THE OUTPUTS OF JACKSONIAN NETWORKS ARE POISSONIAN

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ABSTRACT

An abstract model for the study of Markovian queuing systems is developed and used to obtain a necessary and sufficient condition for a given set of departure processes to be independent of the state of the system. This criterion is used to prove that in equilibrium the outputs of a Jacksonian network are Poisson processes independent of each other and of the state of the network. This result holds also when different classes of customers are routed through the network with different probabilities so long as at each node they are all served at the same rate. This condition cannot be relaxed since the output of an M/M/1 queue with two classes of customers being served at different rates is shown not to be Poisson.

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1. INTRODUCTION

The nature of the output process of a queuing system was first studied by Burke [1] and Reich [2]. They showed that in equilibrium the output of an M/M/m queue is Poisson and, more significantly, that the number of customers in queue (the state of the system) at any time is independent of past departure times. This result which is known as the output theorem holds even when the rate of the exponential server depends upon the state of the system (see [2,3,4].)

The output theorem has some useful consequences for the study of a network of M/M/m queues. For example, consider the simple case of the two-node tandem network of Figure 1 in which node i is a queuing system consisting of a queue and m_i exponential sources. Customers enter the network at node 1, queue up for service and upon completion of service immediately join the queue at node 2. By the output theorem, in equilibrium, the departures from node 1 which are simultaneously the arrivals at node 2 form a Poisson process and so, again by the output theorem, the outputs of the network form a Poisson process. Moreover, since the departure times from node 1 are independent of the number of customers in queue at node 1, therefore

$$P(k_1,k_2) = P_1(k_1)P_2(k_2) \tag{1.1}$$

where $P(k_1,k_2)$ is the steady-state joint probability that there are k_i customers in queue at node i, and the $P_i(k_i)$ are the marginal probabilities.

Consider next a <u>feedforward</u> network as in Figure 2, in which each node i is an $M/M/m_i$ queuing system. Upon leaving node 1 a customer joints the queue at node 2 with probability r_{12} or the queue at node 3 with probability $r_{13} = 1-r_{12}$, whereas upon leaving nodes 2 or 3 a customer immediately queues up at node 4. Once again by the output theorem, in equilibrium, the total departures from node 1 form a Poisson process, and so, because of the independent sampling of these departures, the arrivals at nodes 2 and 3 are also independent Poisson processes. It follows, by the output theorem again, that the arrivals and departures at node 4 are Poisson. The output theorem does not imply however that the number of customers in queue at the various nodes are independent, i.e.,

$$P(k_1,k_2,k_3,k_4) = P_1(k_1)..P_4(k_4), \tag{1.2}$$

where P is the steady-state joint probability and the P_i are the various marginal probabilities.

The product formula (1.2) is a special case of Jackson's theorem [5]. Jackson considered an arbitrary <u>feedback</u> network of n nodes, say, as in Figure 3. Customers enter the system at the nodes. The <u>external</u> arrival process at node i is an independent Poisson process with rate γ_i. The <u>i</u>th node is an $M/M/m_i$ queuing system in which each of the m_i servers is exponential with parameter μ_i. Upon completing service at node i a customer either immediately joints the queue at j with probability r_{ij} or leaves the network from node i with probability $1-\sum_j r_{ij}$. Thus the total flow of customers into node i consists of the external flow with rate γ_i and the flow from the nodes in the network. Let λ_i be the average total rate of flow into i. The λ_i satisfy the equations

$$\lambda_i = \gamma_i + \sum_j r_{ji}\lambda_j, \; i = 1,2,\ldots,n. \tag{1.3}$$

The state of the network at time t is the vector $k(t) = (k_1(t),\ldots,k_n(t))$ where $k_i(t)$ is the queue length at node i. Suppose that (1.3) has a unique solution $\{\lambda_i\}$ and that the stability condition $\lambda_i < m_i\mu_i$ holds at each i. Jackson proved that, in equilibrium,

$$P(k_1,k_2,\ldots) = P_1(k_1)P_2(k_2)\ldots P_n(k_n). \tag{1.4}$$

Moreover the marginal probability $P_i(k_i)$ is the same as the probability of the queue length being k_i in an $M/M/m_i$ system with Poisson arrival of rate λ_i and m_i exponential servers with parameter μ_i.

Although Jackson found his conclusion to be "far from surprising" in view of the Burke-Reich theorem, in fact it is very remarkable in the context of a recent result of Burke [6]. Burke considers the simplest Jacksonian network of Figure 4 consisting of a single M/M/1 queueing system with feedback. Suppose that the arrival process is Poisson with rate γ, the exponential server has parameter μ and that the stability condition $\frac{\gamma}{r} < \mu$ holds. Suppose that the network is in equilibrium. Let E_t, S_t, F_t, D_t respectively denote the total number of arrivals into the node, the total leaving the node, the total fed back, and the total departures from the network, all in the interval $[0,t]$. It is an easy consequence of the output theorem that D_t is Poisson. On the other hand, Burke calculates the interarrival distribution of E_t and finds it not to be exponential so that E_t is not Poisson. Using a very different technique, Brémaud [7] shows that S_t and F_t are not Poisson either. (This result can also be derived from Burke's calculation.)

The Burke-Brémaud result suggests that in a Jacksonian network with feedback the arrivals into a node will not be Poisson. That this is in fact the case is proved in a companion paper [8]; however care must be taken to exclude the situations typified by the tandem queue of Figure 1 where the arrivals at both nodes are indeed Poisson. At the same time the Poisson nature of D_t in Figure 4 suggests that the conclusion of the output theorem might hold for all Jacksonian networks if one were to examine only those departures which <u>leave</u> the network without entering another node. This conjecture is proved here.

The remainder of the paper is organized as follows. In the next section an abstract model of a Markovian queuing network is proposed and a condition is derived which characterizes when a set of arrival or departure processes is independent of the state; as a trivial consequence of the independence it follows that such a set of processes is Poisson in equilibrium. In section 3 it is shown that the external departures from a Jacksonian network satisfy this condition, so that the output theorem conjectured above holds. In section 4 the condition is also verified for state-dependent service rates and in section 5 for Jacksonian networks with several classes of customers so long as at each node all classes of customers are served at the same rate. In section 5 a simple example is given to show that the assumption of class-independent service rate cannot be relaxed.

The method of proof relies heavily on the equations describing the conditional probabilities of the state at time t given that some arrival and departure processes are observed over the interval [0,t]. These equations, known as filtering formulas, are used here in the form given by Brémaud [9,13]. Slightly different versions of the filtering formulas have appeared previously [11,12].

