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VECTOR LYAPUNOV FUNCTIONS:

STABILITY AND STABILIZABILITY PROBLEMS FOR INTERCONNECTED SYSTEMS

by

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Summary

The use of the vector Lyapunov function method as a tool for stability and stabilizability study of large scale interconnected systems is investigated. It is shown that, among the Lyapunov-like methods, the vector approach can provide good results. Of course, the results obtained much depends on a "good" choice of the Lyapunov functions and an algorithm is given which enables in the case of linear time invariant systems, to provide such a good choice. The problem of stabilizability under a decentralized control structure is discussed and some drawbacks of the Lyapunov method approach are pointed out. Necessary conditions for convergence (stabilizability) are given.

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Introduction

Since its first appearance [1] and the pioneering work of Bailey [2], the vector Lyapunov function method has motivated a great deal of works this last decade. This can easily be seen by reading the numerous references given by the two recent books by Michel and Miller [3] and Siljak [4], which themselves make great use of the vector Lyapunov function concept in studying stability properties and control problems for large scale interconnected systems. In [5] is given a critical survey of the Lyapunov method applied to the stability study of large scale systems. It is not the purpose here to undertake a general presentation and survey of the vector Lyapunov function method but merely to discuss some points concerning a comparative study of various approaches of the Lyapunov method for interconnected systems so as to present some improvements that can be made in the case where the subsystems are described by linear time invariant differential equations.

The paper will be divided into three sections. In section I, after a brief exposure of the stability conditions given by the vector Lyapunov function and the weighted sum approach, are said a few words about the respective advantages of the different approaches. It will be shown that in some cases the vector Lyapunov function approach can provide better results. Indeed, the conservativeness of the results much depends on a proper choice of the Lyapunov functions. In section II, the problem of how to find a "good" Lyapunov function for each subsystem is investigated and a parametric optimization problem is defined for that purpose. In the third section, the problem of decentralized control is discussed, in the context of the use of vector Lyapunov functions for stability tests.

I. Stability Analysis

I.1. Vector Lyapunov Functions. The Comparison Principle

Although the remaining is also valid in the case when non-linearities (even non-invariant) are present in the interconnection terms, let us consider the simplest case of linear time invariant interconnected systems, described by

$$\overset{\circ}{x}_i = A_i x_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} A_{ij} x_j \;; \quad i = 1,2,\dots,N \tag{1}$$

with $x_i \in R^{n_i}$; A_i and A_{ij} are respectively $n_i \times n_i$, $n_i \times n_j$ constant matrices; N is the number of subsystems.

It is first assumed that each isolated subsystem (described by $\overset{\circ}{x}_i = A_i x_i$) is asymptotically stable so that for all Q_i matrices $(n_i \times n_i)$

$$Q_i = D_i' D_i \quad (\text{' means transpose}) \tag{2}$$

such that the pair (A_i, D_i) is observable, there exists a positive definite symmetric matrix P_i, which is the solution of the algebraic Lyapunov equation

$$A_i' P_i + P_i A_i = -Q_i \,. \tag{3}$$

For the i^{th} subsystem let us choose the scalar Lyapunov function

$$v_i = (x_i' P_i x_i)^{1/2} \tag{4}$$

which satisfies the inequalities:

$$\begin{aligned} \lambda_m^{1/2}(P_i)\|x_i\| \leq v_i \leq \lambda_M^{1/2}(P_i)\|x_i\| \\ \|\text{grad } v_i\| \leq \lambda_M^{1/2}(P_i)\|x_i\| \end{aligned} \tag{5}$$

where $\|\cdot\|$ is the euclidean norm and $\lambda_m(P)$, $\lambda_M(P)$ represent respectively the minimum and maximum eigenvalues of P. Evaluating the time derivative of v_i along the motion of (1), one gets:

$$\overset{\circ}{v}_i = \frac{1}{2}\,\frac{x_i'(A_i'P_i+P_iA_i)x_i}{v_i} + (\text{grad } v_i)' \sum_j A_{ij}x_j$$
$$\leq -\alpha_i v_i + \lambda_M^{1/2}(P_i) \sum_j \lambda_m^{-1/2}(P_j)\|A_{ij}\|v_j \qquad (6)$$

with
$$\alpha_i = \min_{x_i} \frac{x_i'Q_ix_i}{x_i'P_ix_i} \, .$$

In matrix form, with $V = [v_1, v_2, \ldots, v_N]'$, we get

$$\overset{\circ}{V} \leq PV \; ; \quad P = \{p_{ij} = -\alpha_i\delta_{ij} + (1-\delta_{ij})\lambda_M^{1/2}(P_j)\|A_{ij}\|\} \qquad (7)$$

