GEOMETRIC PROPERTIES OF RESISTIVE NONLINEAR n-PORTS:

TRANSVERSALITY, STRUCTURAL STABILITY, RECIPROCITY AND ANTI-RECIPROCITY

by

L.O. Chua, T. Matsumoto, and S. Ichiraku

Memorandum No. UCB/ERL M78/66

21 August 1978

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

GEOMETRIC PROPERTIES OF RESISTIVE NONLINEAR n-PORTS:

TRANSVERSALITY, STRUCTURAL STABILITY, RECIPROCITY AND ANTI-RECIPROCITY†

by

L.O. Chua, T. Matsumoto, and S. Ichiraku††

ABSTRACT

This paper presents several general properties of resistive nonlinear n-ports from a geometric point of view using recent tools from differential topology. The geometric approach is coordinate-free and hence the results of the paper do not depend on the particular choice of a tree, a loop matrix, a cut set matrix, a set of independent variables, etc.

Firstly, a classification is given of resistive n-ports into logical categories such as weakly regular n-ports, strongly regular n-ports, normal n-ports etc. Transversality of the internal resistor constitutive relations and the Kirchhoff space plays an important role in this paper. Secondly, a structural stability result is given. In this paper, structural stability means the persistence of the configuration space under small perturbations of the internal resistor constitutive relations. Essentially the result asserts that a resistive n-port is structurally stable if and only if the internal resistor constitutive relations are transversal to the Kirchhoff space. Thirdly, two basic perturbation techniques are given which guarantee the transversality of the internal resistor constitutive relations and the Kirchhoff space. The first technique involves element perturbations, i.e., perturbations of the internal resistor constitutive relations. The second technique involves network perturbations, i.e., by augmenting extra ports to an original n-port. Lastly, coordinate-free definitions of reciprocity and anti-reciprocity are given in terms of exterior product and symmetric product of two tensors, respectively, and then some of their properties are investigated.

---

†Research sponsored by the Office of Naval Research Contract N00014-76-0572.

††L.O. Chua and T. Matsumoto (presently on leave from Waseda University, Tokyo 160 Japan) are with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720. S. Ichiraku is with the Department of Mathematics, Yokohama City University, Yokohama 236, Japan.

I. Introduction

This paper presents some properties of resistive nonlinear n-ports from a geometric point of view. The geometric approach has the distinct advantage in that it is coordinate-free, and hence our results do not depend on the particular choice of a tree, a loop matrix, a cut set matrix, a set of independent variables, etc. Therefore, if a property of an n-port, such as reciprocity, is satisfied in terms of one coordinate system, then it is satisfied in terms of every other coordinate system. On the other hand, if a property fails to hold in terms of one choice of coordinates, then whatever coordinates one chooses, this property is not satisfied. Hence properties and results obtained by a geometric method are intrinsic to a given nonlinear n-port.

In Section II we review some basic geometric concepts that are needed for this paper. In Section III we give a classification of resistive nonlinear n-ports. Here, transversality of the internal resistor constitutive relations and the Kirchhoff space plays an important role. In Section IV we discuss structural stability of resistive nonlinear n-ports. Although this important concept has not been invoked in nonlinear circuits, we have found it to be of fundamental importance especially in device modeling. Essentially, the main result asserts that a resistive nonlinear n-port is structurally stable if and only if its internal resistor constitutive relations are transversal to the Kirchhoff space. In Section V we present two basic perturbation techniques which guarantee that the internal resistor constitutive relations are transversal to the Kirchhoff space. The first technique involves element perturbations, i.e., perturbations of the internal resistor constitutive relations. The second technique involves network perturbations, i.e., by augmenting extra ports to an original n-port. In Section VI we first give coordinate-free definitions of reciprocity and anti-reciprocity of resistive n-ports and then derive various criteria for reciprocity and anti-reciprocity in terms of a specific choice of coordinates. For these definitions we need to introduce "exterior product" and "symmetric product" of two tensors. We also show that reciprocity is closely related to the existence of potential functions.

Now, in order to motivate and justify the large hierarchy of n-ports to be introduced in this paper, consider the following examples illustrating how simple elements when interconnected could lead to complicated constitutive relations.

Example 1 Consider the 1-port shown in Fig. 1(a). The composite constitutive relation of this 1-port correponding to eight different combinations of the "internal" resistor constitutive relations are shown in Fig. 1(b)[1], where $i_{R_k} = f_{R_k}(v_{R_k})$ or $v_{R_k} = g_{R_k}(i_{R_k})$, $k = 1, 2$. Observe that some of the constitutive relations of the 1-port have self intersections, while some contain disconnected components.

Example 2 (norator) Consider the 1-port shown in Fig. 2(a) where the internal resistor constitutive relation is given by

$$i_R - i_P = 0. \tag{1}$$

Since the Kirchhoff laws are given by $v_P + v_R = 0$, $i_P - i_R = 0$, one can easily show that the constitutive relation of this 1-port is the whole space $\mathbb{R}^2$ (see Fig. 2(b)). This 1-port is called a norator [1]. Observe that in (1), i_P and i_R are coupled to each other.

Example 3 (nullator) Consider the 1-port shown in Fig. 3(a) where the internal resistor constitutive relations are given by $i_{R_1} - i_{R_2} = 0$, $v_{R_1} = 0$. Since the Kirchhoff laws are $v_P + v_{R_1} = 0$, $v_P - v_{R_2} = 0$, $i_P - i_{R_1} + i_{R_2} = 0$, one sees that the only possible value of (v_P, i_P) is $(0,0)$, i.e., the constitutive relation of this 1-port consists of one point only; namely, the origin (see Fig. 3(b)). This 1-port is called a nullator [1].

Example 4 Consider the 1-port shown in Fig. 4(a) where the constitutive relations of R_1 and R_2 are given by Figs. 4(b) and 4(c), respectively and the constitutive relation of R_3 is given by $i_{R_3} - i_{R_1} = 0$. It is not difficult to show that the constitutive relation of this 1-port includes the shaded area of Fig. 4(d).

Example 5 Consider the 1-port shown in Fig. 5(a) where the internal resistor constitutive relations are given respectively by $i_{R_1} - f_{R_1}(v_{R_1}) = 0$, $i_{R_2} - f_{R_2}(v_{R_2}) = 0$, $v_{R_3} = 0$, $i_{R_4} - i_{R_3} = 0$, $v_{R_5} - i_{R_2} = 0$, and f_{R_1} and f_{R_2} are arbitrary C^1 functions. It is not difficult to show that $v_P = i_{R_1} = f_{R_1}(v_{R_1})$, $i_P = i_{R_2} =$

[1]Observe the polarity of v_P is chosen opposite to the usual convention. This is done to simplify the hypotheses of several theorems in this paper.

$f_{R_2}(v_{R_2})$. Since $v_{R_1} = v_{R_2}$, we can write this as

$$v_P = f_{R_1}(\rho),\ i_P = f_{R_2}(\rho),\ \rho = v_{R_1} = v_{R_2}, \tag{2}$$

i.e., the constitutive relation of this 1-port admits a <u>global parametric representation</u> in terms of ρ. By specifying f_{R_1} and f_{R_2} we can realize an arbitrary 1-port whose constitutive relation is of the form (2). See Fig. 5(b), for example.

<u>Example 6</u> Consider the 1-port shown in Fig. 6(a) where the internal resistor constitutive relations are given respectively by $(v_{R_1}, i_{R_1}) \in \Lambda_{R_1}$, $(v_{R_2}, i_{R_2}) \in \Lambda_{R_2}$, $v_{R_3} - v_{R_1} + v_{R_4} = 0$, $i_{R_4} - i_{R_1} = 0$, where Λ_{R_1} and Λ_{R_2} are as given by Figs. 6(b) and (c) respectively. It is not difficult to show that $v_P = -v_{R_1} = -v_{R_2}$, $i_P = i_{R_1} = i_{R_2}$. Hence the constitutive relation of this 1-port is the intersection of Λ_{R_1} and Λ_{R_2} as shown in Fig. 6(d).

The preceding examples show that very exotic constitutive relations could result from interconnecting resistors with simple constitutive relations. These observations motivate our classifying resistive n-ports into various logical categories to be defined in Section III.

Another important property of n-ports is its <u>structural stability</u>[2] to be defined precisely in Section IV. Roughly speaking, an n-port made up of an interconnection of elements is structurally stable if it is persistent under small perturbations of the internal resistor constitutive relations. Consider, for example, (vi) of Fig. 1(b) where the constitutive relation of the composite 1-port consists of a curve and an isolated point. If one perturbs the constitutive relation of R_2 slightly as shown by the broken curve, then the isolated point disappears and the constitutive relation becomes the union of two curves. Hence a small change of the internal resistor constitutive relation gives rise to an abrupt change of the composite 1-port. It makes sense therefore to call this 1-port structurally unstable. In contrast to this, the other examples

[2]Since resistive n-ports do not have dynamics, structural stability in this paper has a different meaning from that of dynamical systems [2].

in Fig. 1(b) are structurally stable because small perturbations of the internal resistor constitutive relations do not essentially change the constitutive relation of the composite 1-port.

Consider next Example 2. Let the dependent current source be described by a C^1 approximation $\hat{f}(i_P)$ of the original identity function $f(i_P) = i_P$ such that $\hat{f}(i_P) > i_P$ if $i_P > 0$ and $\hat{f}(i_P) < i_P$ if $i_P < 0$. In other words, let the "perturbed" constitutive relation be given by $i_R - \hat{f}(i_P) = 0$. Then, since $i_R \neq i_P$ except at the origin, we see that $i_P \equiv 0$, and the new constitutive relation is the v_P-axis. Since the original constitutive relation is the whole space $\mathbb{R}^2$, this 1-port is structurally unstable.

Consider Example 3. Let $\hat{f}(i_{R_2})$ be a C^1 approximation of the identity function $f(i_{R_2}) = i_{R_2}$ such that

$$(D\hat{f})_{i_{R_2}} - 1 > 0 \text{ for all } i_{R_2} \tag{3}$$

where $(D\hat{f})_{i_{R_2}}$ denotes the derivative of $\hat{f}$ at i_{R_2}. Perturb the constitutive relation of R_2 in such a way that $i_{R_1} - \hat{f}(i_{R_2}) = 0$. Then we have

$$v_P = 0,\ i_P - \left(\hat{f}(i_{R_2}) - i_{R_2}\right) = 0. \tag{4}$$

It follows from (3) that the image of $\hat{f} - i_d$ contains an open interval of the i_p-axis, where i_d is the identity map. Since originally the only possible value of (v_P, i_P) was $(0,0)$, this 1-port is structurally unstable.

Consider Example 4. Let $\hat{f}(i_{R_1})$ be a C^1 approximation of $f(i_{R_1}) = i_{R_1}$, where $\hat{f}(i_{R_1}) > i_{R_1}$ if $i_{R_1} > 0$ and $\hat{f}(i_{R_1}) < i_{R_1}$ if $i_{R_1} < 0$. Perturb the constitutive relation of R_3 by $i_{R_3} - \hat{f}(i_{R_1}) = 0$. Then $i_{R_1} = i_{R_3} = 0$ and hence the constitutive relation of the composite 1-port coincides with that of R_2 which is a 1-dimensional curve. Hence this 1-port is structurally unstable.

Similarly, we will give a rigorous proof later showing the 1-port of Example 5 is structurally stable, while that of Example 6 is structurally unstable. We will also show that transversality of the internal resistor constitutive relations and the Kirchhoff space is crucial for structural stability.

General Remarks For simplicity, we will sometimes abuse our notation with regards to the transpose of a vector or a matrix. To avoid wordiness we will usually refer to the constitutive relation of an n-port instead of the constitutive relation of a "composite" n-port.

II. The Coordinate-Free Approach

For the purpose of this paper, a resistive n-port N is assumed to be an interconnection of "n_R" internal coupled 2-terminal resistors and "n" external terminal pairs which we call ports. We will often view an n-port N as a network $\mathcal{N}$ by terminating the ports of N by norators.[3] Now let $\underset{\sim}{v}_R$ and $\underset{\sim}{v}_P$ denote the voltages of the internal resistors and the external ports, respectively, and let $\underset{\sim}{i}_R$ and $\underset{\sim}{i}_P$ be the currents of the internal resistors and the external ports, respectively, so that $(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \mathbb{R}^{n_R} \times \mathbb{R}^{n_R}$ and $(\underset{\sim}{v}_P,\underset{\sim}{i}_P) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $\underset{\sim}{v} = (\underset{\sim}{v}_R,\underset{\sim}{v}_P)$, $\underset{\sim}{i} = (\underset{\sim}{i}_R,\underset{\sim}{i}_P)$ and $b = n_R+n$. Then $(\underset{\sim}{v},\underset{\sim}{i}) \in \mathbb{R}^b \times \mathbb{R}^b$. Every n-port N in this paper is assumed to satisfy the following properties:

(a) The linear graph which defines the topology of N is connected.

(b) N is time-invariant.

(c) The internal resistor constitutive relations are characterized by

$$(\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda \subset \mathbb{R}^b \times \mathbb{R}^b \tag{5}$$

where Λ is a $(2b-n_R)$-dimensional C^1 submanifold.

Remarks 1. There is no loss of generality in assuming (a) since disconnected subgraphs can be hinged together. Connectedness is necessary for a tree to exist.

2. Most of the results of this paper can be easily generalized to time-varying case under appropriate conditions. We make this assumption simply to avoid introducing complicated notation.

3. Under assumption (c), resistors can be coupled to each other and they need not be voltage or current controlled. Even coupling among $(\underset{\sim}{v}_R,\underset{\sim}{i}_R)$ and $(\underset{\sim}{v}_P,\underset{\sim}{i}_P)$ is allowed. Equation (1) of Example 2 is a case in point. This includes virtually all modes of representation including hybrid and transmission representations. In particular, a broad class of dependent sources is covered by this formulation. We regard independent sources as uncoupled resistors. All multi-terminal elements are represented as coupled 2-terminal elements. To illustrate why the dimension of Λ is chosen equal to $2b-n_R$, consider (i) of Example 1, where

[3] Observe that since norators impose no constraints on the port voltages $\underset{\sim}{v}_P$ and the port currents $\underset{\sim}{i}_P$, they only serve to guarantee the current entering one terminal of each port j is equal to the current leaving the other terminal in the same port j. The relationship between $\underset{\sim}{v}_P$ and $\underset{\sim}{i}_P$ therefore remains the same as that of N.

$$\Lambda = \left\{(\underset{\sim}{v},\underset{\sim}{i}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, i_{R_1} = f_{R_1}(v_{R_1}),\ i_{R_2} = f_{R_2}(v_{R_2})\right\}.$$

Then, clearly, Λ is a 4-dimensional submanifold. Since $b = 3$, $n_R = 2$, one has $4 = 2b-n_R$. A similar remark applies to all other examples.

4. A resistive n-port can be derived naturally from an RLC network upon replacing capacitors and inductors by ports. If we let n_C and n_L be the number of capacitors and inductors, respectively, the result is a resistive (n_C+n_L)-port.

Now, for the convenience of the reader, we will briefly describe some geometric concepts needed in the later sections. Details are found in [3,4]. A subset M in $\mathbb{R}^n$ is called an m-dimensional C^1 submanifold if a neighborhood about each point of M looks like $\mathbb{R}^m$. More precisely, M is an m-dimensional C^1 submanifold of $\mathbb{R}^n$ if for each $\underset{\sim}{x} \in M$ there is a neighborhood U of this point in $\mathbb{R}^n$ and there is a C^1 diffeomorphism $\underset{\sim}{\psi} : M \cap U \to \underset{\sim}{\psi}(M \cap U) \subset \mathbb{R}^m$. For example, each point x of M in Fig. 7 has a neighborhood U such that $M \cap U$ is diffeomorphic to an open interval in $\mathbb{R}$. Hence this is a 1-dimensional submanifold of $\mathbb{R}^2$. The function $\underset{\sim}{\psi}$ is called a <u>local coordinate</u> for M at $\underset{\sim}{x}$ and $\underset{\sim}{\psi}^{-1}$ is called a <u>local parametrization</u> for M at $\underset{\sim}{x}$. A pair $(\underset{\sim}{\psi}, M \cap U)$ is called a <u>local chart</u> for M at $\underset{\sim}{x}$. It should be noted that given a point $\underset{\sim}{x} \in M$, there may be many charts.

There is another way of defining a C^1 submanifold of $\mathbb{R}^n$ which is equivalent to the above definition. A subset M is an m-dimensional C^1 submanifold of $\mathbb{R}^n$ if for each point $\underset{\sim}{x} \in M$, there is a neighborhood U of $\underset{\sim}{x}$ in $\mathbb{R}^n$ and a C^1 function $\underset{\sim}{f} : U \to \mathbb{R}^{n-m}$ such that

$$M \cap U = \{\underset{\sim}{x} \in \mathbb{R}^n \,|\, \underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{0}\} \tag{6}$$

and

$$\text{rank}(\underset{\sim}{Df})_{\underset{\sim}{x}} = n-m \text{ for all } \underset{\sim}{x} \in M \cap U \tag{7}$$

where $(\underset{\sim}{Df})_{\underset{\sim}{x}}$ denotes derivative of $\underset{\sim}{f}$ at $\underset{\sim}{x}$.

The <u>Tangent space</u> of M at $\underset{\sim}{x}$ is a linear approximation of M at $\underset{\sim}{x}$. More precisely the tangent space $T_{\underset{\sim}{x}}M$ is given by

$$T_{\underset{\sim}{x}}M = \text{Ker}(\underset{\sim}{Df})_{\underset{\sim}{x}} \tag{8}$$

where $\underset{\sim}{f}$ is in (6) and Ker denotes kernel of a linear map. It turns out that tangent space is given also by

$$T_{\underset{\sim}{x}}M = \text{Im}(\underset{\sim}{D\psi}^{-1})_{\underset{\sim}{\psi}(\underset{\sim}{x})} \tag{9}$$

where ψ is a local coordinate for M at x and Im denotes the image of a linear map. It can be shown that $T_x M$ does not depend on a particular choice of coordinates.

Let M_1 and M_2 be C^1 submanifolds of $\mathbb{R}^n$ with dimension m_1 and m_2, respectively. A function $F: M_1 \to M_2$ is said to be C^1 if for each $x \in M_1$ there is a chart $(\psi, M_1 \cap U)$ for M_1 at x and there is a chart $(\phi, M_2 \cap V)$ for M_2 at $F(x)$ such that $F(M_1 \cap U) \subset M_2 \cap V$ and such that the function $\phi \circ F \circ \psi^{-1} : \psi(M_1 \cap U) \to \phi(M_2 \cap V)$ is C^1. The derivative $(dF)_x : T_x M_1 \to T_{F(x)} M_2$ of F at x is a linear map defined as follows.[4] For $\xi \in T_x M_1$ let $\hat{\xi}$ be defined by $\xi = (D\psi^{-1})_{\psi(x)} \hat{\xi}$. Since ψ is a diffeomorphism, it follows from (9) that such a $\hat{\xi}$ exists and is unique. Then we define $(dF)_x \xi \triangleq \eta$ where $\eta = (D\phi^{-1})_{\phi \circ F(x)} \hat{\eta}$, $\hat{\eta} = (D\phi \circ F \circ \psi^{-1})_{\psi(x)} \hat{\xi}$. Hence $(dF)_x$ has the following representation in terms of coordinates (see Fig. 8):

$$(dF)_x = (D\phi^{-1})_{\phi \circ F(x)} (D\phi \circ F \circ \psi^{-1})_{\psi(x)} (D\psi)_x. \tag{10}$$

It can be shown that $(dF)_x$ is independent of the choice of coordinates. Since the derivative is a linear map its rank is defined. A C^1 function $F: M_1 \to M_2$ is called an <u>immersion</u> if

$$\operatorname{rank}(dF)_x = m_1 \text{ for all } x \in M_1, \tag{11}$$

where m_1 is the dimension of M_1.

Note that the image of an immersion can have self intersections. For example, the set Γ of Fig. 9(a) is the image of the circle S^1 under some immersion. Since Γ has a self intersection it is not a C^1 submanifold, i.e., at point y, there is no neighborhood U such that $\Gamma \cap U$ is diffeomorphic to an open interval. Even the image of an injective immersion may not be a submanifold. Consider, for example, the function F taking the half infinite interval $(-a,\infty)$ into Γ of Fig. 9(b), where F(x) approaches y as x tends to ∞. Although Γ does not have self intersection, the point y cannot have a neighborhood U such that $\Gamma \cap U$ is diffeomorphic to an open interval. Clearly, the class of C^1 functions which are immersions is very large. In order to guarantee the image of a function to be a submanifold, one needs the stronger concept of an <u>embedding</u>. A function $F: M_1 \to M_2$ is said to be a C^1 embedding if it is an immersion and if it maps M_1 diffeomorphically onto its image $F(M_1) \subset M_2$, where the topology of $F(M_1)$ is the induced topology. Namely, open subset of $F(M_1)$ is defined by $F(M_1) \cap U$, where U is an open subset of M_2. The function F of Fig. 9(b) is not an embedding. To see this let $y_n \in \Gamma$ be a sequence of points following the arrow and approaching the arrowhead y. But $F^{-1}(y_n)$ diverges to ∞. Hence F^{-1} is not continuous at y and therefore F cannot be a

[4] We use the symbol dF when the domain of F is a general manifold while we use DF when the domain is an euclidean space.

diffeomorphism onto its image. This implies that F is not an embedding.