2. A MODEL FOR A MARKOVIAN QUEUING NETWORK

X is a countable set, the state space. E_i, $i \in I = 1,2,\ldots$ are non-empty subsets of X, not necessarily disjoint. For each i,

$T_i : E_i \to X$ is a given function, the state transition function. It is assumed that there is a finite number n such that each x belongs to at most n different E_i. (This ensures that from x at most n different one-step transitions are possible.) $(\Omega,\mathcal{F},\mathcal{P})$ is a probability space on which are given independent Poisson processes $(N_t^i,\ t \geq 0)$ with rate λ^i, $i \in I$; $\sup \lambda^i = \lambda < \infty$ (N^i is a counting process, i.e., N_t^i is the number of events which occur in $[0,t]$.) Also given is the X-valued random variable X_0, the initial state, and X_0 is independent of the (N_t^i). Define the sub-σ-fields

$$\mathcal{F}_t^{N^i} = \sigma(N_s^i, s \leq t),\ \mathcal{F}_t = \sigma(x_0) \vee \mathcal{F}_t^{N^1} \vee \mathcal{F}_t^{N^2} \vee \cdots .$$

The state process $(X_t, t \geq 0)$ is the unique right-continuous piecewise constant solution of the differential equation (2.1) below. For $A \subset X$ let $1(A)$ denote the characteristic function of A, and let

$$\xi_t(A) = 1(X_t \in A),\ \xi_t(x) = 1(X_t = x).$$

$$d\xi_t(x) = \sum_{i\in I} [\xi_{t-}(T_i^{-1}x)-\xi_{t-}(x)]\xi_{t-}(E_i)dN_t^i,\ \xi_0(x) = 1(X_0 = x). \tag{2.1}$$

In (2.1) $T_i^{-1}x = T_i^{-1}\{x\} = \{y \in E_i | T_i(y) = x\}$. Equation (2.1) can be interpreted with the help of the state transition diagram of Figure 5. Suppose that $X_{t-} = y \neq x$ so that $\xi_{t-}(x) = 0$, and that at time t the process N^i jumps, i.e., $dN_t^i = 1$. Then $X_t = x$, or $d\xi_t(x) = \xi_t(x)-\xi_{t-}(x) = 1$, if and only if (i) $y \in E_i$ so that the jump dN_t^i is "enabled" and (ii) $T_i(y) = x$ so that the state transits instantaneously from y to x. (The model will become more comprehensible in subsequent sections where particular examples are studied.) Notice that since any x belongs to at most n of the E_i and since $\lambda_i \leq \lambda$, therefore the probability that the state changes once in an interval of length dt is at most $n\lambda dt + o(dt)$ and the probability that it changes more than once is $o(dt)$.

Now fix $J \subset I$ and define the counting processes

$$S_t^i = \int_0^t \xi_{s-}(E_i)dN_s^i,\ i \in I,\ Y_t = \sum_{j \in J} S_t^j. \tag{2.2}$$

S_t^i is the number of transitions of type T_i which occur in $[0,t]$, while Y_t counts any transition of type T_j, $j \in J$. Our aim is to derive a condition which characterizes the independence of X_t and $\{Y_s, s \leq t\}$. To do this we follow [9] to obtain the conditional probability distribution of X_t given Y_s, $s \leq t$. From (2.1), (2.2) follows

$$d\xi_t(x) = \sum_{i \in I} [\xi_{t-}(T_i^{-1}x) - \xi_{t-}(x)]dS_t^i. \tag{2.3}$$

From the independence of the N^i and X_0 it follows that each N^i is a Poisson process with $(\mathcal{P},\mathcal{F}_t)$-intensity λ^i and so S^i is a counting process with $(\mathcal{P},\mathcal{F}_t)$-intensity $\lambda^i\xi_{t-}(E_i)$, that is, the process

$$S_t^i = \int_0^t \lambda^i\xi_{s-}(E_i)ds$$

is a $(\mathcal{P},\mathcal{F}_t)$-martingale.

Hence (2.3) can be rewritten as a semimartingale

$$\xi_t(x) = \xi_0(x) + \int_0^t f_s(x)ds + M_t(x), \tag{2.4}$$

where

$$f_s(x) = \sum_{i \in I} [\xi_s(T_i^{-1}x) - \xi_s(x)]\lambda^i\xi_s(E_i), \tag{2.5}$$

and $M_t(x)$ is the $(\mathcal{P},\mathcal{F}_t)$-martingale,

$$M_t(x) = \sum_{i\in I} \int_0^t [\xi_{s-}(T_i^{-1}x)-\xi_{s-}(x)](dS_s^i-\lambda^i\xi_s(E_i)ds) \tag{2.6}$$

For any $A \subset X$ and x, denote

$$\hat{\xi}_t(A) = E\{\xi_t(A)|\mathcal{F}_t^Y\} = \mathcal{P}\{X_t \in A|\mathcal{F}_t^Y\},$$

$$\hat{\xi}_t(x) = \hat{\xi}_t(\{x\}),$$

where, as usual, $\mathcal{F}_t^Y = \sigma(Y_s, s \le t)$.

It is known [11,12,13] that the process

$$Y_t - \int_0^t \hat{h}_s ds$$

is a $(\mathcal{P},\mathcal{F}_t^Y)$-martingale where

$$\hat{h}_t = \sum_{j\in J} \lambda^j \hat{\xi}_{t-}(E_j). \tag{2.7}$$

Moreover the process $(\hat{\xi}_t(x))$ is given by

$$\hat{\xi}_t(x) = \hat{\xi}_0(x) + \int_0^t \hat{f}_s(x)ds + \hat{M}_t(x) \tag{2.8}$$

where

$$\hat{f}_t(x) = E\{f_t(x)|\mathcal{F}_t^Y\} \tag{2.9}$$

and $(\hat{M}_t(x))$ is a $(\mathcal{P},\mathcal{F}_t^Y)$-martingale given by

$$\hat{M}_t(x) = \int_0^t k_s(x)(dY_s-\hat{h}_s ds), \tag{2.10}$$

where $k_t(x)$ can be easily calculated using the formulas in [9,12,13] as

$$k_t(x) = -\hat{\xi}_{t-}(x) + (\hat{h}_t)^{-1} \sum_{j\in J} \lambda^j \hat{\xi}_{t-}(T_j^{-1}x). \tag{2.11}$$

To calculate $\hat{f}_s(x)$ observe that if $A \subset X$, $B \subset X$, then

$$\xi_t(A)\xi_t(B) = 1(X_t \in A)1(X_t \in B) = 1(X_t \in A \cap B) = \xi_t(A \cap B),$$

so that from (2.5)

$$f_s(x) = \sum_{i \in I} \lambda^i [\xi_s(T_i^{-1}x) - \xi_s(x \cap E_i)]. \tag{2.12}$$

Here we used the fact that $T_i^{-1}x \subset E_i$ by definition and $x \cap E_i = \{x\} \cap E_i$ to simplify notation. Finally, from (2.12),

$$\hat{f}_s(x) = \sum_{i \in I} \lambda^i [\hat{\xi}_s(T_i^{-1}x) - \hat{\xi}_s(x \cap E_i)]. \tag{2.13}$$

