AN INTERACTIVE RECTANGLE ELIMINATION METHOD FOR BI-OBJECTIVE DECISION-MAKING

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ABSTRACT

This paper presents a new man-machine interactive method for bi-objective decision-making. It is specifically designed to cope with both the ill-defined nature of the decision problem and the high cost of computing points in the tradeoff (Pareto optimal) set. With this method, the decision-maker may efficiently approximate the tradeoff set and/or estimate his preferred objective value.

First, the notion of a finite representation of the tradeoff set by a set of points, called experiments, and a set of rectangles, defined by the experiments, is introduced. Next, a special class of decision-makers is considered. For a decision-maker in this special class, the finite representation of the tradeoff set defines a rectangle of uncertainty which contains the decision-maker's preferred objective value. A measure of the worst-case uncertainty is formulated and minimized to yield an optimal strategy for interactively selecting experiments. Finally, this strategy is employed in a general interactive algorithm that works under minimal assumptions on the tradeoff set and on the decision-maker.

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1. INTRODUCTION

The presence of multiple objectives frequently complicates the solution of engineering design and decision-making problems [1]-[4]. The design or decision objectives are usually incommensurable and cannot be combined into a single objective. Furthermore, the objectives usually conflict with each other and, consequently, there is no one design or decision which is best with respect to all the objectives. Therefore, a compromise design or decision must be obtained.

The simplest situation occurs when only two objectives are present. In this case, the decision-maker has a <u>bi-objective function</u> $f = (f_1, f_2)$, where $f : \mathcal{X} \to \mathbb{R}^2$ and $\mathcal{X}$ is a topological space (commonly, $\mathbb{R}^n$), and he has a <u>set of feasible alternatives</u> $X \subset \mathcal{X}$. Ideally, he would like to find a point $\hat{x} \in X$ such that $f_1(\hat{x}) = \max\{f_1(x) | x \in X\}$ and $f_2(\hat{x}) = \max\{f_2(x) | x \in X\}$. Unfortunately, there is, in general, no such point $\hat{x} \in X$. Nevertheless, the decision-maker must evaluate alternatives in X in terms of their values in the set of <u>feasible objective values</u>

$$Y \triangleq \{y \in \mathbb{R}^2 | \; y = f(x), \; x \in X\} \tag{1.1}$$

and select a single $x^* \in X$ which, in his judgment, represents a reasonable compromise between the conflicting objectives. In making the tradeoffs necessary to reach a compromise, the decision-maker does not need to consider all points in Y but only the subset of nondominated points. The set of <u>nondominated</u> (Pareto optimal, efficient) <u>objective values</u> is

$$\Gamma(Y) \triangleq \{y \in Y | y' \in Y \text{ and } y_1 \leq y_1', \; y_2 \leq y_2' \Rightarrow y = y'\} \tag{1.2}$$

and the corresponding set of nondominated feasible alternatives is

$$\Omega \triangleq \{x \in X | f(x) \in \Gamma(Y)\}. \tag{1.3}$$

The set $\Gamma(Y)$ is often thought of as a "tradeoff curve" between f_1 and f_2, and we shall refer to it as the tradeoff set associated with Y. Since any point in Y that is not in $\Gamma(Y)$ is dominated by some point in $\Gamma(Y)$, the decision-maker needs only to consider $\Gamma(Y)$ when evaluating possible solutions to his problem.

Thus, the original and impossible problem of finding an $\hat{x}$ which simultaneously maximizes the two objective functions is replaced by the subjective tradeoff problem: Given X and $f = (f_1, f_2)$, find the decision-maker's preferred objective value $y^* \in \Gamma(Y)$ and a corresponding alternative $x^* \in \Omega$ with $y^* = f(x^*)$. This problem is mathematically ill-defined in the sense that there is no systematic way of recognizing when a proposed solution is preferred by the decision-maker [5]. The reason for this is that the decision-maker usually finds it difficult to express what he means by a preferred solution until he has seen at least some of $\Gamma(Y)$. Also, his concept of a preferred solution evolves as he explores $\Gamma(Y)$.

For this reason, a customary approach [6] for solving this tradeoff problem calls for first constructing $\Gamma(Y)$ and then presenting $\Gamma(Y)$ to the decision-maker for him to select his preferred objective value y^*. Various characterizations of $\Gamma(Y)$ that are useful for computing points in $\Gamma(Y)$ have been reported [6]-[13]. Usually, $\Gamma(Y)$ consists of infinitely many points and cannot be found explicitly, so the decision-maker must estimate y^* from an approximation to $\Gamma(Y)$ by a finite number of points [11], [12], [14]. An alternate approach to the tradeoff problem, now receiving much attention in the multiple objective optimization literature, does not construct an approximation to $\Gamma(Y)$ but permits the decision-maker to interactively direct the computation to search for y^*. (See, for example, [15]-[20].)

A common difficulty with these two approaches stems from the high computational cost: quite often both the computation of a single point in $\Gamma(Y)$ and each iteration of an interactive search for y^* require the solution of a difficult constrained optimization problem, such as arises in optimal control problems. Therefore, when approximating $\Gamma(Y)$ or when interactively exploring for y^*, we cannot hope to compute a very large number of points in Y.

Thus, we see that solution procedures for the tradeoff problem should meet two requirements. First, because of the ill-defined nature of the problem, they require interaction of some simple form from the decision-maker. Second, because of computational cost, solution procedures should be efficient — they should estimate y^* or approximate $\Gamma(Y)$ accurately from a limited number of points in Y or $\Gamma(Y)$.

In this paper, we introduce the rectangle elimination method[†] for bi-objective optimization. The rectangle elimination method is an efficient and versatile tool both for approximating the tradeoff set $\Gamma(Y)$ and for estimating the decision-maker's preferred point y^*. As a tool for approximating $\Gamma(Y)$, the rectangle elimination method permits the decision-maker to refine interactively selected portions of his picture of $\Gamma(Y)$. As a tool for estimating y^*, the decision-maker uses the method interactively to eliminate sequentially regions of $\Gamma(Y)$ which

[†]This method is of a kindred spirit with interval elimination methods [21], [22] for root finding and maximization of real valued functions of a single variable.

do not contain y*. The decision-maker obtains an estimate of y* in the form of a rectangle which contains y*.

A major advantage of the rectangle elimination method over existing methods is its elementary assumptions about the decision-maker. The rectangle elimination method does not assume the existence of an underlying utility function, such as in [16], [17], [20]. Furthermore, it does not require the decision-maker to respond to or to give quantitative information such as weights or marginal rates of substitution [2] between f_1 and f_2 as, for example, in [14]-[20]. The man-machine interaction required by the method is of very simple nature and is especially suited to graphics display terminals.

In Section 2, we introduce the concept of a finite representation of a tradeoff set by a finite set of points and a finite set of rectangles. We show in Section 3 how this motivates the rectangle elimination method for estimating y*. Next, we propose a model for a special class of decision-makers. This leads in Section 4 to a class of search strategies for estimating y*. In Section 5, we derive an optimal search strategy. This search strategy represents a particular rectangle elimination method which is optimal in a worst-case sense for our special class of decision-makers and a class of tradeoff sets. Finally, in Section 6, we state an algorithm for a very general rectangle elimination method which places minimal assumptions on the decision-maker and the tradeoff set $\Gamma(Y)$.

2. FINITE REPRESENTATIONS OF BI-OBJECTIVE TRADEOFF SETS

To guarantee that solutions exist to various optimization problems that we shall pose in this section and in Section 6, we make the following assumption.

Assumption 2.1: The constraint set X is nonempty and compact, and the objective functions f_1 and f_2 are continuous. ¤

Also, we distinguish three relations ($\leqq$, $\leq$, and $<$) defined for points in $\mathbb{R}^2$, and, for convenience, we introduce notation for representing rectangles in $\mathbb{R}^2$.

Definition 2.1: Let $y, y' \in \mathbb{R}^2$. Then $y \leqq y'$ whenever $y_1 \leq y_1'$ and $y_2 \leq y_2'$. We say that y' dominates y, denoted by $y \leq y'$, if $y \leqq y'$ but $y \neq y'$. We say that y' strictly dominates y, denoted by $y < y'$, whenever $y_1 < y_1'$ and $y_2 < y_2'$. ¤

Notation: If $\underline{z}, \bar{z} \in \mathbb{R}^2$ with $\bar{z}_1 \leq \underline{z}_1$ and $\bar{z}_2 \geq \underline{z}_2$, then the notation $[\bar{z} \backslash \underline{z}]$ denotes the rectangle in $\mathbb{R}^2$, shown in Figure 2.1, with sides parallel to the f_1-f_2 coordinate axes and corners $\bar{z}, \underline{z}$. That is,

$$[\bar{z} \backslash \underline{z}] \triangleq \{y \in \mathbb{R}^2 \mid \bar{z}_1 \leq y_1 \leq \underline{z}_1,\ \bar{z}_2 \geq y_2 \geq \underline{z}_2\}. \quad ¤ \tag{2.1}$$

The rectangle elimination method exploits two simple observations about the structure of tradeoff sets. The first observation is as follows: for given f_1, f_2, and X, the corresponding tradeoff set $\Gamma(Y)$ defined by (1.1) and (1.2) can be enclosed in a rectangle defined by

Proposition 2.1: If $\underline{y} \in \text{Argmax}\{y_1 | y \in Y\}$ and $\bar{y} \in \text{Argmax}\{y_2 | y \in Y\}$, then $\Gamma(Y) \subset [\bar{y} \backslash \underline{y}]$.

Proof: By Assumption 2.1, $\underline{y}$ and $\bar{y}$ exist. Suppose $y \in \Gamma(Y)$. Then $y_1 \leq \underline{y}_1$ and $y_2 \leq \bar{y}_2$. If $y_2 < \underline{y}_2$ or $y_1 < \bar{y}_1$, then $y \leq \underline{y}$ or $y \leq \bar{y}$. But, by the definition of $\Gamma(Y)$ in (1.2), this contradicts the supposition that $y \in \Gamma(Y)$. Thus, we must have $\bar{y}_1 \leq y_1$ and $\underline{y}_2 \leq y_2$, and consequently, $\bar{y}_1 \leq y_1 \leq \underline{y}_1$ and $\underline{y}_2 \leq y_2 \leq \bar{y}_2$. This implies $y \in [\bar{y} \backslash \underline{y}]$ and thus $\Gamma \subset [\bar{y} \backslash \underline{y}]$. ¤

Figure 2.2 shows a typical $\Gamma(Y)$ and the enclosing rectangle $[\bar{y}\backslash \underline{y}]$. In the subsequent discussion, we shall be interested in the class of all tradeoff sets which can be drawn in the rectangle enclosing $\Gamma(Y)$.

Definition 2.2: Let

$$R_0 \triangleq [\bar{y}\backslash \underline{y}] \text{ and } E_0 \triangleq \{\bar{y},\underline{y}\} \tag{2.2}$$

where $\underline{y} \in \text{Argmax}\{y_1|y \in Y\}$ and $\bar{y} \in \text{Argmax}\{y_2|y \in Y\}$. We define the class of all nonempty tradeoff (nondominated) sets in R_0 by

$$\mathcal{G}(R_0) \triangleq \{\Gamma|\Gamma \subset R_0,\ \Gamma \neq \phi;\ y,y' \in \Gamma \text{ and } y \leqq y' \Rightarrow y = y'\}. \ \square \tag{2.3}$$

Note that a $\Gamma \in \mathcal{G}(R_0)$ has the property that no point in Γ dominates any other point in Γ. A tradeoff set $\Gamma \in \mathcal{G}(R_0)$ may be a connected curve between $\underline{y}$ and $\bar{y}$ (Figure 2.3(a)), a set of disconnected arcs separated by gaps (Figure 2.3(b)), or a finite set of points. Clearly, the decision-maker's tradeoff set $\Gamma(Y)$ given by (1.2) is an element of $\mathcal{G}(R_0)$ by Proposition 2.1.

The second observation that the rectangle elimination exploits follows directly from the definition of $\mathcal{G}(R_0)$ and is illustrated in Figure 2.3. Suppose $\Gamma \in \mathcal{G}(R_0)$. If $y \in \Gamma$, as in Figure 2.3(a), then no points of Γ can be in $\{y' \in R_0|y \leq y'\}$, the set of points in R_0 which dominate y, or in $\{y' \in R_0|y' \leq y\}$, the set of points in R_0 which are dominated by y. Thus, Γ is contained in the subset of R_0 obtained from R_0 by eliminating these two sets. The subset of R_0 containing Γ is $[\bar{y}\backslash y] \cup [y\backslash \underline{y}]$, the union of two rectangles.

In the case when Γ is not connected, as in Figure 2.3(b), we can also eliminate two regions of R_0 not containing points in Γ by choosing y so that it neither strictly dominates any point in Γ nor is strictly dominated by any point in Γ. The point y can be in Γ or in a gap, as in Figure 2.3(b). In either case, Γ is contained in the union of two rectangles, $[\bar{y} \setminus y] \cup [y \setminus \underline{y}]$. We now define the set of all $y \in R_0$ for which $\Gamma \subset [\bar{y} \setminus y] \cup [y \setminus \underline{y}]$.

<u>Definition 2.3</u>: Let $\Gamma \in \mathcal{G}(R_0)$. The set $\mathcal{E}(\Gamma,R_0) \subset R_0$ is the set of all points in R_0 which neither strictly dominate nor are strictly dominated by any point in Γ; that is,

$$\mathcal{E}(\Gamma,R_0) \triangleq \{y \in R_0 \mid \{y' \in R_0 \mid y < y' \text{ or } y' < y\} \cap \Gamma = \phi\}. \qquad \square \tag{2.4}$$

As an immediate consequence of (2.3) and (2.4) we have

<u>Proposition 2.2</u>: For all $\Gamma \in \mathcal{G}(R_0)$, $\Gamma \subset \mathcal{E}(\Gamma,R_0)$ and $E_0 \subset \mathcal{E}(\Gamma,R_0)$. $\square$

Figure 2.4 shows the set $\mathcal{E}(\Gamma,R_0)$ for the tradeoff set shown in Figure 2.2. Note that when $\mathcal{E}(\Gamma,R_0) \neq \Gamma$, $\mathcal{E}(\Gamma,R_0)$ is the union of Γ, rectangles, and horizontal and vertical line segments.

Consider now the case when we know a set of K points in $\mathcal{E}(\Gamma,R_0)$. Each of these points allows two rectangular subsets of R_0 not containing points in Γ to be eliminated. Thus, the set of K points and $E_0 = \{\bar{y},\underline{y}\}$ define a subregion of R_0 which encloses Γ. If none of the K points strictly dominates another, then this subregion is the union of K+1 rectangles. Figure 2.5 depicts such a situation.

We now express formally our second observation.

<u>Proposition 2.3</u>: Let $\Gamma \in \mathcal{G}(R_0)$ and let $E \subset \mathcal{E}(\Gamma,R_0)$ be a set of K points. If no point in E strictly dominates any other point in E, that is, if

$$y,y' \in E \text{ and } y \leqq y' \Rightarrow y_1 = y_1' \text{ or } y_2 = y_2', \tag{2.5}$$

then (a) E can be ordered such that $E = \{y^1,y^2,\ldots,y^K\}$ and

$$y_1^1 \leq y_1^2 \leq \cdots \leq y_1^K \quad y_2^1 \geq y_2^2 \geq \cdots \geq y_2^K \tag{2.6}$$

and (b)

$$\Gamma \subset \bigcup_{i=1}^{K+1} R_i \tag{2.7}$$

where $R_1, R_2, \ldots, R_{K+1}$ are rectangles defined by

$$R_i \triangleq [y^{i-1} \backslash y^i] \quad i = 1,2,\ldots,K+1 \tag{2.8}$$

with $y^0 \triangleq \bar{y}$ and $y^{K+1} \triangleq \underline{y}$.

Proof: (a) We can order $E = \{y^1, y^2, \ldots, y^K\}$ so that, for $i = 1,2,\ldots,K-1$, either $y_1^i < y_1^{i+1}$, or $y_1^i = y_1^{i+1}$ and $y_2^i > y_2^{i+1}$. If $y_1^i < y_1^{i+1}$, then $y_2^i \geq y_2^{i+1}$; for otherwise, we have $y^i < y^{i+1}$ which violates property (2.5). Thus (2.5) implies that E satisfies (2.6).

(b) Now let $y \in \Gamma$. Since $E \subset \mathcal{E}(\Gamma, R_0)$, Definition 2.3 implies that for any $y' \in E$ either, $y_1' \leq y_1$ and $y_2' \geq y_2$, or $y_1 \leq y_1'$ and $y_2 \geq y_2'$. Let i be the smallest integer in $\{1,2,\ldots,K+1\}$ such that $y_1 \leq y_1^i$ and $y_2 \geq y_2^i$. Such an i exists because of the ordering (2.6) and the fact that $y \in R_0$ implies $y_1 \leq y_1^{K+1} \triangleq \underline{y}_1$ and $y_2 \geq y_2^{K+1} \triangleq \underline{y}_2$. Then $y_1^{i-1} \leq y_1$ and $y_2^{i-1} \geq y_2$. This gives us $y_1^{i-1} \leq y_1 \leq y_1^i$ and $y_2^{i-1} \geq y_2 \geq y_2^i$, or, equivalently, $y \in R_i$ where R_i is defined by (2.8). Since there is an R_i containing y for each $y \in \Gamma$, (2.7) holds. ¤

This proposition leads us to the next definition.

