(z_i!)^2} \right] H_{2r}\left( z_1\{j\omega_{-1}\}, z_1\{j\omega_1\}, \ldots, z_K\{j\omega_{-K}\}, z_K\{j\omega_K\} \right) \qquad (3.12)$$

where $\overset{\star}{z}_K = r - \sum_{i=1}^{K-1} z_i$,

(See Appendix 5 for a specific example with $K = 3$, $r = 2$)

or the equivalent expression

$$\sum_{k_1=1}^{K} \cdots \sum_{k_2=1}^{k_1} \cdots \sum_{k_\beta=1}^{k_{\beta-1}} \sum_{k_r=1}^{k_{r-1}} \frac{(2r)!}{2^{2r}} \left[ \prod_{i=1}^{K} \frac{|A_i|^{2z_i}}{(z_i!)^2} \right] H_{2r}\left( j\omega_{k_1}, j\omega_{-k_1}, \ldots, j\omega_{k_\beta}, j\omega_{-k_\beta}, \ldots, j\omega_{k_r}, j\omega_{-k_r} \right) \qquad (3.13)$$

where $z_i$ is the number of $j\omega_i$ in the argument of $H_{2r}(\cdot)$.
(See Appendix 6 for a specific example with $K = 3$, $r = 2$).

---

[7]The number of summation signs in (3.12) depends on the number "K" of input frequencies, while the number of summation signs in (3.13) depends on the order $2r$ of the output.

<u>Proof</u>:

To prove (3.12), we will show equation (3.12) contains a sum of all terms which correspond to distinct d.c. module vectors of order $2r$; namely, $\underset{\sim}{Z}_{2r} = (z_{-K}, \ldots, z_{-1}, z_1, \ldots, z_K)$. Since $z_i = z_{-i}, \forall i = 1, \ldots, K$, only the vectors $(z_1, \ldots, z_K)$ are needed. The only constraint on $(z_1, \ldots, z_K)$ are $z_1 + z_2 + \ldots + z_K = r$ and $z_i \geq 0, \forall i = 1, \ldots, K$. These constraints are obviously satisfied for all such vectors contained in (3.12). Now given any $(z_1, z_2, \ldots, z_K)$ satisfying the above two constraints, it can be proved easily that this is one of the vectors produced by (3.12). So (3.12) generates all vectors $(z_1, \ldots, z_K)$ that satisfy the above two constraints. To show that all vectors generated in (3.12) are distinct, let us write a few of these vector generated in a systematic order as follows:

| item | $z_1$ | $z_2$ | $\cdots$ | $z_{K-2}$ | $z_{K-1}$ | $z_K$ |
|------|-------|-------|----------|-----------|-----------|-------|
| 1 | 0 | 0 | $\ldots$ | 0 | 0 | r |
| 2 | 0 | 0 | $\ldots$ | 0 | 1 | r-1 |
| 3 | 0 | 0 | $\ldots$ | 0 | 2 | r-2 |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| r+1 | 0 | 0 | $\ldots$ | 0 | r | 0 |
| r+2 | 0 | 0 | $\ldots$ | 1 | 0 | r-1 |
| r+3 | 0 | 0 | $\ldots$ | 1 | 1 | r-2 |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |

If we look at each vector as an integer of K digits $z_1 z_2 \ldots z_K$ of base[8] $r+1$, then the integers generated form a strictly increasing sequence as we go down the list and hence all are distinct.

To prove (3.13) it suffices to show that the terms in (3.13) contain all distinct combinations $\{j\omega_{k_1}, j\omega_{-k_1}, \ldots, j\omega_{k_r}, j\omega_{-k_r}\}$, i.e., all distinct combinations of $\{k_1, k_2, \ldots, k_r\}$.[9] Given any valid combination $\{k_1, k_2, \ldots, k_r\}$ such that $1 \leq k_i \leq K, \forall i = 1, \ldots, r$, it can be proved easily that this is one of the combinations generated by (3.13). To show that all combinations generated are distinct,

---

[8] If base = 2, i.e., r = 1, then the integer is just a binary number.

[9] For example, {1,1,2,3} and {1,2,3,1} are regarded as the same combination. Hence each distinct combination corresponds to a distinct module vector.

let us list the first few terms generated as follow:

| item | $k_1$ | $k_2$ | $k_3$ | $k_4$ | .... | $k_r$ |
|------|-------|-------|-------|-------|------|-------|
| 1 | 1 | 1 | 1 | 1 | .... | 1 |
| 2 | 2 | 1 | 1 | 1 | .... | 1 |
| 3 | 2 | 2 | 1 | 1 | .... | 1 |
| 4 | 2 | 2 | 2 | 1 | .... | 1 |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |

Observe that $k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_r$ for each row. Similarly, the integers in each column form a nondecreasing sequence as we go down the list. Since no two consecutive terms are equal, all combinations of $\{k_1, k_2, \ldots, k_r\}$ are distinct.

□

## Property 6.

Consider any output frequency $\omega_o = \sum_{i=1}^{K} p_i \omega_i$, where $p_i$ is an integer $\forall i = 1, 2, \ldots, K$, and let $\underset{\sim}{M} = (m_{-K}, \ldots, m_{-1}, m_1, \ldots, m_K)$ is the unique module vector of the lowest order $n = \sum_{i=1}^{K} |p_i|$ (as defined in property 1) such that $\omega_{\underset{\sim}{M}} = \omega_o$. The output with frequency $\omega_o$ at order $n+2r$ is given by the expression

$$
\sum_{z_1=0}^{r} \sum_{z_2=0}^{r-z_1} \cdots \sum_{z_{k-1}=0}^{r-z_1 \cdots z_{k-2}} \frac{(n+2r)!}{2^{n+2r}} \left[ \prod_{\substack{i=-K \\ i \neq 0}}^{K} \frac{A_i^{m_i+z_i}}{(m_i+z_i)!} \right] \cdot
$$
$$
H_{n+2r}\left( (m_{-1}+z_1)\{j\omega_{-1}\}, (m_1+z_1)\{j\omega_1\}, \ldots, (m_{-k}+z_k)\{j\omega_{-k}\}, (m_k+z_k)\{j\omega_k\} \right) e^{j\omega_o t}
$$

$$(3.14)$$

where $z_K = r - \sum_{i=1}^{K-1} z_i$

(See Appendix 7 for a specific example with $K = 3$, $\omega_o = 4\omega_1 - 2\omega_3$)

or by the equivalent expression

$$\sum_{k_1=1}^{K} \sum_{k_2=1}^{k_1} \cdots \sum_{k_\beta=1}^{k_{\beta-1}} \cdots \sum_{k_r=1}^{k_{r-1}} \frac{(n+2r)!}{2^{n+2r}} \left[ \prod_{\substack{i=-K \\ i\neq 0}}^{K} \frac{A_i^{m_i+z_i}}{(m_i+z_i)!} \right] \cdot$$

$$H_{n+2r}\left(m_{-K}\{j\omega_{-K}\},\ldots,m_{-1}\{j\omega_{-1}\},m_1\{j\omega_1\},\ldots,m_K\{j\omega_K\},j\omega_{k_1},j\omega_{-k_1},\ldots,j\omega_{k_r},j\omega_{-k_r}\right) e^{j\omega_o t}$$

$$(3.15)$$

where $z_i$ = number of $j\omega_{k_1}, j\omega_{k_2}, \ldots, j\omega_{k_r}$ which are equal to $j\omega_i$. (See Appendix 8 for a specific example with $K = 3$, $\omega_o = 4\omega_1 - 2\omega_3$.)

Proof:

From property 2, all distinct module vectors of order $n+2r$, such that $\omega_{M'} = \omega_M$, are given by $\underset{\sim}{M} + \underset{\sim}{Z}_{2r}$. In the proof of property 5, it has been shown that either the series of summation signs in (3.12) or that of (3.13) can be used to generate all distinct $\underset{\sim}{Z}_{2r}$. Now since corresponding to each $\underset{\sim}{Z}_{2r}$, there is a distinct $\underset{\sim}{M'}$, the series of summation signs in (3.12) and (3.13) can be used to obtain (3.14) and (3.15) with the help of (3.10).  ¤

Remarks:

1. Equations (3.14) and (3.15) only give the output component at frequency $\omega_o$ and order $n+2r$. The total output at frequency $\omega_o$ can be obtained by adding all components at frequency $\omega_o$ corresponding to all orders. By property 2, the total output signal $y_{\omega_o}(t)$ at frequency $\omega_o$ can be written as

$$y_{\omega_o}(t) = \sum_{z_1=0}^{\infty} \sum_{z_2=0}^{\infty} \cdots \sum_{z_K=0}^{\infty} (n+2r)! \left[ \prod_{\substack{i=-K \\ i\neq 0}}^{K} \frac{(A_i/2)^{m_i+z_i}}{(m_i+z_i)!} \right] \cdot$$

$$H_{n+2r}\left((m_{-1}+z_1)\{j\omega_{-1}\},(m_1+z_1)\{j\omega_1\},\ldots,(m_{-K}+z_K)\{j\omega_{-K}\},(m_K+z_K)\{j\omega_K\}\right) e^{j\omega_o t}$$

where $r = \sum_{i=1}^{K} z_i$ and $(m_{-K},\ldots,m_{-1},m_1,\ldots,m_K)$ is as defined in property 6.

2. The real output waveform at frequency $\omega_o$ and order $n+2r$ can be obtained by adding expression (3.14) or (3.15) and its conjugate.[10] Thus the real output

___

[10] Notice that given any module vector $\underset{\sim}{M} = (m_{-K},\ldots,m_{-1},m_1,\ldots,m_K)$, there exist $\underset{\sim}{M'} = (m'_{-K},\ldots,m'_{-1},m'_1,\ldots,m'_K)$ of the same order defined as $m'_i = m_{-i}$ $\forall i = \pm1,\ldots,\pm K$, such that $y_{\underset{\sim}{M'}}(t) = y^*_{\underset{\sim}{M}}(t)$

signal at frequency $\omega_o$ and order $n+2r$ is

$$|Y| \cos(\omega_o t + \underline{/Y})$$ (3.16)

where $Y = 2\left(\text{coeff. of } e^{j\omega_o t} \text{ in (3.14) or (3.15)}\right)$

## Property 7.

The $\underline{n}$th order output can be given either by

$$y_n(t) = \sum_{m_{-K}=0}^{n} \sum_{m_{-K+1}=0}^{n-m_{-K}} \cdots \sum_{m_{-1}=0}^{n-\sum\limits_{i=-K}^{-2} m_i} \sum_{m_1=0}^{n-\sum\limits_{i=-K}^{-1} m_i} \cdots \sum_{m_{k-1}=0}^{n-\sum\limits_{\substack{i=-K\\i\neq 0}}^{K-2} m_i} \frac{n!}{2^n} \left[ \prod_{\substack{i=-K\\i\neq 0}}^{K} \frac{A_i^{m_i}}{m_i!} \right] \cdot$$

$$H_n\left(m_{-K}\{j\omega_{-K}\},\ldots,m_{-1}\{j\omega_{-1}\},m_1\{j\omega_1\},\ldots,m_K\{j\omega_K\}\right) e^{j\omega_M t}$$ (3.17)

where $m_K = n - \sum\limits_{\substack{i=-K\\i\neq 0}}^{K-1} m_i$ and $\omega_M = \sum\limits_{i=1}^{K} (m_i - m_{-i})\omega_i$,

or by the equivalent expression

$$y_n(t) = \sum_{k_1=-K}^{K} \sum_{k_2=-K}^{k_1} \cdots \sum_{k_n=-K}^{k_{n-1}} \frac{n!}{2^n} \left[ \prod_{\substack{i=-K\\i\neq 0}}^{K} \frac{A_i^{m_i}}{m_i!} \right] H_n(j\omega_{k_1}, j\omega_{k_2}, \ldots, j\omega_{k_n}) e^{j\omega_M t}$$ (3.18)

$$k_i \neq 0 \quad \forall i = 1, \ldots, n$$

where $m_i$ = number of $j\omega_i$'s in the argument of $H_n(\cdot)$.