It will prove useful to note that $P_t(x) = E\hat{\xi}_t(x)$ is just the unconditional probability that $X_t = x$ so that from (2.4), (2.5)

$$\frac{dP_t}{dt}(x) = \sum_{i \in I} \lambda^i [P_t(T_i^{-1}x) - P_t(x \cap E_i)]. \tag{2.14}$$

<u>Lemma 2.1</u> (Independence Criterion.) X_t and $\mathcal{F}_t^Y$ are independent for $t \geq 0$ if and only if for all x and $t \geq 0$

$$\sum_{j \in J} \lambda^j P_t(E_i) P_t(x) = \sum_{j \in J} \lambda^j P_t(T_j^{-1}x). \tag{2.15}$$

<u>Proof</u> The independence of X_t and $\mathcal{F}_t^Y$ is equivalent to

$$\hat{\xi}_t(x) = P_t(x) \text{ for all } x. \tag{2.16}$$

To prove necessity suppose (2.16) holds. Then $\hat{\xi}_t(x)$ must be continuous in t and so $k_t(x) \equiv 0$ in (2.10). From (2.11), (2.7) and (2.16),

$$0 = -P_t(x) + [\sum_{j \in J} \lambda^j P_t(E_j)]^{-1} \sum_{j \in J} \lambda^j P_t(T_j^{-1}x)$$

which is the same as (2.15).

To prove sufficiency suppose (2.15) holds. It is enough to show that $P_t(x)$ solves the equation of conditional probabilities (2.8). Substituting $\hat{\xi}_t(A) = P_t(A)$ into (2.11) and using (2.15) shows that $k_t(x) \equiv 0$, and so $\hat{M}_t(x) \equiv 0$ in (2.8). Since $\hat{\xi}_0(x) = \mathcal{P}\{X_0 = x | \mathcal{F}_0^Y\} = P_0(x)$, it only remains to verify that

$$P_t(x) = P_0(x) + \int_0^t \hat{f}_s(x)ds$$

$$= P_0(x) + \int_0^t \sum_{i\in I} \lambda^i [P_s(T_i^{-1}x) - P_s(x \cap E_i)]ds$$

from (2.13). But the equation above is identical to (2.14). ¤

Corollary 2.1 Suppose the process (X_t) is in equilibrium and let $P_t(x) \equiv P(x)$ be the steady-state distribution. Suppose the independence condition (2.15) holds. Then (Y_t) is a Poisson process with rate $\sum_{j\in J} \lambda^j P(E_j)$.

Proof By the equilibrium assumption and Lemma 2.1 it follows that the process

$$Y_t - \int_0^t \hat{h}_s ds = Y_t - [\sum_{j\in J} \lambda^j P(E_j)]t.$$

is a $(\mathcal{P},\mathcal{F}_t^Y)$-martingale. By Watanabe's thoerem [14,15], (Y_t) must be Poisson with rate $\Sigma_j \lambda^j P(E_j)$. ¤

Note that the independence condition is required to hold only in equilibrium. In fact in every interesting case the condition will not hold outside of equilibrium. The next result gives a condition for the counting processes (S_t^j), $j \in J$, introduced in (2.2) to be all mutually independent Poisson processes. The proof is similar to that of Lemma 2.1.

Lemma 2.2 Suppose the process (X_t) is in equilibrium with steady state distribution $P_t(x) \equiv P(x)$. Suppose that for all j in J and x in X

$$P(x)P(E_j) = P(T_j^{-1}x). \tag{2.17}$$

Then the S^j are independent Poisson processes with rate $\lambda^j P(E_j)$. Moreover X_t and $\{S_s^j | j \in J, s \leq t\}$ are independent.

<u>Proof</u> Let $\mathcal{F}_t^S = \bigvee_{j\in J} \mathcal{F}_t^{S^j}$, and for $A \subset X$ let

$$\tilde{\xi}_t(A) = E\{\xi_t(A)|\mathcal{F}_t^S\},\ \tilde{\xi}_t(x) = \tilde{\xi}_t(\{x\}).$$

It is known [11,12,13] that the process

$$S_t^j - \int_0^t \tilde{h}_s^j\, ds$$

is a $(\mathcal{P},\mathcal{F}_t^S)$-martingale where

$$h_t^j = \lambda^j \tilde{\xi}_{t-}(E_j);$$

moreover,

$$\tilde{\xi}_t(x) = \tilde{\xi}_0(x) + \int_0^t \tilde{f}_s(x)ds + \tilde{M}_t(x) \tag{2.18}$$

where

$$\tilde{f}_t(x) = E\{f_t(x)|\mathcal{F}_t^S\}$$

and $(\tilde{M}_t(x))$ is a $(\mathcal{P},\mathcal{F}_t^S)$-martingale,

$$\tilde{M}_t(x) = \sum_{j\in J} \int_0^t k_s^j(x)(dS_s - \tilde{h}_s^j ds). \tag{2.19}$$

The process $(k_t^j(x))$ can be readily shown to be

$$\begin{aligned} k_t^j(x) &= -\tilde{\xi}_t(x) + (\tilde{h}_t^j)^{-1}\ \lambda^j \tilde{\xi}_t(T_j^{-1}x) \\ &= -\tilde{\xi}_t(x) + [\tilde{\xi}_{t-}(E_j)]^{-1}\ \tilde{\xi}_t(T_j^{-1}x). \end{aligned} \tag{2.20}$$

Also,

$$\tilde{f}_t(x) = \sum_{i\in I} \lambda^i [\tilde{\xi}_t(T_i^{-1}x) - \tilde{\xi}_t(x \cap E_i)]. \tag{2.21}$$

Now suppose (2.17) is satisfied. Then we claim that $\tilde{\xi}_t(x) \equiv P(x)$ solves the conditional probability equations (2.18). Because substituting $\tilde{\xi}_t(x) = P(x)$ in (2.20), and using (2.17), implies $k_t^j(x) \equiv 0$ and so $\tilde{M}_t(x) \equiv 0$; (2.18) then reduces to

$$0 = \int_0^t \sum_{i\in I} \lambda^i [P(T_i^{-1}x) - P(x \cap E_i)],$$

which, from (2.14), certainly holds in equilibrium.