δ_{ij} the kronecker symbol. P is a matrix with non negative off-diagonal elements, so that stability of (6) and consequently (1) can be derived from the stability study of the following linear system, called comparison system [6]

$$\overset{\circ}{W} = PW \qquad (8)$$

since, if $V(t_0) \leq W(t_0)$ then $V(t) \leq W(t)$, $\forall t \geq t_0$; $V(t), W(t) \in R_+^N$. There exist many equivalent stability conditions for (8), [7] among them:

<u>Theorem 1</u>. The system (8) is asymptotically stable

a) iff the leading principal minors of (-P) are positive (this condition is known as the Sevastyanov-Kotelyanski condition);

b) iff there exists a positive vector U_0 (each component positive) such that

$$PU_0 < 0 \; ;$$

c) iff there exists a diagonal matrix $B = \text{diag}(\beta_i)$, $\beta_i > 0$ such that $P'B + BP$ is negative definite.

If P satisfies these conditions (i.e. if P is Hurwitz), then P is called a (-M) matrix.

In the following we will rather consider the matrix

$$\bar{P} = \{p_{ij} = -\alpha_i/\theta_i\delta_{ij} + (1-\delta_{ij})\|A_{ij}\|\} \ ; \quad \theta_p = [\lambda_M(P_i)\lambda_m^{-1}(P_i)]^{1/2} \qquad (9)$$

instead of P.

Theorem 2. P is asymptotically stable iff $\bar{P}$ is asymptotically stable.

The proof is straightforward, noticing that P can be written

$$P = \mathrm{diag}[\lambda_M^{1/2}(P_i)]\,\bar{P}\,\mathrm{diag}[\lambda_m^{-1/2}(P_j)] \ .$$

Then denoting by Δ_J and $\bar{\Delta}_J$ the J^{th} order principal minors of P and $\bar{P}$ respectively, it becomes

$$\Delta_J = \left(\prod_{i=1}^{J} \lambda_M^{1/2}(P_i)\lambda_m^{-1/2}(P_i)\right)\bar{\Delta}_J \ .$$

The form of the $\bar{P}$ matrix is interesting since all the variable terms (i.e. all the terms depending on the choice of P_i) are isolated on the main diagonal. That shows as evidence that a good choice for P_i must be such that it results in a ratio α_i/θ_i as high as possible. This remark will be considered in section II.

I.2. The Weighted Sum Approach

An alternative approach for the stability study of (1) is to try to find a scalar Lyapunov function which is a weighted sum of the individual scalar Lypunov functions v_i (4), i.e.

$$\nu = \sum_{i=1}^{N} \beta_i v_i \ , \quad \beta_i > 0 \ . \qquad (10)$$

The problem is in the determination of the positive numbers β_i such that the time derivative $\overset{\circ}{v}$ be negative definite. Of course, such an approach can be used on inequality (7) and one gets

$$\overset{\circ}{v} \leq V'P'B \ , \quad B = [\beta_1, \beta_2, \ldots, \beta_N]' \ . \tag{11}$$

Existence of B such as $\overset{\circ}{v}$ be negative definite turns out to be the same condition as condition b) of Theorem 1. But such an approach does not need to write a differential inequality of type (7). When considering (5), it is possible to get the differential inequality

$$\overset{\circ}{V} \leq L\Phi \ , \quad \Phi = [\|x_1\|, \|x_2\|, \ldots, \|x_N\|]' \tag{12}$$

with

$$L = \{\ell_{ij} = -\frac{1}{2}\lambda_m(Q_i)\lambda_M^{-1/2}(P_i)\delta_{ij} + (1-\delta_{ij})\lambda_M^{1/2}(P_i)\|A_{ij}\|\} \ .$$

With (12), $\overset{\circ}{v}$ can be written

$$\overset{\circ}{v} \leq \Phi'L'B \tag{13}$$

and $\overset{\circ}{v}$ is negative definite iff L' is a (-M) matrix ($\Leftrightarrow$ L be a (-M) matrix). Here too (in a very similar way as for Theorem 2) stability for L can be stated from that of $\bar{L}$, with

$$\bar{L} = \{\bar{\ell}_{ij} = -\frac{1}{2}\lambda_m(Q_i)\lambda_M^{-1}(P_i)\delta_{ij} + (1-\delta_{ij})\|A_{ij}\|\} \ .$$