## Proof:

The proof is similar to that of property 5 and is therefore omitted.  ¤

-17-

## Property 8.

The number N of new frequencies in the $n$th order output that are not present in any lower order output is given by:

$$N = \sum_{i=1}^{r} 2^i \binom{K}{i}\binom{n-1}{i-1} \tag{3.19}$$

where $r = \min(K,n)$

## Proof:

From property 1, $\omega_o = \sum_{i=1}^{K} p_i \omega_i$ is a new output frequency in the $n$th order output which is not present in any lower order outputs iff $n = \sum_{i=1}^{K} |p_i|$. Hence, it suffices to prove that (3.19) gives the total number of distinct vectors $(p_1, p_2, \ldots, p_K)$ such that $p_i$ is an integer $\forall i = 1, 2, \ldots, K$ and $\sum_{i=1}^{K} |p_i| = n$. Let $q$ of the components of $(p_1, p_2, \ldots, p_K)$ be nonzero and let us denote them by the vector $(p_{k_1}, p_{k_2}, \ldots, p_{k_q})$. The number of distinct vectors $(|p_{k_1}|, |p_{k_2}|, \ldots, |p_{k_q}|)$, such that $\sum_{i=1}^{q} |p_{k_q}| = n$ and $|p_{k_i}|$ is a nonzero integer $\forall i = 1, 2, \ldots, q$, is $\binom{n-1}{q-1}$. Since $p_{k_i}$, $\forall i = 1, \ldots, q$, can either be positive of negative there are $2^q \binom{n-1}{q-1}$ of such distinct vectors $(p_{k_1}, \ldots, p_{k_q})$. As the $q$ components can be any of the $K$ components of $(p_1, p_2, \ldots, p_K)$, there are $\binom{K}{q} 2^q \binom{n-1}{q-1}$ distinct vectors $(p_1, p_2, \ldots, p_K)$ such that $q$ of the components are nonzero and $\sum_{i=1}^{K} |p_i| = n$. Since $q$ can be $1, 2, \ldots$, and up to $\min(K,n)$, the total number of distinct vectors $(p_1, \ldots, p_K)$ such that $\sum_{i=1}^{K} |p_i| = n$ is $\sum_{q=1}^{\min(K,n)} \binom{K}{q} 2^q \binom{n-1}{q-1}$. ∎

We will close this section by presenting two examples which illustrate some useful applications of the preceding properties.

Example 3.1. Deriving the describing function from Volterra Series.

Suppose a system is characterized by symmetric nonlinear transfer functions $H_n(s_1,\ldots,s_n)$, $\forall n \geq 0$. Now we want to find its describing function $N(A,\omega)$, i.e. given a sinusoidal input $u(t) = A\cos\omega t$, its first harmonic is given by $|N(A,\omega)|\cos\left(\omega t + \underline{/N(A,\omega)}\right)$. It follows from property 6 that the unique module vector $\underset{\sim}{M}$ of the lowest order corresponding to an output frequency $\omega_o = \omega$ is given by $\underset{\sim}{M} = (m_{-1}, m_1) = (0,1)$. Using (3.14) or (3.15) the $(\underline{1+2r})$th order component of $e^{j\omega t}$ is given by [11]

$$\frac{(1+2r)!}{2^{1+2r}} \frac{A^{1+2r}}{r!(1+r)!} H_{1+2r}\left(r\{-j\omega\}, (1+r)\{j\omega\}\right) e^{j\omega t}$$

The total coefficient multiplying $e^{j\omega t}$ is therefore given by

$$\sum_{r=0}^{\infty} \frac{(1+2r)!}{2^{1+2r}} \frac{A^{1+2r}}{r!(1+r)!} H_{1+2r}\left(r\{-j\omega\}, (1+r)\{j\omega\}\right) e^{j\omega t}$$

By adding together its conjugate, we obtain the describing function

$$N(A,\omega) = \sum_{r=0}^{\infty} \frac{(1+2r)!}{2^{2r}} \frac{A^{1+2r}}{r!(1+r)!} H_{1+2r}\left(r\{-j\omega\}, (1+r)\{j\omega\}\right) \qquad (3.20)$$

In practice, $N(A,\omega)$ can be approximated by summing only over those orders which are more significant.

Example 3.2. Finding output when $\{\omega_1,\omega_2,\ldots,\omega_K\}$ do not form a frequency base.

Consider an input $u(t) = \sum_{i=1}^{3} |A_i| \cos(\omega_i t + \underline{/A_i})$ where the input frequencies $\{\omega_1,\omega_2,\omega_3\}$ do not form a frequency base. In particular, let $\omega_1 = 1$ rad/sec, $\omega_2 = \sqrt{2}$ rad/sec, $\omega_3 = 1 + \sqrt{2}$ rad/sec. Suppose we are interested in obtaining the output at frequency 1 rad/sec. Below we list the first few distinct vectors $(p_1, p_2, p_3)$ such that $\omega_o = 1$ rad/sec $= \sum_{i=1}^{3} p_i \omega_i$, and the corresponding lowest order module vector $\underset{\sim}{M} = (m_{-3}, \ldots, m_3)$ as defined in property 1

---

[11] In the present case, $K = 1$ and $n = 1$. To use (3.14), observe that the number of summation signs is equal to $K-1 = 0$ and that $z_K = z_1 = r$. To use (3.15), observe that there is only one term which corresponds to $k_1 = k_2 = \ldots = k_r = 1$.

| output order | $P_1$ | $P_2$ | $P_3$ | $\underset{\sim}{M} = (m_{-3}, m_{-2}, m_{-1}, m_1, m_2, m_3)$ |
|---|---|---|---|---|
| n=1 | 1 | 0 | 0 | $\underset{\sim}{M} = (0,0,0,1,0,0)$ |
| n=2 | 0 | -1 | 1 | $\underset{\sim}{M} = (0,1,0,0,0,1)$ |
| n=4 | 2 | 1 | -1 | $\underset{\sim}{M} = (1,0,0,2,1,0)$ |
| . | . | . | . | . |
| . | . | . | . | . |
| . | . | . | . | . |

Now treat $\{\omega_1, \omega_2, \omega_3\}$ as if it were a frequency base, and then apply property 6 for each $\underset{\sim}{M}$ from the preceding table, one at a time. For each $\underset{\sim}{M}$, (3.14) or (3.15) can be used to evaluate the terms at different orders which give rise to the desired component at $\omega_{\underset{\sim}{M}}$. Finally, all components corresponding to each $\underset{\sim}{M}$ are added together to obtain the desired output.

## 4. Symmetrization

Since analytic systems may not have unique nonlinear transfer functions, some of the transfer functions of equivalent systems may have a much simpler form than the others. For example, the 3rd order transfer function $\dfrac{1}{(s_1-a)(s_2-b)(s_3-c)}$ is simpler than its symmetrized form, but the symmetrized form of the 3rd order transfer function $\dfrac{0.1}{s_1-a} + \dfrac{0.9}{s_2-a} - \dfrac{1}{s_3-a}$ is 0. In doing manipulations of transform expressions, especially in a recursive process, it is desirable to choose either the simplest form or some particular forms of nonlinear transfer functions so as to obtain a simple final expression. However, we need to investigate the effects of choosing different nonlinear transfer functions, or output transforms, associated with equivalent systems before we carry out any manipulation of transform expressions.

## Theorem 1. Equivalent Systems

Let H and G be two analytic systems with the same input ensemble and the same output ensemble Let $h_n(\tau_1,\ldots,\tau_n)$, $H_n(s_1,\ldots,s_n)$, $Y_n(s_1,\ldots,s_n)$ and $y_n(t)$ be the nth order kernel, nonlinear transfer function, output transform, and output, respectively, of the system H. Let $g_n(\tau_1,\ldots,\tau_n)$, $G_n(s_1,\ldots,s_n)$, $Z_n(s_1,\ldots,s_n)$ and $z_n(t)$ be the corresponding nth order kernel, nonlinear transfer function, output transform, and output, respectively, of the system G. Under these conditions, the following statements are equivalent.[12]

---

[12] In the second statement, $\bar{h}_n(\cdot)$ may differ from $\bar{g}_n(\cdot)$ trivially.

1. H and G are equivalent.

2. $\bar{h}_n(\tau_1,\ldots,\tau_n) = \bar{g}_n(\tau_1,\ldots,\tau_n)$     $\forall n$

3. $\bar{H}_n(s_1,\ldots,s_n) = \bar{G}_n(s_1,\ldots,s_n)$     $\forall n$

4. $\bar{Y}_n(s_1,\ldots,s_n) = \bar{Z}_n(s_1,\ldots,s_n)$     $\forall n$ and for all inputs of the input ensemble

5. $y_n(t) = z_n(t)$     $\forall n$ and for all inputs of the input ensemble.

Proof:

It follows from the definitions of transform and symmetrization that statements 2,3 and 4 are equivalent to one another. Since statement 1 implies statement 2 [4,7], as mentioned in (2.9), and statement 5 implies statement 1, it is sufficient to show only that statement 2 implies statement 5.

For any input $u(t)$ of the input ensemble, $y_n(t)$ is given by

$$y_n(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n) \prod_{i=1}^{n} u(t-\tau_i)d\tau_i \; .$$

Notice that $y_n(t)$ does not change for any permutation of $\tau_1,\ldots,\tau_n$ in the argument of $h_n(\tau_1,\ldots,\tau_n)$. Thus

$$y_n(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \bar{h}_n(\tau_1,\ldots,\tau_n) \prod_{i=1}^{n} u(t-\tau_i)d\tau_i$$

Similarly $z_n(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \bar{g}_n(\tau_1,\ldots,\tau_n) \prod_{i=1}^{n} u(t-\tau_i)d\tau_i$

Therefore statement 2 implies $y_n(t) = z_n(t)$.         ⌐

Theorem 1 shows that the symbol "$\doteq$" introduced in (2.8)-(2.10) not only means that the symmetrized expressions on both sides of "$\doteq$" are equal, but that they also represent equivalent systems. Sometimes it is helpful to have a method for checking transform expressions of equivalent systems without doing symmetrization. The following lemma provides an easy and useful method which follows immediately from the definition of symmetrization.

Lemma 1.

If $F_n(s_1,\ldots,s_n)$ is a function of $s_1,\ldots,s_n$, then

$$F_n(s_1,\ldots,s_n) \doteq F_n(s_1,\ldots,s_n)\Big|_{\substack{\text{any permutation} \\ \text{of } s_1,\ldots,s_n}} \tag{4.1}$$

<u>Examples</u>

1. If $F_n(s_1,\ldots,s_n) = s_1$, then $F_n(s_1,\ldots,s_n) = s_1 \doteq s_2 \doteq s_3 \doteq \ldots \doteq s_n$.

2. If $F_n(s_1,\ldots,s_n) = \dfrac{1}{s_1-a}$, then $F_n(s_1,\ldots,s_n) = \dfrac{1}{s_1-a} \doteq \dfrac{1}{s_2-a} \doteq \ldots \doteq \dfrac{1}{s_n-a}$

3. If $Fn(s_1,\ldots,s_n) = \left(\dfrac{1}{s_1-a}\right)\left(\dfrac{s_2}{s_3-b}\right)$, then $F_n(s_1,\ldots,s_n) \doteq \left(\dfrac{1}{s_i-a}\right)\left(\dfrac{s_j}{s_k-b}\right)$ as long

   as no two of i, j, k are equal and all $i,j,k \leq n$.

The main advantage of the frequency domain representation is that, for time-invariant systems, almost all types of useful mathematical operations in the time domain are transformed into algebraic operations in the frequency domain. Most of these operations are of the types shown in the following three lemmas. This is true for example in the derivation of nonlinear transfer functions and output transforms from differential equations, from cascade systems, from taking the inverse of systems, from feedback systems, and from nonlinear circuits, etc. These lemmas are useful in determining the effects due to the use of different nonlinear transfer functions and output transforms for equivalent systems under various operations.

<u>Lemma 2.</u>

If $F_n(s_1,\ldots,s_n) \doteq G_n(s_1,\ldots,s_n)$ and $H_n(s_1,\ldots,s_n) \doteq K_n(s_1,\ldots,s_n)$,

then

$$F_n(s_1,\ldots,s_n) \pm H_n(s_1,\ldots,s_n) \doteq G_n(s_1,\ldots,s_n) \pm K_n(s_1,\ldots,s_n) \qquad (4.2)$$

<u>Proof:</u>

$$\overline{F_n(s_1,\ldots,s_n) \pm H_n(s_1,\ldots,s_n)} = \bar{F}_n(s_1,\ldots,s_n) \pm \bar{H}_n(s_1,\ldots,s_n) = \bar{G}_n(s_1,\ldots,s_n)$$

$$\pm \bar{K}_n(s_1,\ldots,s_n) \qquad \qquad ¤$$

<u>Example:</u>

It follows from Lemma 2 that $F_n(s_1,\ldots,s_n) = \dfrac{0.5}{s_1-a} + \dfrac{0.3}{s_2-a} + \dfrac{3.1}{s_4-a} - \dfrac{3.9}{s_n-a} \doteq 0$,

since $\dfrac{1}{s_1-a} \doteq \dfrac{1}{s_2-a} \doteq \dfrac{1}{s_4-a} \doteq \dfrac{1}{s_n-a}$ from Example 2 of Lemma 1.