Since $\tilde{\xi}_t(x) = \mathcal{P}\{X_t = x | \mathcal{F}_t^S\} = P(x)$ it follows that X_t and $\mathcal{F}_t^S$ are independent. Furthermore $\tilde{h}_t^j = \lambda^j P(E_j)$, and so $(S_t^j - \lambda^j P(E_j)t)$ is a $(\mathcal{P}, \mathcal{F}_t^S)$-martingale. Watanabe's theorem [15] now implies that S^j is a Poisson process with rate $\lambda^j P(E_j)$, and for $\tau > t$, the future increment $S_\tau^j - S_t^j$ is independent of $\mathcal{F}_t^S$ from which follows the independence of these processes. ◻

3. OUTPUT THEOREM FOR JACKSONIAN NETWORKS

Consider a feedback network with n nodes. The arrivals of external customers at node i form an independent Poisson process with rate γ_i Node i is an M/M/1 queuing system with service parameter μ_i. A customer who completes service at i joins the queue at node j with probability r_{ij}, $j = 1,\ldots,n$, or leaves the network with probability $r_{i0} = 1 - \sum_j r_{ij}$. Let $\{\lambda_i\}$ be a solution, assumed to be unique, to the equations

$$\lambda_i = \gamma_i + \sum_j \lambda_j r_{ji}, \quad i = 1,\ldots,n.$$

Assume that the stability condition $\rho_i = \frac{\lambda_i}{\mu_i} < 1$ holds. Jackson's result states that, in equilibrium, the probability that there are k_i customers in queue (including the customer in service) at node i is

$$P(k_1,\ldots,k_n) = P_1(k_1)\ldots P_n(k_n), \tag{3.1}$$

$$P_i(k_i) = \rho_i^{k_i}(1-\rho_i). \tag{3.2}$$

Next this description of the network is transposed into the form of the abstract model. It is convenient to partition the transition functions (T_i), the enabling events (E_i) and the Poisson processes (N^i)

into three types corresponding to the different kinds of transitions. The state space is

$$X = \{x = (k_1,\ldots,k_n) \mid k_i \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of nonnegative integers.

(i) Internal transitions. For $1 \le i,j \le n$, let $E_{ij} = \{(k_1,\ldots,k_n) \mid k_i > 0\}$ $T_{ij}(k_1,\ldots,k_n) = (k_1,\ldots,k_i-1,\ldots,k_j+1,\ldots,k_n)$, and N^{ij} an independent Poisson process with rate $\mu_i r_{ij}$.

(ii) External arrivals. For $1 \le i \le n$, let $U_i = X$, $A_i(k_1,\ldots,k_n) = (k_1,\ldots,k_i+1,\ldots,k_n)$, and N^i an independent Poisson process with rate γ_i.

(iii) External departures. For $1 \le i \le n$, let $V_i = \{(k_1,\ldots,k_n) \mid k_i > 0\}$ $D_i(k_1,\ldots,k_n) = (k_1,\ldots,k_i-1,\ldots,k_n)$, and M^i an independent Poisson process with rate $\delta_i = \mu_i r_{i0}$.

Observe that instead of describing the server at node i by a single Poisson process of rate μ_i followed by independent sampling with probabilities $r_{i1},\ldots,r_{in},r_{i0}$ we are using an equivalent description of n+1 independent Poisson processes $N^{i1},\ldots,N^{in}$, M^i with rates $\mu_i r_{i1},\ldots,\mu_i r_{in},\mu_i r_{i0}$.

Theorem 3.1 In equilibrium, the outputs or external departures are Poisson processes independent of each other and of the state.

Proof Let

$$S_t^i = \int_0^t \xi_{s-}(V_i)\,dM_s^i$$

be one of the outputs. According to (2.17) it suffices to show that

$$P(x)P(V_i) = P(D_i^{-1}x) \text{ for all } x. \tag{3.3}$$

Since $V_i = \{(k_1,\ldots,k_n) | k_i > 0\}$, it follows from (3.1), (3.2) that

$$P(V_i) = \rho_i \tag{3.4}$$

On the other hand for $x = (k_1,\ldots,k_n)$, $D_i^{-1}x = (k_1,\ldots,k_{i+1},\ldots,k_n)$ and so from (3.1), (3.2)

$$P(D_i^{-1}x) = \rho_i P(x). \tag{3.5}$$

(3.3) now follows from (3.4), (3.5) and the theorem is proved. ¤

Remark Using Lemma 2.1 it can be seen that the transitions corresponding to the external departures are the only ones which can be independent of the state. Since $T_{ij}(E_{ij}) \subsetneq X$ and $A_i(U_i) \subsetneq X$ therefore there must exist x_{ij} and x_i such that $T_{ij}^{-1}(x_{ij}) = A_i^{-1}(x_i) = \phi$. Hence $P(T_{ij}^{-1}x_{ij}) = 0 \neq P(E_{ij})P(x_{ij}) > 0$, $P(A_i^{-1}x_i) = 0 \neq P(U_i)P(x_i) > 0$, and so the independence criterion (2.15) is not satisfied. Of course, this does not imply that these transitions do not form a Poisson process. For example the external arrivals are certainly Poisson, as are the departures in equilibrium from node 1 in the tandem network of Figure 1.

4. QUEUES WITH STATE DEPENDENT SERVICE RATE

Theorem 3.1 can be shown to hold even when the service rate depends upon the number of customers in queue. To avoid cumbersome notation this is proved here for a single queuing system represented by the birth-death process of Figure 6. Take $X = \mathbb{N}$.

(i) Arrivals. Let $U = X$, $A(k) = k+1$, and M a Poisson process with rate λ.

(ii) Departures. For $i = 1,2,\ldots$ let $E_i = \{i\}$, $T_i(i) = i-1$, and N^i an independent Poisson process with rate μ_i.

Let

$$\pi_0 = 1, \ \pi_j = [\mu_1 \cdots \mu_j]^{-1} \lambda^j \tag{4.1}$$

and assume that $\Sigma\pi_j < \infty$. It is known that the process (X_t) has the steady-state distribution

$$P(X_t = k) = P(k) = [\Sigma\pi_j]^{-1} \pi_k. \tag{4.2}$$

Theorem 4.1 (Reich [2]) In equilibrium the departure process is Poisson and independent of the state.

Proof. The departure process is

$$Y_t = \sum_{i \geq 1} \int_0^t \xi_{s-}(E_i)\, dN_t^i,$$

and so, by Lemma 2.1 and Corollary 2.1, the result is true if and only if

$$\sum_{i \geq 1} \mu_i P(E_i) P(x) = \sum_{i \geq 1} \mu_i P(T_i^{-1} x), \text{ for all } x. \tag{4.3}$$

For $x = k$, $T_i^{-1} x = \phi$ if $i \neq k+1$ and $T_{k+1}^{-1} x = k+1$, so that (4.3) is equivalent to

$$\sum_{i \geq 1} \mu_i P(i) P(k) = \mu_{k+1} P(k+1)$$

which can be readily verified from (4.1), (4.2). ◻

5. NETWORKS WITH SEVERAL CLASSES OF CUSTOMERS

Consider a feedback network with n nodes and L classes of customers. The arrivals of external customers of class ℓ at node i form an independent Poisson process of rate γ_i^ℓ. Node i is an M/M/1 queuing system with service parameter μ_i independent of the class of customer. A customer of class ℓ who completes service at i changes into a customer of class m and either immediately joins the queue at node j with probability $r_{ij}^{\ell m}$ or leaves the network with probability $r_{i0}^{\ell m}$. Naturally

$\sum_{j=0}^{n} \sum_{m=1}^{L} r_{ij}^{\ell m} = 1$. Let $\{\lambda_i^\ell\}$ be a solution, assumed to be unique, to the equations

$$\lambda_i^\ell = \gamma_i^\ell + \sum_{j=1}^{n} \sum_{m=1}^{L} \lambda_j^m r_{ji}^{m\ell}, \quad i = 1,\ldots,n, \quad \ell = 1,\ldots,L.$$

Set $\lambda_i = \sum_\ell \lambda_i^\ell$. Assume the stability condition $\rho_i = \frac{\lambda_i}{\mu_i} < 1$. We can now reformulate the description in terms of the abstract model.