There is no evidence that the stability condition given by using $\bar{L}$ would give better results than those given by using $\bar{P}$ since

$$\alpha_i \geq \lambda_m(Q_i)\lambda_M^{-1}(P_i) \ . \tag{14}$$

In fact it may be invoked at this stage that the computation of the true α_i is a much more involved problem than computing λ_m and λ_{-M} for a given matrix and in fact in some papers α_i is replaced by a lower bound, namely

$\lambda_m(Q_i)\lambda_M^{-1}(P_i)$ so that in that case, indeed $\bar{P}$ provides worse results than $\bar{L}$. However in section II is given a way to try to optimize globally the ratio α_i/θ_i so that the preceding remark still holds.

Another stability condition can be derived using the weighting sum approach with quadratic Lyapunov function. Instead of (4) let us use

$$v_i = x_i' P_i x_i \ . \tag{15}$$

Then, following the same type of calculations as before, one gets

$$\overset{\circ}{v} = \frac{d}{dt}\Big(\sum_{i=1}^{N} \nu_i \beta_i\Big) = \Phi' R \Phi \tag{16}$$

with

$$R = \{r_{ij}\}$$

$$r_{ij} = \begin{cases} -\beta_i \lambda_m(Q_i) \ ; & i = j \\ \beta_i \lambda_M(P_i) \|A_{ij}\| + \beta_j \lambda_M(P_j) \|A_{ji}\| \ ; & i \neq j \end{cases}$$

(16) can be written as

$$\overset{\circ}{v} = \Phi' \Big[\frac{B\bar{R} + \bar{R}' B}{2}\Big] \Phi \tag{17}$$

with

$$\bar{R} = \{\bar{r}_{ij}\}$$

$$\bar{r}_{ij} = \begin{cases} -\lambda_m(Q_i) \ ; & i = j \\ \lambda_M(P_i) \|A_{ij}\| \ ; & i \neq j \ . \end{cases}$$

(17) being negative definite means that $\bar{R}$ is a (-M) matrix (cf. Theorem 1, condition (c)) so that one gets here the same stability condition as with $\bar{L}$ and the same remark holds.

Other motivations for using the vector Lyapunov function approach can be given and this is the purpose of the next two subsections.

I.3. Non Linear Interconnected Systems

We will consider here the problem of the estimation of the stability domains [9] for interconnected non linear systems described by

$$\mathring{x}_i = g_i(x_i,t) + h_i(x,t) \;, \quad i = 1,2,\ldots,N \;, \tag{18}$$

$x_i \in R^{n_i}$, $X \in R^n$, $\sum_i n_i = n$. It is assumed that the origin is an equilibrium point for the overall system and for each isolated subsystem, i.e.

$$g_i(0,t) \equiv 0 \;, \quad h_i(0,t) \equiv 0 \;.$$

There exists, for each isolated subsystem, a scalar Lyapunov function $v_i(x_i,t)$ such that

$$\phi_{i1}(\|x_i\|) \leq v_i(x_i,t) \quad (\leq \phi_{i2}(\|x_i\|) \text{ for uniformity purpose}) \tag{19}$$

and

$$\frac{\partial v_i}{\partial t} + (\text{grad } v_i)' g_i(x_i,t) \leq -\phi_{i3}(\|x_i\|) \tag{20}$$

where ϕ_{i1}, ϕ_{i2}, ϕ_{i3} are functions of class K [8]. Furthermore the interconnection terms are assumed to be such that

$$(\text{grad } v_i)' h_i(x,t) \leq \sum_{j=1}^{N} \gamma_{ij} \phi_{j3}(\|x_j\|) \;. \tag{21}$$

We consider the case when (10), (11) does not hold globally but only in the domain

$$\mathcal{D} = \{x_i : v_i(x_i,t) \leq v_{oi} ;\; i = 1,2,\ldots,N\} \;.$$

Then, using (20) and (21), in $\mathcal{D}$ we obtain

$$\mathring{V} \leq LW \quad \text{with} \quad V = [v_1,v_2,\ldots,v_N]' ,\; W = [\phi_{13},\phi_{23},\ldots,\phi_{N3}]' \tag{22}$$

and

$$L = \{\ell_{ij} = (-1+\gamma_{ii})\delta_{ij} + (1-\delta_{ij})\gamma_{ij}\} \;.$$

Let us now consider the scalar Lyapunov function $\nu = \sum_i \beta_i v_i = V'B$. Then

$$\mathring{\nu} \leq W'L'B \tag{23}$$

which is negative definite if and only if $L'B < 0$. That generally implies (since in most the cases $\gamma_{ij} > 0$, see (21)) L' (and consequently L) be a (-M) matrix.