Now, recall (5). We now consider several important special cases where Λ can be represented in various special forms. Suppose that $(\underset{\sim}{v}_R, \underset{\sim}{i}_R)$ and $(\underset{\sim}{v}_P, \underset{\sim}{i}_P)$ are not coupled to each other and suppose that $(\underset{\sim}{v}_R, \underset{\sim}{i}_R)$ must satisfy

$$(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R \tag{12}$$

where Λ_R is an n_R-dimensional C^1 submanifold of $\mathbb{R}^{n_R} \times \mathbb{R}^{n_R}$. Define

$$\Lambda \triangleq \left\{(\underset{\sim}{v}, \underset{\sim}{i}) \in \mathbb{R}^b \times \mathbb{R}^b \,|\, (\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R\right\}. \tag{13}$$

Then, since the 2n variables $(\underset{\sim}{v}_P, \underset{\sim}{i}_P)$ are free, (13) is a $(2b-n_R)$-dimensional submanifold. In this paper, whenever we discuss Λ_R, we always assume that $(\underset{\sim}{v}_R, \underset{\sim}{i}_R)$ and $(\underset{\sim}{v}_P, \underset{\sim}{i}_P)$ are not coupled to each other.

Definition 1. A submanifold Λ_R is said to be

(i) locally hybrid if there is a C^1 function $\underset{\sim}{f}_R : \mathbb{R}^{n_R} \times \mathbb{R}^{n_R} \to \mathbb{R}^{n_R}$ such that

$$\Lambda_R = \underset{\sim}{f}_R^{-1}(\underset{\sim}{0}) \tag{14}$$

$$\det\left((D\underset{\sim}{f}_R)\underset{\sim}{A}\right)_{(\underset{\sim}{v}_R, \underset{\sim}{i}_R)} \neq 0 \quad \text{for all } (\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R \tag{15}$$

for some fixed $2n_R \times n_R$ matrix $\underset{\sim}{A}$, where each column of $\underset{\sim}{A}$ has either of the following forms:

$$(0,\ldots,0,1,0,\ldots,0,0,\ldots\ldots\ldots\ldots,0)^T$$

$$(\underbrace{0,\ldots\ldots\ldots\ldots,0}_{n_R},\underbrace{0,\ldots,0,1,0,\ldots,0}_{n_R})^T$$

(ii) locally voltage controlled (resp., locally current controlled) if (14) holds and (15) is replaced by

$$\operatorname{rank}(D_{\underset{\sim}{i}_R}\underset{\sim}{f}_R)_{(\underset{\sim}{v}_R, \underset{\sim}{i}_R)} = n_R, \ (\text{resp., } \operatorname{rank}(D_{\underset{\sim}{v}_R}\underset{\sim}{f}_R)_{(\underset{\sim}{v}_R, \underset{\sim}{i}_R)} = n_R) \tag{16}$$

for all $(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R$

where $D_{\underset{\sim}{i}_R}\underset{\sim}{f}_R$ (resp., $D_{\underset{\sim}{v}_R}\underset{\sim}{f}_R$) denotes the derivative with respect to $\underset{\sim}{i}_R$ (resp., $\underset{\sim}{v}_R$),

(iii) globally parametrizable if Λ_R is diffeomorphic to $\mathbb{R}^{n_R}$. In this case we write

$$\left(\underset{\sim}{v}_R(\underset{\sim}{\rho}),\underset{\sim}{i}_R(\underset{\sim}{\rho})\right) \triangleq \underset{\sim}{\psi}^{-1}(\underset{\sim}{\rho}),\ \underset{\sim}{\rho} \in \mathbb{R}^{n_R} \tag{17}$$

where $\underset{\sim}{\psi} : \Lambda_R \to \mathbb{R}^{n_R}$ is a global coordinate,

(iv) represented by generalized port coordinate if Λ_R is represented by

$$\begin{bmatrix} \underset{\sim}{v}_R \\ \underset{\sim}{i}_R \end{bmatrix} = \begin{bmatrix} \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{c} & \underset{\sim}{d} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\xi} \\ \underset{\sim}{\eta} \end{bmatrix},\ \underset{\sim}{\xi} = \underset{\sim}{F}(\underset{\sim}{\eta}) \tag{18}$$

where $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$ are $n_R \times n_R$ matrices, $\begin{bmatrix} \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{c} & \underset{\sim}{d} \end{bmatrix}$ is nonsingular, and $\underset{\sim}{F} : \mathbb{R}^{n_R} \to \mathbb{R}^{n_R}$ is a C^1 function,

(v) globally hybrid if Λ_R is represented by

$$\underset{\sim}{y} = \underset{\sim}{h}(\underset{\sim}{x}) \tag{19}$$

where $\underset{\sim}{y} = (y_1,\ldots,y_{n_R})$, $\underset{\sim}{x} = (x_1,\ldots,x_{n_R})$ and if y_k is the current (resp., voltage) of the k-th resistor, then x_k is the voltage (resp., current) of the k-th resistor. If y_k is the current (resp., voltage), then the k-th resistor is called voltage controlled (resp., current controlled),

(vi) globally voltage controlled (resp., globally current controlled) if in (19), $\underset{\sim}{y} = \underset{\sim}{i}_R$, $\underset{\sim}{x} = \underset{\sim}{v}_R$ (resp., $\underset{\sim}{y} = \underset{\sim}{v}_R$, $\underset{\sim}{x} = \underset{\sim}{i}_R$).

Remarks 1. The matrix $\underset{\sim}{A}$ of (15) interchanges columns of $\underset{\sim}{D}\underset{\sim}{f}_R$. Note that the matrix $((\underset{\sim}{D}\underset{\sim}{f}_R)\underset{\sim}{A})$ cannot contain columns corresponding to the voltage and the current of the same resistor.

2. The following is an example of a locally current controlled Λ_R which is not globally current controlled. Let Λ_R be described by

$$i_{R_1} - e^{v_{R_1}} \cos v_{R_2} = 0\ ,\ i_{R_2} - e^{v_{R_1}} \sin v_{R_2} = 0.$$

This is globally voltage controlled and locally current controlled but not globally current controlled. To see this consider

$$(D_{\underset{\sim}{v}_R}\underset{\sim}{f}_R)_{\underset{\sim}{v}_R} = \begin{bmatrix} -e^{v_{R_1}} \cos v_{R_2} & e^{v_{R_1}} \sin v_{R_2} \\ -e^{v_{R_1}} \sin v_{R_2} & -e^{v_{R_1}} \cos v_{R_2} \end{bmatrix}_{\underset{\sim}{v}_R} .$$

Since $\det(D_{\underset{\sim}{v}_R}\underset{\sim}{f}_R)_{\underset{\sim}{v}_R} = e^{2v_{R_1}} \neq 0$ the inverse function theorem tells us that locally

$\underset{\sim}{v}_R = \underset{\sim}{g}(\underset{\sim}{i}_R)$ at each point of Λ_R. But it is easy to see that the function

$(v_{R_1}, v_{R_2}) \to (e^{v_{R_1}} \cos v_{R_2}, e^{v_{R_1}} \sin v_{R_2})$ is not a global diffeomorphism. Hence there is no global representation $\underset{\sim}{v}_R = \underset{\sim}{g}(\underset{\sim}{i}_R)$.

3. Let $\underset{\sim}{v}_R = (\underset{\sim}{v}_{R_1}, \underset{\sim}{v}_{R_2})$, $\underset{\sim}{i}_R = (\underset{\sim}{i}_{R_1}, \underset{\sim}{i}_{R_2})$ and let

$$\begin{bmatrix} \underset{\sim}{i}_{R_1} \\ \underset{\sim}{v}_{R_2} \end{bmatrix} = \underset{\sim}{h}(\underset{\sim}{v}_{R_1}, \underset{\sim}{i}_{R_2}) \tag{20}$$

be the global hybrid representation of Λ_R where $\underset{\sim}{v}_{R_1}, \underset{\sim}{i}_{R_1} \in \mathbb{R}^k$, $\underset{\sim}{v}_{R_2}, \underset{\sim}{i}_{R_2} \in \mathbb{R}^{n_R-k}$. Then, in terms of the generalized port coordinate representation, (20) can be expressed as follows:

$$\underset{\sim}{a} = \begin{bmatrix} \underset{\sim}{0}_k & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1}_{n_R-k} \end{bmatrix}, \quad \underset{\sim}{b} = \begin{bmatrix} \underset{\sim}{1}_k & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0}_{n_R-k} \end{bmatrix}, \quad \underset{\sim}{c} = \begin{bmatrix} \underset{\sim}{1}_k & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0}_{n_R-k} \end{bmatrix}, \quad \underset{\sim}{d} = \begin{bmatrix} \underset{\sim}{0}_k & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1}_{n_R-k} \end{bmatrix}$$

$$\underset{\sim}{\xi} = (\underset{\sim}{i}_{R_1}, \underset{\sim}{v}_{R_2}), \quad \underset{\sim}{\eta} = (\underset{\sim}{v}_{R_1}, \underset{\sim}{i}_{R_2}), \quad \underset{\sim}{F} = \underset{\sim}{h},$$

where the subscripts k and n_R-k denote the size of matrices.

4. An example of generalized port coordinate is the scattering representation;

$$\underset{\sim}{a} = \operatorname{diag}(\sqrt{r_1}, \ldots, \sqrt{r_{n_R}}), \quad \underset{\sim}{b} = \operatorname{diag}(\sqrt{r_1}, \ldots, \sqrt{r_{n_R}}),$$

$$\underset{\sim}{c} = \operatorname{diag}\left(\frac{1}{\sqrt{r_1}}, \ldots, \frac{1}{\sqrt{r_{n_R}}}\right), \quad \underset{\sim}{d} = \operatorname{diag}\left(-\frac{1}{\sqrt{r_1}}, \ldots, -\frac{1}{\sqrt{r_{n_R}}}\right)$$

where $r_1, \ldots, r_{n_R}$ are real normalization numbers. In this case $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ are called incident voltages and reflected voltages, respectively.

The submanifold Λ describing the internal resistor constitutive relations has been defined in a coordinate-free manner. There is another constraint that must be satisfied by a network; namely, the Kirchhoff laws. Since we would like to describe our results in a coordinate-free manner, we need a coordinate-free description of the Kirchhoff laws. This can be found in the circuit theory literature. See, for example [5] among others. Here, it is enough to know that the set K of all $(\underset{\sim}{v},\underset{\sim}{i})$ satisfying KVL and KCL is a b-dimensional linear subspace of $\mathbb{R}^b \times \mathbb{R}^b$ and K does not depend on a particular choice of a tree, a loop matrix, or a cut set matrix. We call K the Kirchhoff space.

III. Classification and Characterization of Resistive n-ports

Recall Λ and K. Since $(\underset{\sim}{v},\underset{\sim}{i})$ must satisfy the internal resistor constitutive relations and Kirchhoff laws simultaneously, the following must hold:

$$(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma \triangleq \Lambda \cap K. \tag{21}$$

We call Σ the configuration space of an n-port N. Let $\underset{\sim}{\pi}'_p : \mathbb{R}^b \times \mathbb{R}^b \to \mathbb{R}^n \times \mathbb{R}^n$ be defined by the projection map

$$\underset{\sim}{\pi}'_p(\underset{\sim}{v},\underset{\sim}{i}) = (\underset{\sim}{v}_p,\underset{\sim}{i}_p). \tag{22}$$

Let the inclusion map $\underset{\sim}{\iota} : \Sigma \to \mathbb{R}^b \times \mathbb{R}^b$ be defined by

$$\underset{\sim}{\iota}(\underset{\sim}{v},\underset{\sim}{i}) = (\underset{\sim}{v},\underset{\sim}{i}) \tag{23}$$

and let

$$\underset{\sim}{\pi}_p \triangleq \underset{\sim}{\pi}'_p \circ \underset{\sim}{\iota}. \tag{24}$$

Then the set

$$\mathcal{R} \triangleq \underset{\sim}{\pi}_p(\Sigma) \tag{25}$$

is called the constitutive relation of the n-port N. Clearly, the variables $(\underset{\sim}{v}_p,\underset{\sim}{i}_p)$ must satisfy

$$(\underset{\sim}{v}_p,\underset{\sim}{i}_p) \in \mathcal{R}. \tag{26}$$

The set $\mathcal{R}$ is the projection of Σ onto the $(\underset{\sim}{v}_p,\underset{\sim}{i}_p)$-space.

The definitions of Σ and $\mathcal{R}$ are coordinate-free. If Σ is empty, it means that the internal resistor constitutive relations and Kirchhoff laws cannot be satisfied simultaneously. Consider, for example, two independent voltage sources with different voltages connected in parallel. Clearly, then, the constitutive relations disagree with KVL and hence $\Sigma = \Lambda \cap K$ is empty. In the following, we study some important properties of Σ and $\mathcal{R}$. We will see that even if Λ and K are perfectly well-defined C^1 submanifolds, their intersection Σ and hence its projection $\mathcal{R}$ need not be submanifolds and in fact could turn out to be rather complicated if not bizarre geometric objects. The following is the first category of n-ports in our classification.

Definition 2 A resistive n-port N is said to be quasi-weakly regular if

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_\ell$$

where Σ_i is a k_i-dimensional C^1 submanifold, $0 \le k_i \le b$, $k_i \ne k_j$ and $\Sigma_i \cap \Sigma_j = \phi$ if $i \ne j$.

Observe that since $\dim K = b$ and since $\Sigma = \Lambda \cap K$, the dimension of Σ_i cannot exceed b.

Example 7 All the 1-ports of Example 1 are quasi-weakly regular. To see this consider, for example, case (iv). Then this 1-port is described by

$$\left.\begin{aligned} &i_{R_1} - f_{R_1}(v_{R_1}) = 0,\ i_{R_2} - f_{R_2}(v_{R_2}) = 0,\ v_{R_2} + v_{R_1} + v_P = 0, \\ &i_{R_2} - i_{R_1} = 0,\ i_{R_2} - i_P = 0. \end{aligned}\right\} \tag{27}$$

By eliminating i_{R_1} and i_{R_2} we have

$$i_P - f_{R_1}(v_{R_1}) = 0, \tag{28}$$

$$i_P - f_{R_2}(v_{R_2}) = 0, \tag{29}$$

$$v_{R_2} + v_{R_1} + v_P = 0. \tag{30}$$

We first look at (28) and (29). Observe that each defines a 2-dimensional surface in the (i_P, v_{R_1}, v_{R_2})-space. By drawing these two surfaces in the 3-dimensional space, one can see that the intersection of (28) and (29) consists of two connected components as in Fig. 10. Finally, (30) does not change this intersection since (30) does not contain i_P. Therefore the intersection of (28) and (29) gives the configuration space Σ. Since Σ is a union of two connected 1-dimensional submanifolds (in the 4-dimensional space $\mathbb{R}^4$) this is a quasi-weakly regular 1-port. A natural question which arises at this point is how are the port constitutive relations shown in the right hand side of Fig. 1(b) related to Σ? The answer is that they are simply the projection $\mathcal{R} = \pi_P(\Sigma)$ defined by (25). To see this consider (30). Given a value v_P, (30) defines an affine submanifold in the (v_{R_1}, v_{R_2})-space. So if we vary $v_P \in (-\infty, \infty)$, then we have a family of affine submanifolds. Since v_P is represented by $v_P = -v_{R_1} - v_{R_2}$, this means that if we take the v_P-axis as in Fig. 10 and look at Σ from the (i_P, v_P)-plane, we would obtain the curve shown in the right hand side of

Fig. 1(b). Notice that while $\mathcal{R}$ is not a submanifold, Σ is a perfectly well defined submanifold. The other cases are essentially the same.

Definition 3 A resistive n-port N is said to be weakly regular if Σ is a k-dimensional C^1 manifold, $0 \leq k \leq b$.

Clearly, every weakly regular n-port is quasi-weakly regular. The converse is not true, however, as deomonstrated by the following example.

Example 8 Consider (vi) of Example 1. Here, Σ consists of a point and a curve. (Fig. 11). Hence Σ is the union of a 0-dimensional submanifold - an isolated point - and a 1-dimensional submanifold. They are disjoint. Therefore this 1-port is quasi-weakly regular but it is not weakly regular.

The strange object Σ of (vi) in Example 1, stems from the fact that the local maximum of f_{R_1} at $v_{R_{10}}$ coincides with the local minimum of f_{R_2} at $v_{R_{20}}$ and hence the two surfaces meet tangentially at this particular point. So to avoid this situation, we need a nontangential condition for the two submanifolds Λ and K. More precisely, we want Λ and K to be transversal — a basic concept that will play an important role in this paper.

Definition 4 The internal resistor constitutive relations Λ are said to be transversal to the Kirchhoff space K, and is abbreviated by $\Lambda \pitchfork K$, if[5]

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K = \mathbb{R}^b \times \mathbb{R}^b \text{ for all } (\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma. \tag{31}$$

Remarks 1. If Σ is empty, the transversality condition is of course trivially satisfied. However, from the circuit theory point of view, this situation is not meaningful. Consequently, some of our subsequent perturbation results will demand Σ to be nonempty after perturbation.

2. Transversaility is essentially a non tangency condition. For example, in Fig. 12(a), $\Lambda \pitchfork K$ while in Fig. 12(b), $\Lambda \not\pitchfork K$. Since $T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda$ is a linear approximation of Λ at $(\underset{\sim}{v},\underset{\sim}{i})$ and since $T_{(\underset{\sim}{v},\underset{\sim}{i})}K = K$, (31) requires that locally, the internal resistor constitutive relations and Kirchhoff laws do not overlap each other.

3. Observe that (31) is symmetric in the sense that

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K = T_{(\underset{\sim}{v},\underset{\sim}{i})}K + T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda.$$

Hence one can also say that K is transversal to Λ or Λ and K are transversal.

The following theorem shows that transversality is a sufficient condition for weak regularity.

[5] An elementary introduction to the transversality concept can be found in [6].

Theorem 1 If $\Lambda \pitchfork K$ and if $\Lambda \cap K \neq \phi$, then N is weakly regular. In fact $\Sigma = \Lambda \cap K$ is an n-dimensional submanifold of $\mathbb{R}^b \times \mathbb{R}^b$, where n is the number of ports.