Set $\bar{X} = \{\theta\} \cup [\bigcup_{k=1}^{\infty} \{1,\ldots,L\}^k]$. θ denotes the empty string. The state space is $X = \bar{X}^n$. A state is then an n-tuple $x = (x_1,\ldots,x_n)$ where x_i represents the customers in queue at node i with the right-most element in x_i being the customer in service and the left-most the customer who arrived most recently. $x_i = \theta$ means the server is idle. To simplify the description of the transition functions let $a \cdot b$ denote the concatenation of two strings from $\bar{X}$. For example if $a = \langle a^1,\ldots,a^M \rangle$ then $a \cdot \ell = \langle a^1,\ldots,a^M,\ell \rangle$. Also for $x_i \in \bar{X}$ let

$a(x_i)$ = left-most element in x_i, $a(\theta) = 0$,

$d(x_i)$ = right-most element in x_i, $b(\theta) = 0$,

$k^\ell(x_i)$ = number of customers in x_i of class ℓ,

$k(x_i) = \sum_\ell k^\ell(x_i)$ = number of customers in x_i;

and if $k(x_i) > 0$ let $\hat{x}_i$ be obtained from x_i by deleting the right-most element.

(i) Internal transitions. For $1 \le i,j \le n$ and $1 \le \ell,m \le L$, let $E_{ij}^{\ell m} = \{x | d(x_i) = \ell\}$, $T_{ij}^{\ell m}(x_1,\ldots,x_n) = (x_1,\ldots,\hat{x}_i,\ldots,m \cdot x_j,\ldots,x_n)$, and $N_{ij}^{\ell m}$ an independent Poisson process with rate $\mu_i r_{ij}^{\ell m}$.

(ii) External arrivals. For $1 \le i \le n$ and $1 \le \ell \le L$, let $U_i^\ell = X$, $A_i^\ell(x_1,\ldots,x_n) = (x_1,\ldots,\ell \cdot x_i,\ldots,x_n)$, and N_i^ℓ an independent Poisson process with rate γ_i^ℓ.

(iii) External departures. For $1 \leq i \leq n$ and $1 \leq \ell, m \leq L$, let $V_i^{\ell m} = \{x | d(x_i) = \ell\}$, $D_i^{\ell m}(x_1,\ldots,x_n) = (x_1,\ldots,\hat{x}_i,\ldots,x_n)$, and $M_i^{\ell m}$ an independent Poisson process with rate $\mu_i r_{i0}^{\ell m}$.

Thus the transition function $T_{ij}^{\ell m}$ represents a customer of type ℓ who completes service at node i changes to type m and joins the queue at node j whereas $D_i^{\ell m}$ corresponds to this customer departing from the network. A_i^{ℓ} represents a customer of type ℓ arriving at node i from outside the network. To prove the output theorem the following extension of the steady state distribution formulas (3.1), (3.2) is needed.

Lemma 5.1 In equilibrium, the probability distribution of the state process (X_t) is given by

$$P(x_1,\ldots,x_n) = P_1(x_1)\ldots P_n(x_n), \tag{5.1}$$

$$P_i(x_i) = \rho_i^{k(x_i)}(1-\rho_i)\prod_{\ell=1}^{L}(p_i^{\ell})^{k^{\ell}(x_i)}, \tag{5.2}$$

where $p_i^{\ell} = \lambda_i^{\ell}\lambda_i^{-1}$.

Proof See the Appendix. ¤

Note that the distribution (5.1), (5.2) is the same that would prevail if the total arrivals of each class of customers at each node were an independent Poisson process (which is not true in general).

Theorem 5.1 In equilibrium, the outputs or external departures of each class of customers are Poisson processes independent of each other and of the state.

Proof Let

$$S_i^{\ell m}(t) = \int_0^t \varepsilon_{s-}(V_i^{\ell m})\,dM_i^{\ell m}(s)$$

be one of the outputs. According to Lemma 2.2 it suffices to show that

$$P(x)P(V_i^{\ell m}) = P[(D_i^{\ell m})^{-1}x] \quad \text{for all } x. \tag{5.3}$$

Since $V_i^{\ell m} = \{x \mid d(x_i) = \ell\}$, therefore by (5.1), (5.2),

$$P(V_i^{\ell m}) = \sum_{k=1}^{\infty} \rho_i^k (1-\rho_i)\rho_i^{\ell} = \rho_i p_i^{\ell}. \tag{5.4}$$

On the other hand $(D_i^{\ell m})^{-1}x = (x_1,\ldots,x_i \cdot \ell,\ldots,x_n)$ and by (5.1), (5.2)

$$P(x_1,\ldots,x_i \cdot \ell,\ldots,x_n) = P(x)\rho_i p_i^{\ell},$$

which together with (5.4) yields (5.3). ¤

6. EQUALITY OF SERVICE RATES FOR DIFFERENT CUSTOMERS IS NECESSARY

In the previous section the output theorem was proved under the condition that customers of different classes entering the same node are served at the same service rate. Here a simple example is given to show the necessity of this condition. Consider an M/M/1 queue in which the arrivals of customers of class $\ell = 1,2$ form an independent Poisson process of rate $\gamma^{\ell} > 0$. Service is provided on a FCFS basis, and the service time for a customer of class i is exponentially distributed with parameter μ^{ℓ}.

The notation of the preceding section is maintained with the obvious simplification resulting from the fact that there is only one node. In particular there are no internal transitions. The state space is

$$X = \{\theta\} \cup [\bigcup_{k=1}^{\infty} \{1,2\}^k].$$

(i) External arrivals. For $\ell = 1,2$ let $U_{\ell} = X$, $A_{\ell}(x) = \ell \cdot x$ and N^{ℓ} an independent Poisson process with rate γ^{ℓ}.

(ii) External departures. For $\ell = 1,2$ let $V_\ell = \{x | d(x) = \ell\}$, $D_\ell(x) = \hat{x}$, and M^ℓ an independent Poisson process with rate μ^ℓ.