To be able to get a true differential system (the comparison system) from (22) one needs a supplementary condition on the ϕ_{i1}, ϕ_{i2}, ϕ_{i3} functions: namely ϕ_{i1}, ϕ_{i2}, ϕ_{i3}, of the same order of magnitude, i.e.,

$$R_{i1}^{-1}\phi_{i1} = R_{i2}^{-1}\phi_{i2} = \phi_{i3} \quad . \tag{24}$$

Then, (22) can be written

$$\mathring{V} \leq \bar{L}V \;\; ; \quad \bar{L} = \{\bar{\ell}_{ij} = (-R_{i2}^{-1}+\gamma_{ii}R_{i1}^{-1})\delta_{ij}+(1-\delta_{ij})\gamma_{ij}R_{j1}^{-1}\} \quad . \tag{25}$$

Here too one could try to find a scalar Lyapunov function $\nu = \sum \beta_i v_i$. The final stability test would be that $\bar{L}$ be a (-M) matrix but such a way would provide much conservative results than those obtained by using L since

$$\bar{L} = \tilde{L}\cdot \text{diag}(R_{i1}^{-1}) \;\; , \quad \tilde{L} = \{\tilde{\ell}_{ij} = (\gamma_{ii}-R_{i1}R_{i2}^{-1})\delta_{ij}+(1-\delta_{ij})\gamma_{ij}\}$$

and

$$\tilde{L} \geq L \quad \text{(term by term)} \;\; .$$

But, (25) enables one to consider the comparison system

$$\mathring{W} = \bar{L}W \quad . \tag{26}$$

If $\bar{L}$ is a (-M) matrix, i.e., if it exists a positive vector $U_0 = [u_{01},\ldots,u_{0N}]'$ such that $\bar{L}U_0 < 0$, then $v = \max_i v_i \cdot u_{0i}^{-1}$ is a Lyapunov function for (26). It is easy to see that the estimate of the stability

stability domains in the two cases is

$$\sum_i \beta_i v_i \le \max_i \beta_i v_{i0} \quad \text{for (22)}$$

$$v_j u_{0j}^{-1} \le \max_i v_{i0} u_{i0}^{-1} \quad \text{for (26)}$$

It may happen that the estimate obtained for (26) be wider than that obtained for (22) and, even, in the case where U_0 can be chosen such that $U_0 = [v_{10},\ldots,v_{N0}]'$ this estimate actually coincides with the domain of definition $\mathcal{D}$ of the comparison system.

I.4. The Case of Structural Perturbations

Let us consider the system (1) subjected to structural perturbations in the following way. It is assumed that the number of perturbations is finite, say P, and that for each structural perturbation the system is described by

$$\overset{\circ}{x}_i = A_i^k x_i + \sum_{j=1}^{N} A_{ij}^k x_j \ , \quad k = 1,2,\ldots,P; \quad i = 1,2,\ldots,N \ . \tag{27}$$

It will be assumed that the perturbations do not greatly affect the isolated subsystems, i.e. $A_i^k = \overset{\circ}{A}_i$, $k = 1,2,\ldots,P$, such that a unique scalar Lyapunov function can be chosen for each isolated subsystem whatever the perturbation. It can be noticed that such a presentation differs substantially from that used in the concept of connective stability which merely consists in defining a comparison system (the fundamental comparison system) which overevaluates all the comparison systems built for each particular perturbation. Following what has been done in subsection I.1 for each structural perturbation one gets a comparison system

$$\overset{\circ}{v} = P^k V \ , \quad k = 1,2,\ldots,P \tag{28}$$

where

$$P^k = \{p_{ij}^k = \alpha_i^k \delta_{ij} + (1-\delta_{ij}) \lambda_M^{1/2}(P_i) \lambda_m^{-1/2}(P_i) \|A_{ij}^k\|\} \ .$$