Proof If $\Lambda \pitchfork K$ then Σ is a submanifold [3,4] and

$$\text{codim } \Sigma = \text{codim } \Lambda + \text{codim } K$$

where codim denotes the complementary dimension of a submanifold. Since codim $\Lambda = 2b - (2b-n_R) = n_R$, codim $K = 2b - b = b$ we have dim $\Sigma = 2b - (b+n_R) = b - n_R = n$. ¤

We will next give a simple way of checking (31). Recall that $\mathcal{N}$ is a network obtained by terminating the ports of N by norators. (See Section II) Pick any tree $\mathcal{T}$ for $\mathcal{N}$. Let $\underset{\sim}{v}$ and $\underset{\sim}{i}$ be partitioned as $\underset{\sim}{v} = (\underset{\sim}{v}_{\mathcal{L}} \vdots \underset{\sim}{v}_{\mathcal{T}})$, $\underset{\sim}{i} = (\underset{\sim}{i}_{\mathcal{L}} \vdots \underset{\sim}{i}_{\mathcal{T}})$, where $\mathcal{T}$ and $\mathcal{L}$ denote tree and cotree, respectively. Let $\underset{\sim}{B}$ and $\underset{\sim}{Q}$ be the fundamental loop and cut set matrices associated with $\mathcal{T}$, respectively. It is known that $\underset{\sim}{B}$ and $\underset{\sim}{Q}$ assume the following form:

$$\underset{\sim}{B} = [\underset{\sim}{1} \vdots B_{\mathcal{T}}], \quad \underset{\sim}{Q} = [-\underset{\sim}{B}_{\mathcal{T}}^T \vdots \underset{\sim}{1}] \; . \tag{32}$$

Since Λ is a C^1 submanifold of dimension $2b-n_R$, for each point $(\underset{\sim}{v}_0, \underset{\sim}{i}_0) \in \Lambda$, there is a neighborhood $U \subset \mathbb{R}^b \times \mathbb{R}^b$ of this point and there is a C^1 function $\underset{\sim}{f} : U \to \mathbb{R}^{n_R}$ such that (see (6) and (7))

$$\Lambda \cap U = \underset{\sim}{f}^{-1}(\underset{\sim}{0}) \tag{33}$$

and

$$\text{rank}(\underset{\sim}{Df})_{(\underset{\sim}{v},\underset{\sim}{i})} = n_R \text{ for all } (\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda \cap U. \tag{34}$$

Since the Kirchhoff space is represented by $K = \text{Ker } \underset{\sim}{B} \times \text{Ker } \underset{\sim}{Q}$, the set $\Sigma \cap U$ is locally represented by

$$\underset{\sim}{B}\underset{\sim}{v} = \underset{\sim}{0}, \quad \underset{\sim}{Q}\underset{\sim}{i} = \underset{\sim}{0}, \quad \underset{\sim}{f}(\underset{\sim}{v},\underset{\sim}{i}) = \underset{\sim}{0}. \tag{35}$$

Proposition 1 $\Lambda \pitchfork K$ if and only if for each $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$,

$$\text{rank } \underset{\sim}{\mathcal{F}}(\underset{\sim}{v},\underset{\sim}{i}) = n_R \tag{36}$$

where

$$\underset{\sim}{\mathcal{F}}(\underset{\sim}{v},\underset{\sim}{i}) \triangleq \left[D_{\underset{\sim}{v}_{\mathcal{T}}} \underset{\sim}{f} - (D_{\underset{\sim}{v}_{\mathcal{L}}} \underset{\sim}{f}) B_{\mathcal{T}} \vdots D_{\underset{\sim}{i}_{\mathcal{L}}} \underset{\sim}{f} + (D_{\underset{\sim}{i}_{\mathcal{T}}} \underset{\sim}{f}) B_{\mathcal{T}}^T \right]_{(\underset{\sim}{v},\underset{\sim}{i})} . \tag{37}$$

Proof It follows from Fact A of APPENDIX 1 and (35) that $\Lambda \pitchfork K$ if and only if for each $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma \cap U$

$$\operatorname{rank}\begin{bmatrix} \underset{\sim}{B} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{Q} \\ \underset{\sim}{D}_{\underset{\sim}{v}}\underset{\sim}{f} & \underset{\sim}{D}_{\underset{\sim}{i}}\underset{\sim}{f} \end{bmatrix}_{(\underset{\sim}{v},\underset{\sim}{i})} = b + n_R \,. \tag{38}$$

More explicitly, this matrix has the following form:

$$\left[\begin{array}{cc:cc} \underset{\sim}{1} & \underset{\sim}{B}_{\mathcal{T}} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & -\underset{\sim}{B}_{\mathcal{T}}^{T} & \underset{\sim}{1} \\ \underset{\sim}{D}_{\underset{\sim}{v}_{\mathcal{L}}}\underset{\sim}{f} & \underset{\sim}{D}_{\underset{\sim}{v}_{\mathcal{T}}}\underset{\sim}{f} & \underset{\sim}{D}_{\underset{\sim}{i}_{\mathcal{L}}}\underset{\sim}{f} & \underset{\sim}{D}_{\underset{\sim}{i}_{\mathcal{T}}}\underset{\sim}{f} \end{array}\right]_{(\underset{\sim}{v},\underset{\sim}{i})} . \tag{39}$$

By elementary operations, one can show that this matrix has rank $b+n_R$ if, and only if, (36) holds. ¤

Remark It is important to note that transversality of Λ and K is a coordinate-free condition. Hence if (36) holds in terms of a particular tree $\mathcal{T}$ then it holds in terms of every other tree. Conversely, if (36) fails to hold in terms of one tree, then no matter which tree one chooses, (36) does not hold. Therefore one needs to check (36) in terms of only one tree.

We will next give various special cases of (36) corresponding to several common representations of Λ. Suppose that $(\underset{\sim}{v}_R,\underset{\sim}{i}_R)$ and $(\underset{\sim}{v}_P,\underset{\sim}{i}_P)$ are not coupled to each other and Λ is given by (13). Then $\underset{\sim}{f}$ of (33) is independent of $(\underset{\sim}{v}_P,\underset{\sim}{i}_P)$. Let $U_R \triangleq U \cap \mathbb{R}^{n_R} \times \mathbb{R}^{n_R}$ and we define $\underset{\sim}{f}_R : U_R \to \mathbb{R}^{n_R}$ simply by

$$\underset{\sim}{f}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \triangleq \underset{\sim}{f}(\underset{\sim}{v}_R,\underset{\sim}{i}_R). \tag{40}$$

Next, let $\underset{\sim}{\pi}'_R : \mathbb{R}^b \times \mathbb{R}^b \to \mathbb{R}^{n_R} \times \mathbb{R}^{n_R}$ be the projection map

$$\underset{\sim}{\pi}'_R(\underset{\sim}{v},\underset{\sim}{i}) = (\underset{\sim}{v}_R,\underset{\sim}{i}_R) \tag{41}$$

and let

$$\underset{\sim}{\pi}_R \triangleq \underset{\sim}{\pi}'_R \circ \underset{\sim}{\iota} \tag{42}$$

where $\underset{\sim}{\iota}$ is the inclusion map defined by (23). Finally, decompose $\underset{\sim}{v}$ and $\underset{\sim}{i}$ as

$$\underset{\sim}{v} = (\underset{\sim}{v}_{R_{\mathcal{L}}},\underset{\sim}{v}_{P_{\mathcal{L}}} \vdots\, \underset{\sim}{v}_{R_{\mathcal{T}}},\underset{\sim}{v}_{P_{\mathcal{T}}}),\ \underset{\sim}{i} = (\underset{\sim}{i}_{R_{\mathcal{L}}},\underset{\sim}{i}_{P_{\mathcal{L}}} \vdots\, \underset{\sim}{i}_{R_{\mathcal{T}}},\underset{\sim}{i}_{P_{\mathcal{T}}}) \tag{43}$$

where R and P denote resistors and ports, respectively, and $\mathcal{T}$ and $\mathcal{L}$ denote tree and cotree, respectively. Decompose $\underset{\sim}{B}_{\mathcal{T}}$ of (32) accordingly:

$$\underset{\sim}{B}_{\mathcal{T}} = \begin{bmatrix} \underset{\sim}{B}_{RR} & \underset{\sim}{B}_{RP} \\ \underset{\sim}{B}_{PR} & \underset{\sim}{B}_{PP} \end{bmatrix}. \tag{44}$$

__Corollary 1__ Suppose that Λ is given by (13). Then $\Lambda \pitchfork K$ if and only if for each $(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \underset{\sim}{\pi}_R(\Sigma)$,

$$\text{rank}\, \underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = n_R \tag{45}$$

where

$$\underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \triangleq \Big[D_{\underset{\sim}{v}_{R_\mathcal{J}}} \underset{\sim}{f}_R - (D_{\underset{\sim}{v}_{R_\mathcal{L}}} \underset{\sim}{f}_R) \underset{\sim}{B}_{RR} \,\vdots\, -(D_{\underset{\sim}{v}_{R_\mathcal{L}}} \underset{\sim}{f}_R) \underset{\sim}{B}_{RP} \,\vdots$$

$$D_{\underset{\sim}{i}_{R_\mathcal{L}}} \underset{\sim}{f}_R + (D_{\underset{\sim}{i}_{R_\mathcal{J}}} \underset{\sim}{f}_R) \underset{\sim}{B}^T_{RR} \,\vdots\, (D_{\underset{\sim}{i}_{R_\mathcal{J}}} \underset{\sim}{f}_R) \underset{\sim}{B}^T_{PR} \Big]_{(\underset{\sim}{v}_R, \underset{\sim}{i}_R)} \tag{46}$$

__Proof__ Observe that

$$D_{\underset{\sim}{v}_\mathcal{J}} \underset{\sim}{f} = [D_{\underset{\sim}{v}_{R_\mathcal{J}}} \underset{\sim}{f}_R \quad \underset{\sim}{0}],\quad D_{\underset{\sim}{v}_\mathcal{L}} \underset{\sim}{f} = [D_{\underset{\sim}{v}_{R_\mathcal{L}}} \underset{\sim}{f}_R \quad \underset{\sim}{0}]$$

$$D_{\underset{\sim}{i}_\mathcal{J}} \underset{\sim}{f} = [D_{\underset{\sim}{i}_{R_\mathcal{J}}} \underset{\sim}{f}_R \quad \underset{\sim}{0}],\quad D_{\underset{\sim}{i}_\mathcal{L}} \underset{\sim}{f} = [D_{\underset{\sim}{i}_{R_\mathcal{L}}} \underset{\sim}{f}_R \quad \underset{\sim}{0}].$$

Substituting these and (44) into (37), we obtain (46). Since $(\underset{\sim}{v}_R, \underset{\sim}{v}_P, \underset{\sim}{i}_R, \underset{\sim}{i}_P) \in \Sigma$, the vector $(\underset{\sim}{v}_R, \underset{\sim}{i}_R)$ must belong to $\underset{\sim}{\pi}_R(\Sigma)$. ¤

Consider, next, the generalized port coordinate (18) and let

$$\begin{bmatrix} \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{c} & \underset{\sim}{d} \end{bmatrix}^{-1} \triangleq \begin{bmatrix} \underset{\sim}{\alpha} & \underset{\sim}{\beta} \\ \underset{\sim}{\gamma} & \underset{\sim}{\delta} \end{bmatrix}. \tag{47}$$

Then Λ_R is represented by

$$\underset{\sim}{f}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = \underset{\sim}{\alpha}\underset{\sim}{v}_R + \underset{\sim}{\beta}\underset{\sim}{i}_R - \underset{\sim}{F}(\underset{\sim}{\gamma}\underset{\sim}{v}_R + \underset{\sim}{\delta}\underset{\sim}{i}_R) = \underset{\sim}{0}. \tag{48}$$

Recall the partition $\underset{\sim}{v}_R = (\underset{\sim}{v}_{R_\mathcal{L}} \vdots\, \underset{\sim}{v}_{R_\mathcal{J}})$, $\underset{\sim}{i}_R = (\underset{\sim}{i}_{R_\mathcal{L}} \vdots\, \underset{\sim}{i}_{R_\mathcal{J}})$ and partition $\underset{\sim}{\alpha}, \underset{\sim}{\beta}, \underset{\sim}{\gamma}$ and $\underset{\sim}{\delta}$ accordingly;

$$\underset{\sim}{\alpha} = [\underset{\sim}{\alpha}_1 \vdots\, \underset{\sim}{\alpha}_2],\ \underset{\sim}{\beta} = [\underset{\sim}{\beta}_1 \vdots\, \underset{\sim}{\beta}_2],\ \underset{\sim}{\gamma} = [\underset{\sim}{\gamma}_1 \vdots\, \underset{\sim}{\gamma}_2],\ \underset{\sim}{\delta} = [\underset{\sim}{\delta}_1 \vdots\, \underset{\sim}{\delta}_2]. \tag{49}$$

Then we have the following:

__Corollary 2__ Let Λ_R admit a generalized port coordinate representation. Then

$$\underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = \Big[(\underset{\sim}{\alpha}_2 - (\underset{\sim}{D}\underset{\sim}{F})\underset{\sim}{\gamma}_2) - (\underset{\sim}{\alpha}_1 - (\underset{\sim}{D}\underset{\sim}{F})\underset{\sim}{\gamma}_1)\underset{\sim}{B}_{RR} \,\vdots\, -(\underset{\sim}{\alpha}_1 - (\underset{\sim}{D}\underset{\sim}{F})\underset{\sim}{\gamma}_1)\underset{\sim}{B}_{RP} \,\vdots$$

$$(\underset{\sim}{\beta}_1 - (\underset{\sim}{D}\underset{\sim}{F})\underset{\sim}{\delta}_1) + (\underset{\sim}{\beta}_2 - (\underset{\sim}{D}\underset{\sim}{F})\underset{\sim}{\delta}_2)\underset{\sim}{B}^T_{RR} \,\vdots\, (\underset{\sim}{\beta}_2 - (\underset{\sim}{D}\underset{\sim}{F})\underset{\sim}{\delta}_2)\underset{\sim}{B}^T_{PR} \Big]_{(\underset{\sim}{v}_R, \underset{\sim}{i}_R)} \tag{50}$$

In particular, if Λ_R is globally voltage controlled, then

$$\underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = \left[(\underset{\sim}{D}\underset{\sim}{F}) \begin{bmatrix} \underset{\sim}{B}_{RR} \\ -\underset{\sim}{1} \end{bmatrix} \vdots\ (\underset{\sim}{D}\underset{\sim}{F}) \begin{bmatrix} \underset{\sim}{B}_{RP} \\ \underset{\sim}{0} \end{bmatrix} \vdots \begin{bmatrix} \underset{\sim}{1} \\ \underset{\sim}{B}_{RR}^T \end{bmatrix} \vdots \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{B}_{PR}^T \end{bmatrix} \right]_{\underset{\sim}{v}_R} \tag{51}$$

and if Λ_R is globally current controlled, then

$$\underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = \left[\begin{bmatrix} -\underset{\sim}{B}_{RR} \\ \underset{\sim}{1} \end{bmatrix} \vdots \begin{bmatrix} -\underset{\sim}{B}_{RP} \\ \underset{\sim}{0} \end{bmatrix} \vdots -(\underset{\sim}{D}\underset{\sim}{F}) \begin{bmatrix} \underset{\sim}{1} \\ \underset{\sim}{B}_{RR}^T \end{bmatrix} \vdots -(\underset{\sim}{D}\underset{\sim}{F}) \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{B}_{PR}^T \end{bmatrix} \right]_{\underset{\sim}{i}_R} . \tag{52}$$

<u>Proof</u> It follows from (48) that

$$D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \underset{\sim}{\alpha}_1 - (\underset{\sim}{D}\underset{\sim}{F})\underset{\sim}{\gamma}_1,\ D_{\underset{\sim}{v}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R = \underset{\sim}{\alpha}_2 - (\underset{\sim}{D}\underset{\sim}{F})\underset{\sim}{\gamma}_2$$

$$D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \underset{\sim}{\beta}_1 - (\underset{\sim}{D}\underset{\sim}{F})\underset{\sim}{\delta}_1,\ D_{\underset{\sim}{i}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R = \underset{\sim}{\beta}_2 - (\underset{\sim}{D}\underset{\sim}{F})\underset{\sim}{\delta}_2 .$$

Substitution of these into (46) yields (50). If Λ_R is globally voltage controlled, then $\underset{\sim}{\alpha}_1 = \underset{\sim}{0}$, $\underset{\sim}{\alpha}_2 = \underset{\sim}{0}$, $\underset{\sim}{\delta}_1 = \underset{\sim}{0}$, $\underset{\sim}{\delta}_2 = \underset{\sim}{0}$ and

$$\underset{\sim}{\beta}_1 = \begin{bmatrix} \underset{\sim}{1} \\ \underset{\sim}{0} \end{bmatrix},\ \underset{\sim}{\beta}_2 = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{1} \end{bmatrix},\ \underset{\sim}{\gamma}_1 = \begin{bmatrix} \underset{\sim}{1} \\ \underset{\sim}{0} \end{bmatrix},\ \underset{\sim}{\gamma}_2 = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{1} \end{bmatrix} .$$

This gives (51). Similarly, if Λ_R is globally current controlled, then $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$, $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$, $\underset{\sim}{\gamma}_1 = \underset{\sim}{0}$, $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$ and

$$\underset{\sim}{\alpha}_1 = \begin{bmatrix} \underset{\sim}{1} \\ \underset{\sim}{0} \end{bmatrix},\ \underset{\sim}{\alpha}_2 = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{1} \end{bmatrix},\ \underset{\sim}{\delta}_1 = \begin{bmatrix} \underset{\sim}{1} \\ \underset{\sim}{0} \end{bmatrix},\ \underset{\sim}{\delta}_2 = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{1} \end{bmatrix} .$$

This gives (52). ¤

<u>Remark</u> Using (45), (46), (50)-(52), one can give several sufficient conditions for transversality by inspection. Suppose that Λ_R is locally hybrid (see Def. 1) with

$$\det[D_{\underset{\sim}{v}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R \vdots D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R]_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)} \neq 0 \text{ for all } (\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R . \tag{53}$$

Assume also that the resistors in $\mathcal{L}$ form loops exclusively with ports or equivalently, the resistors in $\mathcal{J}$ form cut sets exclusively with ports. Then $\underset{\sim}{B}_{RR} = \underset{\sim}{0}$ and (53) implies (45) and hence $\Lambda \pitchfork K$. If Λ_R is globally voltage controlled and if $(\underset{\sim}{D}\underset{\sim}{F})_{\underset{\sim}{v}_R}$ is positive definite at each $\underset{\sim}{v}_R$, then it is easy to show that the following submatrix of (51) is positive definite:

$$\left[(\underset{\sim}{D}\underset{\sim}{F}) \begin{bmatrix} \underset{\sim}{B}_{RR} \\ -\underset{\sim}{1} \end{bmatrix} \vdots \begin{bmatrix} \underset{\sim}{1} \\ \underset{\sim}{B}_{RR}^{T} \end{bmatrix} \right]_{\underset{\sim}{v}_R}$$

and (45) holds. Therefore $\Lambda \pitchfork K$. A similar statement holds for the globally current controlled case.

Let us give several examples.

Example 9 For the 1-ports (i)-(v), (vii), (viii) of Example 1, $\Lambda \pitchfork K$. To prove this choose $\mathcal{T} = \{R_1, R_2\}$ to be our tree. Then $\underset{\sim}{B}_{RR} = \phi$,[6] $\underset{\sim}{B}_{RP} = \phi$, $\underset{\sim}{B}_{PR} = [1 \;\; 1]$, $\underset{\sim}{B}_{PP} = \phi$. Let

$$\underset{\sim}{f}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = \begin{bmatrix} \underset{\sim}{f}_R^1(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \\ \underset{\sim}{f}_R^2(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \end{bmatrix} \triangleq \begin{bmatrix} i_{R_1} - f_{R_1}(v_{R_1}) \\ i_{R_2} - f_{R_2}(v_{R_2}) \end{bmatrix}.$$

Then

$$\underset{\sim}{D}_{\underset{\sim}{v}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{bmatrix} D_{v_{R_1}} f_R^1 & D_{v_{R_2}} f_R^1 \\ D_{v_{R_1}} f_R^2 & D_{v_{R_2}} f_R^2 \end{bmatrix} = \begin{bmatrix} -Df_{R_1} & 0 \\ 0 & -Df_{R_2} \end{bmatrix}, \quad \underset{\sim}{D}_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \phi$$

and similarly

$$\underset{\sim}{D}_{\underset{\sim}{i}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underset{\sim}{D}_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \phi\,.$$

Hence (46) is given by

$$\underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = \begin{bmatrix} -Df_{R_1} & 0 & \vdots & 1 \\ 0 & -Df_{R_2} & \vdots & 1 \end{bmatrix}_{\underset{\sim}{v}_R}. \tag{54}$$

If one checks (i)-(v), (vii) one sees that for all $(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \underset{\sim}{\pi}_R(\Sigma)$, Df_{R_1} and Df_{R_2} do not vanish simultaneously and hence rank $\underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = 2$ for all $(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \underset{\sim}{\pi}_R(\Sigma)$. Therefore $\Lambda \pitchfork K$ and Σ is a 1-dimensional submanifold. One can use a similar argument to show that for (viii), $\Lambda \pitchfork K$ also. In contrast to these examples, we claim that for (vi) of Example 1, $\Lambda \not\pitchfork K$. To prove this observe that for the value i_R^* in (vi) of Fig. 1(b), we have $i_R^* = f_{R_1}(v_{R_{10}}) = f_{R_2}(v_{R_{20}})$ and $(Df_{R_1})_{v_{R_{10}}} = (Df_{R_2})_{v_{R_{20}}} = 0$. It is clear that the point $(\underset{\sim}{v}_{R_0}, \underset{\sim}{i}_{R_0}) \triangleq (v_{R_{10}}, v_{R_{20}}, i_R^*, i_R^*)$ belongs to $\underset{\sim}{\pi}_R(\Sigma)$. Therefore rank $\underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_{R_0}, \underset{\sim}{i}_{R_0}) = 1 < 2$ and hence $\Lambda \not\pitchfork K$.

---

[6] We denote a 0×0 matrix by ϕ.