<u>Lemma 3.</u>

If $\bar{F}_n(s_1,\ldots,s_n)$ is a symmetric function of $s_1,\ldots,s_n$, and $H_n(s_1,\ldots,s_n) \doteq K_n(s_1,\ldots,s_n)$, then

$$\bar{\bar{F}}_n(s_1,\ldots,s_n) \, H_n(s_1,\ldots,s_n) \doteq \bar{\bar{F}}_n(s_1,\ldots,s_n) \, K_n(s_1,\ldots,s_n) \qquad (4.3)$$

Proof:

$$\bar{\bar{F}}_n(s_1,\ldots,s_n) \, H_n(s_1,\ldots,s_n) = \bar{\bar{F}}_n(s_1,\ldots,s_n) \, \bar{\bar{H}}_n(s_1,\ldots,s_n)$$

$$= \bar{\bar{F}}_n(s_1,\ldots,s_n) \, \bar{\bar{K}}_n(s_1,\ldots,s_n). \qquad \qquad □$$

Example:

Let $H_n(s_1,\ldots,s_n) \doteq K_n(s_1,\ldots,s_n)$. As in the case of differentiation shown in (2.13a)-(2.13c), by choosing $\bar{\bar{F}}_n(s_1,\ldots,s_n) = s_1 + s_2 + \ldots + s_n$, we have

$(s_1+\ldots+s_n) \, H_n(s_1,\ldots,s_n) \doteq (s_1+\ldots+s_n) \, K_n(s_1,\ldots,s_n)$. Observe that although

$s_1+s_2+\ldots+s_n \doteq ns_1$, which follows from Lemma 3 and Example 1 of Lemma 1,

$(s_1+\ldots+s_n) \, H_n(s_1,\ldots,s_n) \neq ns_1 \, H_n(s_1,\ldots,s_n)$.

Lemma 4.

Let
$$A_{k_1}(s_1,\ldots,s_{k_1}) \doteq A'_{k_1}(s_1,\ldots,s_{k_1})$$

$$B_{k_2}(s_1,\ldots,s_{k_2}) \doteq B'_{k_2}(s_1,\ldots,s_{k_2})$$

$$C_{k_3}(s_1,\ldots,s_{k_3}) \doteq C'_{k_3}(s_1,\ldots,s_{k_3})$$

$$\vdots \qquad \qquad \vdots$$

$$N_{k_n}(s_1,\ldots,s_{k_n}) \doteq N'_{k_n}(s_1,\ldots,s_{k_n})$$

and
$$\Lambda_n(s_1,\ldots,s_n) \doteq \Lambda'_n(s_1,\ldots,s_n)$$

Let both $\Lambda_n(s_1,\ldots,s_n)$ and $\Lambda'_n(s_1,\ldots,s_n)$ in the last equation be partially symmetric in the sense that if any of the above equations, say the $\beta_1$, $\beta_2$, $\ldots$, and $\beta_m$ equations, are identical, then $\Lambda_n(\cdot)$ and $\Lambda'_n(\cdot)$ need only be symmetric with respect to all variables other than among those associated

-23-

with the $\beta_1$, $\beta_2,\ldots$, and $\beta_m$ equations.[13]  Under these conditions, we have:

$$A_{k_1}(s_1,\ldots,s_{k_1})B_{k_2}(s_{k_1+1},\ldots,s_{k_1+k_2})\ldots N_{k_n}(s_{k_1+k_2+\ldots+k_{n-1}+1},\ldots,s_{k_1+\ldots+k_n})$$

$$\cdot \Lambda_n\left(\sum_{i=1}^{k_1}s_i, \sum_{i=k_1+1}^{k_1+k_2}s_i,\ldots,\sum_{i=k_1+\ldots+k_{n-1}+1}^{k_1+\ldots+k_n}s_i\right)$$

$$\doteq A'_{k_1}(s_1,\ldots,s_{k_1})B'_{k_2}(s_{k_1+1},\ldots,s_{k_1+k_2})\ldots N'_{k_n}(s_{k_1+\ldots+k_{n-1}+1},\ldots,s_{k_1+\ldots+k_n})$$

$$\cdot \Lambda'_n\left(\sum_{i=1}^{k_1}s_i, \sum_{i=k_1+1}^{k_1+k_2}s_i,\ldots,\sum_{i=k_1+\ldots+k_{n-1}+1}^{k_1+\ldots+k_n}s_i\right) \qquad (4.4)$$

**Proof:**

    For simplicity, we will only prove the case for which the $\beta_1,\beta_2,\ldots,\beta_m$ equations are the only subset of identical equations.  Other cases are similar. Let us denote the left hand side of (4.4) by $F_{k_1+\ldots+k_n}(s_1,\ldots,s_{k_1+\ldots+k_n})$ and the right hand side by $F'_{k_1+\ldots+k_n}(s_1,\ldots,s_{k_1+\ldots+k_n})$.  Let

$H_{k_1+\ldots+k_n}(s_1,\ldots,s_{k_1+\ldots+k_n})$ and $H'_{k_1+\ldots+k_n}(s_1,\ldots,s_{k_1+\ldots+k_n})$ denote

$F_{k_1+\ldots+k_n}(\cdot)$ and $F'_{k_1+\ldots+k_n}(\cdot)$, respectively, after replacing $\Lambda_n(s_1,\ldots,s_n)$ and

$\Lambda'_n(s_1,\ldots,s_n)$ in $F_{k_1+\ldots+k_n}(\cdot)$ and $F'_{k_1+\ldots+k_n}(\cdot)$, respectively, by $\bar{\Lambda}_n(s_1,\ldots,s_n)$.

---

[13]That is to say, if the $\beta_1,\beta_2,\ldots$, and $\beta_m$ equations are the only subset of identical equations, then $\bar{\Lambda}_n(s_1, s_2,\ldots,s_n) = \underset{s_{\beta_1},s_{\beta_2},\ldots,s_{\beta_m}}{\text{sym}} \Lambda_n(\cdot) = \underset{s_{\beta_1},s_{\beta_2},\ldots,s_{\beta_m}}{\text{sym}}$

$\Lambda'_n(\cdot)$, where "$\underset{s_{\beta_1},s_{\beta_2},\ldots,s_{\beta_m}}{\text{sym}}$" denotes the symmetrization operation is with respect to

$s_{\beta_1},s_{\beta_2},\ldots,s_{\beta_m}$ only.  As an example, for n=5, if there are two subset of identical equations(namely, the 1st and 3rd, and the 2nd and 5th) then $\bar{\Lambda}_n(\cdot)$

$= \underset{s_1, s_3}{\text{sym}}\left\{\underset{s_2, s_5}{\text{sym}} \Lambda_n(\cdot)\right\}\cdot$

We first show $H_{k_1+\ldots+k_n}(\cdot) \doteq F_{k_1+\ldots+k_n}(\cdot)$ and $H'_{k_1+\ldots+k_n}(\cdot) \doteq F'_{k_1+\ldots+k_n}(\cdot)$.

To prove this, let us divide $(s_1,\ldots,s_{k_1+\ldots+k_n})$ into subgroups, such that

$(s_1,\ldots,s_{k_1})$ is the 1st subgroup associated with the 1st equation, $(s_{k_1+1},\ldots,s_{k_1+k_2})$ is the 2nd subgroup associated with a 2nd equation, and so on. Now consider each subgroup as a whole and perform a symmetrization on $F_{k_1+\ldots+k_n}(\cdot)$ with respect to the $\beta_1,\beta_2,\ldots,\beta_m$ subgroups. Thus, $F_{k_1+\ldots+k_n}(s_1,\ldots,s_{k_1+\ldots+k_n})$

$$\doteq \underset{\beta_1,\beta_2,\ldots,\beta_m \text{ subgroup}}{\text{sym}} F_{k_1+\ldots+k_n}(s_1,\ldots,s_{k_1+\ldots+k_n})$$

$$= A_{k_1}(\cdot)B_{k_2}(\cdot)\ldots N_{k_n}(\cdot) \underset{\beta_1,\ldots,\beta_m \text{ subgroup}}{\text{sym}} \bar{\Lambda}_n\left(\sum_{i=1}^{k_1} s_i,\ldots,\sum_{i=k_1+\ldots+k_{n-1}+1}^{k_1+\ldots+k_n} s_i\right)$$

This follows from the fact that the $\beta_1,\beta_2,\ldots,$ and $\beta_m$ equations are identical. By the partial symmetric property of $\Lambda_n(s_1,\ldots,s_n)$, we have

$$\underset{\beta_1,\ldots,\beta_m \text{ subgroup}}{\text{sym}} \Lambda_n\left(\sum_{i=1}^{k_1} s_i,\ldots,\sum_{i=k_1+\ldots+k_{n-1}+1}^{k_1+\ldots+k_n} s_i\right) = \Lambda_n\left(\sum_{i=1}^{k_1} s_i,\ldots,\sum_{i=k_1+\ldots+k_{n-1}+1}^{k_1+\ldots+k_n} s_i\right)$$

Thus $H_{k_1+\ldots+k_n}(\cdot) \doteq F_{k_1+\ldots+k_n}(\cdot)$. Similarly, $H'_{k_1+\ldots+k_n}(\cdot) = F'_{k_1+\ldots+k_n}(\cdot)$. Now

$$H_{k_1+\ldots+k_n}(\cdot) \doteq \underset{s_1,\ldots,s_{k_1}}{\text{sym}}\left\{\underset{s_{k_1+1},\ldots,s_{k_1+k_2}}{\text{sym}}\right.$$

$$\cdots \left\{\underset{s_{k_1+\ldots+k_{n-1}+1},\ldots,s_{k_1+\ldots+k_n}}{\text{sym}} H_{k_1+\ldots+k_n}(\cdot)\right\} \cdots \right\}$$

$$\doteq \bar{A}_{k_1}(\cdot)\bar{B}_{k_2}(\cdot)\ldots\bar{N}_{k_n}(\cdot)\bar{\Lambda}_n\left(\sum_{i=1}^{k_1} s_i,\ldots,\sum_{i=k_1+\ldots+k_{n-1}+1}^{k_1+\ldots+k_n} s_i\right)$$

Similarly, this last expression is also equivalent to $H'_{k_1+\ldots+k_n}(\cdot)$. Thus

$$F_{k_1+\ldots+k_n}(\cdot) \doteq H_{k_1+\ldots+k_n}(\cdot) \doteq H'_{k_1+\ldots+k_n}(\cdot) \doteq F'_{k_1+\ldots+k_n}(\cdot). \qquad \square$$

Remarks:

1. In deriving nonlinear transfer functions from cascade systems or from nonlinear circuits, expressions of the type shown in (4.4) will be encountered.

2. As an example illustrating Lemma 3, let $A_2(s_1,s_2) \doteq B_2(s_1,s_2)$, $C_3(s_1,s_2,s_3) \doteq D_3(s_1,s_2,s_3)$ and $E_2(s_1,s_2) \doteq F_2(s_1,s_2)$. It then follows from (4.4) that

$$A_2(s_1,s_2)C_3(s_3,s_4,s_5)E_2(s_1+s_2,s_3+s_4+s_5) \doteq B_2(s_1,s_2)D_3(s_3,s_4,s_5)F_2(s_1+s_2,s_3+s_4+s_5)$$

holds if $F_2(s_1,s_2)$ and $E_2(s_1,s_2)$ are symmetric. However,

$$A_2(s_1,s_2)A_2(s_3,s_4)E_2(s_1+s_2,s_3+s_4) \doteq B_2(s_1,s_2)B_2(s_3,s_4)F_2(s_1+s_2,s_3+s_4)$$

holds even if $E_2(s_1,s_2)$ and $F_2(s_1,s_2)$ are not symmetric because $A_2(s_1,s_2) \doteq B_2(s_1,s_2)$ and $A_2(s_3,s_4) \doteq B_2(s_3,s_4)$ are identical equations.