Then it can be stated

Theorem 3. The system (27) is stable for all structural perturbations if there exists a vector $U_0 > 0$ such that

a) either $(M^k)'U_0 \leq 0 \quad \forall k = 1,2,\ldots,p$

b) or $M^k U_0 \leq 0 \quad \forall k = 1,2,\ldots,P$

The first condition a) expresses the fact that $\nu = V'U_0$ is a scalar Lyapunov function for (27) (weighted sum approach) while for b) the Lyapunov function is $\nu = \max_i v_i u_{0i}^{-1}$, which can only be used if we are dealing with a true differential system like (18). It is easy to see that conditions a) and b) are not equivalent and that furthermore a) may not hold while b) holds.

II. The Choice of the Lyapunov Functions (P_i matrices)

As pointed out in section I, naturally the stability conditions given depend on the choice of the Lyapunov functions associated to each subsystem. In the vector Lyapunov function approach it was seen that the main parameter was the ratio α_i/θ_i. The best stability condition will be achieved if we are able to solve the parametric optimization problem

$$\max_{D_i} \alpha_i \theta_i^{-1}: \; A_i'P_i + P_iA_i + D_i'D_i = 0 \tag{29}$$

where

$$\alpha_i = \frac{1}{2} \min \frac{x_i'D_i'D_ix_i}{x_i'P_ix_i}, \qquad \theta_i = [\lambda_M(P_i)\lambda_m^{-1}(P_i)]^{1/2} .$$

Because of the non differentiability of the cost function this is a very intricate problem and the following will be devoted to derive a noticeably equivalent problem, much simpler to solve.

II.1. An "Equivalent" Problem

First let us remark that the problem (29) is equivalent to

$$\max_{D_i,\beta_i} \beta_i\theta_i^{-1} : (A_i+\beta_i I)'P_i + P_i(A_i+\beta_i I) + D_i'D_i = 0 \ , \quad 0 \le \beta_i \le \sigma_i \tag{30}$$

where I is the $n_i \times n_i$ identity matrix and σ_i the stability degree of the matrix A_i (i.e. $A_i+\sigma_i I$ is stable).

Proof. Let $\alpha_i^0(\theta_i^0)^{-1}$ and $\beta_i^+(\theta_i^+)^{-1}$ be the maximum achieved respectively by problems (23) and (24). First

$$\beta_i^+(\theta_i^+)^{-1} \le \alpha_i^0(\theta_i^0)^{-1} \ . \tag{31}$$

The restriction in (30) can be written $A_i'P_i + P_iA_i + D_i'D_i + 2\beta_i^+P_i = 0$ so that the corresponding α_i is

$$\alpha_i = \frac{1}{2}\min_{x_i}\frac{x_i'[D_i'D_i+2\beta_i^+P_i]x_i}{x_i'P_ix_i} = \beta_i^+ + \frac{1}{2}\min_{x_i}\frac{x_i'D_i'D_ix_i}{x_i'P_ix_i}$$

which obviously implies

$$\beta_i^+(\theta_i^+)^{-1} \le \alpha_i(\theta_i^+)^{-1} \le \alpha_i^0(\theta_i^0)^{-1} \ .$$

But, we also have

$$\alpha_i^0(\theta_i^0)^{-1} \le \beta_i^+(\theta_i^+)^{-1} \ . \tag{32}$$

The restriction in (29) can be written as

$$(A_i+\alpha_i^0 I)'P_i + P_i(A_i+\alpha_i^0 I) + D_i'D_i - 2\alpha_i^0P_i = 0 \ .$$

By definition of α_i^0, the matrix $D_i'D_i-2\alpha_i^0P_i$ is a non negative definite matrix, so that there exists a matrix $\bar{D}_i$ with $\bar{D}_i'\bar{D}_i = D_i'D_i - 2\alpha_i^0P_i$, i.e., there exists β_i and $\bar{D}_i$ corresponding to problem (30) and then

$$\alpha_i^0(\theta_i^0)^{-1} \leq \beta_i(\theta_i^0)^{-1} \leq \beta_i^+(\theta_i^+)^{-1} .$$

(31) together with (32) implies

$$\alpha_i^0(\theta_i^0)^{-1} = \beta_i^+(\theta_i^+)^{-1} .$$

Problem (30) involving explictly $\theta_i = (\lambda_M/\lambda_m)^{1/2}$ is still very difficult to solve in this form. From the following considerations one can propose another problem which is not strictly equivalent but, as far as I know, provides good results in that sense that, most often, it really implies an improvement in the ratio α_i/θ_i.