<u>Example 10</u> Computing $\mathcal{F}(\underset{\sim}{v},\underset{\sim}{i})$ or $\mathcal{F}_{\underset{\sim}{R}}(\underset{\sim}{v}_R,\underset{\sim}{i}_R)$ one can show that for Examples 2,3,4 and 6, $\Lambda \not\pitchfork K$, whereas for Example 5, $\Lambda \pitchfork K$. We will prove this for Examples 5 and 6. Let us first check Example 5. Choose $\mathcal{T} = \{R_4, R_5, P\}$ as our tree. Then

$$\underset{\sim}{B}_{RR} = \begin{array}{cc} v_{R_4} & v_{R_5} \\ \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right. & \left.\begin{array}{c} 0 \\ 1 \\ -1 \end{array}\right] \end{array}, \quad \underset{\sim}{B}_{RP} = \begin{array}{c} v_P \\ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \end{array}, \quad \underset{\sim}{B}_{PR} = \phi, \quad \underset{\sim}{B}_{PP} = \phi$$

$$D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \begin{array}{ccc} v_{R_1} & v_{R_2} & v_{R_3} \\ \left[\begin{array}{c} -Df_{R_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ -Df_{R_2} \\ 0 \\ 0 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{array}\right] \end{array}, \quad D_{\underset{\sim}{v}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{array}{cc} v_{R_4} & v_{R_5} \\ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right. & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}\right] \end{array}$$

$$D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \begin{array}{ccc} i_{R_1} & i_{R_2} & i_{R_3} \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{array}\right] \end{array}, \quad D_{\underset{\sim}{i}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{array}{cc} i_{R_4} & i_{R_5} \\ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{array}\right. & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \end{array}.$$

Therefore

$$\mathcal{F}_{\underset{\sim}{R}}(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = \left[\begin{array}{cc:c:ccc} Df_{R_1} & 0 & 0 & 1 & 0 & 0 \\ Df_{R_2} & Df_{R_2} & Df_{R_2} & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right] \underset{\sim}{v}_R.$$

It is easy to show that rank $\mathcal{F}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = 5$ for any C^1 functions f_{R_1} and f_{R_2}. This implies $\Lambda \pitchfork K$.

To examine Example 6, choose $\mathcal{T} = \{R_3,R_4,P\}$ to be our tree. Then

$$\underset{\sim}{B}_{RR} = \begin{array}{cc} v_{R_3} & v_{R_4} \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}} \end{array}, \quad \underset{\sim}{B}_{RP} = \begin{array}{c} v_P \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}, \quad \underset{\sim}{B}_{PR} = \phi, \; \underset{\sim}{B}_{PP} = \phi .$$

The constitutive relations Λ_{R_1} and Λ_{R_2} corresponding to the portion A of Fig. 6(d) are locally given by $i_{R_1} - 1 = 0$ and $i_{R_2} - 1 = 0$ respectively. Hence, locally we have

$$D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \begin{array}{cc} v_{R_1} & v_{R_2} \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}} \end{array}, \quad D_{\underset{\sim}{v}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{array}{cc} v_{R_3} & v_{R_4} \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}} \end{array}$$

$$D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \begin{array}{cc} i_{R_1} & i_{R_2} \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}} \end{array}, \quad D_{\underset{\sim}{i}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{array}{cc} i_{R_3} & i_{R_4} \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}} \end{array} .$$

Therefore

$$\underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = \left[\begin{array}{cc:c:cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right].$$

Clearly, we have rank $\mathcal{F}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = 3 < 4$ and hence $\Lambda \not\pitchfork K$.

If Λ_R is parametrized by $\underset{\sim}{\rho}$, a different criterion is required. Recall the notation $(\underset{\sim}{v}_R(\underset{\sim}{\rho}),\underset{\sim}{i}_R(\underset{\sim}{\rho}))$ of (17).

<u>Proposition 2</u> Suppose Λ_R is globally parametrized by $\underset{\sim}{\rho}$ as in (17). Then $\Lambda \overline{\pitchfork} K$ if and only if for each $\underset{\sim}{\rho} \in \mathbb{R}^{n_R}$ with $(\underset{\sim}{v}_R(\underset{\sim}{\rho}), \underset{\sim}{i}_R(\underset{\sim}{\rho})) \in \underset{\sim}{\pi}_R(\Sigma)$,

$$\text{rank}\,\underset{\sim}{\mathfrak{J}}^*(\underset{\sim}{\rho}) = n_R \tag{55}$$

where

$$\underset{\sim}{\mathfrak{J}}^*(\underset{\sim}{\rho}) \triangleq \left[\begin{array}{c:cc} \underset{\sim}{D}\underset{\sim}{v}_{R_{\mathcal{L}}} + \underset{\sim}{B}_{RR}\,(\underset{\sim}{D}\underset{\sim}{v}_{R_{\mathcal{T}}}) & -\underset{\sim}{B}_{RP} & \underset{\sim}{0} \\ \underset{\sim}{D}\underset{\sim}{i}_{R_{\mathcal{T}}} - \underset{\sim}{B}_{RR}^T\,(\underset{\sim}{D}\underset{\sim}{i}_{R_{\mathcal{L}}}) & \underset{\sim}{0} & \underset{\sim}{B}_{PR}^T \end{array}\right]_{\underset{\sim}{\rho}} . \tag{56}$$

<u>Proof</u> Let $\underset{\sim}{\sigma} \triangleq (\underset{\sim}{\rho}, \underset{\sim}{v}_P, \underset{\sim}{i}_P)$. Then Λ is globally parametrized by $\underset{\sim}{\sigma}$;

$$\underset{\sim}{v}(\underset{\sim}{\sigma}) \triangleq (\underset{\sim}{v}_{R_{\mathcal{L}}}(\underset{\sim}{\rho}), \underset{\sim}{v}_{P_{\mathcal{L}}} \,\vdots\, \underset{\sim}{v}_{R_{\mathcal{T}}}(\underset{\sim}{\rho}), \underset{\sim}{v}_{P_{\mathcal{T}}}) \tag{57}$$

$$\underset{\sim}{i}(\underset{\sim}{\sigma}) \triangleq (\underset{\sim}{i}_{R_{\mathcal{L}}}(\underset{\sim}{\rho}), \underset{\sim}{i}_{P_{\mathcal{L}}} \,\vdots\, \underset{\sim}{i}_{R_{\mathcal{T}}}(\underset{\sim}{\rho}), \underset{\sim}{i}_{P_{\mathcal{T}}}). \tag{58}$$

It follows from (9) that

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda = \text{Im}\begin{bmatrix} \underset{\sim}{D}\underset{\sim}{v} \\ \underset{\sim}{D}\underset{\sim}{i} \end{bmatrix}_{\underset{\sim}{\sigma}} . \tag{59}$$

Recall that Kirchhoff space is parametrized by $(\underset{\sim}{v}_{\mathcal{T}}, \underset{\sim}{i}_{\mathcal{L}})$:

$$\underset{\sim}{v} = \underset{\sim}{Q}^T \underset{\sim}{v}_{\mathcal{T}}\,,\quad \underset{\sim}{i} = \underset{\sim}{B}^T \underset{\sim}{i}_{\mathcal{L}}\,. \tag{60}$$

This implies that

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}K = \text{Im}\begin{bmatrix} \underset{\sim}{Q}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^T \end{bmatrix} . \tag{61}$$

It follows from (59), (61) and (31) that $\Lambda \overline{\pitchfork} K$ if and only if

$$\text{Im}\begin{bmatrix} \underset{\sim}{D}\underset{\sim}{v} \\ \underset{\sim}{D}\underset{\sim}{i} \end{bmatrix}_{\underset{\sim}{\sigma}} + \text{Im}\begin{bmatrix} \underset{\sim}{Q}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^T \end{bmatrix} = \mathbb{R}^b \times \mathbb{R}^b . \tag{62}$$

This, in turn, holds if and only if

$$\text{rank}\left[\begin{bmatrix} \underset{\sim}{D}\underset{\sim}{v} \\ \underset{\sim}{D}\underset{\sim}{i} \end{bmatrix}_{\underset{\sim}{\sigma}} \vdots \begin{bmatrix} \underset{\sim}{Q}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^T \end{bmatrix}\right] = 2b . \tag{63}$$

This matrix is given more explicitly by

$$
\begin{array}{ccccccccc}
\boldsymbol{\rho} & \mathbf{v}_{P_{\mathcal{L}}} & \mathbf{v}_{P_{\mathcal{J}}} & \mathbf{i}_{P_{\mathcal{L}}} & \mathbf{i}_{P_{\mathcal{J}}} & \mathbf{v}_{R_{\mathcal{L}}} & \mathbf{v}_{P_{\mathcal{J}}} & \mathbf{i}_{R_{\mathcal{L}}} & \mathbf{i}_{P_{\mathcal{J}}}
\end{array}
$$

$$
\left[
\left[\begin{array}{ccccc}
\mathbf{D}\mathbf{v}_{R_{\mathcal{L}}} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \mathbf{1} & \cdot & \cdot & \cdot \\
\mathbf{D}\mathbf{v}_{R_{\mathcal{J}}} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \mathbf{1} & \cdot & \cdot \\
\mathbf{D}\mathbf{i}_{R_{\mathcal{L}}} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \mathbf{1} & \cdot \\
\mathbf{D}\mathbf{i}_{R_{\mathcal{J}}} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \mathbf{1}
\end{array}\right]_{\boldsymbol{\rho}}
\begin{array}{c} \vdots \end{array}
\left[\begin{array}{cccc}
-\mathbf{B}_{RR} & -\mathbf{B}_{RP} & \cdot & \cdot \\
-\mathbf{B}_{PR} & -\mathbf{B}_{PP} & \cdot & \cdot \\
\mathbf{1} & \cdot & \cdot & \cdot \\
\cdot & \mathbf{1} & \cdot & \cdot \\
\cdot & \cdot & \mathbf{1} & \cdot \\
\cdot & \cdot & \cdot & \mathbf{1} \\
\cdot & \cdot & \mathbf{B}_{RR}^T & \mathbf{B}_{PR}^T \\
\cdot & \cdot & \mathbf{B}_{RP}^T & \mathbf{B}_{PP}^T
\end{array}\right]
\right] \tag{64}
$$

where $\cdot$ denotes a zero submatrix of appropriate size. By elementary operations, one can show that this matrix has rank 2b if and only if (55) holds.

¤

<u>Remark</u> Formula (55) holds even if Λ_R is locally parametrized by $\boldsymbol{\rho}$ at each point. In fact (55) holds if and only if rank $\mathbf{J} = 2n_R$ where $\mathbf{J}$ is the matrix defined by Desoer and Wu [7].

If Λ_R admits a generalized port coordinate, then it is globally parametrized by $\boldsymbol{\eta}$;

$$(\mathbf{v}_R(\boldsymbol{\eta}),\mathbf{i}_R(\boldsymbol{\eta})) = (\mathbf{a}\mathbf{F}(\boldsymbol{\eta})+\mathbf{b}\boldsymbol{\eta},\mathbf{c}\mathbf{F}(\boldsymbol{\eta})+\mathbf{d}\boldsymbol{\eta}). \tag{65}$$

Partition $\mathbf{a},\mathbf{b},\mathbf{c}$ and $\mathbf{d}$ in accordance with $(\mathbf{v}_{R_{\mathcal{L}}} \vdots \mathbf{v}_{R_{\mathcal{J}}})$ and $(\mathbf{i}_{R_{\mathcal{L}}} \vdots \mathbf{i}_{R_{\mathcal{J}}})$:

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \cdots \\ \mathbf{a}_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \cdots \\ \mathbf{b}_2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{c}_1 \\ \cdots \\ \mathbf{c}_2 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \cdots \\ \mathbf{d}_2 \end{bmatrix}.$$

By direct substitution we can show the following:

<u>Corollary 3</u>. Let Λ_R admit a generalized port coordinate. Then (56) is given by:

$$\boldsymbol{\mathcal{J}}^*(\boldsymbol{\eta}) = \left[\begin{array}{c:cc} \mathbf{a}_1(\mathbf{D}\mathbf{F})+\mathbf{b}_1+\mathbf{B}_{RR}(\mathbf{a}_2(\mathbf{D}\mathbf{F})+\mathbf{b}_2) & -\mathbf{B}_{RP} & \mathbf{0} \\ \mathbf{c}_1(\mathbf{D}\mathbf{F})+\mathbf{d}_1-\mathbf{B}_{RR}^T(\mathbf{c}_2(\mathbf{D}\mathbf{F})+\mathbf{d}_2) & \mathbf{0} & \mathbf{B}_{PR}^T \end{array}\right]_{\boldsymbol{\eta}} \tag{66}$$

It is important to note that transversality of Λ and K is only a <u>sufficient</u> condition for N to be weakly regular. The following example shows that it is <u>not</u> a necessary condition.

<u>Example 11</u> Consider the 1-port of Fig. 13 where Λ_R is given by $v_{R_1} - R_1 i_{R_1} = 0$, $i_{R_2} - i_{R_1}^3 + 2i_{R_1}^2 - 2i_{R_1} = 0$. Since K is described by $v_P + v_{R_1} + v_{R_2} = 0$, $i_{R_1} - i_P = 0$, $i_{R_2} - i_P = 0$, we have

$$\begin{aligned}\Sigma &= \{(\underset{\sim}{v},\underset{\sim}{i}) \mid i_{R_1} = i_{R_2} = i_P = v_{R_1} = 0,\ v_{R_2} + v_P = 0\} \\ &\quad \cup \{(\underset{\sim}{v},\underset{\sim}{i}) \mid i_{R_1} = i_{R_2} = i_P = 1,\ v_{R_1} = R_1,\ v_P + v_{R_2} + R_1 = 0\} \\ &\triangleq \Sigma_1 \cup \Sigma_2 .\end{aligned} \tag{67}$$

Since Σ is a 1-dimensional submanifold consisting of two connected components, it follows that N is weakly regular. We claim that each point of Σ_2 is a point of nontransversal intersection of Λ and K. To show this choose $\mathcal{T} = \{R_1, R_2\}$ to be our tree. Then $\underset{\sim}{B}_{RR} = \phi$, $\underset{\sim}{B}_{RP} = \phi$, $\underset{\sim}{B}_{PR} = [1\ 1]$, $\underset{\sim}{B}_{PP} = \phi$,

$$D_{\underset{\sim}{v}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \phi$$

$$D_{\underset{\sim}{i}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{bmatrix} -R_1 & 0 \\ -3i_{R_1}^2 + 4i_{R_1} - 2 & 1 \end{bmatrix}, \quad D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \phi .$$

Hence at each $(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \underset{\sim}{\pi}_R(\Sigma_2)$,

$$\begin{aligned}\operatorname{rank} \underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) &= \operatorname{rank} \left[\begin{array}{cc:c} 1 & 0 & -R_1 \\ 0 & 0 & -3i_{R_1}^2 + 4i_{R_1} - 1 \end{array}\right]_{i_{R_1}=1} \\ &= \operatorname{rank} \left[\begin{array}{cc:c} 1 & 0 & -R_1 \\ 0 & 0 & 0 \end{array}\right] = 1 < 2.\end{aligned}$$

Therefore $\Lambda \not\pitchfork K$.

We next discuss the dimension of a weakly regular n-port.

Definition 5 A weakly regular n-port N is said to have dimension[7] m if

$$\operatorname{rank}(d\pi_P)_{(v,i)} = m \text{ for all } (v,i) \in \Sigma \tag{68}$$

where π_P is defined by (24).

Remark In order to check (68), in general, one has to check it in terms of coordinates. Let $(\psi, \Sigma \cap U)$ be a local chart at (v,i). Then (68) holds if and only if

$$\operatorname{rank}(D\pi_P \circ \psi^{-1})_x = m \text{ for all } (v,i) \in \Sigma \tag{69}$$

where $\psi(v,i) = x$. If $\Lambda \overline{\pitchfork} K$, then a more explicit formula can be obtained.

Proposition 3 Let N be a weakly regular n-port with $\Lambda \overline{\pitchfork} K$. Then N has dimension m if and only if for each $(v,i) \in \Sigma$

$$\operatorname{rank} \mathcal{G}(v,i) = m + n_R - n \tag{70}$$

where

$$\mathcal{G}(v,i) \triangleq \left[\begin{array}{c:c} B_{PR} & 0 \\ 0 & B_{RP}^T \\ D_{v_{R_{\mathcal{J}}}} f - (D_{v_{R_{\mathcal{L}}}} f) B_{RR} & D_{i_{R_{\mathcal{L}}}} f + (D_{i_{R_{\mathcal{J}}}} f) B_{RR}^T \end{array}\right]_{(v,i)}. \tag{71}$$

Proof Let $(\psi, \Sigma \cap U)$ be a local chart for Σ at (v,i). We rewrite (35) as

$$\Sigma \cap U = g^{-1}(0) \tag{72}$$

where

$$g(v,i) \triangleq \begin{bmatrix} Bv \\ Qi \\ f(v,i) \end{bmatrix}. \tag{73}$$

Since $\pi_P \circ \psi^{-1}(x) = \pi_P' \circ \iota \circ \psi^{-1}(x)$ (see (24)) we have

$$\operatorname{rank}(D\pi_P \circ \psi^{-1})_x = \dim \operatorname{Im} \, (D\pi_P')_{(v,i)} (d\iota)_{(v,i)} (D\psi^{-1})_x. \tag{74}$$

It follows from (8) and (9) that

$$\operatorname{Im}(D\psi^{-1})_x = \operatorname{Ker}(Dg)_{(v,i)}. \tag{75}$$

[7]This definition is a coordinate-free version of the one defined by Chua and Lam [8].

Next, recall that the linear map $(D\pi'_P)_{(v,i)}$ maps any complement of $\mathrm{Ker}(D\pi'_P)_{(v,i)}$ onto its image space [9]. Since $(d\iota)_{(v,i)}$ is a linear inclusion map, we have from (74) and (75) that

$$\begin{aligned}
\mathrm{rank}(D\pi_P\circ\psi^{-1})_x &= \dim \mathrm{Im}\,(D\pi'_P)_{(v,i)}(D\psi^{-1})_x \\
&= \dim \mathrm{Im}(D\psi^{-1})_x - \dim\left(\mathrm{Im}(D\psi^{-1})_x \cap \mathrm{Ker}(D\pi'_P)_{(v,i)}\right) \\
&= \dim \mathrm{Ker}(Dg)_{(v,i)} - \dim\left(\mathrm{Ker}(Dg)_{(v,i)} \cap \mathrm{Ker}(D\pi'_P)_{(v,i)}\right) \\
&= \left(2b-\mathrm{rank}(Dg)_{(v,i)}\right) - \left(2b-\mathrm{rank}\begin{bmatrix} Dg \\ D\pi'_P \end{bmatrix}_{(v,i)}\right) \\
&= \mathrm{rank}\begin{bmatrix} Dg \\ D\pi'_P \end{bmatrix}_{(v,i)} - \mathrm{rank}(Dg)_{(v,i)}\,. \qquad (76)
\end{aligned}$$

It follows from transversality assumption and (38) that

$$\mathrm{rank}(Dg)_{(v,i)} = b + n_R. \qquad (77)$$

By elementary operations one can show that

$$\mathrm{rank}\begin{bmatrix} Dg \\ D\pi'_P \end{bmatrix}_{(v,i)} = b + n + \mathrm{rank}\,\mathcal{G}(v,i) \qquad (78)$$

where $\mathcal{G}(v,i)$ is defined by (71). Hence (76)-(78) imply

$$\mathrm{rank}(D\pi_P\circ\psi^{-1})_x = n + \mathrm{rank}\,\mathcal{G}(v,i) - n_R. \qquad (79)$$

Therefore N has dimension m if and only if for each $(v,i) \in \Sigma$, $n - n_R + \mathrm{rank}\,\mathcal{G}(v,i) = m$. This gives the result. ¤

If (v_R,i_R) and (v_P,i_P) are not coupled to each other, the condition is simplified in the following manner.

Corollary 4. Let Λ be given by (13) and let $\Lambda \pitchfork K$. Then N has dimension m if and only if for each $(v_R,i_R) \in \pi_R(\Sigma)$,

$$\mathrm{rank}\,\mathcal{G}_R(v_R,i_R) = m + n_R - n \qquad (80)$$

where π_R is defined by (42) and

$$\mathcal{G}_R(v_R, i_R) \triangleq \left[\begin{array}{c:c} B_{PR} & 0 \\ 0 & B_{RP}^T \\ D_{v_{R_g}} f_R - (D_{v_{R_\ell}} f_R) B_{RR} & D_{i_{R_\ell}} f_R + (D_{i_{R_g}} f_R) B_{RR}^T \end{array}\right]_{(v_R, i_R)} . \tag{81}$$

<u>Corollary 5</u> Let Λ admit a generalized port coordinate. Then

$$\mathcal{G}_R(v_R, i_R) = \left[\begin{array}{c:c} B_{PR} & 0 \\ 0 & B_{RP}^T \\ (\alpha_2 - (DF)\gamma_2) - (\alpha_1 - (DF)\gamma_1) B_{RR} & (\beta_1 - (DF)\delta_1) + (\beta_2 - (DF)\delta_2) B_{RR}^T \end{array}\right]_{(v_R, i_R)} \tag{82}$$

where α, β, γ and δ are as in (49). In particular, if Λ_R is globally voltage controlled, then

$$\mathcal{G}_R(v_R, i_R) = \left[\begin{array}{c:c} B_{PR} & 0 \\ 0 & B_{RP}^T \\ (DF)\begin{bmatrix} B_{RR} \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ B_{RR}^T \end{bmatrix} \end{array}\right]_{(v_R, i_R)} \tag{83}$$

and if Λ_R is globally current controlled, then

$$\mathcal{G}_R(v_R, i_R) = \left[\begin{array}{c:c} B_{PR} & 0 \\ 0 & B_{RP}^T \\ \begin{bmatrix} -B_{RR} \\ 1 \end{bmatrix} & -(DF)\begin{bmatrix} 1 \\ B_{RR}^T \end{bmatrix} \end{array}\right]_{i_R} . \tag{84}$$

The following gives a criterion when Λ_R is globally parametrizable.