3. Observe that (4.4) does not hold without the partial symmetric property, even if $\Lambda_n(\cdot)$ and $\Lambda_n'(\cdot)$ are identical.

## Corollary

If $F_m(s_1,\ldots,s_m) \doteq G_m(s_1,\ldots,s_m)$ and $H_r(s_1,\ldots,s_r) \doteq K_r(s_1,\ldots,s_r)$, then

$$F_m(s_1,\ldots,s_m) \, H_r(s_{m+1},\ldots,s_{m+r}) \doteq G_m(s_1,\ldots,s_m) \, K_r(s_{m+1},\ldots,s_{m+r}) \qquad (4.5)$$

## Proof:

This is obtained by choosing n=2 and $\Lambda_2(\cdot) = \Lambda_2'(\cdot) = 1$ in Lemma 4. ◻

## Remarks:

1. The type of operation shown in the left hand side of (4.5) has already been encountered in (2.12) for the multiplication case.

2. Instead of simplifying an expression, sometimes we may want to decompose it into a convenient form, say for synthesis purposes. For example, $P_n(s_1,\ldots,s_n) = \left(\dfrac{a}{s_1-b}\right)\left(\dfrac{s_2}{s_3-d}\right)$ can be decomposed into the form $F_i(s_1,\ldots,s_i)H_{n-i}(s_{i+1},\ldots,s_n)$.

Suppose n=10 and we want i=4. It follows from Example 3 of Lemma 1, that

$P_{10}(s_1,\ldots,s_{10}) \doteq \left(\dfrac{a}{s_1-b}\right)\left(\dfrac{s_5}{s_6-d}\right)$. We now use (4.5) to do the decomposition. Set

$F_4(s_1,\ldots,s_4) = \dfrac{a}{s_1-b}$ and $H_6(s_1,\ldots,s_6) = \dfrac{s_1}{s_2-d}$. It follows from Lemma 1 and 2

that $F_4(s_1,\ldots,s_4) \doteq \dfrac{1}{3}\left(\dfrac{a}{s_1-b} + \dfrac{a}{s_2-b} + \dfrac{a}{s_4-b}\right)$ and $H_6(s_1,\ldots,s_6) \doteq \dfrac{1}{2}\left(\dfrac{s_1}{s_2-d} + \dfrac{s_3}{s_5-d}\right)$.

Applying (4.5), we have $P_{10}(s_1,\ldots,s_{10}) \doteq \dfrac{1}{6}\left(\dfrac{a}{s_1-b} + \dfrac{a}{s_2-b} + \dfrac{a}{s_4-b}\right)\left(\dfrac{s_5}{s_6-d} + \dfrac{s_7}{s_9-d}\right)$.

Most of the operations that we need in the frequency domain are of the types described in the left hand sides of (4.2)-(4.5), and the conditions of Lemmas 2-4 are generally satisfied by the corresponding equivalent component systems. Our next theorem follows from Lemmas 2-4 and the above corollary.

## Theorem 2. Systems combination property

If an expression is obtained by a sequence of operations between transfer functions (or output transforms), such as those shown in the left hand sides of (4.2)-(4.5), and for each operation the conditions of the corresponding lemma or corollary are satisfied, then the _symmetrized_ form of this expression is an invariant of the combined system in the sense that it does not depend upon whether the component functions are symmetrized or not during the intermediate manipulation steps.

## Remark:

It follows from Theorem 1 and 2 that two systems are equivalent if, for each n their nonlinear transfer functions are obtained by performing the same sequence of operations (described in the left hand sides of (4.2)-(4.5)) on corresponding nonlinear transfer functions derived from equivalent subsystems.

We will close this section with an example which illustrates some applications of the above lemmas.

## Example.

Let us look at the cascade system shown in Fig. 4. Systems G and K are differentiators while F and H represent systems whose only nonzero transfer functions are $F_m(s_1, \ldots, s_m) = 1$ and $H_n(s_1, \ldots, s_n) = 1$, respectively. The functions $u(t)$ and $y(t)$ are the input and output, respectively. The relationship between $u(t)$ and $y(t)$ in the time domain is given by $y(t) = \frac{d}{dt}\left(\frac{d}{dt} u^m(t)\right)^n$. Using the formulas from cascade systems [4-7], the overall nonzero transfer function has an order equal to $n \times m$ and is given by:

$$P_{nm}(s_1, \ldots, s_{nm}) \doteq (s_1+s_2+\ldots+s_m)(s_{m+1}+\ldots+s_{2m})(s_{2m+1}+\ldots+s_{3m})\ldots(s_{(n-1)m+1}+\ldots+s_{nm}) \cdot$$

$$(s_1+\ldots+s_{nm}) \qquad (4.6)$$

This is in the form as shown by the left hand side of (4.4), with $\Lambda_n(.) = s_1+s_2+\ldots+s_n$ being symmetric. Set $A_{k_1}(s_1, \ldots, s_{k_1}) = B_{k_2}(s_1, \ldots, s_{k_2}) = \ldots = N_{k_n}(s_1, \ldots, s_{k_n})$ $= s_1+\ldots+s_m$ with $k_1 = k_2 = \ldots = k_n = m$. Since $s_1+s_2+\ldots+s_m \doteq ms_m$, it follows from

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[14] Consider a cascade system with the input of system B being the output of system A. If B is a linear system, the combined $\underline{n}$th order transfer function is $A_n(s_1, s_2, \ldots, s_n)B_1(s_1+s_2+\ldots+s_n)$. If A is a linear system, the combined $\underline{n}$th order transfer function is $A_1(s_1)A_1(s_2)\ldots A_1(s_n)B_n(s_1, s_2, \ldots, s_n)$.

(4.4) that $P_{nm}(s_1, \ldots, s_{nm}) \doteq (ms_m)(ms_{2m}) \ldots (ms_{nm})(s_1 + \ldots + s_{nm})$

$\doteq m^n s_m s_{2m} \ldots s_{nm}(s_1 + \ldots + s_{nm})$. Applying Lemma 4 again, we have $P_{nm}(s_1, \ldots, s_{nm})$

$\doteq m^n s_m s_{2m} \ldots s_{nm} n(s_1 + \ldots + s_m)$. This last expression follows from the fact that

$\Lambda_n(.) = s_1 + \ldots + s_n \doteq ns_1$ need not be symmetric in the present case. Now,

$P_{nm}(s_1, \ldots, s_{nm}) \doteq nm^n s_m s_{2m} \ldots s_{nm}(s_1 + \ldots + s_m) = nm^n s_m^2 s_{2m} \ldots s_{nm} + nm^n(s_1 + \ldots + s_{m-1}) \cdot$

$$s_m s_{2m} \ldots s_{nm}.$$

It follows from Lemma 1 that $s_m^2 s_{2m} \ldots s_{nm} \doteq s_1^2 s_2 s_3 \ldots s_n$ and from (4.5) that
$(s_1 + \ldots + s_{m-1}) s_m s_{2m} \ldots s_{nm} \doteq (m-1) s_1 s_m s_{2m} \ldots s_{nm} \doteq (m-1) s_1 s_2 \ldots s_{n+1}$. Thus

$$P_{nm}(s_1, \ldots, s_{nm}) \doteq nm^n s_1^2 s_2 s_3 \ldots s_n + n(m-1)m^n s_1 s_2 \ldots s_{n+1} \qquad (4.7)$$

Transforming back into the time domain, we have

$$y(t) = nm^n \left(u(t)\right)^{nm-n} \left(\frac{d^2}{dt^2} u(t)\right)\left(\frac{du(t)}{dt}\right)^{n-1} + n(m-1)m^n \left(u(t)\right)^{nm-n-1} \left(\frac{du(t)}{dt}\right)^{n+1} \qquad (4.8)$$

which is precisely the original expression $\frac{d}{dt}\left(\frac{d}{dt} u^m(t)\right)^n$ in expanded form. Using

Theorem 2, (4.8) can also be obtained by simplifying the nonlinear transfer functions or output transforms after each intermediate stage in Fig. 4.

5. <u>Formulation of Nonlinear Transfer Functions for Dynamical Systems Described</u>
   <u>by State Equations</u>

    In this section, we will consider a single input analytic system defined by:

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{f}\left(\underset{\sim}{x}(t), u(t), \frac{du(t)}{dt}, \ldots, \frac{d^N u(t)}{dt^N}\right) \qquad (5.1a)$$

$$y(t) = g\left(\underset{\sim}{x}(t), u(t), \frac{du(t)}{dt}, \ldots, \frac{d^N u(t)}{dt^N}\right) \qquad (5.1b)$$

where $\underset{\sim}{x}(\cdot) \overset{\Delta}{=} \left[x_1(\cdot), x_2(\cdot), \ldots, x_r(\cdot)\right]^T$ is the state vector, $u(\cdot)$ is the input, and $y(\cdot)$ is the output.

<u>Remark:</u>

    The actual input may be $u(t) + \delta u(t)$, where $u(t)$ is a fixed time function and $\delta u(t)$ is a small signal compared with $u(t)$. If $u(t)$ is a constant, then after substituting $u(t) = u(t) + \delta u(t)$ into (5.1a) and (5.1b), the resulting set of

equations is still in the same form as (5.1a) and (5.1b) but with input $\delta u(t)$ instead of $u(t)$. If $u(t)$ is a time-varying function, the resulting system will be a _time-varying_ system with input $\delta u(t)$, and is therefore outside the scope of this paper.

Assume each state variable $x_i(t)$ and the output $y(t)$ are represented by a Volterra Series as follows:

$$x_i(t) = \sum_{j=0}^{\infty} x_{ij}(t) \qquad \forall i = 1,\ldots,r \tag{5.2a}$$

and

$$y(t) = \sum_{j=0}^{\infty} y_j(t), \tag{5.2b}$$

where $x_{ij}(t)$ and $y_j(t)$ are the _j_th order component of $x_i(t)$ and $y(t)$, respectively. The _j_th order transfer function $H_j(s_1,\ldots,s_j)$, $j \geq 0$ of the system corresponding to input $u(t)$ and output $y(t)$ can be obtained recursively by substituting (5.2) into (5.1), and equating the output components of the same orders in both sides of the equations. For the zeroth order output, we set $u(t) = 0$. If then follows from (2.2), that $x(t) = [x_{1,0}, x_{2,0}, \ldots, x_{r,0}]^T \triangleq \underset{\sim}{x}_o$, and $y(t) = y_o$, where $\underset{\sim}{x}_o$ is a constant vector and $y_o$ is a constant scalar. It follows from (5.1) that

$$\underset{\sim}{0} = \underset{\sim}{f}(\underset{\sim}{x}_o, 0, 0, \ldots, 0) \tag{5.3a}$$

$$y_o = g(\underset{\sim}{x}_o, 0, 0, \ldots, 0) \tag{5.3b}$$

Hence, each $\underset{\sim}{x}_o$ satisfying (5.3a) is an equilibrium state of (5.1a). Since the Volterra Series may not converge everywhere, it is desirable to choose an equilibrium state located within the dynamic range of interest. In any event, we can always choose the coordinate with $\underset{\sim}{x}_o$ and $y_o$ as the new origin. Hence without loss of generality, we can assume $\underset{\sim}{x}_o = \underset{\sim}{0}$ and $y_o = 0$.

The procedures for deriving the transfer functions associated with (5.1) can be readily explained via an example. Let (5.1) be a dynamical system defined by

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + p_1 x_2^2(t) + b_1 \dot{u}(t) + e_1 u^2(t) \tag{5.4a}$$

$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + p_2 x_1(t)x_2(t) + b_2 u(t) + e_2 u(t)\dot{u}(t) \tag{5.4b}$$

$$y(t) = c_1 x_1(t) + c_2 x_2(t) + q x_1^2(t) + du(t) \tag{5.4c}$$

which can be rewritten as

$$\dot{x} = Ax + \begin{bmatrix} b_1\ddot{u} \\ b_2 u \end{bmatrix} + \begin{bmatrix} p_1 x_2^2 + e_1 u^2 \\ p_2 x_1 x_2 + e_2 u\dot{u} \end{bmatrix} \quad , \tag{5.5a}$$

$$y = cx + qx_1^2 + du \tag{5.5b}$$

where $\underset{\sim}{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ . $\underset{\sim}{A} \triangleq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\underset{\sim}{c} \triangleq [c_1, c_2]$. Although (5.4) may have more

than one equilibrium state with $u(t) = 0$, let us choose $x_{1,0} = x_{2,0} = 0$ and $y_0 = 0$. In (5.5), and the rest of this section, the time variable t is omitted for convenience.