Definition 2.4: Let $\Gamma \in \mathcal{G}(R_0)$. A finite set $E \subset \mathcal{E}(\Gamma, R_0)$ with property (2.5), namely, no point in E strictly dominates any other point in E, is said to be a set of experiments for Γ. A K-experiment finite representation of Γ is a set $\{y^1, y^2, \ldots, y^K\}$ of K experiments for Γ, satisfying (2.6), together with the corresponding set of K+1 rectangles defined by (2.8). ¤

Note that the intersection of sets of experiments is a set of experiments. However, the union of sets of experiments is not necessarily a set of experiments unless $\Gamma = \mathcal{E}(\Gamma, R_0)$. A comparison of (2.3) and (2.5) gives us

Proposition 2.4: For $\Gamma \in \mathcal{G}(R_0)$, any finite set of points in Γ is a set of experiments for Γ. ¤

Also, we have the following useful result.

Proposition 2.5: Let $\Gamma \in \mathcal{G}(R_0)$ and let $R(E)$ be any rectangle in the finite representation of Γ defined by a set of experiments E. If $E' \subset R(E)$ is a set of experiments for Γ, then $E \cup E'$ is a set of experiments for Γ.

Proof: Let $y \in E$ and $y' \in E' \subset R(E)$. By Definition 2.4, $R(E) = [\bar{z} \setminus \underline{z}]$ and $R(E) \cap (E \cup E_0) = \{\bar{z},\underline{z}\}$ for some $\bar{z},\underline{z} \in E \cup E_0$. Thus, if $y' < y$, then $\bar{z} < y$ or $\underline{z} < y$, which contradicts the fact that E is a set of experiments. Similarly, $y < y'$ implies $y < \bar{z}$ or $y < \underline{z}$ which again contradicts the fact that E is a set of experiments. Therefore, since no point in E strictly dominates or is dominated by any point in E' and since E and E' are set of experiments, it follows from Definition 2.4 that $E \cup E'$ is a set of experiments. ¤

From Proposition 2.3 and Figure 2.5, we infer that a finite representation of a tradeoff set Γ gives an approximate representation of Γ in the following sense: the union of the K+1 rectangles encloses and mimics qualitatively the tradeoff set. If no one R_i, $i \in \{1,2,\ldots,K+1\}$, has a large area relative to R_0, then we can expect that the representation of Γ by $\{y^i\}_{i=1}^{K}$ and $\{R_i\}_{i=1}^{K+1}$ will be good. Furthermore, when $y^i, y^{i-1} \in \Gamma$, the dimensions of R_i give a fuzzy, qualitative characterization of the set $\Gamma \cap R_i$. For example, if $(y_1^i - y_1^{i-1})$ is small, but $(y_2^{i-1} - y_2^i)$ is large, then at the point y^i, little decrease in component y_1 must be traded for a large increase in component y_2. Finally, the finite representation gives a sharper picture of Γ as the number of experiments K is increased.

3. A RECTANGLE ELIMINATION MODEL OF THE DECISION-MAKER

The concept of a finite representation of a tradeoff set, introduced in section 2, suggests a procedure the decision-maker might use interactively to approximate $\Gamma(Y)$ and/or to systematically explore $\Gamma(Y)$ for an estimate of his preferred point y^*. In this section, we first discuss this procedure, called the <u>rectangle elimination method</u>, in its most general form. Then we make some simplifying assumptions and postulate a simple model of a particular class of decision-makers.

3.1 A General Rectangle Elimination Method

We assume that the decision-maker initially possesses no information about $\Gamma(Y)$ other than knowing $\Gamma(Y) \subset R_0$. That is, the decision-maker knows R_0 but has no information with which to distinguish his tradeoff set $\Gamma(Y)$ from any other tradeoff set $\Gamma \in \mathcal{G}(R_0)$. We provide additional information about $\Gamma(Y)$ to the decision-maker by allowing him to observe sets of experiments for $\Gamma(Y)$. By Proposition 2.3, we know that any set of experiments E for $\Gamma(Y)$ gives a finite representation of $\Gamma(Y)$ by defining a set of rectangles whose union contains $\Gamma(Y)$. One of these rectangles, say $R(E)$, contains the decision-maker's preferred point y^*.

Let us now consider a very general interactive decision and search procedure for estimating y^*. The procedure consists of a sequence of stages in which a computer generates a sequence of finite sets of points $E_1, E_2, E_3, \ldots \subset \mathcal{E}(\Gamma(Y), R_0)$ with the property that $\bigcup_{k=1}^{i} E_k$ is a set of experiments for $\Gamma(Y)$ for all $i = 1,2,\ldots$. In the <u>i</u>th stage of the process, the computer displays to the decision-maker the finite representation of $\Gamma(Y)$ defined by $\bigcup_{k=1}^{i} E_k$, the set of all experiments from the first i stages. Since the decision-maker's concept of a preferred solution may still be evolving, he may not, in general, be able to specify $R(\bigcup_{k=1}^{i} E_k)$, <u>the rectangle containing his preferred point</u>. Instead, he may only be able to select several rectangles as possible candidates for containing y*. The decision-maker also may be able to eliminate some rectangles from any further consideration if no point in these rectangles would be an acceptable solution to him.

Thus, the decision-maker's response to $\bigcup_{k=1}^{i} E_k$ consists of specifying the rectangles in the finite representation of $\Gamma(Y)$ defined by $\bigcup_{k=1}^{i} E_k$ in which y* might possibly be and in which he desires more information about $\Gamma(Y)$. The next set of experiments E_{i+1} is then computed to refine the picture of $\Gamma(Y)$ within the rectangles selected by the decision-maker.

At each stage of this process, the decision-maker learns more about $\Gamma(Y)$, and his preferences should become increasingly better defined. The process stops, say after N stages, when the decision-maker eliminates all rectangles except for the rectangle $R(\bigcup_{k=1}^{N} E_k)$ containing his preferred point y* <u>and</u> this rectangle gives a sufficiently good estimate of y*.

3.2 <u>The Decision-Maker DM</u>

A simpler rectangle elimination method than the one described above arises if we restrict the class of decision-makers. Specifically, consider those decision-makers who, at each stage i, can actually specify the rectangle $R(\bigcup_{k=1}^{i} E_k)$ containing their preferred point and thus eliminate all other rectangles in the finite representation defined by $\bigcup_{k=1}^{i} E_k$. We shall let a decision-maker called DM be a generic member of this class of decision-makers.

Formally, we characterize DM by the following model.

Model of Decision-Maker DM: For any $\Gamma \in \mathcal{G}(R_0)$, if DM is presented with a set of experiments E for Γ, then DM selects a rectangle $R(E) \subset R_0$. The correspondence $R(\cdot) : E \rightarrow R(E)$ between sets of experiments for Γ and rectangles in R_0 satisfies the following two assumptions:

Assumption 3.1: For any set of experiments E for Γ, there exist $y, y' \in E \cup E_0$ such that

$$R(E) = [y \backslash y'] \tag{3.1}$$

and

$$(E \cup E_0) \cap R(E) = \{y, y'\}. \tag{3.2}$$

Assumption 3.2: For sets of experiments E and E' for Γ, if $E \subset E'$, then $R(E') \subset R(E)$. ¤

Definition 3.1: A correspondence $R(\cdot) : E \rightarrow R(E)$ between sets of experiments for Γ and rectangles in R_0 satisfying Assumptions 3.1 and 3.2 for all $\Gamma \in \mathcal{G}(R_0)$ is said to be a DM response function. The class of all DM response functions is denoted by $\mathcal{R}$. ¤

Definition 3.2: Let $\Gamma \in \mathcal{G}(R_0)$ and $R \in \mathcal{R}$. Given a set of experiments E for Γ, R(E) said to be the DM rectangle of uncertainty for E, and the area of R(E), denoted by $a(R(E))$, is said to be the DM uncertainty. ¤

3.3 Interpretation and Properties of DM Model

The model of DM has a simple interpretation. We view the rectangle R(E), which DM selects upon observing the set of experiments E for a tradeoff set $\Gamma \in \mathcal{G}(R_0)$, as a rectangle of uncertainty. By choosing R(E), DM specifies that his preferred point y^* is in the subset $\Gamma \cap R(E)$ and thereby eliminates from consideration all of R_0 except for R(E) as a region in which y* can possibly be located. However, he is uncertain of the exact location of y* within R(E) or of the character of Γ within R(E). The set E gives DM only enough information about Γ for him to say that $y^* \in R(E)$, but y* might be any point within R(E).

We view $a(R(E))$ as a measure of the uncertainty in the location of y*. If $a(R(E))$ is small relative to

$$A_0 \triangleq a(R_0) = (\underline{y}_1-\bar{y}_1)(\bar{y}_2-\underline{y}_2), \tag{3.3}$$

the area of the initial rectangle of uncertainty R_0, then y* is in a small subregion of R_0; that is, the uncertainty in the location of y* is small. Thus, $a(R(E))$ indicates how accurately we have estimated y* with the set of experiments E.

Assumption 3.1 merely states that the rectangle of uncertainty selected by DM is defined by two points in $E \cup E_0$ and contains only these two points. In other words, the rectangle R(E) is one of the rectangles in the finite representation defined by E. If DM has observed no experiments (that is, $E = \phi$), then the rectangle of uncertainty is R_0.

Now consider Assumption 3.2. Suppose we perform a set of experiments E, we present E to DM, and DM selects R(E) as his rectangle of uncertainty. If we then present DM with a larger set of experiments E', consisting of E and some additional experiments, then Assumption 3.2 requires DM to select a subset of R(E) as his new rectangle of uncertainty R(E'). Thus, we do not permit DM to change his mind and include in R(E') any region of R_0 which he previously eliminated when he selected R(E). Assumption 3.1 then implies two additional facts: First, R(E') is a proper subset of $R(E)$ when some of the additional experiments lie within $R(E)$. Conversely, if all the additional experiments lie outside of $R(E)$, then $R(E) = R(E')$. These two facts are shown in the next two propositions.

Proposition 3.1: Let $\Gamma \in \mathcal{G}(R_0)$ and $R \in \mathcal{R}$, and let E and E' be sets of experiments for Γ. If $E \subset E'$ and $R(E) \cap (E' \sim E) \neq \phi$, then $R(E')$ is a proper subset of $R(E)$.[†]

Proof: By Assumption 3.2, $R(E') \subset R(E)$. Now we show $R(E') \neq R(E)$. By Assumption 3.1, there exists $y', y'' \in E \cup E_0$ such that $(E \cup E_0) \cap R(E) = \{y', y''\}$. Let $y \in R(E) \cap (E' \sim E)$. Then if $R(E') = R(E)$, we have $(E' \cup E_0) \cap R(E') = \{y', y''\} \cup [R(E) \cap (E' \sim E)] \supset \{y', y'', y\}$. But this contradicts Assumption 3.1, and so $R(E') \neq R(E)$. ⌑

Proposition 3.2: Let $\Gamma \in \mathcal{G}(R_0)$ and $R \in \mathcal{R}$, and suppose E and E' are sets of experiments for Γ. If $E \subset E'$ and $R(E) \cap (E' \sim E) = \phi$, then $R(E') = R(E)$.

Proof: By Assumption 3.2, $R(E') \subset R(E)$. Suppose $R(E) \cap (E' \sim E) = \phi$. Then $(E' \cup E_0) \cap R(E') \subset (E' \cup E_0) \cap R(E) = (E \cup E_0) \cap R(E)$. But then Assumption 3.1 implies that $(E' \cup E_0) \cap R(E') = (E \cup E_0) \cap R(E)$ and, consequently, that $R(E') = R(E)$. ⌑

4. SEARCH STRATEGIES AND AN OPTIMAL SEARCH PROBLEM

4.1 The Rectangle Elimination Method for DM

In this section and Section 5, we consider only those decision-makers who can operate within the limits imposed by the model of DM and, consequently, have a DM response function $R \in \mathcal{R}$. For such

[†]The set $E' \sim E \triangleq \{y \mid y \in E', y \notin E\}$.

decision-makers we can state the first N stages of a rectangle elimination method for any $\Gamma \in \mathcal{G}(R_0)$ in algorithmic form.

Algorithm 4.1

Data: Initial rectangle of uncertainty $R_0 \subset \mathbb{R}^2$.

Step 1: Set i = 1.

Step 2: Compute a set of experiments E_i for Γ such that

$$E_i \subset R_{i-1}. \tag{4.1}$$

Step 3: Solicit the DM response to $\bigcup_{j=1}^{i} E_j$ and set

$$R_i = R(\bigcup_{j=1}^{i} E_j). \tag{4.2}$$

Step 4: If i = N stop; else, set i = i+1 and go to step 2. ¤

Each iteration of Algorithm 4.1 is a stage in the decision process and the index i is the stage counter. Each stage consists of an experiment computation phase (step 2) and a DM interaction phase (step 3). After N stages, the decision-maker obtains an estimate of his preferred point in the form of the rectangle R_N. The set of experiments E_i is required to satisfy (4.1) because, by Proposition 3.2, experiments outside of R_{i-1} do not help in reducing the uncertainty. Also, by Proposition 2.5, (4.1) ensures that $\bigcup_{j=1}^{i} E_j$ is a set of experiments for Γ and, consequently, that the DM response for $\bigcup_{j=1}^{i} E_j$ is defined. By Assumption 3.2 and (4.2)

$$R_i \subset R_{i-1} \quad i = 1,2,\ldots,N. \tag{4.3}$$

If, in addition, one experiment in E_i is distinct from the two experiments defining R_{i-1}, then Proposition 3.1 and (4.1) imply that R_i is a proper subset of R_{i-1}.

4.2 Search Strategies

We note that Algorithm 4.1 maps a given tradeoff set $\Gamma \in \mathcal{G}(R_0)$ and DM response function $R \in \mathcal{R}$ into a sequence of sets of experiments $E_1, E_2, \ldots, E_N$ for Γ.

Definition 4.1: Any map Σ defined on $\mathcal{G} \times \mathcal{R}$ by an algorithm of the form of Algorithm 4.1, mapping (Γ, R) into a sequence of sets of experiments for Γ, $\{E_1, E_2, \ldots, E_N\}$, is said to be an N-stage sequential search strategy. ¤

We further restrict the class of N-stage sequential search strategies under consideration by fixing a priori the number of experiments to be computed in each stage.

Definition 4.2: Let the integers $k_i > 0$, $i = 1,2,\ldots,N$, be given. $\mathcal{S}_N(k_1, k_2, \ldots, k_N)$ is the class of all N-stage sequential search strategies for which the number of experiments in E_i is k_i. ¤

For notational brevity, we shall write $\mathcal{S}_N(k_1, k_2, \ldots, k_N)$ as $\mathcal{S}_N(k)$ when $k_i = k$ for $i = 1,2,\ldots,N$, and as $\mathcal{S}_N(k_i)$ when the k_i's are allowed to be different.

Two special classes of strategies that are of interest are $\mathcal{S}_1(K)$ and $\mathcal{S}_K(1)$. With an $\mathcal{S}_1(K)$ strategy, the decision-maker obtains a finite representation of the tradeoff set, consisting of K experiments and K+1 rectangles, from which to estimate y^*. With an $\mathcal{S}_K(1)$ strategy, the decision-maker interactively explores the tradeoff set, one experiment at a time, until K experiments have been computed.

4.3 Criterion for Search Strategy Selection

We face the problem of estimating a decision-maker's preferred point y^* without knowing explicitly his DM response function R or his tradeoff set Γ. Because of the cost of computing each experiment, we can only allow the decision-maker to observe a limited number of experiments. This obviously constrains the number of stages N and the number of experiments in each stage. Thus, for given N and k_i, we should deploy the experiments so as to estimate y^* as accurately as possible, no matter what R and Γ happen to be.

Our goal, therefore, is to find an $\mathcal{S}_N(k_i)$ strategy which efficiently and accurately estimates the preferred point y^* for any given $\Gamma \in \mathcal{G}(R_0)$ and $R \in \mathcal{R}$. To do this, we need a criterion for comparing the efficiency of different $\Sigma \in \mathcal{S}_N(k_i)$ in estimating y^*. Intuitively, Σ efficiently and accurately estimates y^* if the uncertainty after N stages, $a(R_N)$, is small relative to the initial uncertainty $A_0 = a(R_0)$. Note, however, that $a(R_N)$ depends upon both Γ and R. We cannot evaluate $a(R_N)$ for a given strategy without actually applying that strategy to a particular Γ and R. Also, one strategy may yield a smaller uncertainty $a(R_N)$ than another strategy for some particular Γ and R but a larger $a(R_N)$ for other Γ and R. Since Γ and R are not known explicitly when we begin to search for y^*, $a(R_N)$ is not a suitable criterion to use for comparing and selecting search strategies.

We need a criterion for determining how well a strategy does in estimating y^* which does not depend upon the unknown tradeoff set and DM response function. We obtain such a criterion by considering the <u>worst-case</u> uncertainty possible for all $\Gamma \in \mathcal{G}(R_0)$ and all $R \in \mathcal{R}$.†

†It should be noted that a worst-case analysis of this type is a standard practice in the study of the efficiency of search algorithms (see, for example, [22], [23]).