<u>Proposition 4</u> Let Λ_R be globally parametrized by $\rho \in \mathbb{R}^{n_R}$ and let $\Lambda \pitchfork K$. Then N has dimension m if and only if for each ρ with $(v_R(\rho), i_R(\rho)) \in \pi_R(\Sigma)$,

$$\operatorname{rank} \mathcal{G}^*(\rho) = m + n_R - n \tag{85}$$

where

$$\underset{\sim}{\mathcal{G}}*(\underset{\sim}{\rho}) \triangleq \begin{bmatrix} \underset{\sim}{D}\underset{\sim}{v}_{R_{\mathcal{L}}} + (\underset{\sim}{D}\underset{\sim}{v}_{R_{\mathcal{J}}})\underset{\sim}{B}_{RR} \\ (\underset{\sim}{D}\underset{\sim}{v}_{R_{\mathcal{J}}})\underset{\sim}{B}_{PR} \\ \underset{\sim}{D}\underset{\sim}{i}_{R_{\mathcal{J}}} - (\underset{\sim}{D}\underset{\sim}{i}_{R_{\mathcal{L}}})\underset{\sim}{B}_{RR}^{T} \\ -(\underset{\sim}{D}\underset{\sim}{i}_{R_{\mathcal{L}}})\underset{\sim}{B}_{RP}^{T} \end{bmatrix}_{\underset{\sim}{\rho}} . \tag{86}$$

<u>Proof</u> Recall (59) and (61). Since $\Lambda \pitchfork K$, we have [4]

$$\mathrm{Im}(\underset{\sim}{D}\underset{\sim}{\psi}^{-1})_{\underset{\sim}{x}} = T_{(\underset{\sim}{v},\underset{\sim}{i})}(\Lambda \cap K) = T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda \cap T_{(\underset{\sim}{v},\underset{\sim}{i})}K = \mathrm{Im}\begin{bmatrix} \underset{\sim}{D}\underset{\sim}{v} \\ \underset{\sim}{D}\underset{\sim}{i} \end{bmatrix}_{(\underset{\sim}{v},\underset{\sim}{i})} \cap \mathrm{Im}\begin{bmatrix} \underset{\sim}{Q}^{T} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^{T} \end{bmatrix} \tag{87}$$

where $(\underset{\sim}{v},(\underset{\sim}{\sigma}),\underset{\sim}{i}(\underset{\sim}{\sigma}))$ is as in (57) and (58). It follows from (74) and the first two lines of (76) that

$$\mathrm{rank}(\underset{\sim}{D}\underset{\sim}{\pi}_{P}\circ\underset{\sim}{\psi}^{-1})_{\underset{\sim}{x}} = \dim \mathrm{Im}(\underset{\sim}{D}\underset{\sim}{\psi}^{-1})_{\underset{\sim}{x}} - \dim\left(\mathrm{Ker}(\underset{\sim}{D}\underset{\sim}{\pi}_{P}')_{(\underset{\sim}{v},\underset{\sim}{i})} \cap \mathrm{Im}\begin{bmatrix} \underset{\sim}{D}\underset{\sim}{v} \\ \underset{\sim}{D}\underset{\sim}{i} \end{bmatrix}_{\underset{\sim}{\sigma}} \cap \mathrm{Im}\begin{bmatrix} \underset{\sim}{Q}^{T} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^{T} \end{bmatrix}\right). \tag{88}$$

It is clear that

$$\mathrm{Ker}(\underset{\sim}{D}\underset{\sim}{\pi}_{P}')_{(\underset{\sim}{v},\underset{\sim}{i})} = \mathrm{Im}(\underset{\sim}{D}\underset{\sim}{\pi}_{R}')^{T}_{(\underset{\sim}{v},\underset{\sim}{i})} \tag{89}$$

where $\underset{\sim}{\pi}_{R}'$ is defined by (41). It follows from (64) and (89) that

$$\begin{aligned} \mathrm{Ker}(\underset{\sim}{D}\underset{\sim}{\pi}_{P}')_{(\underset{\sim}{v},\underset{\sim}{i})} \cap \mathrm{Im}\begin{bmatrix} \underset{\sim}{D}\underset{\sim}{v} \\ \underset{\sim}{D}\underset{\sim}{i} \end{bmatrix}_{\underset{\sim}{\sigma}} &= \mathrm{Im}(\underset{\sim}{D}\underset{\sim}{\pi}_{R}')^{T}_{(\underset{\sim}{v},\underset{\sim}{i})} \cap \mathrm{Im}\begin{bmatrix} \underset{\sim}{D}\underset{\sim}{v} \\ \underset{\sim}{D}\underset{\sim}{i} \end{bmatrix}_{\underset{\sim}{\sigma}} \\ &= \mathrm{Im}[\underset{\sim}{D}\underset{\sim}{v}_{R_{\mathcal{L}}} \;\; \underset{\sim}{0} \;\; \underset{\sim}{D}\underset{\sim}{v}_{R_{\mathcal{J}}} \;\; \underset{\sim}{0} \;\; \underset{\sim}{D}\underset{\sim}{i}_{R_{\mathcal{L}}} \;\; \underset{\sim}{0} \;\; \underset{\sim}{D}\underset{\sim}{i}_{R_{\mathcal{J}}} \;\; \underset{\sim}{0}]^{T}_{\underset{\sim}{\sigma}} \triangleq \mathrm{Im}(\underset{\sim}{E})_{\underset{\sim}{\sigma}} . \end{aligned} \tag{90}$$

Equations (88) and (90) imply that

$$\mathrm{rank}(\underset{\sim}{D}\underset{\sim}{\pi}_{P}\circ\underset{\sim}{\psi}^{-1})_{\underset{\sim}{x}} = n - \dim\left(\mathrm{Im}(\underset{\sim}{E})_{\underset{\sim}{\sigma}} \cap \mathrm{Im}\begin{bmatrix} \underset{\sim}{Q}^{T} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^{T} \end{bmatrix}\right). \tag{91}$$

It follows from (91) and formula (which is easy to verify)

$$\dim\left(\mathrm{Im}(\underset{\sim}{E})_{\underset{\sim}{\sigma}} \cap \mathrm{Im}\begin{bmatrix} \underset{\sim}{Q}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^T \end{bmatrix}\right) = \mathrm{rank}(\underset{\sim}{E})_{\underset{\sim}{\sigma}} + \mathrm{rank}\begin{bmatrix} \underset{\sim}{Q}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^T \end{bmatrix} - \mathrm{rank}\left[(\underset{\sim}{E})_{\underset{\sim}{\sigma}} \vdots \begin{matrix} \underset{\sim}{Q}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^T \end{matrix}\right] . \tag{92}$$

that the following holds:

$$\mathrm{rank}(\underset{\sim}{D}\underset{\sim}{\pi}_P \circ \underset{\sim}{\psi}^{-1})_{\underset{\sim}{x}} = n - \mathrm{rank}(\underset{\sim}{E})_{\underset{\sim}{\sigma}} - \mathrm{rank}\begin{bmatrix} \underset{\sim}{Q}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^T \end{bmatrix} + \mathrm{rank}\left[(\underset{\sim}{E})_{\underset{\sim}{\sigma}} \vdots \begin{bmatrix} \underset{\sim}{Q}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^T \end{bmatrix}\right] . \tag{93}$$

Finally it is easy to show that

$$\mathrm{rank}(\underset{\sim}{E})_{\underset{\sim}{\sigma}} = n_R, \quad \mathrm{rank}\begin{bmatrix} \underset{\sim}{Q}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^T \end{bmatrix} = b \text{ and}$$

$$\mathrm{rank}\left[(\underset{\sim}{E})_{\underset{\sim}{\sigma}} \vdots \begin{bmatrix} \underset{\sim}{Q}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}^T \end{bmatrix}\right] = n + n_R + \mathrm{rank}\, \underset{\sim}{\mathcal{G}}^*(\underset{\sim}{\rho}) \tag{94}$$

where $\underset{\sim}{\mathcal{G}}^*(\underset{\sim}{\rho})$ is defined by (86). This and (93) imply the result. ∎

<u>Corollary 6</u> If Λ_R admits a generalized port coordinate, then

$$\underset{\sim}{\mathcal{G}}^*(\underset{\sim}{\eta}) = \left[\begin{array}{c:c} \underset{\sim}{B}_{PR} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_{RP}^T \\ \underset{\sim}{a}_1(\underset{\sim}{D}\underset{\sim}{F})+\underset{\sim}{b}_1+\underset{\sim}{B}_{RR}(\underset{\sim}{a}_2(\underset{\sim}{D}\underset{\sim}{F})+\underset{\sim}{b}_2) & \underset{\sim}{c}_1(\underset{\sim}{D}\underset{\sim}{F})+\underset{\sim}{d}_1-\underset{\sim}{B}_{RR}^T(\underset{\sim}{c}_2(\underset{\sim}{D}\underset{\sim}{F})+\underset{\sim}{d}_2) \end{array}\right]_{\underset{\sim}{\eta}} \tag{95}$$

where $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ and $\underset{\sim}{d}$ are as in (66).

<u>Definition 6</u> A weakly regular n-port N is said to be <u>regular</u> if its dimension is n.

<u>Remarks</u> 1. It is clear that in order for N to be regular it is necessary that $\dim \Sigma \geq n$.

2. Recall the remark after Corollary 2. If Λ_R is locally hybrid as in (53) and if the resistors in $\mathcal{L}$ form loops exclusively with ports, then we saw that $\Lambda \pitchfork K$. It is clear that rank $\mathcal{G}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = n_R$ and hence N has dimension n and is therefore regular. Similarly if Λ_R is globally voltage (resp. current) controlled and if $(\underset{\sim}{DF})_{\underset{\sim}{v}_R}$ (resp., $(\underset{\sim}{DF})_{\underset{\sim}{i}_R}$) is positive definite at each point, then N is regular.

Example 12 Consider (i)-(v) and (vii) of Example 1. Choose $\mathcal{T} = \{R_1,R_2\}$ to be our tree. Then (81) is given by

$$\underset{\sim}{\mathcal{G}}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = \left[\begin{array}{c:c} 1 & 1 \\ -Df_{R_1} & 0 \\ 0 & -Df_{R_2} \end{array}\right]_{\underset{\sim}{v}_R}.$$

Since Df_{R_1} and Df_{R_2} never vanish simultaneously, rank $\mathcal{G}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = 2$ holds. Since $n_R = 2$, $n = 1$, we have $m = \text{rank } \mathcal{G}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) + n - n_R = 2 + 1 - 2 = 1$. Hence this 1-port has dimension 1 and is therefore regular.

Example 13 (Weakly regular 1-port which is not regular) Consider Example 2. The associated configuration space Σ is a 2-dimensional submanifold since it is parametrized by (v_P,i_P); $(v_R,v_P,i_R,i_P) = (-v_P,v_P,i_P,i_P)$. Recall (69). Since $\underset{\sim}{\pi}_P \circ \underset{\sim}{\psi}^{-1}(v_P,i_P) = (v_P,i_P)$, we have rank $(\underset{\sim}{D\pi}_P\circ\underset{\sim}{\psi}^{-1})_{(v_P,i_P)}$ $= \text{rank} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$. Hence this is a 2-dimensional 1-port and therefore is not regular. Consider Example 3. Clearly $\Sigma = \{\underset{\sim}{0}\}$ and this is a weakly regular 1-port. But since $\dim \Sigma = 0 < 1 = n$, it cannot be regular.

Proposition 5 If $\Lambda \pitchfork K$ and if $\underset{\sim}{\pi}_P$ is an immersion, then N is regular.

Proof By Theorem 1, Σ is an n-dimensional submanifold. Since $\underset{\sim}{\pi}_P$ is an immersion, rank $(\underset{\sim}{d\pi}_P)_{(\underset{\sim}{v},\underset{\sim}{i})} = n$ for all $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$, which shows that N has dimension n. ∎

An n-port can be regular without satisfying the transversality condition alluded to earlier. The following example is a case in point:

Example 14 Consider the 1-port of Example 11. For Σ_1 and Σ_2, v_p serves as a coordinate. For Σ_1, $\pi_p \circ \psi^{-1}(v_p) = (v_p,0)$ and hence $(D\pi_p \circ \psi^{-1})_{v_p} = [1 \quad 0]$ which has rank 1. Similarly for Σ_2, $\pi_p \circ \psi^{-1}(v_p) = (v_p,1)$ and hence $(D\pi_p \circ \psi^{-1})_{v_p} = [1 \quad 0]$. Therefore this 1-port is regular even though $\Lambda \not\pitchfork K$ as shown earlier in Example 11.

Definition 7 A regular n-port N is said to be strongly regular if $\mathcal{R} = \pi_p(\Sigma)$ is an n-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.

Example 15 (Regular 1-port which is not strongly regular) Consider (i) - (v) of Example 1. As was shown in Example 12 each is a regular 1-port. But since $\mathcal{R}$ has a self intersection, it is not a submanifold. Hence they are not strongly regular.

This example shows that $\mathcal{R}$ could be a rather complicated object even if Σ is a perfectly well defined submanifold. This is understandable because Σ lies in $\mathbb{R}^6$ while $\mathcal{R}$ is the projected image of Σ onto $\mathbb{R}^2$. If one projects a geometric object in $\mathbb{R}^6$ onto $\mathbb{R}^2$, one naturally loses some "information" concerning that object.

A sufficient condition for strong regularity is the following.

Proposition 6 If $\Lambda \pitchfork K$ and if π_p is an embedding, then N is strongly regular.

Proof By transversality condition, Σ is an n-dimensional submanifold. By definition, the image of a submanifold under an embedding is a submanifold of the same dimension. Hence $\mathcal{R} = \pi_p(\Sigma)$ is an n-dimensional submanifold. $\square$

Definition 8 A strongly regular n-port N is said to be globally strongly regular if $\mathcal{R}$ is globally diffeomorphic to $\mathbb{R}^n$.

The following proposition gives a sufficient condition for N to be globally strongly regular.

Proposition 7 Let Σ be globally diffeomorphic to $\mathbb{R}^n$ and let $\psi : \Sigma \to \mathbb{R}^n$ be a global coordinate. If

(i) $\lim\limits_{\|x\| \to \infty} \|\pi_p \circ \psi^{-1}(x)\| = \infty$

(ii) $\pi_p \circ \psi^{-1}$ is injective

(iii) $\text{rank}(D\pi_p \circ \psi^{-1})_x = n$ for all $x \in \mathbb{R}^n$

the N is globally strongly regular.

Proof Condition (i) is equivalent to saying that $\pi_p \circ \psi^{-1}$ is proper, i.e., preimage of any compact set is compact [3]. Therefore (i)-(iii) imply that $\pi_p \circ \psi^{-1}$ is an injective proper immersion. Such a map is clearly an embedding [3]. Hence $\mathcal{R} = \pi_p \circ \psi^{-1}(\mathbb{R}^n)$ is a diffeomorphic copy of $\mathbb{R}^n$. ¤

Example 16 The 1-ports of (vii) and (viii) of Example 1 are globally strongly regular.

Definition 9 A globally strongly regular n-port N is said to be normal if it admits a generalized port coordinate;

$$\begin{bmatrix} v_p \\ i_p \end{bmatrix} = \begin{bmatrix} a_p & b_p \\ c_p & d_p \end{bmatrix} \begin{bmatrix} \xi_p \\ \eta_p \end{bmatrix}, \quad \xi_p = F_p(\eta_p)$$

where a_p, b_p, c_p and d_p are $n \times n$ matrices and $F_p : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function.

Example 17 The 1-port of (vii) of Example 1 is normal because it is globally voltage controlled. The 1-port of (viii) is not normal because no linear combination of v_p and i_p can be a global coordinate for $\mathcal{R}$.

Sometimes we can choose n among the 2n variables (v_p, i_p) as a global coordinate for Σ. In this case π_p is a global diffeomorphism and hence $\mathcal{R}$ is also globally diffeomorphic to $\mathbb{R}^n$. In fact N turns out to be normal. This sometimes happens when an n-port is derived from an RLC network. For example, consider the n-port seen by the capacitors and inductors of an RLC network. If the capacitor voltages and inductor currents are chosen as global coordinates for the configuration space Σ, then $\mathcal{R}$ of the derived n-port N is globally diffeomorphic to $\mathbb{R}^n$. This means that the n-port as seen by the capacitors and inductors, is a nice n-dimensional submanifold which is gobally diffeomorphic to $\mathbb{R}^n$. We formalize this observation as follows:

Proposition 8 Let $z \in \mathbb{R}^n$ be a subvector of (v_p, i_p). Let $\pi'_z : \mathbb{R}^b \times \mathbb{R}^b \to \mathbb{R}^n$ be defined by $\pi'_z(v,i) = z$ and let $\pi_z : \Sigma \to \mathbb{R}^n$ be defined by

$$\pi_z \triangleq \pi'_z \circ \iota \tag{96}$$

where ι is the inclusion map defined by (23). If π_z is a global diffeomorphism, i.e., z serves as a global coordinate for Σ, then

(i) $\pi_p : \Sigma \to \mathcal{R}$ is a global diffeomorphism,

(ii) $\pi_z \circ (\pi_p^{-1}|\mathcal{R})$ is a global coordinate for $\mathcal{R}$, where $\pi_p^{-1}|\mathcal{R}$ denotes the restriction of π_p^{-1} to $\mathcal{R}$,

(iii) N is normal.

Proof Let us write

$$v_p = (v_a, v_b), \quad i_p = (i_a, i_b), \quad z = (v_a, i_b). \tag{97}$$

By assumption $\underset{\sim}{z}$ globally parametrizes Σ. Hence $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$ is expressible as a C^1 function of $(\underset{\sim}{v}_a,\underset{\sim}{i}_b)$. In particular $(\underset{\sim}{v}_p,\underset{\sim}{i}_p)$ is expressible as a C^1 function of $(\underset{\sim}{v}_a,\underset{\sim}{i}_b)$;

$$(\underset{\sim}{v}_p,\underset{\sim}{i}_p) = \underset{\sim}{\pi}_p \circ \underset{\sim}{\pi}_z^{-1}(\underset{\sim}{v}_a,\underset{\sim}{i}_b) = (\underset{\sim}{v}_a,\underset{\sim}{g}(\underset{\sim}{v}_a,\underset{\sim}{i}_b),\underset{\sim}{h}(\underset{\sim}{v}_a,\underset{\sim}{i}_b),\underset{\sim}{i}_b) \tag{98}$$

where $\underset{\sim}{g}$ and $\underset{\sim}{h}$ are C^1 functions. It is clear that $\underset{\sim}{\pi}_p \circ \underset{\sim}{\pi}_z^{-1}$ is a global diffeomorphism. Hence $\underset{\sim}{\pi}_p$ is a global diffeomorphism and $\underset{\sim}{\pi}_p \circ \underset{\sim}{\pi}_z^{-1}$ is a global parametrization for $\mathcal{R}$. Finally (98) gives $(\underset{\sim}{v}_b,\underset{\sim}{i}_a) = (\underset{\sim}{g}(\underset{\sim}{v}_a,\underset{\sim}{i}_b),\underset{\sim}{h}(\underset{\sim}{v}_a,\underset{\sim}{i}_b))$ which means that $\mathcal{R}$ admits a global hybrid representation. ¤

In table 1 we summarize the various classifications of n-ports given in this section.

IV. Structural Stability of Resistive n-ports

In this section we give a structural stability result for resistive n-ports. By structural stability here we mean the persistence of the configuration space Σ under small perturbations of Λ. The result essentially says that a resistive n-port is structurally stable if and only if $\Lambda \pitchfork K$. Hence, transversality of Λ and K, again, plays a crucial role.