As in section 2, for $A \subset X$ let $\xi_t(A) = 1(X_t \in A)$. Define the counting processes

$$S_t^\ell = \int_0^t \xi_{s-}(V_\ell) dM_s^\ell, \quad \ell = 1,2$$

and $Y_t = S_t^1 + S_t^2$, the total departure process. As before, let

$$\hat{\xi}_t(A) = E\{\xi_t(A) | \mathcal{F}_t^Y\}.$$

Then the process

$$Y_t - \int_0^t \hat{h}_s \, ds$$

is a $(\mathcal{P}, \mathcal{F}_t^Y)$-martingale where

$$\hat{h}_t = \mu^1 \hat{\xi}_{t-}(V_1) + \mu^2 \hat{\xi}_{t-}(V_2).$$

Theorem 6.1 In equilibrium (Y_t) is a Poisson process if and only if $\mu^1 = \mu^2$.

Proof The sufficiency is immediate from Theorem 5.1. To prove the necessity suppose that (Y_t) is Poisson. By Watanabe's theorem $\hat{h}_t$ must be constant and so, in equilibrium,

$$\hat{h}_t = \mu^1 \hat{\xi}_{t-}(V_1) + \mu^2 \hat{\xi}_{t-}(V_2) = \gamma^1 + \gamma^2, \tag{6.1}$$

and hence

$$\mu^1 \hat{\xi}_t(V_1) + \mu^2 \hat{\xi}_t(V_2) = \gamma^1 + \gamma^2 = \gamma \text{ say.} \tag{6.2}$$

Following the development in section 2, the conditional probability $\mathcal{P}\{X_t = x | \mathcal{F}_t^Y\}$ is given by (2.8), which is reproduced here,

$$\hat{\xi}_t(x) = \hat{\xi}_0(x) + \int_0^t \hat{f}_s(x) ds + \int_0^t k_s(x)(dY_s - \hat{h}_s ds) \tag{6.3}$$

From (2.13),

$$\hat{f}_t(x) = \sum_{\ell=1}^{2} \gamma^\ell[\hat{\xi}_t(A_\ell^{-1}x)-\hat{\xi}_t(x \cap U_\ell)] + \sum_{\ell=1}^{2} \mu^\ell[\hat{\xi}_t(D_\ell^{-1}x)-\hat{\xi}_t(x \cap V_\ell)], \tag{6.4}$$

and from (2.11),

$$k_t(x) = -\hat{\xi}_{t-}(x) + (\hat{h}_t)^{-1} \sum_{\ell=1}^{2} \mu^\ell\hat{\xi}_{t-}(D_\ell^{-1}x). \tag{6.5}$$

Substituting from (6.2), (6.4), (6.5) into (6.3), it follows that in between jumps of Y_t, i.e., when $dY_t = 0$,

$$\dot{\hat{\xi}}_t(x) = \frac{d}{dt}\hat{\xi}_t(x) = \sum_{\ell=1}^{2} \gamma^\ell[\hat{\xi}_t(A_\ell^{-1}x)-\hat{\xi}_t(x)] + \sum_{\ell=1}^{2} \mu^\ell[\hat{\xi}_t(D_\ell^{-1}x)-\hat{\xi}_t(x \cap V_\ell)]$$

$$+ \gamma\hat{\xi}_t(x) - \sum_{\ell=1}^{2} \mu^\ell\hat{\xi}_t(D_\ell^{-1}x)$$

$$= \sum_{\ell=1}^{2} [\gamma^\ell\hat{\xi}_t(A_\ell^{-1}x)-\mu^\ell\hat{\xi}_t(x \cap V_\ell)]; \tag{6.6}$$

whereas at a jump of Y_t, i.e., when $dY_t = 1$,

$$\hat{\xi}_t(x) = \gamma^{-1} \sum_{\ell=1}^{2} \mu^\ell\hat{\xi}_{t-}(D_\ell^{-1}x). \tag{6.7}$$

Note that $V_\ell = \{x \cdot \ell \mid x \in X\}$. Set $V_{\ell m} = \{x \cdot \ell \cdot m \mid x \in X\}$. Then it follows from (6.6) that, <u>before</u> the first jump of (Y_t),

$$\dot{\hat{\xi}}_t(\theta) = 0,$$

$$\dot{\hat{\xi}}_t(V_\ell) = \sum_{x\in V_\ell} \dot{\hat{\xi}}_t(x) = (\gamma-\mu^\ell)\hat{\xi}_t(V_\ell) + \gamma^\ell\hat{\xi}_t(\theta),$$

$$\dot{\hat{\xi}}_t(\ell) = \gamma^\ell\hat{\xi}_t(\theta) - \mu^\ell\hat{\xi}_t(\ell),$$

$$\dot{\hat{\xi}}_t(V_{\ell m}) = (\gamma-\mu^m)\hat{\xi}_t(V_{\ell m}) + \gamma^\ell\hat{\xi}_t(m).$$

This system of equations can be considered recursively and its general solution is

$$\hat{\xi}_t(V_\ell) = a_\ell \exp(\gamma-\mu^\ell)t + b_\ell, \tag{6.8}$$

$$\hat{\xi}_t(V_{\ell m}) = a_{\ell m} \exp(\gamma-\mu^m)t + b_{\ell m} \exp - \mu^m t + c_{\ell m}. \tag{6.9}$$

The constants can be found by observing that if $\gamma < \mu^\ell$ the system is stable and then (6.8), (6.9) must converge to the unique equilibrium which can be easily computed. This gives

$$b_\ell = \gamma^\ell P(\theta)(\mu^\ell-\gamma)^{-1}, \tag{6.10}$$

$$c_{\ell m} = \gamma^\ell \gamma^m P(\theta)[\mu^m(\mu^m-\gamma)]^{-1}. \tag{6.11}$$

Substituting (6.8), (6.10) into (6.2) gives

$$\gamma = \sum_{\ell=1}^{2} \mu^\ell \gamma^\ell (\mu^\ell-\gamma)^{-1} P(\theta). \tag{6.12}$$

Next, at the first jump of Y_t, we obtain from (6.7)

$$\gamma\hat{\xi}_t(V_\ell) = \sum_{m=1}^{2} \mu^m \hat{\xi}_{t-}(V_{\ell m}),$$

and so, from (6.2),

$$(\gamma)^2 = \gamma \sum_{\ell=1}^{2} \mu^\ell \hat{\xi}_{t-}(V_\ell) = \sum_{\ell,m} \mu^\ell \mu^m \hat{\xi}_{t-}(V_{\ell m}) \tag{6.13}$$

Hence, before the first jump of Y_t,

$$(\gamma)^2 \equiv \sum_{\ell,m} \mu^\ell \mu^m \hat{\xi}_t(V_{\ell m}). \tag{6.14}$$

Substituting (6.9), (6.11) into (6.14) implies

$$(\gamma)^2 = \sum_{\ell,m} \mu^\ell \gamma^\ell \gamma^m (\mu^m-\gamma)^{-1} P(\theta)$$

and substituting for $P(\theta)$ from (6.12) gives

$$\gamma \sum_\ell \mu^\ell \gamma^\ell (\mu^\ell-\gamma)^{-1} = \sum_{\ell,m} \mu^\ell \gamma^\ell \gamma^m (\mu^m-\gamma)^{-1}$$

Substituting $\gamma^1 + \gamma^2$ for γ on the left leads, after some simplification, to $\mu^1 = \mu^2$ as required. □

7. CONCLUSION

The Burke-Reich output theorem has been generalized to show that, in equilibrium, customers leaving from any node of a feedback network of M/M/1 queues form a Poisson process which is independent of the state of the network. Moreover the departures from different nodes are independent. The result is true even if there are several classes of customers and even if the service rate is state-dependent so long as at each node the service rate is independent of the customer class. This final condition is necessary. The techniques of proof rest heavily upon recent formulas of the conditional probability of the state given that a subset of the transitions are observed. Although flows internal to the network as not generally independent of the state they may nevertheless be Poisson. These flows are characterized in terms of the network topology is a companion paper.