For all positive definite $n \times n$ P matrix, one has

$$1 \leq \frac{1}{n^2} \operatorname{Tr}(P)\operatorname{Tr}(P^{-1}) \leq \theta^2 \leq \operatorname{Tr}(P)\operatorname{Tr}(P^{-1})$$

and

$$\frac{1}{n^2} \operatorname{Inf}[\operatorname{Tr}(P)\operatorname{Tr}(P^{-1})] = \operatorname{Inf}(\theta^2) = 1 \quad \text{iff} \quad \lambda_i(P) = \lambda = \text{etc.}$$
$$i = 1,2,\ldots,n \quad (\lambda_i \text{ eigenvalue of } P)$$

Problem (30) can now be written as

$$\min_{0 \leq \beta_i \leq \sigma_i} \left\{\frac{1}{\beta_i} \min_{D_i} \theta_i(P_i) : (A_i+\beta_i I)'P_i+P_i(A_i+\beta_i I)+D_i'D_i = 0\right\}$$

and, from the above considerations, let us deal with the problem:

$$\min_{0 \leq \beta_i \leq \sigma_i} \left\{\frac{1}{\beta_i}\theta(P_i);\ P_i \text{ given by } \min_{D_i} J(D_i) = \operatorname{Tr}(P_i)\operatorname{Tr}(P_i^{-1}) : (A_i+\beta_i I)'P_i+P_i(A+\beta_i I)+D_i'D_i = 0\right\} \qquad (32)$$

II.2. An Algorithm for Solving (32) [10]

Solution of (32) can be searched for by gradient techniques in the following way. For a given β_i, solve the problem $\min_{D_i} J(D_i)$ which amounts to calculating the gradient matrix dJ/dD_i. The minimization with respect to β_i is a problem of one-dimensional optimization performed by a step-by-step search using the knowledge at each step of the function θ/β.

The determination of the gradient matrix dJ/dD_i is easily derived from the following: Let $f(x)\colon R^{n\times m} \to R$ be such that

$$f(x+\varepsilon\Delta x) = f(x) + \varepsilon \mathrm{Tr}[M(x)\cdot\Delta x] \quad \text{as} \quad \varepsilon \to 0 \ .$$

Then

$$\frac{dF}{dx} = M'(x) \ .$$

We get:

Theorem 4. The gradient matrix of the function $J(D_i) = \mathrm{Tr}(P_i)\mathrm{Tr}(P_i^{-1})$, where P_i is the solution of the Lyapunov equation

$$(A_i'+\beta_i I)P_i + P_i(A_i+\beta_i I) + D_i'D_i = 0 \ , \quad 0 \le \beta_i \le \sigma_i \ ,$$

with respect to D_i is such that:

$$\begin{aligned} \frac{dJ}{dD_i} &= 2D_i S \\ (A_i+\beta_i I)S + S(A_i+\beta_i I)' + T &= 0 \\ T &= \mathrm{Tr}[P_i^{-1}]I - \mathrm{Tr}[P_i]P_i^{-2} \end{aligned} \tag{33}$$

Proof. $J[D_i+\varepsilon\Delta D_i] = \mathrm{Tr}\{P(D_i+\varepsilon\Delta D_i)\}\mathrm{Tr}\{P^{-1}(D_i+\varepsilon\Delta D_i)\}$ where $P(D_i+\varepsilon\Delta D_i)$ is the solution of

$$F'P(D_i+\varepsilon\Delta D_i) + P(D_i+\varepsilon\Delta D_i)F + (D_i+\varepsilon\Delta D_i)'(D_i+\varepsilon\Delta D_i) = 0 \ , \quad F = A_i + \beta_i I \tag{34}$$

Restricting the calculations to the first degree in ε it becomes

$$P(D_i+\varepsilon\Delta D_i) = P(D_i) + \varepsilon\Delta P(D_i,\Delta D_i)$$

with

$$\Delta P(D_i+\varepsilon\Delta D_i) = \int_0^\infty e^{F't}(D_i'\Delta D_i+\Delta D_i'D_i)e^{Ft}dt \ .$$