We first give a precise definition of perturbation. Let M be a C^1 submanifold of $\mathbb{R}^n$ and let $C^1(M,\mathbb{R}^n)$ be the set of all C^1 functions from M into $\mathbb{R}^n$. Let $\underset{\sim}{F} \in C^1(M,\mathbb{R}^n)$ and consider

$$\mathcal{U}\left(\underset{\sim}{F};\varepsilon(\cdot)\right) \triangleq \left\{ \underset{\sim}{G} : M \to \mathbb{R}^n \;\middle|\; \begin{array}{l} \underset{\sim}{G} \in C^1(M,\mathbb{R}^n) \\ \|\underset{\sim}{F}(\underset{\sim}{x})-\underset{\sim}{G}(\underset{\sim}{x})\| + \|(d\underset{\sim}{F})_{\underset{\sim}{x}} - (d\underset{\sim}{G})_{\underset{\sim}{x}}\| < \varepsilon(\underset{\sim}{x}) \\ \text{for all } \underset{\sim}{x} \in M \end{array} \right\} \tag{99}$$

where $\varepsilon(\underset{\sim}{x})$ is an arbitrary continuous function from M into the set of all positive numbers. The <u>Whitney C^1 topology</u> or the <u>strong C^1 topology</u> on $C^1(M,\mathbb{R}^n)$ is generated by sets of the form (99), i.e., $\mathcal{U}\left(\underset{\sim}{F},\varepsilon(\cdot)\right)$ is a neighborhood of $\underset{\sim}{F}$ and any open subset of $C^1(M,\mathbb{R}^n)$ is expressible as a union of sets of the form (99). Observe that if a neighborhood $\mathcal{U}\left(\underset{\sim}{F};\varepsilon(\cdot)\right)$ is small and if $\underset{\sim}{G} \in \mathcal{U}\left(\underset{\sim}{F};\varepsilon(\cdot)\right)$, then $\|\underset{\sim}{F}(\underset{\sim}{x})-\underset{\sim}{G}(\underset{\sim}{x})\| + \|(d\underset{\sim}{F})_{\underset{\sim}{x}} - (dG)_{\underset{\sim}{x}}\| \to 0$ as $\|\underset{\sim}{x}\| \to \infty$. (See Fig. 14) This is the reason why the strong C^1 topology can control the behavior of functions over a noncompact set. We need this property since our Λ is generally unbounded. One of the greatest advantages of the strong C^1 topology is that the set $\text{Emb}^1(M,\mathbb{R}^n)$ of all C^1 embeddings of M into $\mathbb{R}^n$ is <u>open</u> with respect to this topology [3]. If M is a C^1 submanifold of $\mathbb{R}^n$ then the inclusion map $\underset{\sim}{\imath}_M$ is an embedding. Hence there is a neighborhood $\mathcal{V}(\underset{\sim}{\imath}_M)$ of $\underset{\sim}{\imath}_M$ such that all elements of $\mathcal{V}(\underset{\sim}{\imath}_M)$ are embeddings.

Definition 10. A C^1 perturbation $\hat{M}$ of M is defined by $\hat{M} \triangleq \underset{\sim}{G}(M)$, where $\underset{\sim}{G} \in \mathcal{V}(\underset{\sim}{\imath}_M)$.

Making the neighborhood small, one can obtain arbitrarily small C^1 perturbations of M.

Remark The set of embeddings of M into $\mathbb{R}^n$ is not open in the weak C^1 topology which is generated by sets of the form;

$$\mathcal{U}_W(\underset{\sim}{F};\varepsilon) \triangleq \left\{ \underset{\sim}{G} : M \to \mathbb{R}^n \;\middle|\; \begin{array}{l} \underset{\sim}{G} \in C^1(M,\mathbb{R}^n) \\ \|\underset{\sim}{F}(\underset{\sim}{x})-\underset{\sim}{G}(\underset{\sim}{x})\| + \|(d\underset{\sim}{F})_{\underset{\sim}{x}} - (d\underset{\sim}{G})_{\underset{\sim}{x}}\| < \varepsilon \\ \text{for all } \underset{\sim}{x} \in M \end{array} \right\}$$

where $\varepsilon > 0$ is a constant. In order to see that the set of embeddings is not open in this topology, consider the 1-dimensional submanifold $M \subset \mathbb{R}^2$ as shown in Fig. 15, where the two "tails" get closer and closer to each other. Since M is a submanifold, the inclusion map $\imath_M$ is an embedding. Now, in an arbitrary neighborhood of $\imath_M$ with respect to the weak C^1 topology, one can find a map G such that G(M) has "tails" touching each other. Such a map is certainly not an embedding. On the other hand in the strong C^1 topology, if one chooses $\varepsilon(\cdot)$ in an appropriate manner, then all the elements of the neighborhood are embeddings. We are now ready to state our structural stability result. In the following we assume that Λ is closed. This assumption of course entails no loss of generality for electrical networks.

Theorem 2. Given a resistive n-port N assume that Λ is closed and $\Lambda \cap K \neq \phi$.

(i) If $\Lambda \pitchfork K$, then N is structurally stable in the sense that for any small C^1 perturbation $\hat{\Lambda}$ of Λ, the configuration space $\hat{\Sigma} = \hat{\Lambda} \cap K$ persists to be an n-dimensional submanifold. In fact, $\hat{\Sigma}$ is diffeomorphic to Σ.

(ii) If $\Lambda \not\pitchfork K$, then N is structurally unstable in the following sense:

(a) If Σ is not an n-dimensional submanifold, then there is an arbitrarily small C^1 perturbation $\hat{\Lambda}$ of Λ such that $\hat{\Sigma}$ is an n-dimensional submanifold.

(b) If Σ is an n-dimensional submanifold, then there is an arbitrarily small C^1 perturbation $\hat{\Lambda}$ of Λ such that $\hat{\Sigma}$ contains an (n+k)-dimensional submanifold for some $k > 0$.

Remarks 1. Recall from Theorem 1 that if $\Lambda \pitchfork K$ then Σ is an n-dimensional submanifold.

2. When $\Lambda \not\pitchfork K$, there are two cases which could happen; (a) Σ is not a submanifold of dimension n as in (vi) of Example 1, and (b), Σ is a submanifold of dimension n as in Example 11. Statement (ii-b) of Theorem 2 says that if (b) happens, then one can perturb Λ in such a way that $\hat{\Lambda} \cap K$ contains a higher dimensional sub-submanifold. To see this, recall Example 11. Observe that the constitutive relation $f_2(i_{R_1}, i_{R_2}) = i_{R_2} - i_{R_1}^3 + 2i_{R_1}^2 - 2i_{R_1} = 0$ and KCL $i_{R_1} = i_{R_2} = i_P$ imply $i_P(i_P-1)^2 = 0$. Next look at the graph of the function $g(i_P) \triangleq i\,(i_P-1)^2$. It is clear that one can give a small C^1 perturbation to f_2 in such a way that the corresponding graph of the perturbation $\hat{g}(i_P)$ of $g(i_P)$ has a flat portion $[a,b]$ as in Fig. 16. Then the perturbation $\hat{\Sigma}_2$ of Σ_2 is given by

$$\hat{\Sigma}_2 = \left\{ (\underset{\sim}{v}, \underset{\sim}{i}) \,\middle|\, \begin{array}{l} i_{R_1} = i_{R_2} = i_P = \alpha,\ v_{R_1} = \alpha R_1 \\ v_{R_2} + v_P + \alpha R_1 = 0, \alpha \in [a,b] \end{array} \right\} . \tag{100}$$

This set contains a 2-dimensional submanifold because it is parametrized by (α, v_P);

$$(v_{R_1}, v_{R_2}, v_P, i_{R_1}, i_{R_2}, i_P) = (\alpha R_1, -v_P - \alpha R_1, v_P, \alpha, \alpha, \alpha),\ v_P \in \mathbb{R},\ \alpha \in [a,b].$$

Hence, even though Σ is a submanifold of the correct dimension, $\Lambda \not\pitchfork K$ implies structural instability.

3. Structural stability as used in Section I was vague not only because the term "perturbation" was not defined rigorously, but also because we looked at Σ through the projection $\mathcal{R}$ while discussing concepts of persistence and abrupt changes. Sometimes it might be more appropriate to consider the persistence of $\mathcal{R}$ rather than the persistence of Σ. To do this, however, one might have to assume that $\mathcal{R}$ is a submanifold, a more stringent condition.

For the proof of Theorem 2 we need the following four lemmas whose proofs are given in APPENDIX 2.

Lemma 1. Let M_1 and M_2 be C^1 submanifolds of $\mathbb{R}^n$. Then one can obtain an arbitrarily small C^1 perturbation $\hat{M}_1$ of M_1 such that $\hat{M}_1 \pitchfork M_2$. ($\hat{M}_1 \cap M_2$ may be empty).

Remark A serious drawback of Lemma 1 is that one cannot guarantee $\hat{M}_1 \cap M_2 \neq \phi$ even if $M_1 \cap M_2 \neq \phi$. To be specific, let $M_1 = \Lambda$ and $M_2 = K$ be chosen such that $\Lambda \cap K \neq \phi$, i.e., the configuration space is nonempty. After perturbation, we may end up with $\hat{\Lambda} \pitchfork K$ but $\hat{\Lambda} \cap K = \phi$. For example, consider the situation shown in Fig. 17(a). By giving an arbitrarily small C^1 perturbation to M_1, one can pull the two submanifolds apart as in Fig. 17(b). Hence transversality is trivially satisfied in this case but serves no useful purpose. However, there is another small C^1 perturbation of M_1 as in Fig. 17(c) where $\hat{M}_1 \pitchfork M_2$ and $\hat{M}_1 \cap M_2 \neq \phi$. This latter perturbation is what we would like to have and the following lemmas characterize such perturbations.

<u>Lemma 2</u> Let X and Y be two linear subspaces of $\mathbb{R}^n$ with dim X = n_X, dim Y = n_Y and

$$n_X + n_Y \geq n. \tag{101}$$

Then there is a nonsingular n×n matrix $\underset{\sim}{A}$ such that the matrix norm $\|\underset{\sim}{A}-\underset{\sim}{1}\|$ is arbitrarily small (See Fig. 18) and

$$X + \underset{\sim}{A}(Y) = \mathbb{R}^n, \; X \cap \underset{\sim}{A}(Y) \neq \phi. \tag{102}$$

i.e.,

$$X \pitchfork \underset{\sim}{A}(Y), \; X \cap \underset{\sim}{A}(Y) \neq \phi. \tag{103}$$

<u>Lemma 3</u> Let $\underset{\sim}{f}$ and $\underset{\sim}{g} : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 functions with $\underset{\sim}{f}(\underset{\sim}{x}_0) = \underset{\sim}{g}(\underset{\sim}{x}_0)$ and $(\underset{\sim}{D}\underset{\sim}{f})_{\underset{\sim}{x}_0} = (\underset{\sim}{D}\underset{\sim}{g})_{\underset{\sim}{x}_0}$ for some point $\underset{\sim}{x}_0 \in \mathbb{R}^n$. Then there are neighborhoods U_1 and U_2 of $\underset{\sim}{x}_0$ with $\bar{U}_1 \subset U_2$ where $\bar{U}_1$ is the closure of U_1, and there is a C^1 function $\hat{\underset{\sim}{g}} : \mathbb{R}^n \to \mathbb{R}^n$ such that

(i) $\hat{\underset{\sim}{g}} = \underset{\sim}{f}$ on U_1

(ii) $\hat{\underset{\sim}{g}} = \underset{\sim}{g}$ off U_2

(iii) $\hat{\underset{\sim}{g}}$ is arbitrarily close to $\underset{\sim}{g}$ in the strong C^1 topology. (See Fig. 19).

<u>Lemma 4</u> Let $\underset{\sim}{A}$ be an n×n matrix such that $\|\underset{\sim}{A}-\underset{\sim}{1}\|$ is arbitrarily small. Then there are neighborhoods U_1 and U_2 of the origin with $\bar{U}_1 \subset U_2$ and there is a diffeomorphism $\underset{\sim}{G}$ of $\mathbb{R}^n$ such that

(i) $\underset{\sim}{G} = \underset{\sim}{A}$ on U_1

(ii) $\underset{\sim}{G} = i_d$ off U_2

(iii) $\underset{\sim}{G}$ is arbitrarily close to the identity map i_d in the strong C^1 topology.

To prove <u>Theorem 2</u>, we also need to define the transversality of a <u>function</u> to a manifold.

<u>Definition 11</u>. Let M_1 and M_2 be C^1 submanifolds of $\mathbb{R}^n$ and let $\underset{\sim}{F} : M_1 \to \mathbb{R}^n$ be a C^1 function. Then $\underset{\sim}{F}$ is said to be transversal to M_2 and is abbreviated by $\underset{\sim}{F} \pitchfork M_2$ if

$$\mathrm{Im}(\underset{\sim}{dF})_{\underset{\sim}{x}} + T_{\underset{\sim}{F}(\underset{\sim}{x})}M_2 = \mathbb{R}^n \tag{104}$$

for all $\underset{\sim}{x}$ satisfying $F(\underset{\sim}{x}) \in M_2$.

<u>Remark</u> <u>Transversality</u> of a function is a generalization <u>of transversality</u> of two

submanifolds. Take $\underset{\sim}{F} = \underset{\sim}{\iota}_{M_1}$, the inclusion map of M_1, then $\mathrm{Im}(d\underset{\sim}{F})_{\underset{\sim}{x}} = T_{\underset{\sim}{x}}M_1$. Hence (104) is reduced to transversality of M_1 and M_2. Therefore, if $\underset{\sim}{F}: M_1 \to \mathbb{R}^n$ is an embedding transversal to M_2, then $\underset{\sim}{F}(M_1) \pitchfork M_2$.

Proof of Theorem 2 (i) Assume $\Lambda \cap K \neq \phi$ and $\Lambda \pitchfork K$. Since Λ is assumed to be closed and since K is closed, the set of all C^1 functions from Λ into $\mathbb{R}^b \times \mathbb{R}^b$ which are transversal to K is open [3]. Let $\underset{\sim}{\iota}_\Lambda : \Lambda \to \mathbb{R}^b \times \mathbb{R}^b$ be the inclusion map. By assumption $\underset{\sim}{\iota}_\Lambda \pitchfork K$. Hence there is a neighborhood $\mathcal{U}(\underset{\sim}{\iota}_\Lambda)$ of $\underset{\sim}{\iota}_\Lambda$ in $C^1(\Lambda, \mathbb{R}^b \times \mathbb{R}^b)$ such that all elements of $\mathcal{U}(\underset{\sim}{\iota}_\Lambda)$ are transversal to K. On the other hand, since the set of all C^1 embeddings is open in $C^1(\Lambda, \mathbb{R}^b \times \mathbb{R}^b)$, there is a neighborhood $\mathcal{V}(\underset{\sim}{\iota}_\Lambda)$ whose elements are embeddings. Set $\mathcal{W}(\underset{\sim}{\iota}_\Lambda) \triangleq \mathcal{U}(\underset{\sim}{\iota}_\Lambda) \cap \mathcal{V}(\underset{\sim}{\iota}_\Lambda)$. Then $\mathcal{W}(\underset{\sim}{\iota}_\Lambda)$ is a neighborhood of $\underset{\sim}{\iota}_\Lambda$ consisting of embeddings of Λ which are transversal to K. For any $G \in \mathcal{W}(\underset{\sim}{\iota}_\Lambda)$ set $\hat{\Lambda} = \underset{\sim}{G}(\Lambda)$. Then, as in the proof of Lemma 1 we have that $\hat{\Lambda}$ is a $(2b-n_R)$-dimensional submanifold and $\hat{\Lambda} \pitchfork K$. Hence $\hat{\Sigma} \triangleq \hat{\Lambda} \cap K$ is an n-dimensional submanifold. The proof of the fact that $\hat{\Sigma}$ is diffeomorphic to Σ is technically involved. It is given in APPENDIX 2.

(ii-a). Let $(\underset{\sim}{v}, \underset{\sim}{i}) \in \Lambda \cap K$ be a point of nontransversal intersection, i.e.,

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K \neq \mathbb{R}^b \times \mathbb{R}^b. \tag{105}$$

In order to simplify notation assume, without loss of generality, that $(\underset{\sim}{v}, \underset{\sim}{i})$ is the origin of $\mathbb{R}^b \times \mathbb{R}^b$. For the general case one can simply translate the origin to $(\underset{\sim}{v}, \underset{\sim}{i})$. Recall that $\dim \Lambda = 2b-n_R$ and $\dim K = b$ so that $\dim T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + \dim T_{(\underset{\sim}{v},\underset{\sim}{i})}K = \dim \Lambda + \dim K = 3b-n_R \geq 2b$. Hence Lemma 2 is applicable with $X = T_{(\underset{\sim}{v},\underset{\sim}{i})}K$, $Y = T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda$ and we have

$$\underset{\sim}{A}(T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda) + T_{(\underset{\sim}{v},\underset{\sim}{i})}K = \mathbb{R}^b \times \mathbb{R}^b \tag{106}$$

where $\underset{\sim}{A}$ is a $2b \times 2b$ nonsingular matrix such that $\|\underset{\sim}{A} - \underset{\sim}{1}\|$ is arbitrarily small. Let $\mathcal{U}(i_d)$ be a small enough neighborhood of the identity map i_d of $\mathbb{R}^b \times \mathbb{R}^b$ such that elements of $\mathcal{U}(i_d)$ are diffeomorphisms of $\mathbb{R}^b \times \mathbb{R}^b$. It follows from Lemma 4 that there is a diffeomorphism $\underset{\sim}{G}_1 \in \mathcal{U}(i_d)$ such that for some neighborhoods U_1 and U_2 of $(\underset{\sim}{v}, \underset{\sim}{i})$ with $\bar{U}_1 \subset U_2$, the following hold:

(a) $\underset{\sim}{G}_1 = \underset{\sim}{A}$ on U_1 (107)

(b) $\underset{\sim}{G}_1 = i_d$ off U_2 (108)

(c) $\underset{\sim}{G}_1$ is arbitrarily close to i_d in the strong C^1 topology.

Now let $\underset{\sim}{G}_2$ be the restriction of $\underset{\sim}{G}_1$ to Λ;

$$\underset{\sim}{G}_2 \triangleq \underset{\sim}{G}|\Lambda . \tag{109}$$

Since $\underset{\sim}{G}_1$ is a diffeomorphism of $\mathbb{R}^b \times \mathbb{R}^b$, $\underset{\sim}{G}_2$ is an embedding and hence $\underset{\sim}{G}_2(\Lambda)$ is a $(2b-n_R)$-dimensional C^1 submanifold. The function $\underset{\sim}{G}_2$ locally perturbes Λ on U_1 in such a way that (106) holds at $(\underset{\sim}{v},\underset{\sim}{i})$ and leaves $\Lambda \cap (\mathbb{R}^b \times \mathbb{R}^b - U_2)$ unchanged. Therefore there is a neighborhood U_3 of $(\underset{\sim}{v},\underset{\sim}{i})$ such that

$$(\underset{\sim}{G}_2(\Lambda) \cap U_3) \overline{\pitchfork} (K \cap U_3). \tag{110}$$

It follows from (107) that $\underset{\sim}{G}_2(\underset{\sim}{v},\underset{\sim}{i}) = (\underset{\sim}{v},\underset{\sim}{i})$. Since $(\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda \cap K \neq \phi$, we have

$$(\underset{\sim}{G}_2(\Lambda) \cap U_3) \cap (K \cap U_3) \neq \phi. \tag{111}$$

Now, although we have eliminated the particular nontransversal intersection $(\underset{\sim}{v},\underset{\sim}{i})$ there may be some more nontransversal intersections remaining, or by applying $\underset{\sim}{G}_2$, we might have created new nontransversal intersections. We now apply Lemma 1 with $M_1 = \underset{\sim}{G}_2(\Lambda)$, $M_2 = K$ and obtain a small C^1 perturbation $\hat{\Lambda}$ such that

$$\hat{\Lambda} \overline{\pitchfork} K. \tag{112}$$

By (110) and (111) we know that $\underset{\sim}{G}_2(\Lambda) \cap U_3$ and $K \cap U_3$ have nonempty transversal intersection. Hence if $\hat{\Lambda}$ is close enough to $\underset{\sim}{G}_2(\Lambda)$, then local nonemptiness is not destroyed. Namely, there is a neighborhood $U_4 \subset U_3$ of $(\underset{\sim}{v},\underset{\sim}{i})$ such that

$$(\hat{\Lambda} \cap U_4) \cap (K \cap U_4) \neq \phi. \tag{113}$$

Hence $\hat{\Lambda} \cap K \neq \phi$. This, (112), and Theorem 1 imply the result. By choosing neighborhoods small enough, one can make $\hat{\Lambda}$ arbitrarily close to Λ.