REFERENCES

[1] P.J. Burke, "The output of a queuing system," Operations Research, vol. 4, no. 6, 699-704, Dec. 1956.

[2] E. Reich, "Waiting times when queues are in tandem," Ann. Math. Statist., vol. 28, 527-530, 1957.

[3] H. Kobayashi and A.G. Konheim, "Queueing models for computer communications system analysis," IEEE Trans. on Communications, vol. COM-25, no. 1, 2-29, Jan. 1977.

[4] P. Brémaud, "On the output theorem of queueing theory, via filtering," preprint, 1977.

[5] J.R. Jackson, "Networks of waiting lines," Operations Research, vol. 5, no.4, 518-521, Aug. 1957.

[6] P.J. Burke, "Proof of a conjecture on the interarrival-time distribution in an M/M/1 queue with feedback," IEEE Transactions on Communications, vol. COM-24, no. 5, 575-576, May 1976.

[7] P. Brémaud, "Streams of a M/M/1 feedback queue in statistical equilibrium," preprint, Sept. 1977.

[8] J. Walrand and P. Varaiya, "When is a flow in a Jacksonian network Poisson?" Electronics Research Laboratory Memo ERL M78/59, August 1978.

[9] P. Brémaud, "La méthode des semi-martingales en filtrage lorsque l'observation est un processus ponctuel marqué," Séminaire de Probabilités X, Univ. de Strasbourg, Lecture Notes in Math. No. 511, Berlin: Springer, 1-18, 1976.

[10] P. Brémaud, "Estimation de l'état d'une file d'attente et du temps de panne d'une machine par la méthode de semi-martingales," Adv. Appl. Prob., vol. 7, 845-863, 1975.

[11] R.K. Boel, P. Varaiya, and E. Wong, "Martingales on jump processes II: Applications," SIAM J. Control, vol. 13, no. 5, 1022-1061, Aug. 1975.

[12] J.H. Van Schuppen, "Filtering, prediction and smoothing for counting process observations, a martingale approach," SIAM J. Appl. Math., vol. 32, no. 3, 552-570, May 1977.

[13] P. Brémaud and J. Jacod, "Processus ponctuels et martingales: résultats récents sur la modélisation et le filtrage," Adv. Appl. Prob., vol. 9, no. 2, 362-416, June 1977.

[14] S. Watanabe, "On discontinuous additive functionals and Lévy measures of Markov processes," Japanese J. Math., vol. 34, 53-70, 1964.

[15] H. Kunita and S. Watanabe, "On square integrable martingales," Nagoya Math. J., vol. 30, 209-245, 1967.

APPENDIX. Proof of Lemma 5.1.

The unconditional probabilities $P_t(x) = P(X_t = x)$ are given by the differential equation (2.14). In the notation of Section 5 the differential equation is

$$\frac{dP_t}{dt}(x) = \sum_{i,j=1}^{n} \sum_{\ell,m=1}^{L} \mu_i r_{ij}^{\ell m} \{P_t[(T_{ij}^{\ell m})^{-1}x] - P_t(x \cap E_{ij}^{m})\}$$

$$+ \sum_{i=1}^{n} \sum_{\ell=1}^{L} \gamma_i^{\ell}\{P_t[(A_i^{\ell})^{-1}x] - P_t(x \cap U_i^{\ell})\}$$

$$+ \sum_{i=1}^{n} \sum_{\ell,m=1}^{L} \mu_i r_{i0}^{\ell m} \{P_t[(D_i^{\ell m})^{-1}x] - P_t(x \cap V_i^{\ell m})\}$$

$$= \sigma_t(x), \text{ say.} \qquad (A.1)$$

It will first be shown that $\sigma_t(x) \equiv 0$ for all x when $P_t(x) \equiv P(x)$ where P(x) is given by (5.1), (5.2). From (5.1), (5.2) and the definitions of the various transition functions the following evaluations are obtained.

$$P[(T_{ij}^{\ell m})^{-1}x] = \rho_i p_i^{\ell}(\rho_j p_j^{m})^{-1} \, 1(a(x_j) = m) \, P(x),$$

$$P(x \cap E_{ij}^{\ell m}) = 1(d(x_i) = \ell) \, P(x),$$

$$P[(A_i^{\ell})^{-1}x] = (\rho_i p_i^{\ell})^{-1} \, 1(a(x_i) = \ell) \, P(x),$$

$$P(x \cap U_i^{\ell}) = P(x),$$

$$P[(D_i^{\ell m})^{-1}x] = \rho_i p_i^{\ell} \, P(x),$$

$$P[x \cap V_i^{\ell m}] = 1(d(x_i) = \ell) \, P(x).$$

Here 1(•) is the indicator function of the set (•). Using these formulas, and recalling that $\rho_i = \lambda_i \mu_i^{-1}$, $p_i^{\ell} = \lambda_i^{\ell} \lambda_i^{-1}$, we get

$$\sigma_t(x)[P_t(x)]^{-1} = \sum_{i,j,\ell,m} r_{ij}^{\ell m} \{\lambda_i^\ell \mu_j (\lambda_j^m)^{-1} 1(a(x_j) = m) - \mu_i 1(d(x_i) = \ell)\}$$

$$+ \sum_{i,\ell} \gamma_i^\ell \{\mu_i (\lambda_i^\ell)^{-1} 1(a(x_i) = \ell) - 1\}$$

$$+ \sum_{i,\ell,m} r_{i0}^{\ell m} \{\lambda_i^\ell - \mu_i 1(d(x_i) = \ell)\}.$$

Next substituting from the relations

$$\sum_{i,\ell} r_{ij}^{\ell m} \lambda_i^\ell = \lambda_j^m - \gamma_j^m , \quad \sum_{i,\ell,m} r_{i0}^{\ell m} \lambda_i^\ell = \sum_{i,\ell} \gamma_i^\ell,$$

where the second relation expresses the equality between total input and output rates, gives

$$\sigma_t(x)[P_t(x)]^{-1} = \sum_{j,m} (\lambda_j^m - \lambda_j^m)\mu_j(\lambda_j^m)^{-1} 1(a(x_j) = m) - \sum_{i,j,\ell,m} r_{ij}^{\ell m} \mu_i 1(d(x_i) = \ell)$$