Definition 4.3: Let $\Gamma \in \mathcal{G}(R_0)$ and suppose the strategy $\Sigma \in \mathcal{S}_N(k_i)$ generates the sequence $\{E_1,E_2,\ldots,E_N\}$ of sets of experiments for Γ. Then the worst-case uncertainty with respect to the DM response, or more simply, the worst-case DM response uncertainty $A(\Sigma,\Gamma)$ is defined by

$$A(\Sigma,\Gamma) \triangleq \sup_{R\in\mathcal{R}} a(R(\bigcup_{i=1}^{N} E_i)). \qquad ¤ \qquad (4.4)$$

Note that a sequence $\{E_1,E_2,\ldots,E_N\}$ of sets of experiments for Γ, generated by an $\mathcal{S}_N(k_i)$ strategy Σ, defines a finite representation of Γ consisting of $1 + \sum_{i=1}^{N} k_i$ rectangles. The rectangle $R(\bigcup_{i=1}^{N} E_i)$ is one of these rectangles and $A(\Sigma,\Gamma)$ is the largest area it can have.

By considering the largest worst-case DM response uncertainty for $\Gamma \in \mathcal{G}(R_0)$, we obtain the worst-case uncertainty, which is the largest uncertainty that can arise for any $\Gamma \in \mathcal{G}(R_0)$ and any $R \in \mathcal{R}$.

Definition 4.4: For a strategy $\Sigma \in \mathcal{S}_N(k_i)$, the worst-case uncertainty is $\sup_{\Gamma\in\mathcal{G}(R_0)} A(\Sigma,\Gamma)$. ¤

The worst-case uncertainty depends only upon the search strategy, not upon the tradeoff set or response function.

We may now compare strategies in $\mathcal{S}_N(k_i)$ using the criterion of worst-case uncertainty and choose a strategy that minimizes worst-case uncertainty. For this purpose, we pose the following

Optimal Search Problem: Given integers $N > 0$ and $k_i > 0$, $i = 1,2,\ldots,N$, find $\hat{\Sigma} \in \mathcal{S}_N(k_i)$ such that

$$\sup_{\Gamma\in\mathcal{G}(R_0)} A(\hat{\Sigma},\Gamma) \leq \sup_{\Gamma\in\mathcal{G}(R_0)} A(\Sigma,\Gamma) \quad \text{for all } \Sigma \in \mathcal{S}_N(k_i). \qquad (4.5)$$

¤

Definition 4.5: A search strategy $\hat{\Sigma} \in \mathcal{S}_N(k_i)$ satisfying (4.5) is said to be an optimal $\mathcal{S}_N(k_i)$ search strategy. The minimum worst-case uncertainty $\hat{A}_N(k_i)$ for $\mathcal{S}_N(k_i)$ strategies is then defined by

$$\hat{A}_N(k_i) \triangleq \inf_{\Sigma \in \mathcal{S}_N(k_i)} \sup_{\Gamma \in \mathcal{G}(R_0)} A(\Sigma,\Gamma) \qquad \square \qquad (4.6)$$

5. OPTIMAL SEARCH STRATEGIES

We now study the solution of the optimal search problem. In order to simplify the derivation and statement of the optimal search strategy, we impose the following assumption in this section.

Assumption 5.1: Each $\Gamma \in \mathcal{G}(R_0)$ is a connected curve between $\bar{y}$ and $\underline{y}$; that is, for each $\Gamma \in \mathcal{G}(R_0)$, there exists a continuous function $\gamma : [0,1] \to R_0$ such that $\gamma(0) = \bar{y}$, $\gamma(1) = \underline{y}$, and $\Gamma = \{\gamma(t) \mid t \in [0,1]\}$. $\square$

This assumption restricts the tradeoff sets under consideration to those having no gaps. We shall drop this assumption in Section 6 where we consider the use of optimal search strategies with tradeoff sets having gaps.

5.1 Partitions of a Rectangle

To derive an optimal $\mathcal{S}_N(k_i)$ search strategy, we first study the properties of a special set of points and of a particular partition in an arbitrary rectangle. Let R' be a rectangle in $\mathbb{R}^2$ defined by

$$R' \triangleq [\bar{z} \setminus \underline{z}] \qquad (5.1)$$

for some $\bar{z},\underline{z} \in \mathbb{R}^2$ and let

$$\Gamma' \triangleq \{y \in R' \mid y = (1-t)\bar{z} + t\underline{z},\ t \in [0,1]\}. \qquad (5.1)$$

The set Γ' is the straight line segment between $\bar{z}$ and $\underline{z}$. The rectangle R' and line segment Γ' are shown in Figure 5.1. R' may be viewed as a typical rectangle in the sequence of N rectangles of uncertainty generated by an $\mathcal{S}_N(k_i)$ strategy.

Any k points $y^1, y^2, \ldots, y^k \in \Gamma'$, with $y_1^1 \le y_1^2 \le \ldots \le y_1^k$, and the points $y^0 \triangleq \bar{z}$ and $y^{k+1} \triangleq \underline{z}$ define k+1 rectangles in R': $[y^i \backslash\, y^{i+1}]$ $i = 0,1,\ldots,k$ (see Figure 5.1). The following lemma gives a lower bound on the area of the largest such rectangle.

Lemma 5.1: Let $[y^i \backslash\, y^{i+1}]$, $i = 0,1,\ldots,k$, be rectangles in R' such that $y^0, y^1, \ldots, y^{k+1} \in \Gamma'$ with $y^0 = \bar{z}$ and $y^{k+1} = \underline{z}$. Then, the area of the rectangle $[y^i \backslash\, y^{i+1}]$ with largest area is bounded from below by $a(R')/(k+1)^2$, that is,

$$\frac{a(R')}{(k+1)^2} \le \max_{0 \le i \le k} \{a([y^i \backslash\, y^{i+1}])\}. \tag{5.3}$$

Proof: Since $y^0, y^1, \ldots, y^{k+1} \in \Gamma'$, we have $y^i = (1-t_i)\bar{z} + t_i\underline{z}$, $i = 0,1,\ldots,k+1$, for some $t_i \in [0,1]$ and with $t_0 = 0$ and $t_{k+1} = 1$. Thus

$$a([y^i \backslash\, y^{i+1}]) = (y_1^{i+1}-y_1^i)(y_2^i-y_2^{i+1}) = (t_{i+1}-t_i)^2(\underline{z}_1-\bar{z}_1)(\bar{z}_2-\underline{z}_2)$$
$$= (t_{i+1}-t_i)^2 a(R'). \tag{5.4}$$

Let $\mu \triangleq \max\{(t_{i+1}-t_i)^2\}$. Then $\mu^{1/2} \ge |t_{i+1}-t_i| \ge t_{i+1}-t_i$, for $i = 0,1,\ldots,k$, and thus $(k+1)\mu^{1/2} \ge \sum_{i=0}^{k} (t_{i+1}-t_i) = 1$. Therefore, $\mu \ge 1/(k+1)^2$. This fact, the definition of μ, and (5.4) imply (5.3).

¤

Now consider a partition of $R' = [\bar{z} \backslash\, \underline{z}]$ defined by k parallel lines which have slope $(\bar{z}_2-\underline{z}_2)/(\underline{z}_1-\bar{z}_1)$ and which cut Γ' into k+1 segments of equal length. Figure 5.2 depicts such a partition for the case when k = 3. For any given k, the k lines intersect Γ' at the points

$$\hat{y}^i = \bar{z} + \frac{i}{k+1}(\underline{z}-\bar{z}) \qquad i = 1,2,\ldots,k. \tag{5.5}$$

The equation for the ith line is then

$$\frac{y_1-\hat{y}_1^i}{\underline{z}_1-\bar{z}_1}+\frac{\hat{y}_2^i-y_2}{\bar{z}_2-\underline{z}_2}=0. \tag{5.6}$$

Substituting (5.5) into (5.6), the desired partition of R' is defined by the family of k equations

$$g(y;\bar{z},\underline{z}) = b_i \qquad i = 1,2,\ldots,k \tag{5.7}$$

where

$$g(y;\bar{z},\underline{z}) \triangleq \frac{y_1-\bar{z}_1}{\underline{z}_1-\bar{z}_1}+\frac{\bar{z}_2-y_2}{\bar{z}_2-\underline{z}_2} \tag{5.8}$$

and

$$b_i \triangleq \frac{2i}{k+1}\,. \tag{5.9}$$

An important property of this partition is given by the next fact.

Lemma 5.2: If $y^i \in R'$ satisfies $g(y^i;\bar{z},\underline{z}) = b_i$, for $i = 1,2,\ldots,k$, and $y^0 = \bar{z}$, $y^{k+1} = \underline{z}$, then

$$(y_1^{i+1}-y_1^i)(y_2^i-y_2^{i+1}) \le \frac{a(R')}{(k+1)^2} \qquad i = 0,1,\ldots,k. \tag{5.10}$$

Proof: Let $g(y^i;\bar{z},\underline{z}) = b_i$ for $i = 1,2,\ldots,k$ and let $y^0 = \bar{z}$ and $y^{k+1} = \underline{z}$.

Observe that

$$(y_1^{i+1}-y_1^i)(y_2^i-y_2^{i+1}) = a(R')\left(\frac{y_1^{i+1}-y_1^i}{\underline{z}_1-\bar{z}_1}\right)\left(\frac{y_2^i-y_2^{i+1}}{\bar{z}_2-\underline{z}_2}\right). \tag{5.11}$$

Applying first the identity $ab = (a+b)^2/4 - (a-b)^2/4$ and then (5.8), we obtain

$$(y_1^{i+1}-y_1^i)(y_2^i-y_2^{i+1}) \le \frac{a(R')}{4} \left[\frac{y_1^{i+1}-y_1^i}{\underline{z}_1-\bar{z}_1} + \frac{y_2^i-y_2^{i+1}}{\bar{z}_2-\underline{z}_2}\right]^2$$

$$= \frac{a(R')}{4} \left[g(y^{i+1};\bar{z},\underline{z}) - g(y^i;\bar{z},\underline{z})\right]^2 . \qquad (5.12)$$

Since $g(y^i;\bar{z},\underline{z}) = b_i$ for $i = 0,1,\ldots,k+1$ and $b_{i+1}-b_i = 2/(k+1)$, (5.12) implies (5.10). ¤

Next suppose R' is actually a rectangle of uncertainty in a typical stage of an $\mathcal{S}_N(k_i)$ strategy. As the next lemma shows, the partition defined by (5.7) can be used to deploy k experiments in R' when Assumption 5.1 holds: the intersection of each of the k lines with $\Gamma \cap R'$ defines one experiment. Also, by Lemma 5.2, the resulting uncertainty is bounded from above by $a(R')/(k+1)^2$.

Lemma 5.3: Let $\Gamma \in \mathcal{G}(R_0)$ and let R' be defined by (5.1) with $\bar{z},\underline{z} \in \Gamma$. Under Assumption 5.1, the following hold:

(a) For each $b \in [0,2]$, there exists a unique $y(b) \in \Gamma \cap R'$ such that $g(y(b);\bar{z},\underline{z}) = b$. Moreover, if $0 \le b < b' \le 2$, then $y_1(b) < y_1(b')$ and $y_2(b) > y_2(b')$.

(b) If, for $i = 0,1,\ldots,k+1$, $y^i \in \Gamma \cap R'$ satisfies $g(y^i;\bar{z},\underline{z}) = b_i$, where b_i is defined by (5.9), then

$$a([y^i \backslash\, y^{i+1}]) \le \frac{a(R')}{(k+1)^2} \qquad i = 0,1,2,\ldots,k. \qquad (5.13)$$

Proof: (a) Let $b \in [0,2]$. From (5.8), we see that $g(\bar{z};\bar{z},\underline{z}) - b \le 0$ and $g(\underline{z};\bar{z},\underline{z}) - b \ge 0$. Since $g(\cdot;\bar{z},\underline{z})$ is continuous and $\bar{z},\underline{z} \in \Gamma$, Assumption 5.1 implies that there exists a $y(b) \in \Gamma \cap R'$ satisfying $g(y(b);\bar{z},\underline{z}) = b$. Suppose $y' \in \Gamma \cap R'$ and $g(y';\bar{z},\underline{z}) = b$. Since $g(y;\bar{z},\underline{z}) = b$ represents a straight line with positive slope, either $y(b) \leqq y'$ or $y' \leqq y(b)$. But $y',y(b) \in \Gamma$ implies $y' = y(b)$, and so, $y(b)$ is unique. Now suppose $0 \leq b < b' \leq 2$. Since $y(b)$ and $y(b') \in \Gamma$, either $y_1(b) < y_1(b')$ and $y_2(b) > y_2(b')$, or $y_1(b') < y_1(b)$ $y_2(b') > y_2(b)$. But

$$\frac{y_1(b')-y_1(b)}{\underline{z}_1-\bar{z}_1} + \frac{y_2(b)-y_2(b')}{\bar{z}_2-\underline{z}_2} = g(y(b');\bar{z},\underline{z}) - g(y(b);\bar{z},\underline{z}) = b'-b > 0. \tag{5.14}$$

Therefore it must be true that $y_1(b) < y_1(b')$ and $y_2(b) > y_2(b')$.

(b) From part (a), there exists a unique $y^i \in \Gamma \cap R'$ satisfying $g(y^i;\bar{z},\underline{z}) = b_i$, for $i = 0,1,\ldots,k+1$. Since $b_i < b_{i+1}$, $y_1^i < y_1^{i+1}$ and $y_2^i > y_2^{i+1}$. Thus, we can define the rectangle $[y^i\backslash\, y^{i+1}]$ which, by Lemma 5.2, has area no greater than $a(R')/(k+1)^2$. ¤

5.2 An Optimal $\mathcal{S}_N(k_i)$ Search Strategy

Motivated by Lemma 5.3, we define an $\mathcal{S}_N(k_i)$ strategy below. Note that in this strategy, the set E_i is computed by partitioning R_{i-1} with k_i parallel lines defined by (5.15) similar to the manner we partitioned R'.

Definition 5.1: Let the integers $N > 0$ and $k_i > 0$, $i = 1,2,\ldots,N$, be given. The strategy $\hat{\Sigma}_N(k_i)$ is defined for all $\Gamma \in \mathcal{G}(R_0)$ and all $R \in \mathcal{R}$ by the following

Algorithm 5.1

Data: R_0, N, and $k_1,k_2,\ldots,k_N$.

Step 1: Set $i = 1$, $\bar{y}^0 = \bar{y}$ and $\underline{y}^0 = \underline{y}$.

Step 2: Compute $E_i = \{y^1,y^2,\ldots,y^{k_i}\} \subset \Gamma \cap R_{i-1}$ satisfying

$$g(y^j;\bar{y}^{i-1},\underline{y}^{i-1}) = \frac{2j}{k_i+1} \qquad j = 1,2,\ldots,k_i. \tag{5.15}$$

Step 3: Solicit the DM response to $\bigcup_{k=1}^{i} E_k$ and set

$$R_i = [\bar{y}^i \setminus \underline{y}^i] \triangleq R\left(\bigcup_{k=1}^{i} E_k\right). \qquad (5.16)$$

Step 4: If $i = N$ stop; else, set $i = i+1$ and go to step 2. ¤

The application of the strategy $\hat{\Sigma}_N(k_i)$ for the case when $N = 2$ and $k_1 = 2$, $k_2 = 3$ is shown in Figure 5.3. It is interesting to note that, besides being a strategy for a rectangle elimination method, $\hat{\Sigma}_N(k_i)$ may also be viewed as a particular strategy for a method of displaced ideal [24]. The points $(\underline{y}_1^i, \bar{y}_2^i)$, $i = 0,1,2,\ldots,N$, in Algorithm 5.3 may be viewed as the decision-maker's ideal points; $\hat{\Sigma}_N(k_i)$ is then a strategy for sequentially determining and displacing the ideal point.

We now show that $\hat{\Sigma}_N(k_i)$ is an optimal $\mathcal{S}_N(k_i)$ strategy.

Theorem 5.1: The minimum worst-case uncertainty for an $\mathcal{S}_N(k_i)$ strategy is

$$\hat{A}_N(k_i) = A_0 \prod_{i=1}^{N} \frac{1}{(k_i+1)^2}. \qquad (5.17)$$

Moreover, the strategy $\hat{\Sigma}_N(k_i)$, given by Algorithm 5.1, is an optimal $\mathcal{S}_N(k_i)$ strategy.