Furthermore,

$$P^{-1}(D_i+\varepsilon\Delta D_i) = P(D_i) - \varepsilon P^{-1}(D_i)\Delta P(D_i,\Delta D_i)P^{-1}(D_i)$$

so that

$$\begin{aligned} J(D_i+\varepsilon\Delta D_i) = J(D_i) &+ \varepsilon Tr[\Delta P(D_i,\Delta D_i)]Tr[P^{-1}(D_i)] \\ &- \varepsilon Tr[P(D_i)]Tr[P^{-1}(D_i)\Delta P(D_i,\Delta D_i)P^{-1}(D_i)] \end{aligned} \tag{35}$$

since

$$Tr[\Delta P(D_i,\Delta D_i)] = 2Tr[\int_0^\infty e^{Ft}e^{F't}D_i'dt\circ\Delta D_i]$$

and

$$Tr[P^{-1}(D_i)\Delta P(D_i,\Delta D_i)P^{-1}(D_i)] = 2Tr[\int_0^\infty e^{Ft}P^{-2}(D_i)e^{F't}D_i'dt\circ\Delta D_i]$$

one gets

$$\frac{dJ}{dD_i} = 2D_i\int_0^\infty e^{Ft}Te^{F't}dt = 2D_iS \ . \qquad \text{Q.E.D.}$$

Remark. It is clear that the proposed algorithm enables one to give only a local minimum. Furthermore it is not claimed that the solution of problem (32) always results in an increase of the ratio β/θ; however from numerous numerical experiments, chosen at random, it seems that this is the general case.

As an example, let us consider

$$\min_D J(D) = Tr[P]Tr[P^{-1}]: A'P + PA + D'D = 0$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix}$$

Starting with an initial condition on D being the unity matrix, the final "optimal value for D" was found to be

$$D_{opt} = \begin{bmatrix} 0.08 & -0.07 & 0.18 & 0.05 \\ -0.03 & 0.59 & 0.15 & 0.55 \\ 0.26 & -0.01 & 0.05 & 0.38 \\ 0.16 & 0.96 & 0.76 & 1.20 \end{bmatrix}$$

With D the unity matrix $\theta^2 \cong 84$ and with D_{opt} $\theta^2 = 6.5$.

III. Multilevel and Decentralized Control Schemes

Let us now consider the system

$$\dot{x}_i = A_i x_i + B_i u_i + \sum_{j=1}^{N} A_{ij} x_j \,, \quad i = 1,2,\ldots,N \tag{36}$$

and the problem of the design of a control $u = \{u_i;\ i=1,2,\ldots,N\}$ in order to stabilize the overall controlled system (36), the stability being checked by means of the vector Lyapunov function approach.

In several papers, i.e. [11], a two level control scheme was proposed with the following features: the first level (local) is designed in order to insure a certain stability degree to each isolated subsystem, the second one (higher hierarchical level) in order to decrease the amplitude of the interconnection terms:

$$u_i = K_{i\ell} x_i + \sum_{j}^{N} K_{ij} x_j \,, \quad i = 1,2,\ldots,N \tag{37}$$

where $K_{i\ell}$ is the matrix gain of the local control, and K_{ij} the matrix gain of the control delivered by the higher level. A way for the choice of the K_{ij} matrices is the use of the Moore-Penrose pseudo-inverse which is known to minimize $\|A_{ij}-B_iK_{ij}\|$. It is not the purpose to discuss the quality of such a design but to point out that, once the higher level is fixed, one is faced with a decentralized control problem, namely, the choice of

$$u_i = -K_i x_i \ , \quad i = 1,2,\ldots,N$$

such that system (36) is asymptotically stable. The following is restricted to such a problem, recalling that the method used to test the stability properties is the vector Lyapunov function method.

Taking into account the results of section II, one can propose the following algorithm:

1st step: Determination of the norms of the interconnection matrices. Let

$$H = \{h_{ij}: h_{ij} = \|A_{ij}\|;\ i,j = 1,2,\ldots,N\}$$

2nd step: For each subsystem, design of a control $u_i = -K_i x_i$ so as to give a stability degree equal to σ_i.

3rd step: Determination of the P_i matrices (section II).

4th step: Stability test (section I); if not satisfied increase of σ_i and go to step 2.

In fact, the choice itself of the K_i matrices should affect the results obtained in step 3 and henceforth step 4, and there is also a problem of the best choice of the K_i matrices according to the structural properties of the matrices A_i and B_i in order to get the best stability conditions. This problem is not investigated here, where it is assumed (as in all the papers on the subject) an a priori choice of the design method for K_i; for instance, pole placement techniques, or linear quadratic optimization.