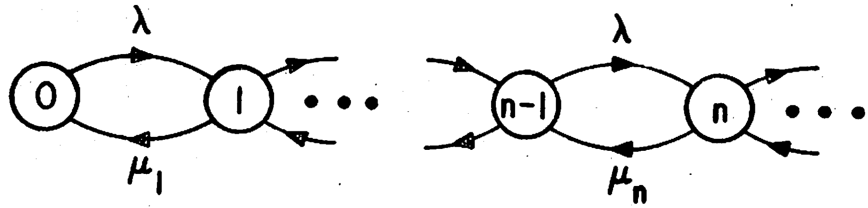
$$+ \sum_{i,\ell} \gamma_i^\ell \mu_i (\lambda_i^\ell)^{-1} 1(a(x_i) = \ell) - \sum_{i,\ell} \gamma_i^\ell$$

$$+ \sum_{i,\ell} \gamma_i^\ell - \sum_{i,\ell,m} r_{i0}^{\ell m} \mu_i 1(d(x_i) = \ell)$$

$$= \sum_{j,m} \mu_j 1(a(x_j) = m) - \sum_{i,\ell} \mu_i 1(d(x_i) = \ell) = 0.$$

Hence $P_t(x) \equiv P(x)$ is indeed a solution of (A.1). It can be verified that $\sum_x P(x) = 1$ and so the lemma is proved. □

FIGURE CAPTIONS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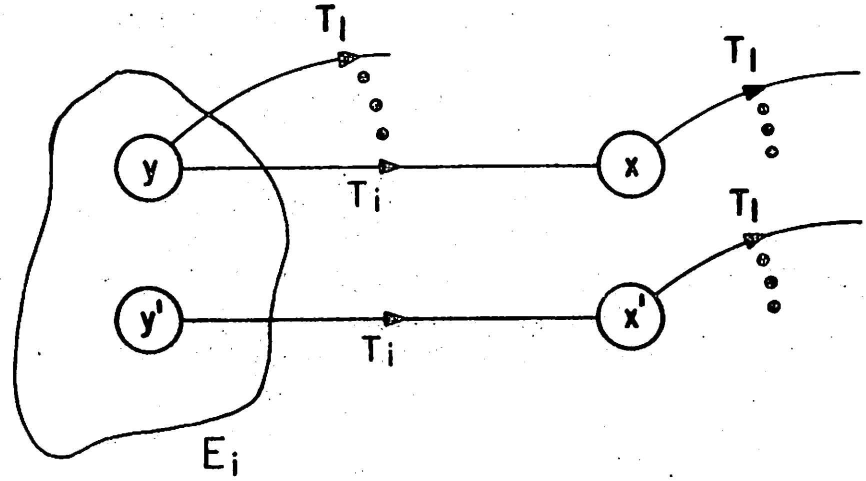
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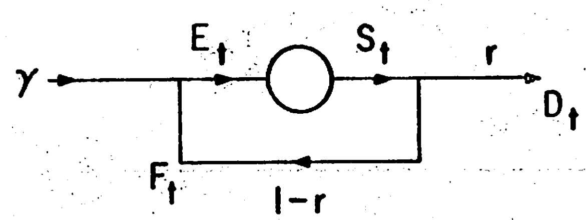
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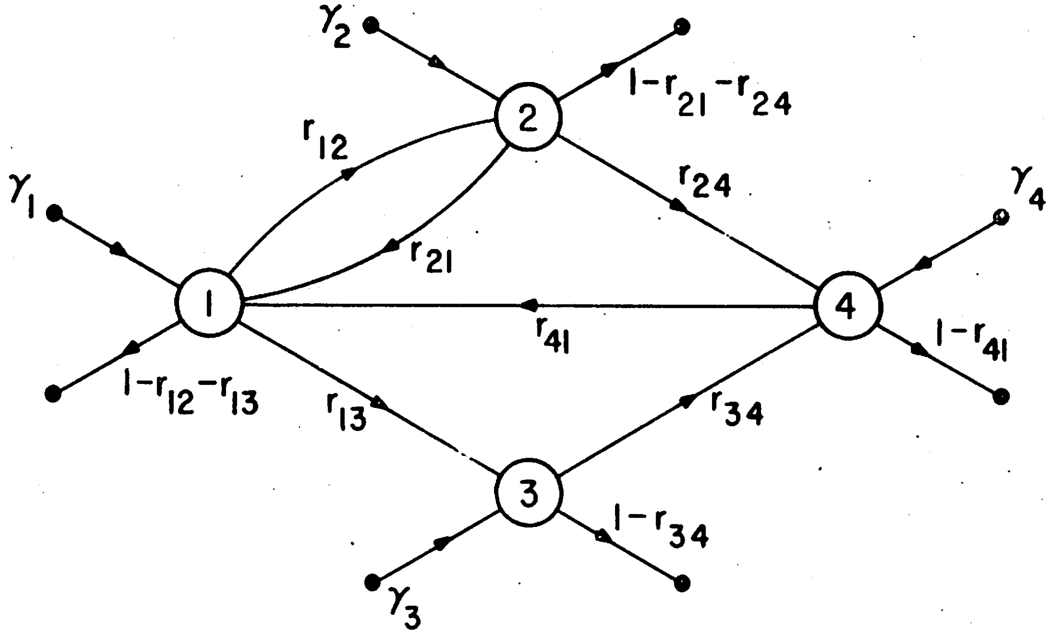
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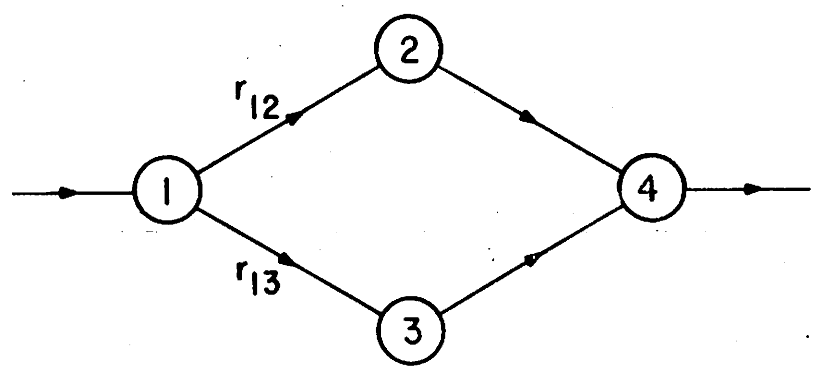
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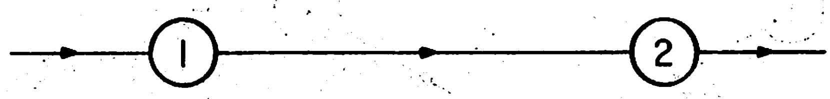
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