Proof: Define

$$\bar{\Gamma} = \{y \in R_0 \mid y = (1-t)\bar{y} + t\underline{y},\ t \in [0,1]\}. \qquad (5.18)$$

Clearly $\bar{\Gamma} \in \mathcal{G}(R_0)$. Suppose a strategy $\Sigma \in \mathcal{S}_N(k_i)$ is applied to $\bar{\Gamma}$ and some $R \in \mathcal{R}$. In stage i, the set E_i of k_i experiments is computed in $\bar{\Gamma} \cap R_{i-1}$. The DM response R_i is one of the k_i+1 rectangles in the finite representation of $\bar{\Gamma} \cap R_{i-1}$ defined by E_i. The worst-case DM response uncertainty at stage i is $\sup_{R \in \mathcal{R}} a(R_i)$ and, by Lemma 5.1, satisfies

$$\frac{a(R_{i-1})}{(k_i+1)^2} \leq \sup_{R \in \mathcal{R}} a(R_i) \qquad i = 1,2,\ldots,N. \tag{5.19}$$

Applying (5.19) recursively, we have

$$A_0 \prod_{i=1}^{N} \frac{1}{(k_i+1)^2} \leq \sup_{R \in \mathcal{R}} a(R_N) \tag{5.20}$$

and so, by Definition 4.3,

$$A_0 \prod_{i=1}^{N} \frac{1}{(k_i+1)^2} \leq A(\Sigma,\bar{\Gamma}) \leq \sup_{\Gamma \in \mathcal{G}(R_0)} A(\Sigma,\Gamma) \quad \text{for all } \Sigma \in \mathcal{S}_N(k_i). \tag{5.21}$$

Now let us apply the strategy $\hat{\Sigma}_N(k_i)$ to a given $\Gamma \in \mathcal{G}(R_0)$ and $R \in \mathcal{R}$. By (5.15) and Lemma 5.3, the set E_i in Algorithm 5.1 is well defined and the relationship between the uncertainty at stages i and i-1 is

$$a(R_i) \leq \frac{a(R_{i-1})}{(k_i+1)^2} \qquad i = 1,2,\ldots,N. \tag{5.22}$$

Then (5.22) implies

$$a(R_N) \leq A_0 \prod_{i=1}^{N} \frac{1}{(k_i+1)^2} \tag{5.23}$$

and, consequently,

$$A(\hat{\Sigma}_N(k_i),\Gamma) = \sup_{R \in \mathcal{R}} a(R_N) \leq A_0 \prod_{i=1}^{N} \frac{1}{(k_i+1)^2} \quad \text{for all } \Gamma \in \mathcal{G}(R_0). \tag{5.24}$$

From (5.21) and (5.24), we obtain

$$\sup_{\Gamma \in \mathcal{G}(R_0)} A(\hat{\Sigma}_N(k_i),\Gamma) \leq A_0 \prod_{i=1}^{N} \frac{1}{(k_i+1)^2} \leq \inf_{\Sigma \in \mathcal{S}_N(k_i)} \ \sup_{\Gamma \in \mathcal{G}(R_0)} A(\Sigma,\Gamma). \tag{5.25}$$

This proves (5.17) and optimality of $\hat{\Sigma}_N(k_i)$. ¤

5.3 Discussion

Despite its very conservative nature, the worst-case uncertainty criterion leads to a strategy which reduces the uncertainty quite rapidly.

At stage i of $\hat{\Sigma}_N(k_i)$, (5.22) guarantees that the uncertainty is reduced by a factor of $(k_i+1)^2$. Theorem 5.1 says that the actual uncertainty after N stages, $a(R_N)$, will always be reduced from A_0 to, at most, $\hat{A}_N(k_i)$ when $\hat{\Sigma}_N(k_i)$ is used. In practice, $a(R_N)$ may be much less than $\hat{A}_N(k_i)$ since $a(R_N)$ depends upon the given tradeoff set and DM response function.

However, it is still quite possible that $a(R_N)$ is equal or approximately equal to $\hat{A}_N(k_i)$ when using $\hat{\Sigma}_N(k_i)$. Note that (5.21) indicates that for $\bar{\Gamma}$, the straight line segment between $\bar{y}$ and $\underline{y}$, the uncertainty $a(R_N)$ will be exactly $\hat{A}_N(k_i)$. Thus, $\bar{\Gamma}$ is a <u>worst-case tradeoff set</u>. Moreover, as indicated by Lemma 5.1, if, for given Γ, $\Gamma \cap R_{i-1}$ is a straight line between $\bar{y}^{i-1}$ and $\underline{y}^{i-1}$, then $a(R_i) = a(R_{i-1})/(k_i+1)^2$ when $\hat{\Sigma}_N(k_i)$ is used. That is, the reduction in uncertainty is the worst it can be, and Γ is a worst-case tradeoff set for stage i. Intuitively, the closer $\Gamma \cap R_{i-1}$ is to being a straight line segment, the less the reduction in uncertainty will be in stage i. For many practical bi-objective problems, portions of the tradeoff set may be approximately linear. In fact, $\Gamma(Y)$ is piecewise linear curve when f_1 and f_2 are linear and $X \subset \mathbb{R}^n$ is a polyhedron, as with bi-objective linear programming problems [25]. Therefore, conditions which are worst-case, or approximately worse-case, do not arise only from some pathological bi-objective problems but are typically present. This observation justifies the use of the optimal search strategy $\hat{\Sigma}_N(k_i)$.

5.4 <u>Optimal $\mathcal{S}_1(K)$ and $\mathcal{S}_K(1)$ Search Strategies</u>

We immediately obtain optimal $\mathcal{S}_1(K)$ and $\mathcal{S}_K(1)$ strategies as corollaries to Theorem 5.1.

Corollary 5.1: For any integer $K > 0$,

$$\hat{A}_1(K) = \frac{A_0}{(K+1)^2}, \tag{5.26}$$

and an optimal $\mathcal{S}_1(K)$ strategy is $\hat{\Sigma}_1(K)$: compute experiments $y^1, y^2, \ldots, y^K \in \Gamma$ satisfying

$$g(y^i; \bar{y}, \underline{y}) = \frac{2i}{K+1} \qquad i = 1,2,\ldots,K. \quad ¤ \tag{5.27}$$

Corollary 5.2: For any integer $K > 0$,

$$\hat{A}_K(1) = \frac{A_0}{4^K}, \tag{5.28}$$

and an optimal $\mathcal{S}_K(1)$ strategy is $\hat{\Sigma}_K(1)$ given by:

Algorithm 5.2

Data: R_0 and K

Step 1: Set $i = 1$, $\bar{y}^0 = \bar{y}$, $\underline{y}^0 = \underline{y}$

Step 2: Compute $y^i \in \Gamma$ satisfying

$$g(y^i; \bar{y}^{i-1}, \underline{y}^{i-1}) = 1. \tag{5.29}$$

Step 3: Solicit the DM response to $\{y^1, y^2, \ldots, y^i\}$ and set

$$R_i = [\bar{y}^i \setminus \underline{y}^i] \triangleq R(\{y^1, y^2, \ldots, y^i\}). \tag{5.30}$$

Step 4: If $i = K$ stop; else, set $i = i+1$ and go to step 2. ¤

Figure 5.4 shows an example of the application of $\hat{\Sigma}_k(1)$. The strategy $\hat{\Sigma}_K(1)$ places the experiment y^i at the unique point of intersection of Γ with the diagonal line segment between the points $(\bar{y}_1^{i-1}, \underline{y}_2^{i-1})$ and $(\underline{y}_1^{i-1}, \bar{y}_2^{i-1})$. This diagonal is defined by $g(y; \bar{y}^{i-1}, \underline{y}^{i-1}) = 1$ and bisects R_{i-1} into two congruent triangles. Thus, the rectangle elimination method employing strategy $\hat{\Sigma}_K(1)$ may be thought of as a "bisection" method for bi-objective decision-making.

There is another optimal $\mathcal{S}_K(1)$ strategy which can be viewed as a bisection method. This strategy, defined by Algorithm 5.3 below, sequentially bisects the line $\bar{\Gamma}$, given by (5.18), instead of bisecting rectangles of uncertainty.

Algorithm 5.3 (Strategy $\tilde{\Sigma}_K(1)$)

Data: R_0 and K.

Step 1: Set $i = 1$, $\bar{\lambda}_0 = 0$, and $\underline{\lambda}_0 = 2$.

Step 2: Set $\lambda_i = \bar{\lambda}_{i-1} + (\underline{\lambda}_{i-1} - \bar{\lambda}_{i-1})/2$ and compute $y^i \in \Gamma$ satisfying

$$g(y^i;\bar{y},\underline{y}) = \lambda_i. \tag{5.31}$$

Step 3: Solicit the DM response to $\{y^1,y^2,\ldots,y^i\}$ and set $R_i = R(\{y^1,y^2,\ldots,y^i\})$.

Step 4: Set

$$(\bar{\lambda}_i,\underline{\lambda}_i) = (\bar{\lambda}_{i-1},\lambda_i) \quad \text{if} \quad R_i \subset G_i^- \triangleq \{y \in R_0 | g(y;\bar{y},\underline{y}) - \lambda_i \le 0\}$$

$$(\bar{\lambda}_i,\underline{\lambda}_i) = (\lambda_i,\underline{\lambda}_{i-1}) \quad \text{if} \quad R_i \subset G_i^+ \triangleq \{y \in R_0 | 0 \le g(y;\bar{y},\underline{y}) - \lambda_i\}. \tag{5.32}$$

If $i = K$, stop; else, set $i = i+1$, and go to step 2. ¤

Figure 5.5 demonstrates the use of the strategy $\tilde{\Sigma}_K(1)$. The next result proves the optimality of $\tilde{\Sigma}_K(1)$.

Theorem 5.2: The strategy $\tilde{\Sigma}_K(1)$ defined by Algorithm 5.3 is an optimal $\mathcal{S}_K(1)$ search strategy.

Proof: Let $R_i \triangleq [\bar{y}^i \backslash \underline{y}^i]$, with $\bar{y}^i$, $\underline{y}^i \in \{y^1,y^2,\ldots,y^i\} \cup E_0$, $i = 0,1,\ldots,K$, be the sequence of rectangles of uncertainty generated by $\tilde{\Sigma}_K(1)$ for a given $\Gamma \in \mathcal{G}(R_0)$ and $R \in \mathcal{R}$. To prove the optimality of $\tilde{\Sigma}_K(1)$, we need only to show that $a(R_K) \le A_0/4^K$. For this purpose, we now show by induction that

$$g(\bar{y}^K;\bar{y},\underline{y}) = \bar{\lambda}_K \quad g(\underline{y}^K;\bar{y},\underline{y}) = \underline{\lambda}_K, \tag{5.33}$$

$$\underline{\lambda}_K - \bar{\lambda}_K = 1/2^{K-1}. \tag{5.34}$$

Clearly, (5.33) and (5.34) hold for $K = 0$. Suppose they hold for $K = i$. Since $\lambda_{i+1} = \bar{\lambda}_i + (\underline{\lambda}_i - \bar{\lambda}_i)/2$, Lemma 5.3(a) and $g(y^{i+1};\bar{y},\underline{y}) = \lambda_{i+1}$ imply $y^{i+1} \in R_i$. Thus, by Proposition 3.1, R_{i+1} is a proper subset of R_i. Suppose first that $R_{i+1} \subset G^-_{i+1}$. Then $(\bar{y}^{i+1},\underline{y}^{i+1}) = (\bar{y}^i,y^{i+1})$ and, by (5.32), $(\bar{\lambda}_{i+1},\underline{\lambda}_{i+1}) = (\bar{\lambda}_i,\lambda_{i+1})$. Thus, $g(\underline{y}^{i+1};\bar{y},\underline{y}) = \underline{\lambda}_{i+1}$, and, by the induction hypothesis, $g(\bar{y}^{i+1},\bar{y},\underline{y}) = \bar{\lambda}_{i+1}$ and $\underline{\lambda}_{i+1} - \bar{\lambda}_{i+1} = \lambda_{i+1} - \bar{\lambda}_i = (\underline{\lambda}_i - \bar{\lambda}_i)/2 = 1/2^i$. Next, suppose $R_{i+1} \subset G^+_{i+1}$. In this case, $(\bar{y}^{i+1},\underline{y}^{i+1}) = (y^{i+1},\underline{y}^{i+1})$ and $(\bar{\lambda}_{i+1},\underline{\lambda}_{i+1}) = (\lambda_{i+1},\underline{\lambda}_i)$. Thus, $g(\bar{y}^{i+1},\bar{y},\underline{y}) = \bar{\lambda}_{i+1}$, and, by the induction hypothesis, $g(\underline{y}^{i+1};\bar{y},\underline{y}) = \underline{\lambda}_{i+1}$ and $\underline{\lambda}_{i+1} - \bar{\lambda}_{i+1} = \underline{\lambda}_i - \lambda_{i+1} = (\underline{\lambda}_i - \bar{\lambda}_i)/2 = 1/2^i$. Therefore, (5.33) and (5.34) are true by induction. Now note that

$$a(R_K) = (\underline{y}_1^K - \bar{y}_1^K)(\bar{y}_2^K - \underline{y}_2^K) = \frac{A_0}{4}\left(\frac{\underline{y}_1^K - \bar{y}_1^K}{\underline{y}_1 - \bar{y}_1}\right)\left(\frac{\bar{y}_2^K - \underline{y}_2^K}{\bar{y}_2 - \underline{y}_2}\right). \tag{5.35}$$

Applying the identity $ab = (a+b)^2/4 - (a-b)^2/4$ to (5.35), we obtain

$$a(R_K) \le \frac{A_0}{4}\left[\frac{\underline{y}_1^K - \bar{y}_1^K}{\underline{y}_1 - \bar{y}_1} + \frac{\bar{y}_2^K - \underline{y}_2^K}{\bar{y}_2 - \underline{y}_2}\right]^2 = \frac{A_0}{4}\,[g(\underline{y}^K;\bar{y},\underline{y}) - g(\bar{y}^K;\bar{y},\underline{y})]^2. \tag{5.36}$$

Then (5.33) and (5.34) applied to (5.36) yield $a(R_K) \le A_0/4^K$. ¤

It is interesting to note that, unlike $\hat{\Sigma}_K(1)$, $\tilde{\Sigma}_K(1)$ does not guarantee that $a(R_i) \le a(R_{i-1})/4$ for $i \ge 2$.[†] $\tilde{\Sigma}_K(1)$ only ensures that $a(R_i) \le A_0/4^i$.

[†]This can easily be shown by constructing a $\Gamma \in \mathcal{G}(R_0)$ for which $a(R_2) > a(R_1)/4$.

5.5 Selection of an Optimal Strategy

Two factors govern the decision-maker's selection of the integers N and $k_1,k_2,\ldots,k_N$ when he applies the optimal strategy $\hat{\Sigma}_N(k_i)$ to his tradeoff set $\Gamma(Y)$. The first factor is computational cost, which constrains the total number of experiments that can be computed. Thus, the number of stages N and the number of experiments k_i in stage i, $i = 1,2,\ldots,N$, must satisfy the constraint $\sum_{i=1}^{N} k_i \le \bar{K}$, where $\bar{K}$ is the total number of experiments allowed.

An important property of $\hat{\Sigma}_N(k_i)$ is that the calculation of experiments in the first j stages does not require the values of N and $k_{j+1},\ldots,k_N$ to be known; the first j stages of $\hat{\Sigma}_N(k_i)$ comprise the optimal $\mathcal{S}_j(k_i)$ strategy $\hat{\Sigma}_j(k_i)$ for all $j \le N$. Thus, the decision maker does not need to specify N and $k_1,k_2,\ldots,k_N$ <u>before</u> computation begins. Rather, he can select $k_1,k_2,\ldots$ sequentially, choosing k_j just before computing E_j. Of course, for each stage j, he is only free to choose k_j such that $1 \le k_j \le \bar{K} - \sum_{i=1}^{j-1} k_i$. The process terminates at the stage N in which either $\sum_{i=1}^{N} k_i = \bar{K}$ or the decision-maker decides $a(R_N)$ is small enough. Then, no matter what N and $k_1,k_2,\ldots,k_N$ are, the decision-maker will have used an optimal $\mathcal{S}_N(k_i)$ strategy.

If the decision-maker has $\bar{K}$ experiments to expend, then there are various possible values of N and $k_1,k_2,\ldots,k_N$ he might use and still have $\sum_{i=1}^{N} k_i = \bar{K}$. The second factor governing the selection of N and k_i, $i = 1,2,\ldots,N$, is the number of experiments per stage for which the decision-maker can be reasonably modeled as DM; that is, the k_i experiments in R_{i-1} must give the decision-maker enough information about $\Gamma(Y) \cap R_{i-1}$ so that he can make a DM response. Obviously, the larger the decision-maker chooses k_i, the more information he will receive about $\Gamma(Y) \cap R_{i-1}$. However, the price he pays for this information may be a larger final uncertainty. For example, if $\bar{K} = 8$, then three possible schemes for expending the eight experiments are: three stages with $\{k_1,k_2,k_3\} = \{4,3,1\}$ and $\hat{A}_3(k_1,k_2,k_3) = A_0/40$, three stages with $\{k_1,k_2,k_3\} = \{3,3,2\}$ and $\hat{A}_3(k_1,k_2,k_3) = A_0/48$, and four stages with $\{k_1,k_2,k_3,k_4\} = \{2,2,2,2\}$ and $\hat{A}_4(k_1,k_2,k_3,k_4) = A_0/81$. Note that there is a tradeoff between the number of stages and experiments per stage and the final uncertainty.

Roughly speaking, the fewer experiments per stage that the decision-maker requires in order to make a DM response, the smaller will be the final uncertainty. If the decision-maker is able or is forced to make a DM response after each experiment, that is, if he uses the strategy $\hat{\Sigma}_{\bar{K}}(1)$ or $\tilde{\Sigma}_{\bar{K}}(1)$, then the final uncertainty will be the smallest. If the decision-maker cannot make a DM response until he has seen all $\bar{K}$ experiments, then he uses the strategy $\hat{\Sigma}_1(\bar{K})$ and the final uncertainty is largest. Thus, we have upper and lower bounds on $\hat{A}_N(k_i)$.