Now substitute the Volterra series $x_1 = \sum_{i=1}^{\infty} x_{1i}$, $x_2 = \sum_{i=1}^{\infty} x_{2i}$ and $y = \sum_{i=1}^{\infty} y_i$

into (5.5), and let $\underset{\sim}{x}_i \triangleq [x_{1i}, x_{2i}]^T$.

$$\sum_{i=1}^{\infty} \dot{\underset{\sim}{x}}_i = \sum_{i=1}^{\infty} A\underset{\sim}{x}_i + \begin{bmatrix} b_1\dot{u} \\ b_2 u \end{bmatrix} + \begin{bmatrix} p_1\left(\sum_{i=1}^{\infty} x_{2i}\right)^2 + e_1 u^2 \\ p_2\left(\sum_{i=1}^{\infty} x_{1i}\right)\left(\sum_{i=1}^{\infty} x_{2i}\right) + e_2 u\dot{u} \end{bmatrix} \tag{5.6a}$$

$$\sum_{i=1}^{\infty} y_i = \sum_{i=1}^{\infty} c\underset{\sim}{x}_i + q\left(\sum_{i=1}^{\infty} x_{1i}\right)^2 + du \tag{5.6b}$$

Since both sides of (5.6) represent equivalent systems, it follows from Theorem 1 of section 5 that the output components of the same order can be equated from both sides of (5.6). It is clear from (2.2) that the nth order terms on both sides of (5.6) can be identified as the coefficients of $\varepsilon^n$ after substituting $\varepsilon u(t)$ for u(t). In general the order of a term can be determined by inspection. For example,

if we let $a(t) = \sum_{i=0}^{\infty} a_i(t)$, $b(t) = \sum_{i=0}^{\infty} b_i(t)$ and $c(t) = \sum_{i=0}^{\infty} c_i(t)$ be the

Volterra series with respect to input u(t), then $a_i(t)\, b_j(t)$ is a term of order $(i+j)$ and $a_i(t)\, b_j(t)\, c_k(t)$ is of order $(i+j+k)$. Thus each term in (2.12a) is obviously of order n. Although (2.12) is only for multiplication of two Volterra series, for multiplication of more than two Volterra series we just apply (2.12) more than once. For example, the total nth order term of $a(t)\, b(t)\, c(t)$ is

$$\sum_{i=0}^{n} e_i(t)c_{n-i}(t) = \sum_{i=0}^{n} \sum_{j=0}^{i} a_j(t)b_{i-j}(t)c_{n-i}(t), \text{ where } e_i(t) \text{ is as defined in}$$

(2.12a).

Now let us start by equating the <u>first order output terms</u> in (5.6):

$$\dot{\underset{\sim}{x}}_1 = \underset{\sim}{A}\underset{\sim}{x}_1 + \begin{bmatrix} b_1\dot{u} \\ b_2 u \end{bmatrix} \qquad (5.7a)$$

$$y_1 = \underset{\sim}{c}\underset{\sim}{x}_1 + du \qquad (5.7b)$$

The first order transform of (5.7) is given by:

$$\underset{\sim}{X}_1(s_1) = (s_1\underset{\sim}{I}-\underset{\sim}{A})^{-1} \begin{bmatrix} b_1 s_1 \\ b_2 \end{bmatrix} U(s_1) \qquad (5.8a)$$

$$Y_1(s_1) = \left\{ \underset{\sim}{c}(s_1\underset{\sim}{I}-\underset{\sim}{A})^{-1} \begin{bmatrix} b_1 s_1 \\ b_2 \end{bmatrix} + d \right\} U(s_1) \qquad (5.8b)$$

It follows from (5.8b) that the <u>first order transfer function</u> is simply

$$H_1(s_1) = \underset{\sim}{c}(s_1\underset{\sim}{I}-\underset{\sim}{A})^{-1} \begin{bmatrix} b_1 s_1 \\ b_2 \end{bmatrix} + d.$$

Next, equating the second order output terms of (5.6) and using (2.11)-(2.13) we obtain

$$\dot{\underset{\sim}{x}}_2 = \underset{\sim}{A}\underset{\sim}{x}_2 + \begin{bmatrix} p_1 x_{21}^2 + e_1 u^2 \\ p_2 x_{11} x_{21} + e_2 u\dot{u} \end{bmatrix} \qquad (5.9a)$$

$$y_2 = \underset{\sim}{c}\underset{\sim}{x}_2 + q x_{11}^2 \qquad (5.9b)$$

It follows from (2.11b), (2.12b) and (2.13b) that the 2nd order transform of (5.9a) is given by:

$$(s_1+s_2)\, \underset{\sim}{X}_2(s_1,s_2) = \underset{\sim}{A}\underset{\sim}{X}_2(s_1,s_2) + \begin{bmatrix} p_1 X_{21}(s_1) X_{21}(s_2) + e_1 U(s_1)U(s_2) \\ p_2 X_{11}(s_1) X_{21}(s_2) + e_2 s_2 U(s_1)U(s_2) \end{bmatrix} \qquad (5.10)$$

Applying Lemma 2 from section 4, we have:

$$[(s_1+s_2)\underset{\sim}{I}-\underset{\sim}{A}]\, \underset{\sim}{X}_2(s_1,s_2) = \begin{bmatrix} p_1 X_{21}(s_1) X_{21}(s_2) + e_1 U(s_1)U(s_2) \\ p_2 X_{11}(s_1) X_{21}(s_2) + e_2 s_2 U(s_1)U(s_2) \end{bmatrix}$$

Using an analogy to Lemma 3 from section 4, $X_2(s_1,s_2)$ is obtained by multiplying by

$[(s_1+s_2)\underset{\sim}{1}-\underset{\sim}{A}]^{-1}$ on both sides:

$$\underset{\sim}{X}_2(s_1,s_2) \doteq \left[(s_1+s_2)\underset{\sim}{1}-\underset{\sim}{A}\right]^{-1} \begin{bmatrix} p_1 X_{21}(s_1)X_{21}(s_2) + e_1 U(s_1)U(s_2) \\ p_2 X_{11}(s_1)X_{21}(s_2) + e_2 s_2 U(s_1)U(s_2) \end{bmatrix} \tag{5.11a}$$

Observe that (5.11a) can be obtained from (5.10) by considering (5.10) as a set of two scalar equations in two unknowns: $X_{21}(s_1,s_2)$ and $X_{22}(s_1,s_2)$. Applying Lemmas 2 and 3 of section 4, a sequence of additions and subtractions between the two scalar equations, along with multiplications on both sides of the two scalar equations by symmetric functions, can be carried out to obtain (5.11a).

Now from (5.9b), we obtain

$$Y_2(s_1,s_2) \doteq \underset{\sim}{c}\underset{\sim}{X}_2(s_1,s_2) + q X_{11}(s_1) \, X_{11}(s_2) \tag{5.11b}$$

Hence, the second-order transfer function is simply

$$H_2(s_1,s_2) = \frac{Y_2(s_1,s_2)}{U(s_1)U(s_2)} \tag{5.11c}$$

From (2.12) the nth order terms contained in $x_2^2$, $x_1 x_2$ and $x_1^2$ are given

by $\displaystyle\sum_{i=1}^{n-1} x_{2,i} x_{2,n-i}$, $\displaystyle\sum_{i=1}^{n-1} x_{1,i} {}_{2,n-i}$ and $\displaystyle\sum_{i=1}^{n-1} x_{1,i} x_{1,n-i}$, respectively. Thus by

equating the nth order terms on both sides of (5.6) for $n \geq 3$, we obtain

$$\underset{\sim}{\dot{x}}_n = \underset{\sim}{A}\underset{\sim}{x}_n + \begin{bmatrix} p_1 \displaystyle\sum_{i=1}^{n-1} x_{2,i} x_{2,n-i} \\ p_2 \displaystyle\sum_{i=1}^{n-1} x_{1,i} x_{2,n-i} \end{bmatrix} \tag{5.12a}$$

$$y_n = \underset{\sim}{c}\underset{\sim}{x}_n + q \sum_{i=1}^{n-1} x_{1,i} x_{1,n-i} \tag{5.12b}$$

Following the same procedure as above, the nth order transforms are given by

$$\underset{\sim}{X}_n(s_1,\ldots,s_n) \doteq \left[(s_1+\ldots+s_n)\underset{\sim}{1}-\underset{\sim}{A}\right]^{-1} \begin{bmatrix} p_1 \displaystyle\sum_{i=1}^{n-1} X_{2,i}(s_1,\ldots,s_i)X_{2,n-i}(s_{i+1},\ldots,s_n) \\ p_2 \displaystyle\sum_{i=1}^{n-1} X_{1,i}(s_1,\ldots,s_i)X_{2,n-i}(s_{i+1},\ldots,s_n) \end{bmatrix}$$

$$\tag{5.13a}$$

$$Y_n(s_1,\ldots,s_n) \doteq \underline{c}X_n(s_1,\ldots,s_n) + q \sum_{i=1}^{n-1} X_{1,i}(s_1,\ldots,s_i)X_{1,n-i}(s_{i+1},\ldots,s_n) \tag{5.13b}$$

Hence the <u>nth order transfer function</u> is simply

$$H_n(s_1,\ldots,s_n) = \frac{Y_n(s_1,\ldots,s_n)}{U(s_1)\ldots U(s_n)} \tag{5.13c}$$

Observe that the right hand side of (5.13a) only involves transforms of state responses of <u>orders less than n</u>. Consequently, all order output transforms and transfer functions can be obtained <u>recursively</u>.

In general, normal form equations such as (5.1) can be decomposed into the following form:

$$\underline{\dot{x}} = \underline{A}\underline{x} + \underline{B}\left(u, \frac{du}{dt}, \ldots, \frac{d^N u}{dt^N}\right) + \underline{F}\left(\underline{x}, u, \frac{du}{dt}, \ldots, \frac{d^N u}{dt^N}\right) \tag{5.14a}$$

$$y = \underline{c}\underline{x} + D\left(u, \frac{du}{dt}, \ldots, \frac{d^N u}{dt^N}\right) + G\left(\underline{x}, u, \frac{du}{dt}, \ldots, \frac{d^N u}{dt^N}\right) \tag{5.14b}$$

where $D(\cdot)$ and each component of $\underline{B}(\cdot)$ are linear combinations of $u, \frac{du}{dt}, \ldots, \frac{d^N u}{dt^N}$,

and $\underline{F}(\cdot)$ and $G(\cdot)$ only contain nonlinear terms. By changing coordinates in (5.1) if necessary, we can always choose the zeroth order component $\underline{x}_o = \underline{0}$ and $y_o = 0$ so that both $\underline{F}(\cdot)$ and $G(\cdot)$ vanish when $\underline{x} = \underline{0}$ and $u = 0$. Using the same procedure as above, we obtain the following general relationships:

<u>1st order component:</u>

$$X_1(s_1) = (s_1\underline{1}-\underline{A})^{-1} \underline{B}(1,s_1,\ldots,s_1^N)U(s_1) \tag{5.15a}$$

$$Y_1(s_1) = \underline{c}X_1(s_1) + D(1,s_1,\ldots,s_1^N)U(s_1) \tag{5.15b}$$

$$H_1(s_1) = \underline{c}(s_1\underline{1}-\underline{A})^{-1} \underline{B}(1,s_1,\ldots,s_1^N) + D(1,s_1,\ldots,s_1^N) \tag{5.15c}$$

<u>nth order component:</u>

$$\underline{X}_n(s_1,\ldots,s_n) \doteq \left[(s_1+\ldots+s_n)\underline{1}-\underline{A}\right]^{-1} \begin{bmatrix} \text{nth order transform terms of } \underline{F}(\cdot), \text{ which} \\ \text{only involve input and state responses} \\ \text{of order lower than n} \end{bmatrix} \tag{5.16a}$$

$$Y_n(s_1,\ldots,s_n) \doteq c\underset{\sim}{X}_n(s_1,\ldots,s_n) + \begin{bmatrix} \text{nth order transform terms of } \underset{\sim}{G}(\cdot) \\ \text{which only involve input and state} \\ \text{responses of order lower than n} \end{bmatrix}$$

<div align="right">(5.16b)</div>

$$H_n(s_1,\ldots,s_n) \doteq \frac{Y_n(s_1,\ldots,s_n)}{U(s_1)\ldots U(s_n)}$$

<div align="right">(5.16c)</div>

Hence, all higher order output transforms and transfer functions can be obtained recursively.