(ii-b) Let $(\underset{\sim}{v},\underset{\sim}{i})$ be a point of nontransversal intersection. Then (105) implies

$$\dim (T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K) = 2b-k \tag{114}$$

for some $k > 0$. The following is an elementary fact in linear algebra:

$$\dim(T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda \cap T_{(\underset{\sim}{v},\underset{\sim}{i})}K) = \dim T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + \dim T_{(\underset{\sim}{v},\underset{\sim}{i})}K - \dim(T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda + T_{(\underset{\sim}{v},\underset{\sim}{i})}K).$$

This and (114) imply

$$\dim(T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda \cap T_{(\underset{\sim}{v},\underset{\sim}{i})}K) = 2b-n_R+b-(2b-k) = n+k \tag{115}$$

for some $k > 0$. The number k is the extra dimension due to nontransversality. Applying Lemma 3 we push Λ onto $T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda$ locally. To this end recall that any submanifold is locally expressible as the graph of a function. In particular, there is a neighborhood U of $(\underset{\sim}{v},\underset{\sim}{i})$ in $\mathbb{R}^b \times \mathbb{R}^b$ and there is a C^1 function

$$\underset{\sim}{F} : T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda \cap U \to (T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda)^{\perp} \cap U$$

such that

$$\Lambda \cap U = \text{graph } \underset{\sim}{F}$$

$$= \{ (\underset{\sim}{x},\underset{\sim}{y}) \in (T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda) \times (T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda)^{\perp} \cap U | \underset{\sim}{y} = \underset{\sim}{F}(\underset{\sim}{x}) \}$$

where $(T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda)^{\perp}$ is the orthogonal complement of $T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda$ in $\mathbb{R}^b \times \mathbb{R}^b$. Without loss of generality one can assume that $(\underset{\sim}{v},\underset{\sim}{i})$ is the origin. Therefore we have

$$\underset{\sim}{F}(\underset{\sim}{0}) = \underset{\sim}{0}. \tag{116}$$

Since $T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda = \mathbb{R}^{2b-n_R}$, $(T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda)^{\perp} = \mathbb{R}^{n_R}$, we think Λ lies in $\mathbb{R}^{2b-n_R} \times \mathbb{R}^{n_R}$. Now let V be another neighborhood of $(\underset{\sim}{v},\underset{\sim}{i})$ with $\bar{V} \subset U$. Take $\underset{\sim}{f} \equiv \underset{\sim}{0}$ and $\underset{\sim}{g} = \underset{\sim}{F}$ in Lemma 3. Then there are neighborhoods U_1 and U_2 of $\underset{\sim}{0}$ in $\mathbb{R}^{2b-n_R}$ with $\bar{U}_1 \subset U_2 \subset \mathbb{R}^{2b-n_R} \cap V$ and there is a C^1 function $\hat{\underset{\sim}{F}} : \mathbb{R}^{2b-n_R} \cap V \to \mathbb{R}^{n_R} \cap V$ such that

(a) $\hat{\underset{\sim}{F}} = \underset{\sim}{0}$ on U_1

(b) $\hat{\underset{\sim}{F}} = \underset{\sim}{F}$ off U_2

(c) $\hat{\underset{\sim}{F}}$ is arbitrarily close to $\underset{\sim}{F}$ in the strong C^1 topology.

Let $\underset{\sim}{G}$: (graph $\underset{\sim}{F}$) $\cap V \to V$ be defined by

$$G(\underset{\sim}{x},\underset{\sim}{y}) \triangleq \left(\underset{\sim}{x},\hat{\underset{\sim}{F}}(\underset{\sim}{x})\right) \tag{117}$$

and let $\underset{\sim}{H} : \Lambda \to \mathbb{R}^{2b-n_R} \times \mathbb{R}^{n_R}$ be defined by

$$\underset{\sim}{H}(\underset{\sim}{x},\underset{\sim}{y}) \triangleq \lambda(\underset{\sim}{x},\underset{\sim}{y})\ \underset{\sim}{G}(\underset{\sim}{x},\underset{\sim}{y}) + \left(1-\lambda(\underset{\sim}{x},\underset{\sim}{y})\right)(\underset{\sim}{x},\underset{\sim}{y}) \tag{118}$$

where $\lambda : \mathbb{R}^{2b-n_R} \times \mathbb{R}^{n_R} \to [0,1]$ is a C^1 function satisfying

$$\lambda(\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{on } V \\ 0 & \text{off } U. \end{cases}$$

Set

$$\hat{\Lambda} \triangleq \underset{\sim}{H}(\Lambda). \tag{119}$$

Now $\underset{\sim}{H}$ is the inclusion map off $\Lambda \cap U$. On the other hand $\underset{\sim}{H}$ locally flatters Λ onto $T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda$ so that there is a neighborhood W of $(\underset{\sim}{v},\underset{\sim}{i})$ in $\mathbb{R}^{2b-n_R} \times \mathbb{R}^{n_R}$ such that

$$\hat{\Lambda} \cap K \cap W = T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda \cap T_{(\underset{\sim}{v},\underset{\sim}{i})}K \cap W. \tag{120}$$

It follows from (115) that the set (120) is an (n+k)-dimensional submanifold. If all the neighborhoods and perturbations are small enough, then $\hat{\Lambda}$ will be a small C^1 perturbation of Λ. ¤

Remark Our reason for requiring Λ to be closed is as follows. Let M_1 and M_2 be submanifolds of $\mathbb{R}^n$ with M_1 and M_2 closed. We used the fact that the set of all functions from M_1 into $\mathbb{R}^n$ transversal to M_2 is open. If M_1 is not closed, this is not true. Suppose, for example, that $M_1 = \{(x_1,x_2) | x_2 = 0, x_1 \in (0,1)\}$ and that M_2 is as in Fig. 20. Note that $(0,0) \notin \iota_{M_1}(M_1) \cap M_2$ and $\iota_{M_1} \pitchfork M_2$. It is clear that given any neighborhood of ι_{M_1}, one can find an embedding G of M_1 such that $G(M_1)$ meets M_2 tangentially near the origin, i.e., there is a 1-dimensional submanifold $\hat{M}_1 = G(M_1)$ which is close to M_1 and touches M_2 in a tangential manner near the origin. Hence $G(M_1) \not\pitchfork M_2$. Therefore closedness of M_1 cannot be relaxed.

V. Constructing Weakly Regular n-ports via Perturbation

Given a resistive n-port N with $\Lambda \not\pitchfork K$, we ask if it is possible to perturb Λ in such a manner that the perturbed n-port $\hat{N}$ has $\hat{\Lambda}$ with $\hat{\Lambda} \pitchfork K$. We will show that the answer is affirmative. Moreover, we will give a second method for transversalizing Λ and K by creating extra ports instead of perturbing Λ. The first method is called element perturbation and consists of perturbing the existing constitutive relations Λ. The second method is called network perturbation and consists of creating extra ports by "pliers-type entry" or "soldering-iron entry." Note that element perturbation gives rise to a new $\hat{\Lambda}$ but it keeps K unchanged, while network perturbation gives rise to a new ambient space $\mathbb{R}^{b+\hat{n}} \times \mathbb{R}^{b+\hat{n}}$ and hence a new $\hat{\Lambda}$ and a new $\hat{K}$, where $\hat{n}$ is the number of extra ports created. Recall that the norator of Example 2 imposes no constraint in so far as the constitutive relation is concerned. Therefore network perturbation is equivalent to inserting norators by pliers-type entry or soldering-iron entry.

Remark In the case of RLC networks, network perturbation usually consists of addition of parasitic capacitors and inductors at appropriate locations. In particular, we augment the network by adding arbitrarily small linear inductors and arbitrarily large linear capacitors by pliers-type entry, and arbitrarily large linear inductors and arbitrarily small linear capacitors by soldering-iron entry. Hence in the limit we recover the original network.

We first give a transversalization result via element perturbation.

<u>Theorem 3</u> Given a resistive n-port N suppose $\Lambda \cap K \neq \phi$ and $\Lambda \not\pitchfork K$. Then we can find a perturbation $\hat{\Lambda}$ of Λ arbitrarily close to Λ such that $\hat{\Lambda} \cap K \neq \phi$ and $\hat{\Lambda} \pitchfork K$. Hence the perturbed n-port $\hat{N}$ is weakly regular and structurally stable.

<u>Proof</u> The proof of (ii-a) of <u>Theorem 2</u> is applicable here because it does not use the fact that Σ is not an n-dimensional submanifold. It uses only the fact that $\Lambda \not\pitchfork K$. $\square$

We give next another transversalization result obtained by network perturbation. Recall that the network $\mathcal{N}$ is obtained by terminating the ports of N by norators and that $\mathcal{N}$ has $b = n + n_R$ branches.

<u>Theorem 4</u> Given a resistive n-port N let $\Lambda \cap K \neq 0$ and $\Lambda \not\pitchfork K$. Let $\mathcal{T}$ be an arbitrary tree for $\mathcal{N}$ and let $\mathcal{L}$ be its associated cotree. Create an extra port in parallel with each branch of $\mathcal{T}$ and create an extra port in series with each branch of $\mathcal{L}$. Then the perturbed $\hat{N}$ is an <u>(n+b)-port</u> and it has the <u>following</u> properties: (i) $\hat{\Lambda} \cap \hat{K} \neq \phi$, (ii) $\hat{\Lambda} \pitchfork \hat{K}$. Moreover $\hat{N}$ is weakly regular and structurally stable, where $\hat{\Lambda}$ and $\hat{K}$ are the constitutive relations and the Kirchhoff space of $\hat{N}$, respectively.

<u>Proof</u> (i) Let $\mathcal{T}_1$ denote the branches representing the extra ports inserted in parallel with $\mathcal{T}$ and let $\mathcal{L}_1$ denote the branches representing the extra ports inserted in series with $\mathcal{L}$. Let $\hat{\mathcal{N}}$ be the network obtained from $\hat{N}$ by terminating ports by norators. Then $\hat{\mathcal{T}} \triangleq \mathcal{L} \cup \mathcal{T}_1$ is a tree for $\hat{\mathcal{N}}$ and $\hat{\mathcal{L}} \triangleq \mathcal{T} \cup \mathcal{L}_1$ is its associated cotree. Let

$$\hat{\mathbf{v}} = (\mathbf{v}_{\mathcal{T}}, \mathbf{v}_{\mathcal{L}_1} \vdots \mathbf{v}_{\mathcal{L}}, \mathbf{v}_{\mathcal{T}_1}) \quad \hat{\mathbf{i}} = (\mathbf{i}_{\mathcal{T}}, \mathbf{i}_{\mathcal{L}_1} \vdots \mathbf{i}_{\mathcal{L}}, \mathbf{i}_{\mathcal{T}_1})$$

be the variables of $\hat{\mathcal{N}}$. Now let $(\mathbf{v}_0, \mathbf{i}_0) \in \Lambda \cap K \neq \phi$. We claim that with

$$\hat{\mathbf{v}}_0 \triangleq (\mathbf{v}_{0\mathcal{T}}, \mathbf{0} \vdots \mathbf{v}_{0\mathcal{L}}, \mathbf{v}_{0\mathcal{T}_1}) \quad \hat{\mathbf{i}}_0 \triangleq (\mathbf{i}_{0\mathcal{T}}, \mathbf{i}_{0\mathcal{L}_1} \vdots \mathbf{i}_{0\mathcal{L}}, \mathbf{0})$$

we have

$$(\hat{\mathbf{v}}_0, \hat{\mathbf{i}}_0) \in \hat{K}\,. \tag{121}$$

This follows from the fact that $(\hat{\underset{\sim}{v}}_0, \hat{\underset{\sim}{i}}_0)$ corresponds to open circuiting $\mathcal{T}_1$ and short circuiting $\mathcal{L}_1$ and the fact that such situation is certainly contained in the Kirchhoff space $\hat{K}$. Next, since no resistors are added, $\hat{\Lambda}$ is given by

$$\hat{\Lambda} = \{(\hat{\underset{\sim}{v}}, \hat{\underset{\sim}{i}}) \mid (\underset{\sim}{v}, \underset{\sim}{i}) \in \Lambda\} \tag{122}$$

and hence $(\hat{\underset{\sim}{v}}_0, \hat{\underset{\sim}{i}}_0) \in \hat{\Lambda}$. This and (121) imply (i).

(ii) We compute $\hat{\underset{\sim}{\mathcal{F}}}(\hat{\underset{\sim}{v}}, \hat{\underset{\sim}{i}})$ for $\hat{N}$. Observe that the main part $\hat{\underset{\sim}{B}}_{\mathcal{T}}$ of the fundamental loop matrix $\hat{\underset{\sim}{B}}$ of $\hat{\mathcal{N}}$ assumes the following form:

$$\hat{\underset{\sim}{B}} = \begin{array}{cc} \begin{array}{cc} \underset{\sim}{v}_{\mathcal{L}} & \underset{\sim}{v}_{\mathcal{T}_1} \end{array} & \\ \begin{bmatrix} \underset{\sim}{0} & -\underset{\sim}{1} \\ \underset{\sim}{1} & \underset{\sim}{B}_{\mathcal{T}} \end{bmatrix} & \begin{array}{c} \underset{\sim}{v}_{\mathcal{T}} \\ \underset{\sim}{v}_{\mathcal{L}_1} \end{array} \end{array} \tag{123}$$

where $\underset{\sim}{B}_{\mathcal{T}}$ is the main part of $\underset{\sim}{B}$ for $\mathcal{N}$. The signs of the identity matrices are chosen just for convenience and such choice of signs involves no loss of generality. Next observe that $\hat{\underset{\sim}{f}}(\hat{\underset{\sim}{v}}, \hat{\underset{\sim}{i}}) = \underset{\sim}{f}(\underset{\sim}{v}, \underset{\sim}{i})$ and that

$$\underset{\sim}{D}_{\hat{\underset{\sim}{v}}_{\mathcal{T}}} \hat{\underset{\sim}{f}} = [\underset{\sim}{D}_{\underset{\sim}{v}_{\mathcal{L}}} \underset{\sim}{f} : \underset{\sim}{0}], \ \underset{\sim}{D}_{\hat{\underset{\sim}{v}}_{\mathcal{L}}} \hat{\underset{\sim}{f}} = [\underset{\sim}{D}_{\underset{\sim}{v}_{\mathcal{T}}} \underset{\sim}{f} : \underset{\sim}{0}], \ \underset{\sim}{D}_{\hat{\underset{\sim}{i}}_{\mathcal{T}}} \underset{\sim}{f} = [\underset{\sim}{D}_{\underset{\sim}{i}_{\mathcal{L}}} \underset{\sim}{f} : \underset{\sim}{0}], \ \underset{\sim}{D}_{\hat{\underset{\sim}{i}}_{\mathcal{L}}} \hat{\underset{\sim}{f}} = [\underset{\sim}{D}_{\underset{\sim}{i}_{\mathcal{T}}} \underset{\sim}{f} : \underset{\sim}{0}].$$

Substituting these and (123) into (37) we obtain

$$\hat{\underset{\sim}{\mathcal{F}}}(\hat{\underset{\sim}{v}}, \hat{\underset{\sim}{i}}) = [\underset{\sim}{D}_{\underset{\sim}{v}_{\mathcal{L}}} \underset{\sim}{f} : \underset{\sim}{D}_{\underset{\sim}{v}_{\mathcal{T}}} \underset{\sim}{f} : \underset{\sim}{D}_{\underset{\sim}{i}_{\mathcal{T}}} \underset{\sim}{f} : \underset{\sim}{D}_{\underset{\sim}{i}_{\mathcal{L}}} \underset{\sim}{f}]_{(\underset{\sim}{v}, \underset{\sim}{i})} = (\underset{\sim}{D}\underset{\sim}{f})_{(\underset{\sim}{v}, \underset{\sim}{i})}.$$

It follows from (34) that this matrix has rank n_R for all $(\underset{\sim}{v}, \underset{\sim}{i}) \in \Lambda$. It follows from (122) that for any $(\hat{\underset{\sim}{v}}, \hat{\underset{\sim}{i}}) \in \hat{\Lambda} \cap \hat{K}$, the subvector $(\underset{\sim}{v}, \underset{\sim}{i})$ must belong to Λ. By Proposition 1, $\hat{\Lambda} \pitchfork \hat{K}$. ¤

Remarks 1. Theorems 3 and 4 say that given any constitutive relations Λ provided that it is a $(2b-n_R)$-dimensional C^1 submanifold, one can always transversalize Λ and K by either element perturbation or network perturbation.

2. In the proof of Theorem 4 we took advantage of the fact that transversality is a coordinate-free property and hence we need to check it in terms of only one particular tree.

3. Recall that the transversality condition (31) requires the vector space $\mathbb{R}^b \times \mathbb{R}^b$ be spanned by the algebraic sum of $T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda$ and $T_{(\underset{\sim}{v},\underset{\sim}{i})}K$. The augmentation procedure of Theorem 4 is to provide more vectors for $T_{(\underset{\sim}{v},\underset{\sim}{i})}\Lambda$ and $T_{(\underset{\sim}{v},\underset{\sim}{i})}K$ so that their algebraic sum spans the ambient space.

Observe that in Theorem 4 the number of extra ports provided was b. We will show, next, that if Λ has simpler forms, then the number of extra ports can be reduced.

Proposition 9 Given an n-port N let $\Lambda \cap K \neq \phi$, $\Lambda \not\pitchfork K$ and let Λ be described by (13). Let $\mathcal{T}$ be an arbitrary tree for $\mathcal{N}$, the network obtained by terminating the ports of N by norators, and let $\mathcal{L}$ be associated cotree. Decompose $\mathcal{T}$ and $\mathcal{L}$ as $\mathcal{T} = R_{\mathcal{T}} \cup P_{\mathcal{T}}$ and $\mathcal{L} = R_{\mathcal{L}} \cup P_{\mathcal{L}}$, respectively, where R and P denote resistors and ports, respectively. Create an extra port in parallel with each branch of $R_{\mathcal{T}}$ and create an extra port in series with each branch of $R_{\mathcal{L}}$. Then the perturbed $\hat{N}$ is an $(n+n_R)$-port having properties (i) and (ii) of Theorem 4. Hence $\hat{N}$ is weakly regular and structurally stable.

Proof Let $\mathcal{T}_1$ be the branches of the extra ports created in parallel with $R_{\mathcal{T}}$ and $\mathcal{L}_1$ be the branches of the extra ports created in series with $R_{\mathcal{L}}$. Then $\hat{\mathcal{T}} \triangleq R_{\mathcal{L}} \cup P_{\mathcal{T}} \cup \mathcal{T}_1$ is a tree for $\hat{\mathcal{N}}$ and $\hat{\mathcal{L}} \triangleq R_{\mathcal{T}} \cup P_{\mathcal{L}} \cup \mathcal{L}_1$ is its associated cotree. Let

$$\hat{\underset{\sim}{v}} = \left(\underset{\sim}{v}_{R_{\mathcal{T}}}, \underset{\sim}{v}_{P_{\mathcal{L}}}, \underset{\sim}{v}_{\mathcal{L}_1} \vdots \underset{\sim}{v}_{R_{\mathcal{L}}}, \underset{\sim}{v}_{P_{\mathcal{T}}}, \underset{\sim}{v}_{\mathcal{T}_1}\right), \quad \hat{\underset{\sim}{i}} = \left(\underset{\sim}{i}_{R_{\mathcal{T}}}, \underset{\sim}{i}_{P_{\mathcal{L}}}, \underset{\sim}{i}_{\mathcal{L}_1} \vdots \underset{\sim}{i}_{R_{\mathcal{L}}}, \underset{\sim}{i}_{P_{\mathcal{T}}}, \underset{\sim}{i}_{\mathcal{T}_1}\right)$$

be the variables of $\hat{\mathcal{N}}$. The proof of Property (i) is similar to that of Theorem 4. To prove (ii) observe that

$$\hat{\underset{\sim}{B}}_{\mathcal{T}} = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} \underset{\sim}{v}_{R_{\mathcal{L}}} & \underset{\sim}{v}_{P_{\mathcal{T}}} & \underset{\sim}{v}_{\mathcal{T}_1} \\ \end{array}$$

$$\hat{\underset{\sim}{B}}_{\mathcal{T}} = \begin{array}{c|c} & \begin{array}{ccc} \underset{\sim}{v}_{R_{\mathcal{L}}} & \underset{\sim}{v}_{P_{\mathcal{T}}} & \underset{\sim}{v}_{\mathcal{T}_1} \end{array} \\ \begin{array}{c} \underset{\sim}{v}_{R_{\mathcal{T}}} \\ \underset{\sim}{v}_{P_{\mathcal{L}}} \\ \underset{\sim}{v}_{\mathcal{L}_1} \end{array} & \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & -\underset{\sim}{1} \\ \underset{\sim}{0} & \underset{\sim}{B}_{PP} & \underset{\sim}{B}_{PR} \\ \underset{\sim}{1} & \underset{\sim}{B}_{RP} & \underset{\sim}{B}_{RR} \end{bmatrix} \end{array}$$

$$\hat{\underset{\sim}{f}}_R = \underset{\sim}{f}_R, \quad \hat{\underset{\sim}{B}}_{RR} = \underset{\sim}{0}, \quad \hat{\underset{\sim}{B}}_{RP} = [\underset{\sim}{0} \;\; -\underset{\sim}{1}], \quad B_{PR}^T = [\underset{\sim}{0} \;\; \underset{\sim}{1}]$$

$$D_{\hat{\underset{\sim}{v}}_{R_{\mathcal{T}}}} \hat{\underset{\sim}{f}}_R = D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R, \quad D_{\hat{\underset{\sim}{v}}_{R_{\mathcal{L}}}} \hat{\underset{\sim}{f}}_R = D_{\underset{\sim}{v}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R$$

where the submatrices $\underset{\sim}{B}_{RR}$, $\underset{\sim}{B}_{PR}$, $\underset{\sim}{B}_{RP}$ and $\underset{\sim}{B}_{PP}$ are those of $\mathcal{N}$ (see (44)). Substituting these into (46) we have

$$\hat{\underset{\sim}{\mathcal{F}}}_R(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) = \left[D_{\underset{\sim}{v}_{R_\mathcal{L}}} \underset{\sim}{f}_R \vdots -(D_{\underset{\sim}{v}_{R_\mathcal{J}}} \underset{\sim}{f}_R)[\underset{\sim}{0}\ -\underset{\sim}{1}] \vdots D_{\underset{\sim}{i}_{R_\mathcal{J}}} \underset{\sim}{f}_R \vdots (D_{\underset{\sim}{i}_{R_\mathcal{L}}} \underset{\sim}{f}_R)[\underset{\sim}{0}\ \underset{\sim}{1}] \right]_{(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R)} . \tag{124}$$

It follows from (34) and (40) that

$$\text{rank } (D\underset{\sim}{f}_R)_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)} = n_R \text{ for all } (\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R . \tag{125}$$

Let $\hat{\pi}_R$ be the projection for $\hat{N}$ defined by (42). Then, since $\hat{\pi}_R(\Sigma) \subset \Lambda_R$ and since $(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) = (\underset{\sim}{v}_R,\underset{\sim}{i}_R)$, (125) implies that the matrix of (124) has rank n_R for all $(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) \in \hat{\underset{\sim}{\pi}}_R(\Sigma)$. By <u>Corollary 1</u>, $\hat{\Lambda} \,\bar{\pitchfork}\, \hat{K}$.