<u>Proposition 5.1</u>: If $\bar{K} > 0$ is the total number of experiments which can be computed, and if the integers $N > 0$ and k_i, $i = 1,2,\ldots,N$, satisfy $\sum_{i=1}^{N} k_i = \bar{K}$, then

$$\frac{A_0}{4^{\bar{K}}} = \hat{A}_{\bar{K}}(1) \leq \hat{A}_N(k_i) \leq \hat{A}_1(\bar{K}) = \frac{A_0}{(\bar{K}+1)^2} \, . \tag{5.37}$$

<u>Proof</u>: Since $\bar{K}+1 = \sum_{i=1}^{N} k_i+1 \leq \prod_{i=1}^{N} (k_i+1)$, (5.17) and (5.26) imply $\hat{A}_N(k_i) \leq \hat{A}_1(\bar{K}) = A_0/(\bar{K}+1)^2$. Now note that $(k_i+1) \leq 2^{k_i}$ and so $\prod_{i=1}^{N} (k_i+1) \leq 2^{\bar{K}}$. Then (5.17) and (5.28) imply $A_0/4^{\bar{K}} = \hat{A}_{\bar{K}}(1) \leq \hat{A}_N(k_i)$. ◻

6. A GENERALIZATION FOR TRADEOFF SETS WITH GAPS

In Section 5, we derived optimal strategies for rectangle elimination under the assumption that the decision-maker makes DM responses and under the assumption that the tradeoff set is a connected curve between $\bar{y}$ and $\underline{y}$. Now, in this section, we state a general interactive algorithm for the decision-maker who does not necessarily respond like DM. We also allow for the possibility that the tradeoff set $\Gamma(Y)$ may not be a connected curve between $\bar{y}$ and $\underline{y}$. The deployment of experiments is motivated by $\hat{\Sigma}_N(k_i)$, but the possible existence of gaps in $\Gamma(Y)$ forces us to modify our characterization of the experiments prescribed by $\hat{\Sigma}_N(k_i)$.

6.1 Characterization of Experiments

The optimal search strategies derived in Section 5 prescribe selecting each experiment for $\Gamma(Y)$ by solving a problem of the following form.

Problem 6.1: Given a rectangle $R' \subset R_0$ defined by

$$R' \triangleq [\bar{z} \backslash \underline{z}] \qquad \bar{z}, \underline{z} \in R_0, \ \bar{z}_1 < \underline{z}_1, \ \bar{z}_2 > \underline{z}_2 \tag{6.1}$$

and given b, $0 < b < 2$, find a $\hat{y} \in \Gamma(Y) \cap R'$ (if it exists) such that

$$g(\hat{y};\bar{z},\underline{z}) = b \tag{6.2}$$

where $g(\cdot;\bar{z},\underline{z})$ is defined by (5.8). ◻

In particular, for the jth experiment in the ith stage of $\hat{\Sigma}_N(k_i)$, $\underline{z} = \underline{y}^{i-1}$, $\bar{z} = \bar{y}^{i-1}$, and $b = 2j/(k_i+1)$.

Since we do not assume $\Gamma(Y)$ to be connected, there is no guarantee that there exists a point of intersection between $\Gamma(Y) \cap R'$ and the line defined by $g(y;\bar{z},\underline{z}) = b$. Consequently, Problem 6.1 may have no solution and we must modify our statement of the optimal strategies. Since we want to ensure the appropriate reduction in uncertainty, we shall force $\hat{y}$ to satisfy (6.2) but relax the requirement that $\hat{y}$ be in $\Gamma(Y)$. We only require $\hat{y}$ to be in the set $\mathcal{E}(\Gamma(Y),R_0)$ defined by Definition 2.3. We now look for a solution to the next problem.

Problem 6.2: Given $R' \subset \mathbb{R}^2$ defined by (6.1) and $0 < b < 2$, find a $\hat{y} \in \mathcal{E}(\Gamma(Y),R_0) \cap R'$ satisfying (6.2). ¤

For the purpose of solving Problem 6.2, we define the function $v(\cdot,b;\bar{z},\underline{z}) : \mathbb{R}^2 \to \mathbb{R}$ by

$$v(y,b;\bar{z},\underline{z}) \triangleq \max\left\{b - \frac{y_1-\bar{z}_1}{\underline{z}_1-\bar{z}_1} \; , \; \frac{\bar{z}_2-y_2}{\bar{z}_2-\underline{z}_2}\right\}. \tag{6.3}$$

Level sets of $v(\cdot,b;\bar{z},\underline{z})$ are shown in Figure 6.1 along with the line defined by $g(y;\bar{z},\underline{z}) = b$. Note that $g(y;\bar{z},\underline{z}) = b$ if, and only if, the two maximands in (6.3) are equal. This gives us a fact we shall need later.

Lemma 6.1: For any $\hat{y} \in \mathbb{R}^2$ satisfying $g(\hat{y};\bar{z},\underline{z}) = b$ and any $y \in \mathbb{R}^2$,

$$v(y,b;\bar{z},\underline{z}) = v(\hat{y},b;\bar{z},\underline{z}) + \max\left\{\frac{\hat{y}_1-y_1}{\underline{z}_1-\bar{z}_1} \; , \; \frac{\hat{y}_2-y_2}{\bar{z}_2-\underline{z}_2}\right\}. \tag{6.4}$$

Proof: From (6.3)

$$v(y,b;\bar{z},\underline{z}) = \max\left\{b - \frac{\hat{y}_1-\bar{z}_1}{\underline{z}_1-\bar{z}_1} + \frac{\hat{y}_1-y_1}{\underline{z}_1-\bar{z}_1} \; , \; \frac{\bar{z}_2-\hat{y}_2}{\bar{z}_2-\underline{z}_2} + \frac{\hat{y}_2-y_2}{\bar{z}_2-\underline{z}_2}\right\} \tag{6.5}$$

But $g(\hat{y};\underline{z},\bar{z}) = b$ implies

$$v(\hat{y},b;\bar{z},\underline{z}) = b - \frac{\hat{y}_1-\bar{z}_1}{\underline{z}_1-\bar{z}_1} = \frac{\bar{z}_2-\hat{y}_2}{\bar{z}_2-\underline{z}_2} \,. \tag{6.6}$$

Using (6.6) we can write (6.5) as (6.4). ¤

We propose to solve Problem 6.2 by minimizing $v(\cdot,b;\bar{z},\underline{z})$ over Y to obtain the point $\tilde{y}$ shown in Figure 6.1. We then determine the point $\hat{y} \in \mathcal{E}(\Gamma(Y),R_0) \cap R'$ satisfying (6.2) by finding where the level set $\{y \in \mathbb{R}^2 \mid v(y,b;\bar{z},\underline{z}) = v(\tilde{y},b;\bar{z},\underline{z})\}$ intersects the line defined by $g(y;\bar{z},\underline{z}) = b$. Therefore, we pose the following mathematical programming problem.

Problem 6.3: Given $\bar{z},\underline{z} \in R_0$, with $\bar{z}_1 < \underline{z}_1$ and $\bar{z}_2 > \underline{z}_2$, and given b, $0 < b < 2$,

$$\begin{aligned} &\text{minimize } v(y,b;\bar{z},\underline{z}) \\ &\text{subject to } y \in Y. \quad ¤ \end{aligned} \tag{6.7}$$

Let $\hat{Y}(b;\bar{z},\underline{z})$ be the set of solutions of Problem 6.3, that is,

$$\hat{Y}(b;\bar{z},\underline{z}) \triangleq \text{Argmin}\{v(y,b;\bar{z},\underline{z}) \mid y \in Y\} \tag{6.8}$$

and let

$$\hat{v}(b;\bar{z},\underline{z}) \triangleq \min\{v(y,b;\bar{z},\underline{z}) \mid y \in Y\}. \tag{6.9}$$

Assumption 2.1 implies $\hat{Y}(b;\bar{z},\underline{z})$ is nonempty. Proposition 6.1 below gives the relationship between Problems 6.3 and 6.1.

Proposition 6.1: If there exists a $\hat{y} \in \Gamma(Y)$ satisfying $g(\hat{y};\bar{z},\underline{z}) = b$, then $\hat{y}$ is the unique solution of Problem 6.3, that is, $\hat{Y}(b;\bar{z},\underline{z}) = \{\hat{y}\}$.

Proof: Let $\hat{y} \in \Gamma(Y)$ and $g(\hat{y};\bar{z},\underline{z}) = b$. By Lemma 6.1, if $y \in Y$ and $v(y,b;\bar{z},\underline{z}) \leq v(\hat{y},b;\bar{z},\underline{z})$, then $\hat{y} \leqq y$. Since $\hat{y} \in \Gamma(Y)$, we must have $\hat{y} = y$. Therefore, $y \in Y$ and $\hat{y} \neq y$ imply $v(\hat{y},b;\bar{z},\underline{z}) < v(y,b;\bar{z},\underline{z})$. Hence $\hat{Y}(b;\bar{z},\underline{z}) = \{\hat{y}\}$. ¤

Note that when $\Gamma(Y) \cap R'$ is a connected curve between $\bar{z}$ and $\underline{z}$, Proposition 6.1 implies Problem 6.3 has a unique solution and this solution solves Problem 6.1. However, in general, a solution of Problem 6.3

does not solve Problem 6.1. In fact, solutions of Problem 6.3 need not be elements of $\Gamma(Y)$. The solutions are elements of the set

$$\tilde{\Gamma}(Y) \triangleq \{y \in Y \mid y' \in Y \text{ and } y \leqq y' \Rightarrow y_1 = y_1' \text{ or } y_2 = y_2'\}. \tag{6.10}$$

From (6.10) and (1.2), we see that $\Gamma(Y) \subset \tilde{\Gamma}(Y)$. Figure 2.2 illustrates $\tilde{\Gamma}(Y)$ and $\Gamma(Y)$.

Lemma 6.2: If $\tilde{y} \in \tilde{\Gamma}(Y)$, then there exists a $y \in \Gamma(Y)$ such that $\tilde{y} \leqq y$, with $\tilde{y}_1 = y$ or $\tilde{y}_2 = y_2$.

Proof: Let $\tilde{y} \in \tilde{\Gamma}(Y)$. By Assumption 2.1, there exists a $y \in \text{Argmax}\{y_1+y_2 \mid y \in Y,\ \tilde{y} \leqq y\}$. It must be true that $y \in \Gamma(Y)$. Otherwise, there exists a $y' \in Y$ such that $y \leq y'$ and, consequently, such that $y_1+y_2 < y_1'+y_2'$ and $\tilde{y} \leqq y'$. Since $y \in \Gamma(Y) \subset Y$ and $\tilde{y} \in \tilde{\Gamma}(Y)$, (6.10) implies $\tilde{y}_1 = y_1$ or $\tilde{y}_2 = y_2$. ¤

Proposition 6.2: The solutions of Problem 6.3 are in $\tilde{\Gamma}(Y)$ and at least one solution is in $\Gamma(Y)$; that is, $\hat{Y}(b;\bar{z},\underline{z}) \subset \tilde{\Gamma}(Y)$ and $\hat{Y}(b;\bar{z},\underline{z}) \cap \Gamma(Y) \neq \phi$.

Proof: Suppose $\tilde{y} \in \hat{Y}(b;\bar{z},\underline{z})$. If $\tilde{y} \notin \tilde{\Gamma}(Y)$, then there exists a $y' \in Y$ such that $\tilde{y} < y'$. But this implies that $v(y',b;\bar{z},\underline{z}) < v(\tilde{y};b,\bar{z},\underline{z})$ or, equivalently, the contradiction that $\tilde{y} \notin \hat{Y}(b;\bar{z},\underline{z})$. Thus, we must have $\tilde{y} \in \tilde{\Gamma}(Y)$. By Lemma 6.2, there exists a $y \in \Gamma(Y)$ such that $\tilde{y} \leq y$. This implies $v(y,b;\bar{z},\underline{z}) \leq v(\tilde{y},b;\bar{z},\underline{z}) = \hat{v}(b;\bar{z},\underline{z})$. The equality $v(y,b;\bar{z},\underline{z}) = \hat{v}(b;\bar{z},\underline{z})$ must hold since $\tilde{y} \in \hat{Y}(b;\bar{z},\underline{z})$. Thus $y \in \hat{Y}(b;\bar{z},\underline{z})$ and so $\hat{Y}(b;\bar{z},\underline{z}) \cap \Gamma(Y) \neq \phi$. ¤

The set $\tilde{\Gamma}(Y)$ has the following important property:

Proposition 6.3: $\tilde{\Gamma}(Y) \cap R_0 \subset \mathcal{E}(\Gamma(Y),R_0)$ and any finite set of points in $\tilde{\Gamma}(Y) \cap R_0$ is a set of experiments for $\Gamma(Y)$.

$\underline{\text{Proof}}$: Let $y \in \tilde{\Gamma}(Y) \cap R_0$ and suppose $y \notin \mathcal{E}(\Gamma(Y),R_0)$. Then, by Definition 2.3, there exists a $y' \in \Gamma(Y)$ such that either $y < y'$ or $y' < y$. But, by (6.10) and the fact that $\Gamma(Y) \subset \tilde{\Gamma}(Y)$, this gives the contradiction that $y \notin \tilde{\Gamma}(Y)$. Therefore, we have $\tilde{\Gamma}(Y) \cap R_0 \subset \mathcal{E}(\Gamma(Y),R_0)$. The second part of the proposition is verified by comparing (2.5) and (6.10) and applying Definition 2.4. ⌑

We now establish conditions under which a solution to Problem 6.2 can be constructed by solving Problem 6.3. First, we need two preliminary lemmas which establish conditions for bounding $\hat{v}(b;\bar{z},\underline{z})$.

$\underline{\text{Lemma 6.3}}$: If $\bar{z} \leqq \bar{z}'$ and $\underline{z} \leqq \underline{z}'$ for some $\bar{z}',\underline{z}' \in Y$, then $v(b;\bar{z},\underline{z}) \leq \min\{1,b\}$.

$\underline{\text{Proof}}$: By the definition of $v(\cdot,b;\bar{z},\underline{z})$ in (6.3), the fact that $\bar{z} \leqq \bar{z}'$, $\underline{z} \leqq \underline{z}'$, and $0 < b < 2$ implies $v(\bar{z}',b;\bar{z},\underline{z}) \leq v(\bar{z},b;\bar{z},\underline{z}) = \max\{b,0\} = b$ and $v(\underline{z}',b;\bar{z},\underline{z}) \leq v(\underline{z},b;\bar{z},\underline{z}) = \max\{b-1,1\} = 1$. Thus, if $\bar{z}',\underline{z}' \in Y$, $\hat{v}(b;\bar{z},\underline{z}) \leq \min\{b,1\}$. ⌑

$\underline{\text{Lemma 6.4}}$: If $\underline{z},\bar{z} \in \mathcal{E}(\Gamma(Y),R_0)$, then $\max\{0,b-1\} \leq \hat{v}(b;\bar{z},\underline{z})$.

$\underline{\text{Proof}}$: Suppose $\bar{z},\underline{z} \in \mathcal{E}(\Gamma(Y),R_0)$. By Proposition 6.2, there exists a $\hat{y} \in \Gamma(Y)$ such that $v(\hat{y},b;\bar{z},\underline{z}) = \hat{v}(b;\bar{z},\underline{z})$. This fact and (6.3) imply

$$\begin{aligned} \bar{z}_1 + (b-\hat{v}(b;\bar{z},\underline{z}))(\underline{z}_1-\bar{z}_1) &= \underline{z}_1 + (b-1-\hat{v}(b;\bar{z},\underline{z}))(\underline{z}_1-\bar{z}_1) \leq \hat{y}_1 \\ \bar{z}_2 - \hat{v}(b;\bar{z},\underline{z})(\bar{z}_2-\underline{z}_2) &= \underline{z}_2 + (1-\hat{v}(b;\bar{z},\underline{z}))(\bar{z}_2-\underline{z}_2) \leq \hat{y}_2 . \end{aligned} \tag{6.11}$$

If $\hat{v}(b;\bar{z},\underline{z}) < 0$, then $0 < b-\hat{v}(b;\bar{z},\underline{z})$, and (6.11) implies $\bar{z} < \hat{y}$. But, since $\hat{y} \in \Gamma(Y)$, this contradicts the assumption that $\bar{z} \in \mathcal{E}(\Gamma(Y),R_0)$. Thus $0 \leq \hat{v}(b;\bar{z},\underline{z})$. If $\hat{v}(b;\bar{z},\underline{z}) < b-1$, then also $\hat{v}(b;\underline{z},\bar{z}) < 1$ since $0 < b < 2$. Then (6.11) implies $\underline{z} < \hat{y}$, which contradicts the assumption that $\underline{z} \in \mathcal{E}(\Gamma(Y),R_0)$. Therefore, $b-1 \leq \hat{v}(b;\bar{z},\underline{z})$. ⌑

Now, as the next result shows, we can find a solution $\hat{y}$ to Problem 6.2 by first solving Problem 6.3, determining the minimum value $\hat{v}(b;\bar{z},\underline{z})$, and then computing $\hat{y}$ with (6.12) below.