## 6. Generalization to Multi-input Systems

Extension of the results in the preceding sections to multi-input systems are generally easy and straightforward. By means of vector representations, we can write expressions which are similar in form to those derived for single input cases. Indeed if $\underset{\sim}{u}(t) = [u_1(t),u_2(t),\ldots,u_N(t)]^T$ is the input, then (2.14) can be written as

$$y(t) = \sum_{\underset{\sim}{n}=(0,0,\ldots,0)}^{(\infty,\infty,\ldots,\infty)} y_{\underset{\sim}{n}}(t).$$

where $\underset{\sim}{n} \triangleq (n_1,n_2,\ldots,n_N)$ and $\displaystyle\sum_{\underset{\sim}{n}=(0,\ldots,0)}^{(\infty,\ldots,\infty)} \triangleq \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_N=0}^{\infty}$

We can call $y_{\underset{\sim}{n}}(t)$ the output of order $\underset{\sim}{n}$, which is given by (2.15). The kernel at order $\underset{\sim}{n}$ is $h_{\underset{\sim}{n}}(\underset{\sim}{\tau}_1;\underset{\sim}{\tau}_2;\ldots;\underset{\sim}{\tau}_N)$, where $\underset{\sim}{\tau}_i \triangleq (\tau_{i1},\tau_{i2},\ldots,\tau_{in_i})$. The transfer function at order $\underset{\sim}{n}$ can also be written as $H_{\underset{\sim}{n}}(\underset{\sim}{s}_1;\underset{\sim}{s}_2;\ldots;\underset{\sim}{s}_N)$, where $\underset{\sim}{s}_i \triangleq (s_{i1},s_{i2},\ldots,s_{in_i})$. Module vectors and other expressions can be similarly written. As an example we show below how property 3 and property 4 from section 3 can be generalized for the multi-input case.

Let each input component $u_i(t) = \sum_{k=1}^{K_i} |A_{ik}| \cos(\omega_{ik} + \angle A_{ik})$, $i = 1,\ldots,N$,

and let all input frequencies form a frequency base. Let $\underset{\sim}{r} \triangleq (r_1,\ldots,r_N)$, where $r_i, \forall i = 1,\ldots,N$, are nonnegative integers. We then have:

1. All output frequencies of order $\underset{\sim}{n} = (n_1,n_2,\ldots,n_N)$ are present in the output of order $\underset{\sim}{n}+2\underset{\sim}{r} = (n_1+2r_1,n_2+2r_2,\ldots,n_N+2r_N)$.

2. Output components of order $\underset{\sim}{n} = (n_1,n_2,\ldots,n_N)$ and output components of order $\underset{\sim}{n}+\underset{\sim}{r}$ can never contain identical output frequency, if one of the components, $r_i$ in $\underset{\sim}{r}$ is an odd integer.

Further extensions to allow system combinations, such as those of (2.11)-(2.13), for the multi-input case are also straightforward. Consequently, the nonlinear transfer function formulations given in section 5 can also be generalized.

## 7. Conclusion

We have derived the relationships between the input and output frequencies and amplitudes at steady state. Expressions for the amplitude and phase of a certain output frequency of a certain order are given. Relationships between the output frequencies of different orders are also given.

We show that symmetrized transfer functions are not required when operations are carried out between transfer functions of corresponding equivalent systems. Lemmas in section 4 also provide conditions under which the simplification or expansion of transfer functions to equivalent forms can be done. Finally, we show a method for obtaining transfer functions from state equation representations recursively.

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This appendix contains specific examples showing the detailed expansion of various general formulae given in section 3.

1.  For (3.2), let K=2, n=2

$$y_n(t) = \sum_{k_1=1}^{2} \sum_{k_2=1}^{2} A_{k_1} A_{k_2} H_n(j\omega_{k_1}, j\omega_{k_2}) e^{j(\omega_{k_1}+\omega_{k_2})t}$$

$$= \sum_{k_2=1}^{2} A_1 A_{k_2} H_n(j\omega_1, j\omega_{k_2}) e^{j(\omega_1+\omega_{k_2})t}$$

$$+ \sum_{k_2=1}^{2} A_2 A_{k_2} H_n(j\omega_2, j\omega_{k_2}) e^{j(\omega_2+\omega_{k_2})t}$$

$$= A_1 A_1 H_n(j\omega_1, j\omega_1) e^{2j\omega_1 t} + A_1 A_2 H_n(j\omega_1, j\omega_2) e^{j(\omega_1+\omega_2)t}$$

$$+ A_2 A_1 H_n(j\omega_2, j\omega_1) e^{j(\omega_2+\omega_1)t} + A_2 A_2 H_n(j\omega_2, j\omega_2) e^{2j\omega_2 t}$$

2.  For (3.5) let K=5, $\underset{\sim}{M} = (m_1, m_2, m_3, m_4, m_5) = (2,0,1,3,5)$

$$\therefore \quad n = m_1 + m_2 + m_3 + m_4 + m_5 = 11$$

$$\therefore \quad y_{\underset{\sim}{M}}(t) = 11! \left( \prod_{k=1}^{5} \frac{A_k^{m_k}}{m_k!} \right) H_{11}\left( 2\{j\omega_1\}, 0\{j\omega_2\}, 1\{j\omega_3\}, 3\{j\omega_4\}, 5\{j\omega_5\} \right)$$

$$\cdot e^{j(2\omega_1 + 0\omega_2 + \omega_3 + 3\omega_4 + 5\omega_5)t}$$

$$= 11! \frac{A_1^2}{2!} \frac{A_2^0}{0!} \frac{A_3^1}{1!} \frac{A_4^3}{3!} \frac{A_5^5}{5!} H_{11}(j\omega_1, j\omega_1, j\omega_3, j\omega_4, j\omega_4, j\omega_4, j\omega_5, j\omega_5, j\omega_5, j\omega_5, j\omega_5) \cdot$$

$$\cdot e^{j(2\omega_1 + \omega_3 + 3\omega_4 + 5\omega_5)t}$$

3.  For (3.10), let K=2, $M = (m_{-2}, m_{-1}, m_1, m_2) = (2,0,2,3)$

$$\therefore \quad n = m_{-2} + m_{-1} + m_1 + m_2 = 7, \quad \omega_{\underset{\sim}{M}} = (m_1 - m_{-1})\omega_1 + (m_2 - m_{-2})\omega_2 = 2\omega_1 + \omega_2$$

$$y_{\underline{M}}(t) = \frac{7!}{2^7} \left( \prod_{\substack{i=-2 \\ i\neq 0}}^{2} \frac{A_i^{m_i}}{m_i!} \right) H_7(2\{j\omega_{-2}\},0\{j\omega_{-1}\},2\{j\omega_1\},3\{j\omega_2\}) e^{j\omega_{\underline{M}}t}$$

$$= \frac{7!}{2^7} \frac{A_{-2}^2}{2!} \frac{A_{-1}^0}{0!} \frac{A_1^2}{2!} \frac{A_2^3}{3!} H_7(j\omega_{-2},j\omega_{-2},j\omega_1,j\omega_1,j\omega_2,j\omega_2,j\omega_2) e^{j(2\omega_1+\omega_2)t}$$

$$= \frac{7!A_1^2|A_2|^4 A_2}{2^7 \times 2! \times 2! \times 3!} H_7(j\omega_{-2},j\omega_{-2},j\omega_1, j\omega_1,j\omega_2,j\omega_2,j\omega_2) e^{j(2\omega_1+\omega_2)t}$$

4. For (3.11), let K=3, $\omega_o = \sum_{i=1}^{3} p_i\omega_i = 4\omega_1 + 0\omega_2 + (-2)\omega_3$. Thus, the lowest

order $n = |4| + |0| + |-2| = 6$, and the unique lowest order $\underline{M}$ which gives rise

to $\omega_o$ is

$$\underline{M} = (m_{-3},m_{-2},m_{-1},m_1,m_2,m_3) = (2,0,0,4,0,0)$$

5. For (3.12), let K=3, r=2.

The total d.c. output at order 2r=4 is:

$$\sum_{z_1=0}^{2} \sum_{z_2=0}^{2-z_1} \frac{4!}{2^4} \left[ \prod_{i=1}^{3} \frac{|A_i|^{2z_i}}{(z_i!)^2} \right] H_4\left(z_1\{j\omega_{-1}\},z_1\{j\omega_1\},z_2\{j\omega_{-2}\},z_2\{j\omega_2\},z_3\{j\omega_{-3}\},z_3\{j\omega_3\}\right)$$

$$\text{where } z_3 = r - \sum_{i=1}^{2} z_i = 2-z_1-z_2$$

$$= \sum_{z_2=0}^{2-0} \frac{4!}{2^4} \frac{|A_1^0|}{(0!)^2} \frac{|A_2|^{2z_2}}{(z_2!)^2} \frac{|A_3|^{2(2-z_2)}}{[(2-z_2)!]^2} H_4\left(0\{j\omega_{-1}\},0\{j\omega_1\},z_2\{j\omega_{-2}\},z_2\{j\omega_2\},(2-z_2)\{j\omega_{-3}\},(2-z_2)\{j\omega_3\}\right)$$

$$+ \sum_{z_2=0}^{2-1} \frac{4!}{2^4} \frac{|A_1|^2}{(1!)^2} \frac{|A_2|^{2z_2}}{(z_2!)^2} \frac{|A_3|^{2(2-1-z_2)}}{[(2-1-z_2)!]^2} H_4\left(1\{j\omega_{-1}\},1\{j\omega_1\},z_2\{j\omega_{-2}\},z_2\{j\omega_2\},(2-1-z_2)\{j\omega_{-3}\},(2-1-z_2)\{j\omega_3\}\right)$$

$$+ \sum_{z_2=0}^{2-2} \frac{4!}{2^4} \frac{|A_1|^4}{(2!)^4} \frac{|A_2|^{2z_2}}{[(z_2!)]^2} \frac{|A_3|^{2(2-2-z_2)}}{[(2-2-z_2)!]^2} H_4\left(2\{j\omega_{-1}\},2\{j\omega_1\},z_2\{j\omega_{-2}\},z_2\{j\omega_2\},(2-2-z_2)\{j\omega_{-3}\},(2-2-z_2)\{j\omega_3\}\right)$$

$$= \frac{4!}{2^4} \frac{|A_3|^4}{(2!)^2} H_4(j\omega_{-3},j\omega_{-3},j\omega_3,j\omega_3) + \frac{4!}{2^4} \frac{|A_2|^2}{(1!)^2} \frac{|A_3|^2}{(1!)^2} H_4(j\omega_{-2},j\omega_2,j\omega_{-3},j\omega_3)$$

$$+ \frac{4!}{2^4} \frac{|A_2|^4}{(2!)^2} H_4(j\omega_{-2},j\omega_{-2},j\omega_2,j\omega_2) + \frac{4!}{2^4} \frac{|A_1|^2}{(1!)^2} \frac{|A_3|^2}{(1!)^2} H_4(j\omega_{-1},j\omega_1,j\omega_{-3},j\omega_3)$$

$$+ \frac{4!}{2^4} \frac{|A_1|^2}{(1!)^2} \frac{|A_2|^2}{(1!)^2} H_4(j\omega_{-1},j\omega_1,j\omega_{-2},j\omega_2) + \frac{4!}{2^4} \frac{|A_1|^4}{(2!)^2} H_4(j\omega_{-1},j\omega_{-1},j\omega_1,j\omega_1)$$