The question which can now be asked is: "Is it possible to give classes of systems for which the preceding algorithm will work, i.e., will it provide us with a decentrally stabilizing control?"

The answer is: generally, no (save for very trivial cases such as B_i a regular $n_i \times n_i$ matrix). The main reason for such an impossibility comes from the fact that the vector Lyapunov function approach needs a fairly great degree of aggregation to get equations like (7) (in some papers, the scalar Lyapunov functions v_i are called aggregation functions). The aggregation is particularly hard on the interaction terms. Indeed, there is a great loss of structural information when passing from A_{ij} to $\|A_{ij}\|$. This loss of information is deduced by the fact that to a given comparison system such as (8) there can be associated an infinite number of systems of type (1) or (36), from which it can be derived. Let us denote by (S) the set of all possible systems giving the same comparison system (with the same choice of the vector Lyapunov function V).

To the contrary, from such a remark it is often easily possible to give sufficient conditions for the preceding algorithm not to work, or, which is the same, necessary conditions for the algorithm to work. This can be done by simply trying to find in (S) systems which are known not to be stabilizable.

For instance, let the system (36) be constituted by A_i matrices in comparison form and B_i the column vector $[0,0,\ldots,1]'$. Then

<u>Theorem 5</u>. A necessary condition for the algorithm to work is that (H-I) be an asymptotically stable matrix.

The proof comes from the fact that for all P_i, $\alpha_i \theta_i^{-1} \leq 1$. This can be seen by considering the system

$$\overset{\circ}{x} = \begin{bmatrix} A_i - B_i K_i & A_{i2} \\ A_{2i} & A_i - B_i K_i \end{bmatrix} x \qquad (38)$$

The comparison system is written

$$\overset{\circ}{W} = \begin{bmatrix} -\alpha_i \theta_i^{-1} & \|A_{i2}\| \\ \|A_{2i}\| & -\alpha_i \theta_i^{-1} \end{bmatrix} w$$

which is stable whenever $\alpha_i \theta_i^{-1} \geq \sqrt{\|A_{i2}\| \|A_{2i}\|}$. Choosing $A_{i2} = A_{2i} = \begin{bmatrix} 01000\cdots \\ \hline 0 \end{bmatrix}$. The system (38) having one zero fixed mode cannot be asymptotically stabilized which implies necessarily that

$$\alpha_i \theta_i^{-1} \leq 1 \ .$$

This result can be extended to the multivariable case when (A_i, B_i) is controllable, by using similar transformations, but this involves many calculations.

In general, the following can be stated:

<u>Theorem 6</u>. Let I_{i0} be the set of indices of the null rows of matrices B_i, $i = 1,2,\ldots,N$, and assume that

$$I_{i0} \neq \Phi \quad \forall i = 1,2,\ldots,N \ .$$

A necessary condition for the convergence of the algorithm is that the matrix $[H - \mathrm{diag}(r_i)]$ is asymptotically stable.

$$r_i = \min_{j \in I_{i0}} \sqrt{\sum_{k=1}^{n_i} a_{jk}^2} \ ,$$

i.e. r_i is the minimum of the norms of the rows of A_i corresponding to the null rows of B_i.

The proof is similar to the above one and is omitted.

Conclusion

This paper was concerned with the use of the vector Lyapunov function method for some problems of interest in the field of interconnected systems: stability, stabilizability. It has been shown that the use of the vector Lyapunov approach (together with the comparison principle) can provide in some cases better results than the weighted sum approach, although the obtention of a eomparison system generally implies a degree of approximation higher than for the weighting sum approach. This can be explained by the fact that the true differential system, which is the comparison system, enables one to use a wider class of techniques for its study than differential inequalities (used in the weighted sum approach). In some sense there is a tradeoff between the two approaches. It is quite obvious that the stability conditions depend on the choice of each Lyapunov function, associated to each subsystem. In the case of linear time invariant systems, a problem was defined in order to try to get "good" Lyapunov functions. The problem is a parametrical optimization problem which can easily be solved by the gradient method. In the last section the problem of decentralized control is discussed within the framework of the use of vector Lyapunov functions methods. This method which is conceptually simple suffers from the major drawback of needing a relatively high amount of approximation, which results in a loss of structural information. This point is discussed, giving rise to some simple necessary conditions for stabilizability under decentralized structure.

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