Proposition 6.4: The unique $\hat{y} \in \mathbb{R}^2$ satisfying $g(\hat{y};\bar{z},\underline{z}) = b$ and $\hat{v}(y,b;\bar{z},\underline{z}) = \hat{v}(b;\bar{z},\underline{z})$ is given by

$$\begin{aligned} \hat{y}_1 &= \bar{z}_1 + (b-\hat{v}(b;\bar{z},\underline{z}))(\underline{z}_1-\bar{z}_1) \\ \hat{y}_2 &= \underline{z}_2 + (1-\hat{v}(b;\bar{z},\underline{z}))(\bar{z}_2-\underline{z}_2) \end{aligned} \tag{6.12}$$

and has the following properties: (a) $\hat{y} \leqq \tilde{y}$ with $\hat{y}_1 = \tilde{y}_1$ or $\hat{y}_2 = \tilde{y}_2$ for all $\tilde{y} \in \hat{Y}(b;\bar{z},\underline{z})$, and (b) if $\bar{z},\underline{z} \in \mathcal{E}(\Gamma(Y),R_0)$ and $\bar{z} \leqq \bar{z}'$, $\underline{z} \leqq \underline{z}'$ for some $\bar{z}',\underline{z}' \in Y$, then $\hat{y} \in \mathcal{E}(\Gamma(Y),R_0) \cap [\bar{z} \backslash \underline{z}]$, that is, $\hat{y}$ solves Problem 6.2.

Proof: If $g(\hat{y};\bar{z},\underline{z}) = b$ and $v(\hat{y},b;\bar{z},\underline{z}) = \hat{v}(b;\bar{z},\underline{z})$, then

$$\hat{v}(b;\bar{z},\underline{z}) = b - \frac{\hat{y}_1-\bar{z}_1}{\underline{z}_1-\bar{z}_1} = \frac{\bar{z}_2-\hat{y}_2}{\bar{z}_2-\underline{z}_2} . \tag{6.13}$$

Solving the two equations in (6.13) for $\hat{y}$ in terms of $\hat{v}(b;\bar{z},\underline{z})$, we obtain (6.12). Now let $\tilde{y} \in \hat{Y}(b;\bar{z},\underline{z})$. Then $\hat{v}(b;\bar{z},\underline{z}) = v(\tilde{y},b;\bar{z},\underline{z})$ $= v(\hat{y},b;\bar{z},\underline{z})$ and, from Lemma 6.1, $\max\{(\hat{y}_1-\tilde{y}_1)/(\underline{z}_1-\bar{z}_1),(\hat{y}_2-\tilde{y}_2)/(\bar{z}_2-\underline{z}_2)\} = 0$. This implies $\hat{y} \leqq \tilde{y}$ with $\hat{y}_1 = \tilde{y}_1$ or $\hat{y}_2 = \tilde{y}_2$, and thus proves property (a). To prove (b), suppose $\bar{z},\underline{z} \in \mathcal{E}(\Gamma(Y),R_0)$ and $\bar{z} \leqq \bar{z}'$, $\underline{z} \leqq \underline{z}'$ for some $\bar{z}',\underline{z}' \in Y$. By Lemmas 6.3 and 6.4, we have $0 \leq \hat{v}(b;\bar{z},\underline{z}) \leq 1$ and $0 \leq b-\hat{v}(b;\bar{z},\underline{z}) \leq 1$. This fact and (6.12) imply $\bar{z}_1 \leq \hat{y}_1 \leq \underline{z}_1$ and $\underline{z}_2 \leq \hat{y}_2 \leq \bar{z}_2$ or, equivalently, $\hat{y} \in [\bar{z} \backslash \underline{z}]$. Next, let $y \in \Gamma(Y)$. If $\hat{y} < y$, then (6.3) implies $v(y,b;\bar{z},\underline{z}) < v(\hat{y},b;\bar{z},\underline{z}) = \hat{v}(b;\bar{z},\underline{z})$. This contradicts the minimality of $\hat{v}(b;\bar{z},\underline{z})$. If $y < \hat{y}$, then $y < \tilde{y}$ since, by property (a), $\hat{y} \leqq \tilde{y}$. This contradicts the fact that $y \in \Gamma(Y)$. Since we cannot have $y \in \Gamma(Y)$ and either $y < \hat{y}$ or $\hat{y} < y$, we must have $\hat{y} \in \mathcal{E}(\Gamma(Y),R_0)$. ∎

In solving Problem 6.3 to obtain a solution $\hat{y}$ to Problem 6.2, we also obtain a solution $\tilde{y}$ to Problem 6.3 which satisfies $\hat{y} \leqq \tilde{y}$. Note that if $\hat{y} \neq \tilde{y}$, then $\Gamma(Y)$ is not connected. Thus, we have a means of detecting gaps in $\Gamma(Y)$.

6.2 A General Algorithm

Proposition 6.4 motivates Algorithm 6.1, which can handle a tradeoff set $\Gamma(Y)$ that is not connected. Briefly, the algorithm works as follows. In stage i, the algorithm (step 2d) computes a set of experiments $\hat{E}_i \subset R_{i-1}$ prescribed by $\hat{\Sigma}_N(k_i)$. The algorithm obtains the points in $\hat{E}_i$ by solving problems in the form of Problem 6.2; it obtains each point in $\hat{E}_i$ by first solving an associated Problem 6.3 (step 2c). An additional set of points $\tilde{E}_i \subset \tilde{\Gamma}(Y)$ results from solving the problems of the Problem 6.3 type. The set $(\bigcup_{k=1}^{i} \tilde{E}_k) \cup (\bigcup_{k=1}^{i} \hat{E}_k)$ is a set of experiments for $\Gamma(Y)$ and thus defines a finite representation of $\Gamma(Y)$. It is this finite representation, rather than the one defined by $\bigcup_{k=1}^{i} \hat{E}_k$, that the algorithm presents to the decision-maker in stage i (step 3). The algorithm does not require the decision-maker to give a DM response at each stage; thus it is not necessarily true that $R_{i+1} \subset R_i$. However, when the decision-maker gives DM responses, he selects rectangles so that $R_0 \supset R_1 \ldots \supset R_N$ and the algorithm guarantees that $a(R_N) \leq \hat{A}_N(k_i)$.

Algorithm 6.1

Data: $R_0 = [\bar{y} \backslash \underline{y}]$ (see Definition 2.2); the maximum number of experiments allowed, $\bar{K}$.

Step 1: Set $i = 1$, $\bar{y}^0 = \bar{y}$, $\underline{y}^0 = \underline{y}$

Step 2a: Solicit from the decision-maker the number of experiments $k_i \in [1, \bar{K} - \sum_{j=1}^{i-1} k_j]$ to be computed in R_{i-1}.

Step 2b: Set $\hat{E}_i = \phi$, $\tilde{E}_i = \phi$, and $j = 1$.

Step 2c: Set

$$b_{ij} = 2j/(k_i+1). \tag{6.14}$$

Compute $x^j \in \text{Argmin}\{v(f(x),b_{ij};\bar{y}^{i-1},\underline{y}^{i-1})\,|\,x \in X\}$, $y^j = f(x^j)$, and $\hat{v}_{ij} = v(y^j,b_{ij};\bar{y}^{i-1},\underline{y}^{i-1})$. Set $\tilde{E}_i = \tilde{E}_i \cup (\{y^j\} \cap R_0)$.

Step 2d: Compute $\hat{y}^j$:

$$\begin{aligned} \hat{y}_1^j &= \bar{y}_1^{i-1} + (b_{ij}-\hat{v}_{ij})(\underline{y}_1^{i-1}-\bar{y}_1^{i-1}) \\ \hat{y}_2^j &= \underline{y}_2^{i-1} + (1-\hat{v}_{ij})(\bar{y}_2^{i-1}-\underline{y}_2^{i-1}). \end{aligned} \tag{6.15}$$

Set $\hat{E}_i = \hat{E}_i \cup \{\hat{y}^j\}$.

Comment: By Proposition 6.4, the construction of $\hat{y}^j$ by (6.15) guarantees that $g(\hat{y}^j,\bar{y}^{i-1},\underline{y}^{i-1}) = b_{ij}$. Thus $\hat{E}_i$ is the set of experiment prescribed by $\hat{\Sigma}_N(k_i)$ for R_{i-1} (see (5.15)).

Comment: If $\hat{y}^j \neq y^j$, then a gap in $\Gamma(Y)$ has been detected, that is, $\Gamma(Y)$ is not a connected curve between $\bar{y}^{i-1}$ and $\underline{y}^{i-1}$.

Step 2e: If $j = k_i$, then set $E_i = \hat{E}_i \cup \tilde{E}_i$ and go to step 3; else, set $j = j+1$ and go to step 2c.

Step 3: Display the finite representation of $\Gamma(Y)$ defined by $\bigcup_{k=1}^{i} E_k$ and ask the decision-maker to select a (nondegenerate) rectangle $R_i = [\bar{y}^i \setminus \underline{y}^i]$ from this representation in which he wants the next set of experiments.

Comment: Theorem 6.1 below shows that $\bigcup_{k=1}^{i} E_k$ is a set of experiments for $\Gamma(Y)$. By Proposition 6.4(a), $\hat{y}_1^j = y_1^j$ or $\hat{y}_2^j = y_2^j$; thus some rectangles defined by $\bigcup_{k=1}^{i} E_k \cup E_0$ will be degenerate, that is, line segments or points.

Comment: If the decision-maker gives a DM response in step 3, then $R_i \subset R_{i-1}$.

Step 4: If $\sum_{j=1}^{i} k_i = \bar{K}$, stop; else, set i = i+1 and go to step 2a.

Comment: The decision-maker may also terminate the algorithm in step 4 under two other conditions. First, if the decision-maker's goal is to approximate $\Gamma(Y)$, then he may stop when the points in $\bigcup_{k=1}^{i} E_k$ give to him what he considers an adequate picture of $\Gamma(Y)$. Second, if the decision-maker's goal is to estimate his preferred point y^* and he chooses R_i such that $y^* \in R_i$, then he may stop when he considers $a(R_i)$ small enough. ¤

When using Algorithm with a $\Gamma(Y)$ having gaps, it may happen that $\Gamma(Y) \cap R_i = \phi$ in some stage i. This anomaly arises because the experiments do not necessarily belong to $\Gamma(Y)$ and R_i lies in a gap of $\Gamma(Y)$. This situation will be indicated when all points of $\hat{E}_{i+1}$ are on the boundary of R_i. Note that in this case the decision-maker would not make a DM response; that is, he would not select a rectangle $R_{i+1} \subset R_i$. Rather, he would need to pick a rectangle outside of R_i in which to place the next set of experiments $\hat{E}_{i+2}$.

The man-machine interaction in Algorithm 6.1 is probably best aided by a color graphics display terminal. The finite representation of $\Gamma(Y)$ can then be readily displayed to the decision-maker, and color can be used to distinguish different pieces of information. For example, the sets $\hat{E}_i$ and $\tilde{E}_i$ can be distinguished by different colors. Moreover, rectangles which possibly contain or lie in gaps of $\Gamma(Y)$ can be colored red to alert the decision-maker to the existence of gaps.

6.3 Analysis of the Algorithm

We now employ Propositions 6.2, 6.3, and 6.4 to verify that Algorithm 6.1 does indeed generate sets of experiments for $\Gamma(Y)$.

Lemma 6.5: If step 2d of Algorithm 6.1 generates the set $\hat{E}_i = \{\hat{y}^1, \hat{y}^2, \ldots, \hat{y}^{k_i}\}$, then $\hat{y}_1^j \leq \hat{y}_1^{j+1}$ and $\hat{y}_2^j \geq \hat{y}_2^{j+1}$, for $j = 1,2,\ldots,k_i-1$, and, consequently, $\hat{E}_i$ has the property that $y,y' \in \hat{E}_i$ and $y \leqq y'$ imply $y_1 = y_1'$ or $y_2 = y_2'$.

Proof: Note that (6.14) and (6.3) imply $v(y,b_{ij};\bar{y}^{i-1},\underline{y}^{i-1}) \leq v(y,b_{ij+1};\bar{y}^{i-1},\underline{y}^{i-1})$ and $v(y,b_{ij+1};\bar{y}^{i-1},\underline{y}^{i-1}) - b_{ij+1} \leq v(y,b_{ij};\bar{y}^{i-1},\underline{y}^{i-1}) - b_{ij}$ for all $y \in \mathbb{R}^2$. Thus, in step 2c, $\hat{v}_{ij} \leq \hat{v}_{ij+1}$ and $b_{ij} - \hat{v}_{ij} \leq b_{ij+1} - \hat{v}_{ij+1}$. This fact and (6.15) imply $\hat{y}_1^j \leq \hat{y}_1^{j+1}$ and $\hat{y}_2^j \geq \hat{y}_2^{j+1}$ for $j = 1,2,\ldots,k_i-1$. Thus, for $y,y' \in E_i$ such that $y \leqq y'$ we must have $y_1 = y_1'$ or $y_2 = y_2'$. ¤

Theorem 6.1: Suppose Algorithm 6.1 generates $E_i = \hat{E}_i \cup \tilde{E}_i$, $i = 1,2,\ldots,N$. Then, for each $i = 1,2,\ldots,N$, $\hat{E}_i \subset R_{i-1}$ and $\bigcup_{k=1}^{i} E_k$ is a set of experiments for $\Gamma(Y)$.

Proof: First note that, for $i = 1,2,\ldots,N$, $\tilde{E}_i \subset \tilde{\Gamma}(Y)$ by Proposition 6.2, and consequently, $\tilde{E}_i$ is a set of experiments for $\Gamma(Y)$ by Proposition 6.3. The proof now proceeds inductively.

Since $\bar{y}^0,\underline{y}^0 \in Y \cap \mathcal{E}(\Gamma(Y),R_0)$, Proposition 6.4(b) implies $\hat{E}_1 \subset \mathcal{E}(\Gamma(Y),R_0)$. This and Lemma 6.5 imply that $\hat{E}_1$ is a set of experiments for $\Gamma(Y)$. Suppose $\hat{y} \in \hat{E}_1$ and $\tilde{y} \in \tilde{E}_1$. Associated with $\hat{y}$ is a $\tilde{y}' \in \tilde{\Gamma}(Y)$, computed in step 2c, such that $\hat{y} \leqq \tilde{y}'$ by Proposition 6.4(a). If $\tilde{y} < \hat{y}$, then $\tilde{y} < \tilde{y}'$, which contradicts the fact that $\tilde{y} \in \tilde{\Gamma}(Y)$. On the other hand, if $\hat{y} < \tilde{y}$, then the fact that $\tilde{y} \in \tilde{\Gamma}(Y)$ and Lemma 6.2 imply there is a $y \in \Gamma(Y)$ such that $\hat{y} < \tilde{y} \leqq y$. But this contradicts the fact that $\hat{y}$ is an experiment for $\Gamma(Y)$. Thus $\hat{y} \leqq \tilde{y}$ or $\tilde{y} \leqq \hat{y}$ implies $\hat{y}_1 = \tilde{y}_1$ or $\hat{y}_2 = \tilde{y}_2$. This proves $E_1 = \hat{E}_1 \cup \tilde{E}_1$ is a set of experiments for $\Gamma(Y)$.

Now suppose $\bigcup_{k=1}^{i-1} E_k$ is a set of experiments for $\Gamma(Y)$. From step 3, $\bar{y}^{i-1}, \underline{y}^{i-1} \in \bigcup_{k=1}^{i-1} E_k$ and so, by Definition 2.4, $\bar{y}^{i-1}, \underline{y}^{i-1} \in \mathcal{E}(\Gamma(Y), R_0)$. There exist $\bar{y}', \underline{y}' \in \tilde{\Gamma}(Y)$, associated with $\bar{y}^{i-1}$ and $\underline{y}^{i-1}$ and computed in step 2c, which satisfy $\bar{y}^{i-1} \leqq \bar{y}'$ and $\underline{y}^{i-1} \leqq \underline{y}'$ by Proposition 6.4(a). Then, by Proposition 6.4(b), we obtain $\hat{E}_i \subset \mathcal{E}(\Gamma(Y), R_0) \cap R_{i-1}$. This, Lemma 6.5, and Definition 2.4 imply $\hat{E}_i$ is a set of experiment for $\Gamma(Y)$. Then, by Proposition 2.5, $\hat{E}_i \cup (\bigcup_{k=1}^{i-1} E_k)$ is a set of experiments for $\Gamma(Y)$.

Finally, we need to show that the union of the sets of experiments $\tilde{E}_i$ and $\hat{E}_i \cup (\bigcup_{k=1}^{i-1} E_k)$ is a set of experiments. Suppose $\hat{y} \in \hat{E}_i \cup (\bigcup_{k=1}^{i-1} E_k)$ and $\tilde{y} \in \tilde{E}_i$. Then there is a $\tilde{y}' \in \tilde{\Gamma}(Y)$, computed in step 2c, for which $\hat{y} \leqq \tilde{y}'$ by Proposition 6.4(a). If $\tilde{y} < \hat{y}$, then $\tilde{y} < \tilde{y}'$, which implies the contradiction that $\tilde{y} \notin \tilde{\Gamma}(Y)$. If $\hat{y} < \tilde{y}$, then Lemma 6.2 and the fact that $\tilde{y} \in \tilde{\Gamma}(Y)$ imply there is a $y \in \Gamma(Y)$ such that $\hat{y} < \tilde{y} \leqq y$. This contradicts the fact that $\hat{y}$ is an experiment for $\Gamma(Y)$. Thus $\hat{y} \leqq \tilde{y}$ or $\tilde{y} \leqq \hat{y}$ implies $\hat{y}_1 = \tilde{y}_1$ or $\hat{y}_2 = \tilde{y}_2$. This implies $\bigcup_{k=1}^{i} E_k$ is a set of experiments for $\Gamma(Y)$.
$\square$

Theorem 6.1 ensures that Algorithm 6.1 constructs finite representations for $\Gamma(Y)$. The next theorem implies that when the decision-maker responds like DM, then Algorithm 6.1 guarantees the minimum worst-case uncertainty.