6.  For (3.13), let $K=3$, $r=2$ (same as in Appendix 5).  The total d.c. output at order $2r=4$ is

$$\sum_{k_1=1}^{3} \sum_{k_2=1}^{k_1} \frac{4!}{2^4} \left[ \prod_{i=1}^{3} \frac{|A_i|^{2z_i}}{(z_i!)^2} \right] H_4(j\omega_{k_1}, j\omega_{-k_1}, j\omega_{k_2}, j\omega_{-k_2})$$

where $z_i$ is the number of $j\omega_i$ in the argument of $H_4(\cdot)$

$$= \sum_{k_2=1}^{1} \frac{4!}{2^4} \left[ \prod_{i=1}^{3} \frac{|A_i|^{2z_i}}{(z_i!)^2} \right] H_4(j\omega_1, j\omega_{-1}, j\omega_{k_2}, j\omega_{-k_2})$$

$$+ \sum_{k_2=1}^{2} \frac{4!}{2^4} \left[ \prod_{i=1}^{3} \frac{|A_i|^{2z_i}}{(z_i!)^2} \right] H_4(j\omega_2, j\omega_{-2}, j\omega_{k_2}, j\omega_{-k_2})$$

$$+ \sum_{k_2=1}^{3} \frac{4!}{2^4} \left[ \prod_{i=1}^{3} \frac{|A_i|^{2z_i}}{(z_i!)^2} \right] H_4(j\omega_3, j\omega_{-3}, j\omega_{k_2}, j\omega_{-k_2})$$

$$= \frac{4!}{2^4} \frac{|A_1|^4}{(2!)^2} \frac{|A_2|^0}{(0!)^2} \frac{|A_3|^0}{(0!)^2} H_4(j\omega_1, j\omega_{-1}, j\omega_1, j\omega_{-1})$$

$$+ \frac{4!}{2^4} \frac{|A_1|^2}{(1!)^2} \frac{|A_2|^2}{(1!)^2} \frac{|A_3|^0}{(0!)^2} H_4(j\omega_2, j\omega_{-2}, j\omega_1, j\omega_{-1})$$

$$+ \frac{4!}{2^4} \frac{|A_1|^0}{(0!)^2} \frac{|A_2|^4}{(2!)^2} \frac{|A_3|^0}{(0!)^2} H_4(j\omega_2, j\omega_{-2}, j\omega_2, j\omega_{-2})$$

$$+ \frac{4!}{2^4} \frac{|A_1|^2}{(1!)^2} \frac{|A_2|^0}{(0!)^2} \frac{|A_3|^2}{(1!)^2} H_4(j\omega_3, j\omega_{-3}, j\omega_1, j\omega_{-1})$$

$$+ \frac{4!}{2^4} \frac{|A_1|^0}{(0!)^2} \frac{|A_2|^2}{(1!)^2} \frac{|A_3|^2}{(1!)^2} H_4(j\omega_3, j\omega_{-3}, j\omega_2, j\omega_{-2})$$

$$+ \frac{4!}{2^4} \frac{|A_1|^0}{(0!)^2} \frac{|A_2|^0}{(0!)^2} \frac{|A_3|^4}{(2!)^4} H_4(j\omega_3, j\omega_{-3}, j\omega_3, j\omega_{-3})$$

This gives the same answer as in Appendix 5, as it should.

7. For (3.14), let $K=3$ and $\omega_o = 4\omega_1 - 2\omega_3$, as in Appendix 4. So $n=6$ and
$$\underset{\sim}{M} = (m_{-3}, m_{-2}, m_{-1}, m_1, m_2, m_3) = (2,0,0,4,0,0)$$

As in Appendix 5, let $r=2$. Thus output with frequency $\omega_o$ at order 10 is:

$$\sum_{z_1=0}^{2} \sum_{z_2=0}^{2-z_1} \frac{(6+4)!}{2^{6+4}} \left[ \prod_{\substack{i=-3 \\ i \neq 0}}^{3} \frac{A_i^{m_i+z_i}}{(m_i+z_i)!} \right] \cdot$$

$$H_{10}\Big((0+z_1)\{j\omega_{-1}\}, (4+z_1)\{j\omega_1\}, (0+z_2)\{j\omega_{-2}\}, (0+z_2)\{j\omega_2\}, (2+z_3)\{j\omega_{-3}\}, (0+z_3)\{j\omega_3\}\Big) e^{j\omega_o t}$$

where $z_3 = 2 - z_1 - z_2$

$$= \sum_{z_2=0}^{2-0} \frac{10!}{2^{10}} \frac{A_{-3}^{(2+z_3)}}{(2+z_3)!} \frac{A_{-2}^{(0+z_2)}}{(0+z_2)!} \frac{A_{-1}^{(0+0)}}{(0+0)!} \frac{A_1^{(4+0)}}{(4+0)!} \frac{A_2^{(0+z_2)}}{(0+z_2)!} \frac{A_3^{(0+z_3)}}{(0+z_3)!} \cdot$$

$$H_{10}\Big((0+0)\{j\omega_{-1}\}, (4+0)\{j\omega_1\}, (0+z_2)\{j\omega_{-2}\}, (0+z_2)\{j\omega_2\}, (2+z_3)\{j\omega_{-3}\}, (0+z_3)\{j\omega_3\}\Big) e^{j\omega_o t}$$

$$+ \sum_{z_2=0}^{2-1} \frac{10!}{2^{10}} \frac{A_{-3}^{(2+z_3)}}{(2+z_3)!} \frac{A_{-2}^{(0+z_2)}}{(0+z_2)!} \frac{A_{-1}^{(0+1)}}{(0+1)!} \frac{A_1^{(4+1)}}{(4+1)!} \frac{A_2^{(0+z_2)}}{(0+z_2)!} \frac{A_3^{(0+z_3)}}{(0+z_3)!} \cdot$$

$$H_{10}\Big((0+1)\{j\omega_{-1}\}, (4+1)\{j\omega_1\}, (0+z_2)\{j\omega_{-2}\}, (0+z_2)\{j\omega_2\}, (2+z_3)\{j\omega_{-3}\}, (0+z_3)\{j\omega_3\}\Big) e^{j\omega_o t}$$

$$+ \sum_{z_2=0}^{2-2} \frac{10!}{2^{10}} \frac{A_{-3}^{(2+z_3)}}{(2+z_3)!} \frac{A_{-2}^{(0+z_2)}}{(0+z_2)!} \frac{A_{-1}^{(0+2)}}{(0+2)!} \frac{A_1^{(4+2)}}{(4+2)!} \frac{A_2^{(0+z_2)}}{(0+z_2)!} \frac{A_3^{(0+z_3)}}{(0+z_3)!} \cdot$$

$$H_{10}\Big((0+2)\{j\omega_{-1}\}, (4+2)\{j\omega_1\}, (0+z_2)\{j\omega_{-2}\}, (0+z_2)\{j\omega_2\}, (2+z_3)\{j\omega_{-3}\}, (0+z_3)\{j\omega_3\}\Big) e^{j\omega_o t}$$

$$= \frac{10!}{2^{10}} \frac{A_{-3}^4}{4!} \frac{A_1^4}{4!} \frac{A_3^2}{2!} H_{10}\Big(4\{j\omega_1\}, 4\{j\omega_{-3}\}, 2\{j\omega_3\}\Big) e^{j(4\omega_1-2\omega_3)t}$$

$$+ \frac{10!}{2^{10}} \frac{A_{-3}^3}{3!} \frac{A_{-2}}{1!} \frac{A_1^4}{4!} \frac{A_2}{1!} \frac{A_3}{1!} H_{10}\Big(4\{j\omega_1\}, j\omega_2, j\omega_{-2}, j\omega_3, 3\{j\omega_{-3}\}\Big) e^{j(4\omega_1-2\omega_3)t}$$

$$+ \frac{10!}{2^{10}} \frac{A_{-3}^2}{2!} \frac{A_{-2}^2}{2!} \frac{A_1^4}{4!} \frac{A_2^2}{2!} H_{10}\left(4\{j\omega_1\},2\{j\omega_2\},2\{j\omega_{-2}\},2\{j\omega_{-3}\}\right)e^{j(4\omega_1-2\omega_3)t}$$

$$+ \frac{10!}{2^{10}} \frac{A_{-3}^3}{3!} \frac{A_{-1}}{1!} \frac{A_1^5}{5!} \frac{A_3}{1!} H_{10}\left(j\omega_{-1},5\{j\omega_1\},j\omega_3,3\{j\omega_{-3}\}\right)e^{j(4\omega_1-2\omega_3)t}$$

$$+ \frac{10!}{2^{10}} \frac{A_{-3}^2}{2!} \frac{A_{-2}}{1!} \frac{A_{-1}}{1!} \frac{A_1^5}{5!} \frac{A_2}{1!} H_{10}\left(j\omega_{-1},5\{j\omega_1\},j\omega_{-2},j\omega_2,2\{j\omega_{-3}\}\right)e^{j(4\omega_1-2\omega_3)t}$$

$$+ \frac{10!}{2^{10}} \frac{A_{-3}^2}{2!} \frac{A_{-1}^2}{2!} \frac{A_1^6}{6!} H_{10}\left(2\{j\omega_{-1}\},6\{j\omega_1\},2\{j\omega_{-3}\}\right)e^{j(4\omega_1-2\omega_3)t} .$$

8.  For (3.15), let K=3, and $\omega_o = 4\omega_1-2\omega_3$ (same as in Appendix 7).
Thus n=6 and $\underset{\sim}{M} = (m_{-3},m_{-2},m_{-1},m_1,m_2,m_3) = (2,0,0,4,0,0)$. For r=2, the output
with frequency $\omega_o$ at order n+2r = 10 is:

$$\sum_{k_1=1}^{3} \sum_{k_2=1}^{k_1} \frac{10!}{2^{10}} \prod_{\substack{i=-3 \\ i\neq 0}}^{3} \frac{A_i^{m_i+z_i}}{(m_i+z_i)!} \quad .$$

$$H_{10}\left(2\{j\omega_{-3}\},0\{j\omega_{-2}\},0\{j\omega_{-1}\},4\{j\omega_1\},0\{j\omega_2\},0\{j\omega_3\},j\omega_{k_1},j\omega_{-k_1},j\omega_{k_2},j\omega_{-k_2}\right)e^{j\omega_o t}$$

where $z_i$ = number of $j\omega_{k_1},j\omega_{k_2}$ equal to $j\omega_i$, $\forall i = 1,2,3$.

$$= \sum_{k_2=1}^{1} \frac{10!}{2^{10}} \frac{A_{-3}^{2+z_3}}{(2+z_3)!} \frac{A_{-2}^{0+z_2}}{(0+z_2)!} \frac{A_{-1}^{0+z_1}}{(0+z_1)!} \frac{A_1^{4+z_1}}{(4+z_1)!} \frac{A_2^{0+z_2}}{(0+z_2)!} \frac{A_3^{0+z_3}}{(0+z_3)!} .$$

$$H_{10}\left(2\{j\omega_{-3}\},4\{j\omega_1\},j\omega_1,j\omega_{-1},j\omega_{k_2},j\omega_{-k_2}\right)e^{j\omega_o t}$$

$$+ \sum_{k_2=1}^{2} \frac{10!}{2^{10}} \frac{A_{-3}^{2+z_3}}{(2+z_3)!} \frac{A_{-2}^{0+z_2}}{(0+z_2)!} \frac{A_{-1}^{0+z_1}}{(0+z_1)!} \frac{A_1^{4+z_1}}{(4+z_1)!} \frac{A_2^{0+z_2}}{(0+z_2)!} \frac{A_3^{0+z_3}}{(0+z_3)!} .$$

$$H_{10}\left(2\{j\omega_{-3}\},4\{j\omega_1\},j\omega_2,j\omega_{-2},j\omega_{k_2},j\omega_{-k_2}\right)e^{j\omega_o t}$$

$$+ \sum_{k_2=1}^{3} \frac{10!}{2^{10}} \frac{A_{-3}^{2+z_3}}{(2+z_3)!} \frac{A_{-2}^{0+z_2}}{(0+z_2)!} \frac{A_{-1}^{0+z_1}}{(0+z_1)!} \frac{A_{1}^{4+z_1}}{(4+z_1)!} \frac{A_{2}^{0+z_2}}{(0+z_2)!} \frac{A_{3}^{0+z_3}}{(0+z_3)!} \cdot$$

$$H_{10}\left(2\{j\omega_{-3}\},4\{j\omega_1\},j\omega_3,j\omega_{-3},j\omega_{k_2},j\omega_{-k_2}\right)e^{j\omega_o t}$$

$$= \frac{10!}{2^{10}} \frac{A_{-3}^2}{2!} \frac{A_{-1}^2}{2!} \frac{A_{1}^6}{6!} H_{10}\left(2\{j\omega_{-3}\},4\{j\omega_1\},j\omega_1,j\omega_{-1},j\omega_1,j\omega_{-1}\right)e^{j\omega_o t}$$

$$+ \frac{10!}{2^{10}} \frac{A_{-3}^2}{2!} \frac{A_{-2}}{1!} \frac{A_{-1}}{1!} \frac{A_{1}^5}{5!} \frac{A_{2}}{1!} H_{10}\left(2\{j\omega_{-3}\},4\{j\omega_1\},j\omega_2,j\omega_{-2},j\omega_1,j\omega_{-1}\right)e^{j\omega_o t}$$