<u>Theorem 6.2</u>: If the decision-maker gives a DM response in stage i of Algorithm 6.1 (step 3), that is, if $R_i \subset R_{i-1}$, then $a(R_i) \leq a(R_{i-1})/(k_i+1)^2$.

Proof: By Theorem 6.1, $\hat{E}_i \subset R_{i-1} = [\bar{y}^{i-1} \setminus \underline{y}^{i-1}]$. Thus, if $R_i \subset R_{i-1}$, then by construction in step 3 and by Lemma 6.5, $R_i \subset [\hat{y}^j \setminus \hat{y}^{j+1}]$ for some $j \in \{0,1,\ldots,k_i\}$ and with $\hat{y}^0 \triangleq \bar{y}^{i-1}$ and $\hat{y}^{k_i+1} \triangleq \underline{y}^{i-1}$. Since $\hat{y}^j$ and $\hat{y}^{j+1}$ satisfy $g(\hat{y}^j;\bar{y}^{i-1},\underline{y}^{i-1}) = b_{ij}$ and $g(\hat{y}^{j+1};\bar{y}^{i-1},\underline{y}^{i-1}) = b_{ij+1}$, Lemma 5.2 gives $a(R_i) \leq a([\hat{y}^j \setminus \hat{y}^{j+1}])$ $\leq a(R_{i-1})/(k_i+1)^2$. ¤

6.4 Selection of a Preferred Point

Suppose that the decision-maker uses Algorithm 6.1 and claims that a rectangle of uncertainty $R_N = [\bar{y}^N \setminus \underline{y}^N]$ contains his preferred point y^*. If his problem actually requires him to select y^* and a corresponding $x^* \in \Omega$, then he needs to compute an $x^* \in \Omega$ which gives $y^* = f(x^*) \in R_N$. He may try to do this using any of the approaches in [9]-[12]. For example, he may select an $\varepsilon \in [0,1]$ and solve the problem [11]

$$\begin{aligned} &\text{maximize } f_2(x) \\ &\text{subject to } x \in \{x \in X \mid \bar{y}_1^N + \varepsilon(\underline{y}_1^N - \bar{y}_1^N) \leq f_1(x)\}. \end{aligned} \tag{6.16}$$

Under suitable conditions, given in [11], a solution $x^*(\varepsilon)$ to this problem gives $y^*(\varepsilon) = f(x^*(\varepsilon)) \in \Gamma(Y)$. If it happens that $\Gamma(Y) \cap R_N = \phi$, then $y^*(\varepsilon)$ will not be in R_N for any $\varepsilon \in [0,1]$. In this case, the decision-maker learns that he is trying to force his preferred point to be in a gap; he must adjust his preferences and reconsider rectangles that he previously eliminated in arriving at R_N. Alternatively, he may wish to modify the specification of X, and possibly f, and apply Algorithm 6.1 to the modified problem.

7. CONCLUSION

The rectangle elimination method has a number of features which make it attractive for bi-objective decision-making. First, the method is a man-machine interactive procedure that can cope with the ill-defined nature of the decision problem. Second, it makes very simple assumptions about the decision-maker's ability to process information and to express his preferences. The method does not require the decision-maker to have a utility function or to deal with numerical tradeoff ratios. Third, the method affords the decision-maker much flexibility in its use. He may choose to approximate the entire tradeoff set or only selected portions of it, or he may concentrate on finding his preferred objective value. Fourth, the man-machine dialogue is ideally done via a graphics display terminal. Then the decision-maker can easily view the display of the finite representation of his tradeoff set, select those rectangles in which he wants more information, and say how many experiments he wants in these rectangles. And finally, as we have demonstrated, there are strategies for selecting experiments which make the method computationally efficient. With these strategies, the decision-maker has a priori bounds on the number of experiments he needs in a given rectangle to obtain a desired reduction in uncertainty.

While our method is designed for bi-objective decision problems, some of its underlying principles may be generalized for problems with three or more objectives. Such generalizations will be the subject of a subsequent paper. We hope that the rectangle elimination method and its generalizations will become important tools for computer-aided decision-making and design.

REFERENCES

[1] K.R. MacCrimmon, "An overview of multiple objective decision-making," in Multiple Criteria Decision Making, J.L. Cochrane and M. Zeleny, Eds. Columbia, S.C.: USC Press, 1973, pp. 18-44.

[2] R.L. Keeney and H. Raiffa, Decisions with Multiple Objectives: Preferences and Value Tradeoffs. New York: John Wiley and Sons, 1976.

[3] E. Polak and A.N. Payne, "On multicriteria optimization," in Directions in Large-Scale Systems, Y.C. Ho and S.K. Mitter, Eds. New York: Plenum, 1976, pp. 77-94.

[4] B. Roy, "Problems and methods with multiple objective functions," Mathematical Programming, vol. 1, pp. 239-266, Nov. 1971.

[5] W. R. Reitman, "Heuristic decision procedures, open constraints, and the structure of ill-defined problems," in Human Judgments and Optimality, M.W. Shelly and G.L. Bryan, Eds. New York: John Wiley and Sons, 1964, Ch. 15.

[6] L.A. Zadeh, "Optimality and non-scalar-valued performance criteria," IEEE Trans. Automat. Contr., vol.AC-8, pp. 59-60, Jan. 1963.

[7] N.O. DaCunha and E. Polak, "Constrained minimization under vector-valued criteria in linear topological spaces," in Mathematical Theory of Control, A.V. Balakrishnan and L.W. Neustadt, Eds. New York: Academic Press, 1967, pp. 96-108.

[8] A.M. Geoffrion, "Solving bicriterion mathematical programs," Operations Research, vol. 15, pp. 39-54, Jan./Feb. 1967.

[9] J.G. Lin, "Multiple-objective problems: Pareto-optimal solutions by the method of proper equality constraints," IEEE Trans. Automat.

Contr., vol. AC-21, pp. 641-650, Oct. 1976.

[10] J.G. Lin, "Three methods for determining Pareto-optimal solutions of multiple-objective problems, " in Directions in Large-Scale Systems, Y.C. Ho and S.K. Mitter, Eds. New York: Plenum, 1976, pp. 117-138.

[11] H.J. Payne, E. Polak, D.C. Collins, and W.S. Meisel, "An algorithm for bicriteria optimization based on the sensitivity function, " IEEE Trans. Automat. Contr., vol. AC-20, pp. 546-548, Aug. 1975.

[12] E. Polak, "On the approximation of solutions to multiple criteria decision making problems," in Multiple Criteria Decision Making: Kyoto 1975, M. Zeleny, Ed. Berlin: Springer-Verlag, 1976,pp. 271-282.

[13] F.W. Gembicki and Y.Y. Haimes, "Approach to performance and sensitivity multiobjective optimization: the goal attainment method," IEEE Trans. Automat. Contr., vol. AC-20, pp. 769-771, Dec. 1975.

[14] Y.Y. Haimes and W.A. Hall, "Multiobjectives in water resources systems analysis: the surrogate worth trade-off method," Water Resources Res., vol. 10, pp. 615-624, Aug. 1974.

[15] R. Benayoun, J. de Montgolfier, J. Tergny, and O. Laritchev, "Linear programming with multiple objective functions: STEP method (STEM)," Mathematical Programming, vol.1, pp. 366-375, Dec. 1971.

[16] A.M. Geoffrion, J.S. Dyer, and A. Feinberg, "An interactive approach for multi-criterion optimization with an application to the operation of an academic department," Management Science,

vol.19, pp. 357-368, Dec. 1972.

[17] J.S. Dyer, "Interactive goal programming," Management Science, vol. 19, pp. 62-70, Sept. 1972.

[18] S.M. Belenson and K.C. Kapur, "An algorithm for solving multicriterion linear programming problems with examples," Operational Research Quarterly, vol. 24, pp. 65-77, Mar. 1973.

[19] J. Wallenius, "Comparative evaluation of some interactive approaches to multicriterion optimization," Management Science, vol. 21, pp. 1387-1396, Aug. 1975.

[20] S. Zionts and J. Wallenius, "An interactive programming method for solving the multiple criteria problem," Management Science, vol. 22, pp. 652-663, Feb. 1976.

[21] D.J. Wilde and C.S. Beightler, Foundations of Optimization. Englewood Cliffs, N.J.: Prentice Hall, 1967.

[22] J. Kiefer, "Optimum sequential search and approximation methods under minimum regularity assumptions," J. Soc. Indust. Appl. Math., vol.5, pp. 105-136, Sept. 1957.

[23] A.V. Aho, J.E. Hopcroft, and J.D. Ullman, The Design and Analysis of Computer Algorithms. Reading, Mass.: Addison-Wesley, 1974.

[24] M. Zeleny, "The theory of the displaced ideal," in Multiple Criteria Decision Making: Kyoto 1975, M. Zeleny, Ed. Berlin: Springer-Verlag, 1976, pp. 152-206.

[25] M. Zeleny, Linear Multiobjective Programming, Berlin: Springer-Verlag, 1974.

FIGURE CAPTIONS

Fig. 2.1. The rectangle $[\bar{z} \backslash \underline{z}]$.

Fig. 2.2. A typical tradeoff set $\Gamma(Y)$ and its enclosing rectangle $[\bar{y} \backslash \underline{y}]$.

Fig. 2.3. (a) A connected tradeoff set Γ and a point $y \in \Gamma$ such that $\Gamma \subset [\bar{y} \backslash y] \cup [y \backslash \underline{y}]$. (b) A tradeoff set Γ with gaps and a point $y \notin \Gamma$ such that $\Gamma \subset [\bar{y} \backslash y] \cup [y \backslash \underline{y}]$.

Fig. 2.4. An example of the set $\mathcal{E}(\Gamma, R_0)$.

Fig. 2.5. A 7-experiment finite representation of the tradeoff set Γ.

Fig. 5.1. The rectangle R' and line segment Γ'.

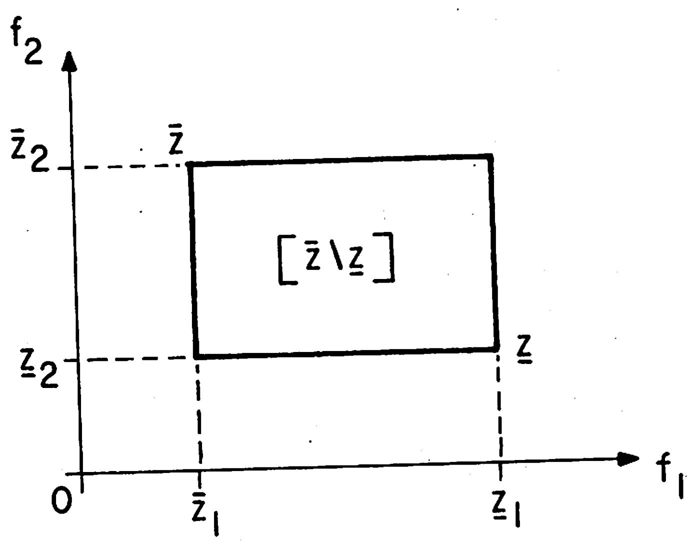
Fig. 5.2. A partition of R' by three parallel lines cutting Γ' into four segments of equal length.

Fig. 5.3. Application of the strategy $\hat{\Sigma}_2(2,3)$ for a typical tradeoff set Γ and a typical DM response.

Fig. 5.4. Application of the strategy $\hat{\Sigma}_3(1)$ for a typical tradeoff set Γ and a typical DM response.

Fig. 5.5. Application of the strategy $\tilde{\Sigma}_3(1)$ for a typical tradeoff set Γ and a typical DM response.

Fig. 6.1. Level sets of the function $v(\cdot, b; \bar{z}, \underline{z})$.



f_2
$\bar{z}_2$
$\bar{z}$
$[\bar{z} \backslash \underline{z}]$
$\underline{z}_2$
$\underline{z}$
0
$\bar{z}_1$
$\underline{z}_1$
f_1

Figure 2.1

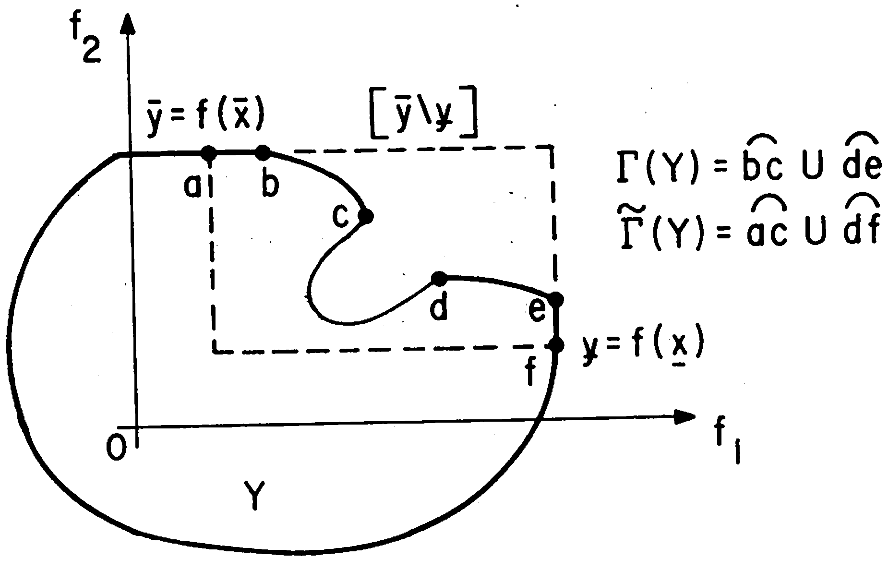
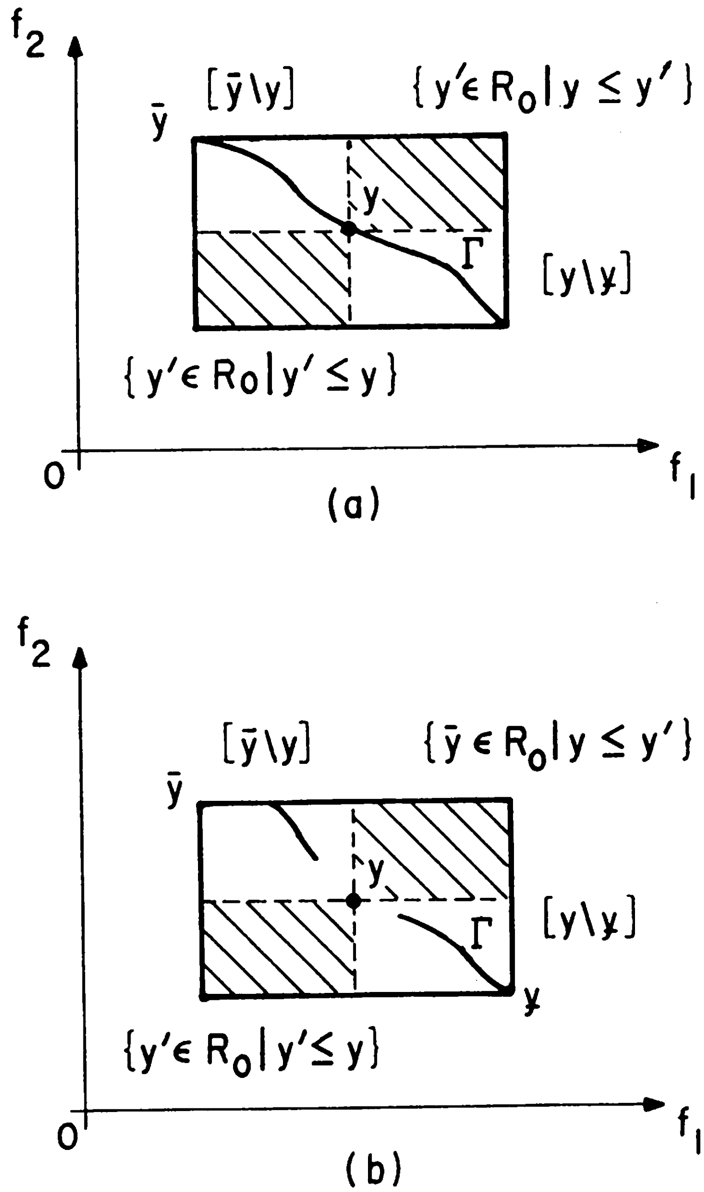
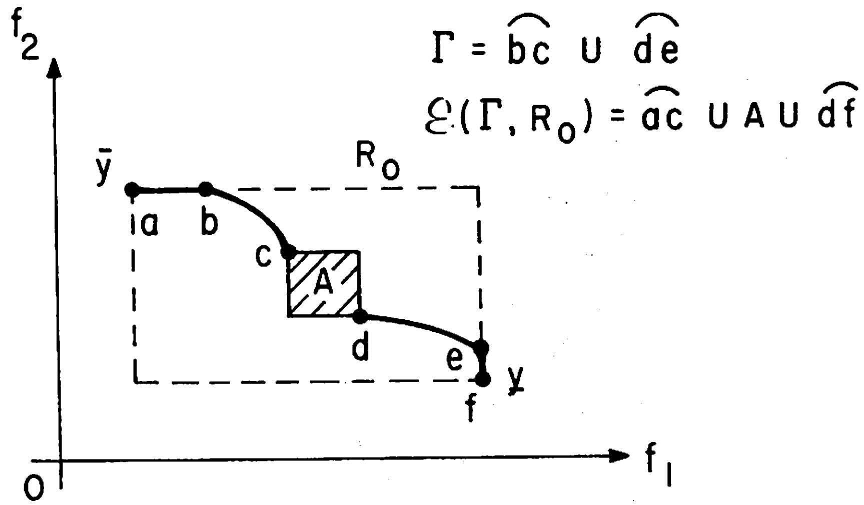


Figure 2.2



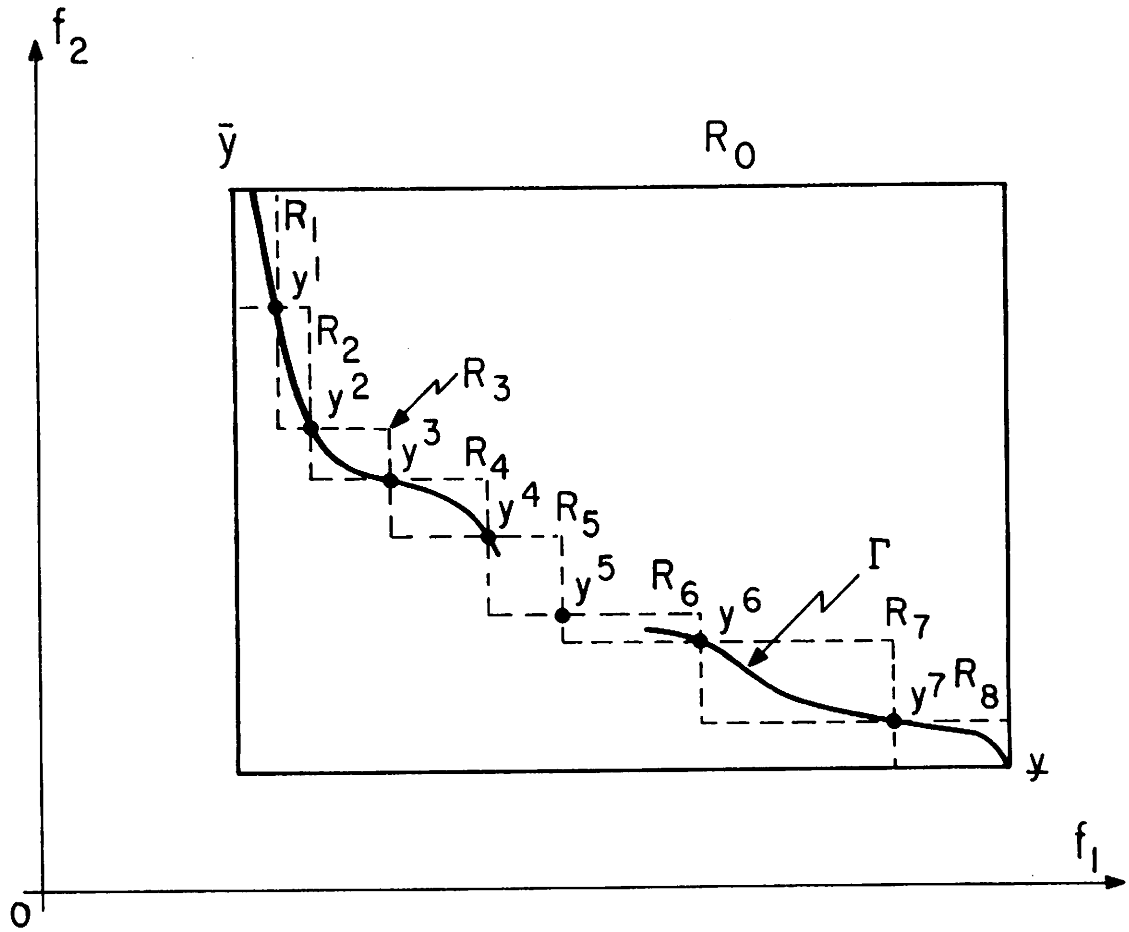
f_2
$[\bar{y}\backslash y]$
$\{y' \in R_0 | y \leq y'\}$
$\bar{y}$
y
Γ
$[y\backslash \underline{y}]$
$\{y' \in R_0 | y' \leq y\}$
0
f_1
(a)
f_2
$[\bar{y}\backslash y]$
$\{\bar{y} \in R_0 | y \leq y'\}$
$\bar{y}$
y
$[y\backslash \underline{y}]$
Γ
$\underline{y}$
$\{y' \in R_0 | y' \leq y\}$
0
f_1
(b)

Figure 2.3



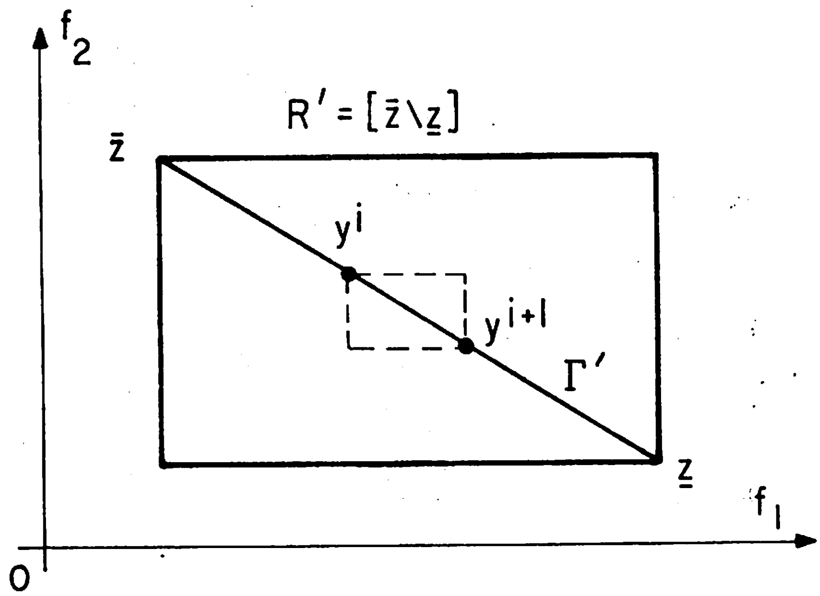
$\Gamma = \widehat{bc} \cup \widehat{de}$
$\mathcal{E}(\Gamma, R_0) = \widehat{ac} \cup A \cup \widehat{df}$
f_2
$\bar{y}$
R_0
a
b
c
A
d
e
f
$\underline{y}$
O
f_1

Figure 2.4



f_2
$\bar{y}$
R_0
R_1
y^1
R_2
y^2
R_3
y^3
R_4
y^4
R_5
y^5
R_6
y^6
Γ
R_7
y^7
R_8
$\underline{y}$
f_1
0

Figure 2.5



f_2
$R' = [\bar{z} \backslash \underline{z}]$
$\bar{z}$
y^i
y^{i+1}
Γ'
$\underline{z}$
f_1
0

Figure 5.1

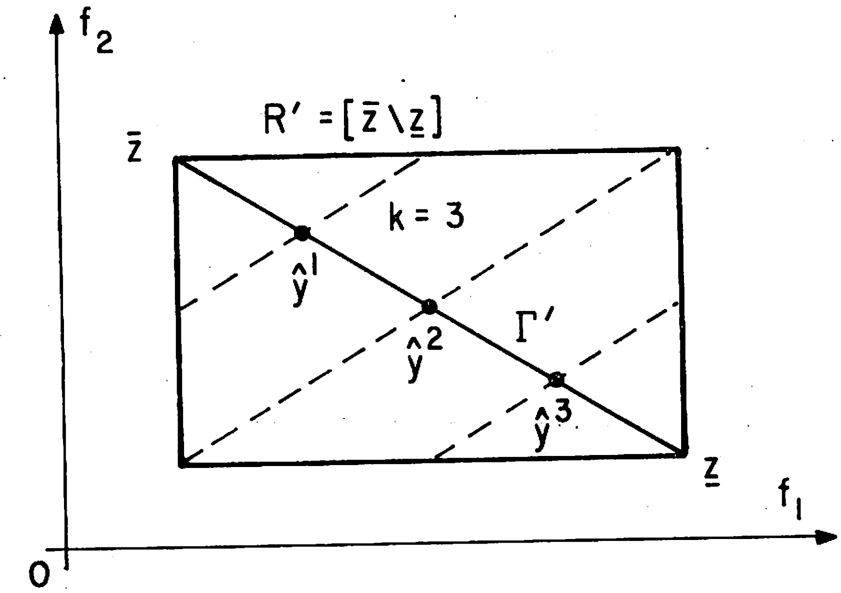
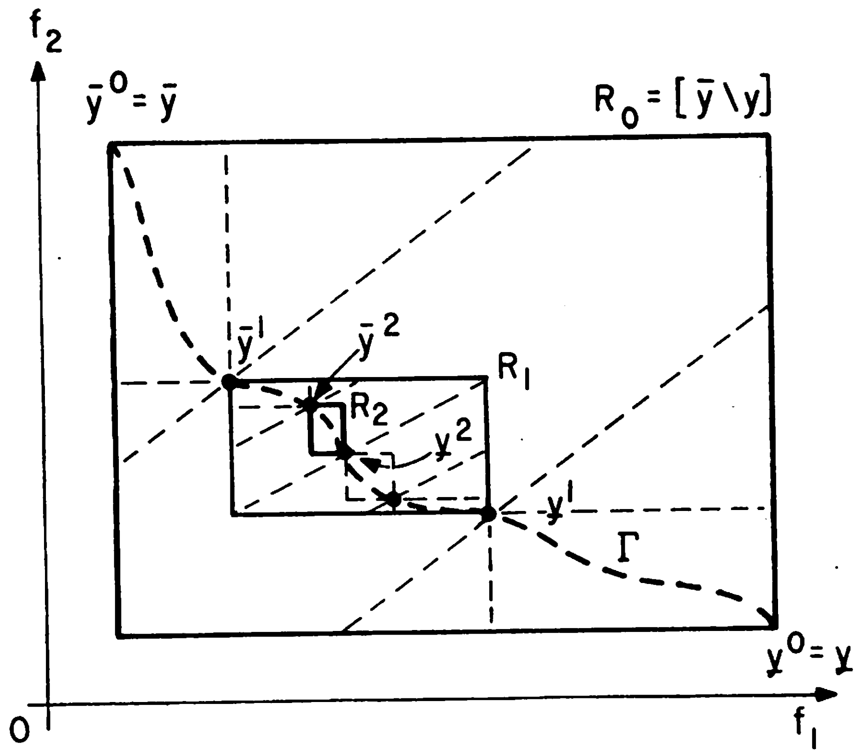


Figure 5.2



f_2
$\bar{y}^0 = \bar{y}$
$R_0 = [\bar{y} \backslash \underline{y}]$
$\bar{y}^1$
$\bar{y}^2$
R_1
R_2
$\underline{y}^2$
$\underline{y}^1$
Γ
$\underline{y}^0 = \underline{y}$
O
f_1

Figure 5.3

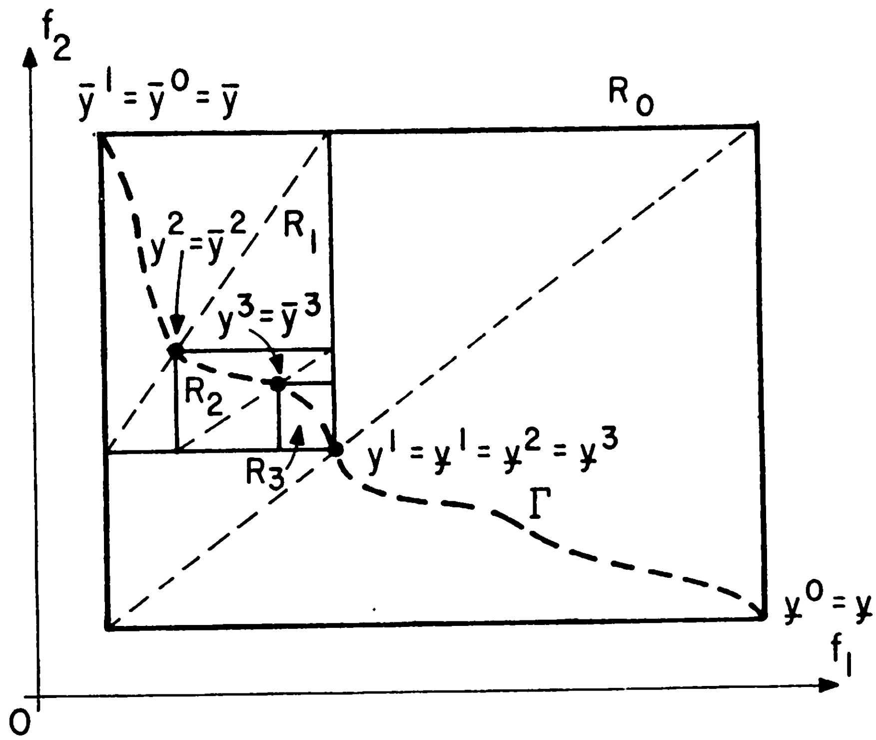


Figure 5.4

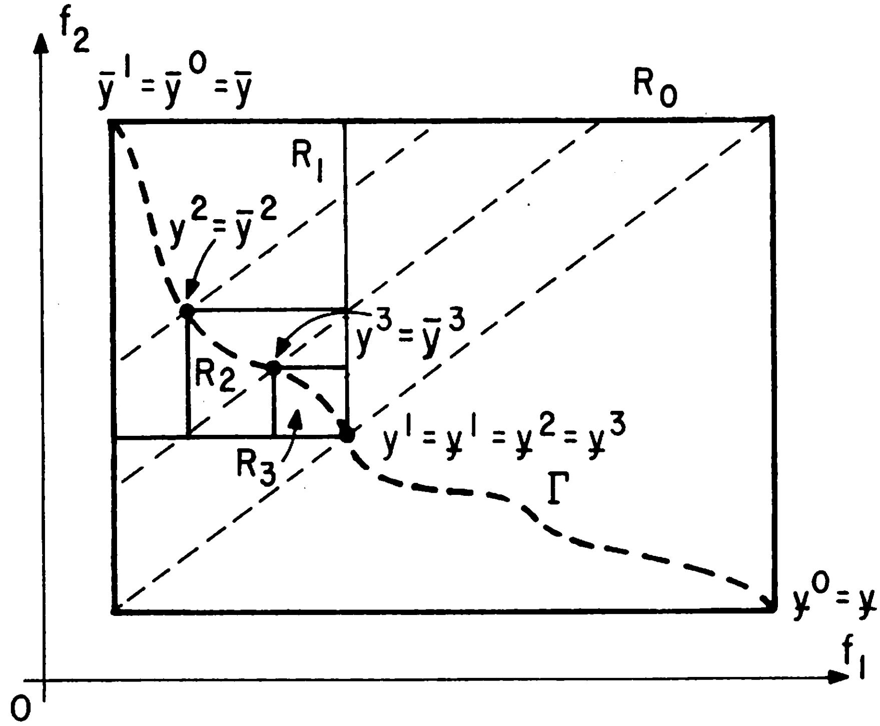


Figure 5.5

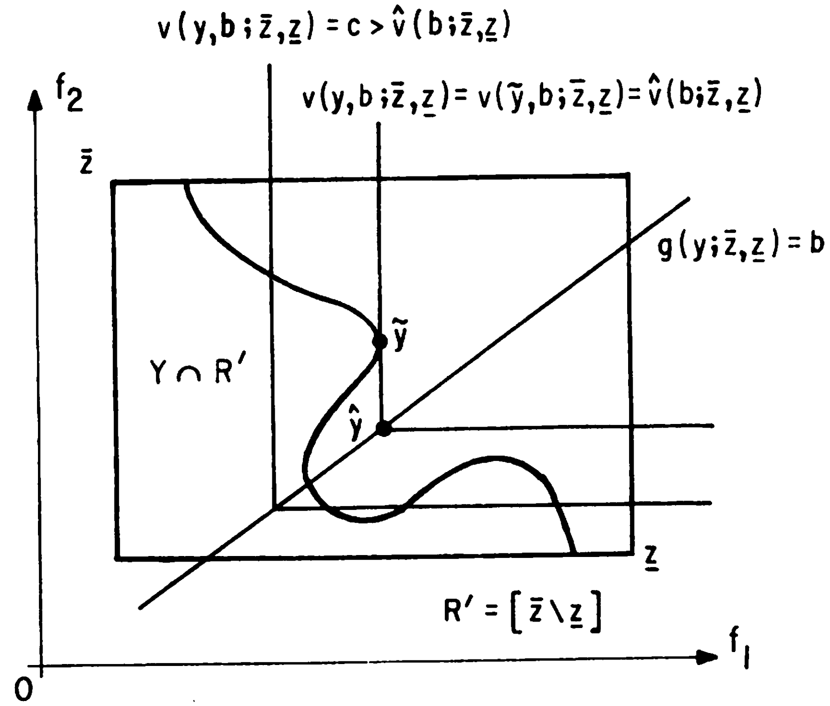


Figure 6.1