$$+ \frac{10!}{2^{10}} \frac{A_{-3}^2}{2!} \frac{A_{-2}^2}{2!} \frac{A_{1}^4}{4!} \frac{A_{2}^2}{2!} H_{10}\left(2\{j\omega_{-3}\},4\{j\omega_1\},j\omega_2,j\omega_{-2},j\omega_2,j\omega_{-2}\right)e^{j\omega_o t}$$

$$+ \frac{10!}{2^{10}} \frac{A_{-3}^3}{3!} \frac{A_{-1}}{1!} \frac{A_{1}^5}{5!} \frac{A_{3}}{1!} H_{10}\left(2\{j\omega_{-3}\},4\{j\omega_1\},j\omega_3,j\omega_{-3},j\omega_1,j\omega_{-1}\right)e^{j\omega_o t}$$

$$+ \frac{10!}{2^{10}} \frac{A_{-3}^3}{3!} \frac{A_{-2}}{1!} \frac{A_{1}^4}{4!} \frac{A_{2}}{1!} \frac{A_{3}}{1!} H_{10}\left(2\{j\omega_{-3}\},4\{j\omega_1\},j\omega_3,j\omega_{-3},j\omega_2,j\omega_{-2}\right)e^{j\omega_o t}$$

$$+ \frac{10!}{2^{10}} \frac{A_{-3}^4}{4!} \frac{A_{1}^4}{4!} \frac{A_{3}^2}{2!} H_{10}\left(2\{j\omega_{-3}\},4\{j\omega_1\},j\omega_3,j\omega_{-3},j\omega_3,j\omega_{-3}\right)e^{j\omega_o t}$$
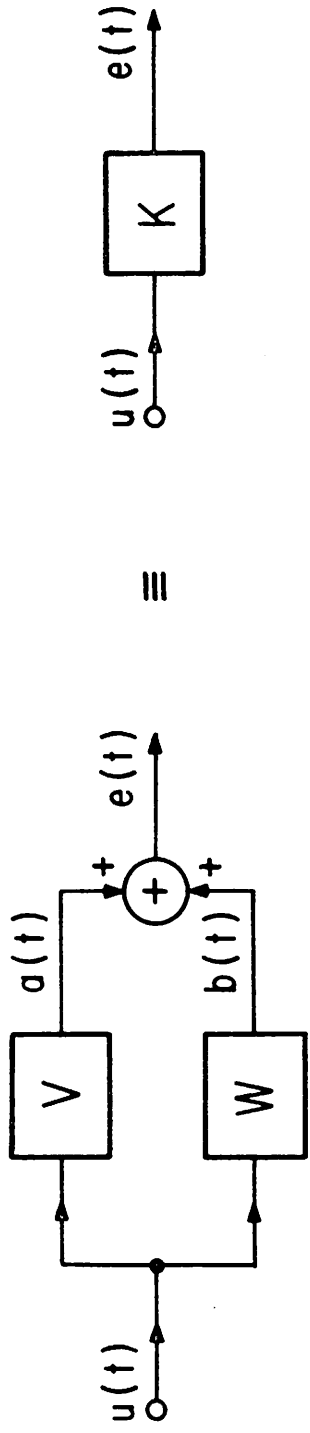
This answer is the same as that of Appendix 7, as it should.

**Table 1. Steady-State Response of a 2-tone Input**

| Order of Output | Item | Module Vector $\underline{M}$ | | | | Output Frequency $\omega_M$ | Complex Amplitude of Output |
|---|---|---|---|---|---|---|---|
| | | $m_{-2}$ | $m_{-1}$ | $m_1$ | $m_2$ | | |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | $H_0$ |
| 1 | 1 | 0 | 0 | 0 | 1 | $\omega_2$ | $\frac{1}{2} A_2 H_1(j\omega_2)$ |
| | 2 | 0 | 0 | 1 | 0 | $\omega_1$ | $\frac{1}{2} A_1 H_1(j\omega_1)$ |
| | 3 | 0 | 1 | 0 | 0 | $-\omega_1$ | $\frac{1}{2} A_1^* H_1(-j\omega_1)$ |
| | 4 | 1 | 0 | 0 | 0 | $-\omega_2$ | $\frac{1}{2} A_2^* H_1(-j\omega_2)$ |
| 2 | 1 | 0 | 0 | 0 | 2 | $2\omega_2$ | $\frac{1}{4} A_2^2 H_2(j\omega_2,j\omega_2)$ |
| | 2 | 0 | 0 | 1 | 1 | $\omega_1+\omega_2$ | $\frac{1}{2} A_1 A_2 H_2(j\omega_1,j\omega_2)$ |
| | 3 | 0 | 0 | 2 | 0 | $2\omega_1$ | $\frac{1}{4} A_1^2 H_2(j\omega_1,j\omega_1)$ |
| | 4 | 0 | 1 | 0 | 1 | $\omega_2-\omega_1$ | $\frac{1}{2} A_1^* A_2 H_2(-j\omega_1,j\omega_2)$ |
| | 5 | 0 | 1 | 1 | 0 | 0 | $\frac{1}{2}|A_1|^2 H_2(-j\omega_1,j\omega_1)$ |
| | 6 | 0 | 2 | 0 | 0 | $-2\omega_1$ | $\frac{1}{4} A_1^{*2} H_2(-j\omega_1,-j\omega_1)$ |
| | 7 | 1 | 0 | 0 | 1 | 0 | $\frac{1}{2}|A_2|^2 H_2(-j\omega_2,-j\omega_2)$ |
| | 8 | 1 | 0 | 1 | 0 | $\omega_1-\omega_2$ | $\frac{1}{2} A_1 A_2^* H_2(j\omega_1,-j\omega_2)$ |
| | 9 | 1 | 1 | 0 | 0 | $-\omega_1-\omega_2$ | $\frac{1}{2} A_1^* A_2^* H_2(-j\omega_1,-j\omega_2)$ |
| | 10 | 2 | 0 | 0 | 0 | $-2\omega_2$ | $\frac{1}{4} A_2^{*2} H_2(-j\omega_2,-j\omega_2)$ |
| 3 | 1 | 0 | 0 | 0 | 3 | $3\omega_2$ | $\frac{1}{8} A_2^3 H_3(j\omega_2,j\omega_2,j\omega_2)$ |
| | 2 | 0 | 0 | 1 | 2 | $\omega_1+2\omega_2$ | $\frac{3}{8} A_1 A_2^2 H_3(j\omega_1,j\omega_2,j\omega_2)$ |
| | 3 | 0 | 0 | 2 | 1 | $2\omega_1+\omega_2$ | $\frac{3}{8} A_1^2 A_2 H_3(j\omega_1,j\omega_1,j\omega_2)$ |
| | 4 | 0 | 0 | 3 | 0 | $3\omega_1$ | $\frac{1}{8} A_1^3 H_3(j\omega_1,j\omega_1,j\omega_1)$ |
| | 5 | 0 | 1 | 0 | 2 | $-\omega_1+2\omega_2$ | $\frac{3}{8} A_1^* A_2^2 H_3(-j\omega_1,j\omega_2,j\omega_2)$ |
| | 6 | 0 | 1 | 1 | 1 | $\omega_2$ | $\frac{3}{4}|A_1|^2 A_2 H_3(-j\omega_1,j\omega_1,j\omega_2)$ |
| | 7 | 0 | 1 | 2 | 0 | $\omega_1$ | $\frac{3}{8}|A_1|^2 A_1 H_3(-j\omega_1,j\omega_1,j\omega_1)$ |
| | 8 | 0 | 2 | 0 | 1 | $-2\omega_1+\omega_2$ | $\frac{3}{8} A_1^{*2} A_2 H_3(-j\omega_1,-j\omega_1,j\omega_2)$ |
| | 9 | 0 | 2 | 1 | 0 | $-\omega_1$ | $\frac{3}{8} A_1^*|A_1|^2 H_3(-j\omega_1,-j\omega_1,j\omega_1)$ |
| | 10 | 0 | 3 | 0 | 0 | $-3\omega_1$ | $\frac{1}{8} A_1^{*3} H_3(-j\omega_1,-j\omega_1,-j\omega_1)$ |
| | 11 | 1 | 0 | 0 | 2 | $\omega_2$ | $\frac{3}{8} A_2|A_2|^2 H_3(-j\omega_2,j\omega_2,j\omega_2)$ |
| | 12 | 1 | 0 | 1 | 1 | $\omega_1$ | $\frac{3}{4} A_1|A_2|^2 H_3(j\omega_1,-j\omega_2,j\omega_2)$ |
| | 13 | 1 | 0 | 2 | 0 | $2\omega_1-\omega_2$ | $\frac{3}{8} A_1^2 A_2^* H_3(j\omega_1,j\omega_1,-j\omega_2)$ |
| | 14 | 1 | 1 | 0 | 1 | $-\omega_1$ | $\frac{3}{4} A_1^*|A_2|^2 H_3(-j\omega_1,-j\omega_2,j\omega_2)$ |
| | 15 | 1 | 1 | 1 | 0 | $-\omega_2$ | $\frac{3}{4}|A_1|^2 A_2^* H_3(-j\omega_1,j\omega_1,-j\omega_2)$ |
| | 16 | 1 | 2 | 0 | 0 | $-2\omega_1-\omega_2$ | $\frac{3}{8} A_1^{*2} A_2^* H_3(-j\omega_1,-j\omega_1,-j\omega_2)$ |
| | 17 | 2 | 0 | 0 | 1 | $-\omega_2$ | $\frac{3}{8}|A_2|^2 A_2^* H_3(-j\omega_2,-j\omega_2,j\omega_2)$ |
| | 18 | 2 | 0 | 1 | 0 | $\omega_1-2\omega_2$ | $\frac{3}{8} A_1 A_2^{*2} H_3(j\omega_1,-j\omega_2,-j\omega_2)$ |
| | 19 | 2 | 1 | 0 | 0 | $-\omega_1-2\omega_2$ | $\frac{3}{8} A_1^* A_2^{*2} H_3(-j\omega_1,-j\omega_2,-j\omega_2)$ |
| | 20 | 3 | 0 | 0 | 0 | $-3\omega_2$ | $\frac{1}{8} A_2^{*3} H_3(-j\omega_2,-j\omega_2,-j\omega_2)$ |

FIGURE CAPTIONS

Figure 1.    Equivalent system resulting from the addition operation.

Figure 2.    Equivalent system resulting from the multiplication operation.

Figure 3.    Equivalent system resulting from the differentiation operation.

Figure 4.    An example of a composite system obtained by cascading various

subsystems.
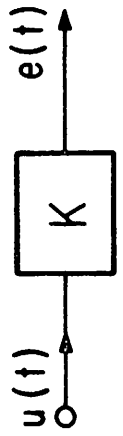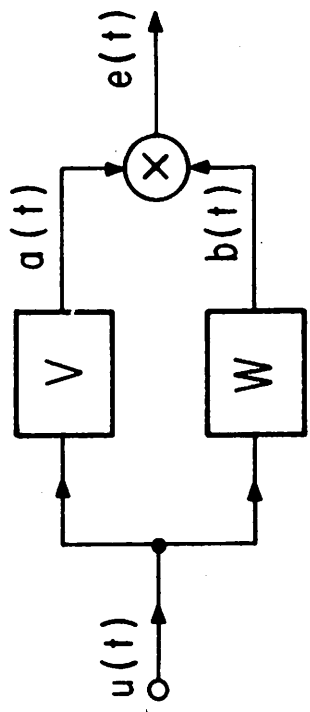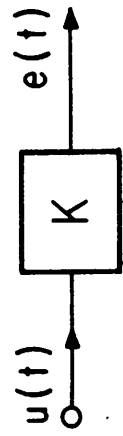
u(t)

e(t)

K

≡

Fig. 1

u(t)

a(t)

V

W

b(t)

+

+

e(t)

u(t)

e(t)

K

≡

Fig. 2

u(t)

a(t)

V

W

b(t)

×

e(t)

## Fig. 3

u(t) → [V] → a(t) → (S) differentiator → e(t)

≡

u(t) → [K] → e(t)

**Fig. 3**

## Fig. 4

u(t) → [F] → (G) differentiator → [H] → (K) differentiator → y(t)

$F_m(s_1, \cdots, s_m) = 1$

$F_r(s_1, \cdots, s_r) = 0, \forall r \neq m$

$H_n(s_1, \cdots, s_n) = 1$

$H_r(s_1, \cdots, s_r) = 0, \forall r \neq n$

**Fig. 4**