<u>Proposition 10</u> Given an n-port N let $\Lambda \cap K \neq \phi$, $\Lambda \,\bar{\not\pitchfork}\, K$ and let Λ_R be locally voltage controlled (See Def. 1). Create an extra port in parallel with each branch of the tree resistors $R_\mathcal{J}$. Then the perturbed N is an $(n+n_{R_\mathcal{J}})$-port and it satisfies (i) and (ii) of <u>Theorem 4</u>, where $n_{R_\mathcal{J}}$ is the number of branches in $R_\mathcal{J}$. Hence $\hat{N}$ is weakly regular and structurally stable.

<u>Proof</u> The proof for (i) is similar to that of <u>Theorem 4</u>. (ii) It is clear that $\hat{\mathcal{J}} \triangleq P_\mathcal{J} \cup \mathcal{J}_1$ is a tree for $\hat{\mathcal{N}}$ and $\hat{\mathcal{L}} \triangleq R_\mathcal{J} \cup R_\mathcal{L} \cup P_\mathcal{L}$ is its associated cotree, where $\mathcal{J}_1$ denotes the ports created. It is easy to show the following:

$$\hat{\underset{\sim}{B}}_\mathcal{J} = \begin{array}{c} \begin{array}{cc} \underset{\sim}{v}_{P_\mathcal{J}} & \underset{\sim}{v}_{\mathcal{J}_1} \end{array} \\ \left[\begin{array}{cc} \underset{\sim}{0} & -\underset{\sim}{1} \\ \underset{\sim}{B}_{RP} & \underset{\sim}{B}_{RR} \\ \underset{\sim}{B}_{PP} & \underset{\sim}{B}_{PR} \end{array}\right] \end{array} \begin{array}{c} \\ \underset{\sim}{v}_{R_\mathcal{J}} \\ \underset{\sim}{v}_{R_\mathcal{L}} \\ \underset{\sim}{v}_{P_\mathcal{L}} \end{array}$$

$$\hat{\underset{\sim}{B}}_{RR} = \phi,\ \hat{\underset{\sim}{B}}_{RP} = \begin{bmatrix} \underset{\sim}{0} & -\underset{\sim}{1} \\ \underset{\sim}{B}_{RP} & \underset{\sim}{B}_{RR} \end{bmatrix},\ \hat{\underset{\sim}{B}}_{PR} = \underset{\sim}{B}_{PP},\ \hat{\underset{\sim}{B}}_{PP} = \underset{\sim}{B}_{PR}$$

$$D_{\hat{\underset{\sim}{v}}_{R_\mathcal{J}}} \hat{\underset{\sim}{f}}_R = \phi,\ D_{\hat{\underset{\sim}{v}}_{R_\mathcal{L}}} \hat{\underset{\sim}{f}}_R = D_{\underset{\sim}{v}_R} \underset{\sim}{f}_R,\ D_{\hat{\underset{\sim}{i}}_{R_\mathcal{J}}} \hat{\underset{\sim}{f}}_R = \phi,\ D_{\hat{\underset{\sim}{i}}_{R_\mathcal{L}}} \hat{\underset{\sim}{f}}_R = D_{\underset{\sim}{i}_R} \underset{\sim}{f}_R.$$

Substituting these equations into (46) we obtain

$$\hat{\mathcal{F}}_{\underset{\sim}{R}}(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) = [-(D_{\underset{\sim}{v}_R}\underset{\sim}{f}_R)\hat{\underset{\sim}{B}}_{RP} \vdots D_{\underset{\sim}{i}_R}\underset{\sim}{f}_R]_{(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R)}. \tag{126}$$

Since Λ_R is locally voltage controlled (16) implies that the matrix of (126) has rank n_R for all $(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R$. Since $(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) = (\underset{\sim}{v}_R,\underset{\sim}{i}_R)$ and since $\hat{\pi}_R(\Sigma) \subset \Lambda_R$, condition (45) of Corollary 1 is satisfied. This implies $\hat{\Lambda} \pitchfork \hat{K}$. ¤

Example 18 Consider (vi) of Example 1. In Example 9 we showed that $\Lambda \not\pitchfork K$. Choose $\mathcal{T} = \{R_2,P\}$ as our tree. Create extra port $\hat{P}$ in parallel with R_2 as in Fig. 21. Then $\hat{\mathcal{T}} = \{P,\hat{P}\}$ is the tree chosen in Proposition 9. Since

$$\hat{\underset{\sim}{B}}_{RP} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

we have from (126) that

$$\hat{\mathcal{F}}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = \left[-(D_{\underset{\sim}{v}_R}\underset{\sim}{f}_R)\hat{\underset{\sim}{B}}_{RP} \vdots D_{\underset{\sim}{i}_R}\underset{\sim}{f}_R\right]_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)} = \begin{bmatrix} -Df_{R_1} & Df_{R_1} & \vdots & 1 & 0 \\ 0 & -Df_{R_2} & \vdots & 0 & 1 \end{bmatrix}_{\underset{\sim}{v}_R}.$$

This matrix has rank 2. This implies $\hat{\Lambda} \pitchfork \hat{K}$.

A dual argument to Proposition 10 gives the following:

Proposition 11 Given an n-port N let $\Lambda \cap K \neq \phi$, $\Lambda \not\pitchfork K$ and let Λ_R be locally current controlled. Create an extra port in series with each branch of the cotree resistors $R_{\mathcal{L}}$. Then the perturbed $\hat{N}$ is an $(n+n_{R_{\mathcal{L}}})$-port and it satisfies (i) and (ii) of Theorem 4, where $n_{R_{\mathcal{L}}}$ is the number of branches in $R_{\mathcal{L}}$. Hence $\hat{N}$ is weakly regular and structurally stable.

Remark A natural question that arises at this point is: Suppose $\Lambda \pitchfork K$ and $\Lambda \cap K \neq \phi$ and hence Σ is an n-dimensional submanifold. Is this n-port structurally stable under network perturbation in the sense that after creating an extra port, the configuration space $\Sigma = \hat{\Lambda} \cap \hat{K}$ of the perturbed $\hat{N}$ is an (n+1)-dimensional submanifold? The answer is negative as demonstrated by the following example.

<u>Example 19</u> Consider the 1-port of Fig. 22 where the constitutive relations of R_1 and R_2 are given by (vi) of Example 1 and the constitutive relation of R_3 is given by $i_{R_3} = f_{R_3}(v_{R_3})$. Choose $\mathcal{T} = \{P, R_1\}$ as our tree. Then $\underset{\sim}{B}_{RR} = [1\ 0]^T$, $\underset{\sim}{B}_{RP} = [0\ 1]^T$, $\underset{\sim}{B}_{PR} = \underset{\sim}{B}_{PP} = \phi$ and

$$D_{v_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{bmatrix} -Df_{R_1} \\ 0 \\ 0 \end{bmatrix}, \quad D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \begin{bmatrix} 0 & 0 \\ -Df_{R_2} & 0 \\ 0 & -Df_{R_3} \end{bmatrix},$$

$$D_{i_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Substituting these into (46) we have

$$\underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = \left[\begin{array}{c|c|cc} -Df_{R_1} & 0 & 1 & 0 \\ Df_{R_2} & 0 & 1 & 0 \\ 0 & Df_{R_3} & 0 & 1 \end{array}\right] \underset{\sim}{v}_R .$$

It is clear that

$$\Sigma = \left\{ (\underset{\sim}{v}, \underset{\sim}{i}) \in \mathbb{R}^4 \times \mathbb{R}^4 \;\middle|\; \begin{array}{l} v_{R_1} = v_{R_2} = i_{R_1} = i_{R_2} = 0,\ v_P = -v_{R_3} \\ i_{R_3} = f_{R_3}(v_{R_3}),\ v_{R_3} \in \mathbb{R} \end{array} \right\}$$

and therefore $\underset{\sim}{\pi}_R$ maps Σ onto $\mathcal{R}$ diffeomorphically. It is clear that rank $\underset{\sim}{\mathcal{F}}_R(0,0,v_{R_3},0,0,i_{R_3}) = 3$. Therefore $\Lambda \pitchfork K$ and Σ is a 1-dimensional submanifold. Now let N_1 be the 1-port consisting of the port P and R_3 and let N_2 be the 0-port consisting of R_1 and R_2, Then the 1-port N of our interest consists of N_1 and N_2 hinged together. Next, insert an extra port $\hat{P}$ by pliers-type entry between R_1 and R_2. The resulting 2-port $\hat{N}$ consists of N_1 and the 1-port (call it $\hat{N}_2$) of (vi) in Example 1, hinged together. We saw, in Example 9, that the configuration space of N_2 is not a 1-dimensional submanifold. It is clear, then, that the configuration space of $\hat{N}$ is not a 2-dimensional submanifold. Hence N is not persitent under network perturbations.

VI. Reciprocity and Anti-Reciprocity

Our objective in this section is to generalize the definition of "reciprocity" [13,11] and "anti-reciprocity" [8] for the more general classes of resistive n-ports considered in this paper.

In order to define these two basic circuit-theoretic concepts [17] in a coordinate-free manner, let us briefly review some properties of "differential forms" and "induced maps". A differential 1-form or simply 1-form $\underset{\sim}{\eta}$ on $\mathbb{R}^2$ is a collection of functions given at each $(x_1,x_2) \in \mathbb{R}^2$ by

$$\underset{\sim}{\eta}_{(x_1,x_2)} = f_1(x_1,x_2)dx_1 + f_2(x_1,x_2)dx_2$$

where f_1 and f_2 are real-valued functions, and $dx_1 = [1\ 0]$ and $dx_2 = [0\ 1]$ are 1×2 row vectors. Hence the values of $\underset{\sim}{\eta}_{(x_1,x_2)}$ at $[1\ 0]^T$ and $[0\ 1]^T$ are given respectively by $\underset{\sim}{\eta}_{(x_1,x_2)}([1\ 0]^T) = f_1(x_1,x_2)$, $\underset{\sim}{\eta}_{(x_1,x_2)}([0\ 1]^T) = f_2(x_1,x_2)$. Therefore $\underset{\sim}{\eta}$ can be thought of as a vector-valued function (f_1,f_2). If $y = f(x_1,x_2)$ where f is a real-valued function, then

$$dy = D_{x_1} f\, dx_1 + D_{x_2} f\, dx_2 \ . \tag{127}$$

The exterior product $\wedge$ of two 1-forms has the following property. If $\underset{\sim}{\eta} = f_1 dx_1 + f_2 dx_2$, $\underset{\sim}{\xi} = g_1 dx_1 + g_2 dx_2$, then

$$\underset{\sim}{\eta} \wedge \underset{\sim}{\xi} = (f_1g_2 - f_2g_1)dx_1 \wedge dx_2 \tag{128}$$

and

$$dx_1 \wedge dx_1 = dx_2 \wedge dx_2 = \underset{\sim}{0}, \quad dx_1 \wedge dx_2 = -dx_2 \wedge dx_1 \ . \tag{129}$$

Exterior product of two 1-forms is a 2-form [4]. Special caution must be exercised when one discusses 1-forms on general manifolds instead of on euclidean spaces, because manifolds are generally nonlinear. See [4] for details. Next, let $\underset{\sim}{\eta}$ be a 1-form on $\mathbb{R}^2$ and let $\underset{\sim}{F}: \mathbb{R}^k \to \mathbb{R}^2$ be a C^1 function where, $k \geq 1$. Then $\underset{\sim}{F}$ induces a 1-form $\underset{\sim}{\zeta}$ on $\mathbb{R}^k$ by the formula $\underset{\sim}{\zeta}_{(\underset{\sim}{u})}(\underset{\sim}{w}) \triangleq \underset{\sim}{\eta}_{\underset{\sim}{F}(\underset{\sim}{u})}((\underset{\sim}{DF})_{\underset{\sim}{w}})$, $\underset{\sim}{w} \in \mathbb{R}^k$, $\underset{\sim}{u} \in \mathbb{R}^k$. We write this as $\underset{\sim}{\zeta} \triangleq \underset{\sim}{F}^*\underset{\sim}{\eta}$ and we call F^* the induced map of $\underset{\sim}{F}$. A similar argument holds for 2-forms also. Again, care must be taken when we discuss 2-forms on general manifolds. We are now ready to define reciprocity.

Definition 12. A weakly regular n-port N is said to be reciprocal if

$$\underset{\sim}{\imath}^*\left(\sum_{k=1}^{n} di_{P_k} \wedge dv_{P_k}\right) = \underset{\sim}{0} \tag{130}$$

where $\underset{\sim}{\iota}^*$ is the induced map of the inclusion map $\underset{\sim}{\iota}$ defined in (23).

Remarks: 1. We need weak regularity of N because the differential 2-form (130) must be well defined.

2. Observe that $\sum_{k=1}^{n} di_{P_k} \wedge dv_{P_k}$ is defined on $\mathbb{R}^b \times \mathbb{R}^b$. However, the map $\underset{\sim}{\iota}^*$ pulls this 2-form back to Σ and defines it on Σ.

3. Although $\sum_{k=1}^{n} di_{P_k} \wedge dv_{P_k}$ is related only to the exterior port variables, condition (130) depends on the internal resistor variables through $\underset{\sim}{\iota}^*$. In fact the following holds.

Proposition 12. A weakly regular n-port N is reciprocal if and only if

$$\underset{\sim}{\iota}^*\Big(\sum_{k=1}^{n_R} di_{R_k} \wedge dv_{R_k}\Big) = \underset{\sim}{0} . \tag{131}$$

Proof. Using Tellegen's theorem we obtain [10,11]

$$\underset{\sim}{\iota}^*\Big(\sum_{k=1}^{n_R} i_{R_k} dv_{R_k} + \sum_{k=1}^{n} i_{P_k} dv_{P_k}\Big) = \underset{\sim}{0}$$

which implies

$$\underset{\sim}{\iota}^*\Big(\sum_{k=1}^{n_R} i_{R_k} dv_{R_k}\Big) = -\underset{\sim}{\iota}^*\Big(\sum_{k=1}^{n} i_{P_k} dv_{P_k}\Big) . \tag{132}$$

Taking exterior derivative d [4] of both sides of (132) we have[8]

$$\underset{\sim}{\iota}^*\Big(\sum_{k=1}^{n_R} di_{R_k} \wedge dv_{R_k}\Big) = -\underset{\sim}{\iota}^*\Big(\sum_{k=1}^{n} di_{P_k} \wedge dv_{P_k}\Big) . \tag{133}$$

This shows that the 2-forms (130) and (131) must vanish simultaneously. ¤

The importance of reciprocity lies in the fact that it is closely related to the existence of potential functions.

Proposition 13. Let N be weakly regular and let $\underset{\sim}{\beta}$ be any 1-form on Σ satisfying

$$d\underset{\sim}{\beta} = \underset{\sim}{\iota}^*\Big(\sum_{k=1}^{n} di_{P_k} \wedge dv_{P_k}\Big) . \tag{134}$$

If N is reciprocal and if Σ is simply connected [4], then there is a function $P: \Sigma \to \mathbb{R}$ such that

$$\underset{\sim}{\beta} = dP . \tag{135}$$

[8] If $\underset{\sim}{\eta} = g dh$ is a 1-form, then $d\underset{\sim}{\eta} = dg \wedge dh + g d^2 h = dg \wedge dh$ since $d^2 h = \underset{\sim}{0}$ [4].

Remark. For a real-valued function P, the exterior derivative d and the ordinary derivative $\underset{\sim}{d}$ coincide [12]. Hence there is no inconsistency.

Proof. By reciprocity and by (134), we have $d\underset{\sim}{\beta} = \underset{\sim}{0}$, i.e., $\underset{\sim}{\beta}$ is a closed 1-form [4] on Σ. Since Σ is simply connected, it is exact [4]. Hence there is a real-valued function P satisfying (135). ¤

In RLC networks reciprocity plays an important role in the sense that the dynamics gives rise to a gradient dynamical system [11]. Observe that our reciprocity definition (130) is coordinate-free. We will next give a method for checking reciprocity in terms of some specific coordinates. Of course we need to check it in terms of only one convenient choice of coordinates.

Proposition 14. Let N be weakly regular with dim $\Sigma = m$ and let $(\underset{\sim}{\psi}, \Sigma \cap U)$ be a local chart at $(\underset{\sim}{v}, \underset{\sim}{i})$ for Σ. Then N is reciprocal if and only if for each $(\underset{\sim}{v}, \underset{\sim}{i}) \in \Sigma$, the following m×m matrix is symmetric:

$$(\underset{\sim}{D}\underset{\sim}{v}_P)^T_{\underset{\sim}{x}}(\underset{\sim}{D}\underset{\sim}{i}_P)_{\underset{\sim}{x}} \tag{136}$$

where

$$(\underset{\sim}{v}_P(\underset{\sim}{x}), \underset{\sim}{i}_P(\underset{\sim}{x})) \triangleq \underset{\sim}{\pi}_P \circ \underset{\sim}{\psi}^{-1}(\underset{\sim}{x}) \tag{137}$$

and $\underset{\sim}{\pi}_P$ is defined by (24).

Proof. It follows from (127) and (129) that in terms of the coordinate $\underset{\sim}{\psi}$, the 2-form on the left hand side of (130) is expressed by

$$\begin{aligned}(\underset{\sim}{\iota} \circ \underset{\sim}{\psi}^{-1})^*\left(\sum_{k=1}^{n} di_{P_k} \wedge dv_{P_k}\right) &= \sum_{k=1}^{n}\left(\sum_{j=1}^{m}\frac{\partial i_{P_k}}{\partial x_j}(\underset{\sim}{x})\,dx_j\right) \wedge \left(\sum_{\ell=1}^{m}\frac{\partial v_{P_k}}{\partial x_\ell}(\underset{\sim}{x})\,dx_\ell\right) \\ &= \sum_{k=1}^{n}\sum_{j<\ell}\left(\frac{\partial i_{P_k}}{\partial x_j}(\underset{\sim}{x})\frac{\partial v_{P_k}}{\partial x_\ell}(\underset{\sim}{x}) - \frac{\partial i_{P_k}}{\partial x_\ell}(\underset{\sim}{x})\frac{\partial v_{P_k}}{\partial x_j}(\underset{\sim}{x})\right) dx_j \wedge dx_\ell .\end{aligned} \tag{138}$$

It is easy to show that this 2-form vanishes if and only if the following m×m matrix is a zero matrix: $(\underset{\sim}{D}\underset{\sim}{v}_P)^T_{\underset{\sim}{x}}(\underset{\sim}{D}\underset{\sim}{i}_P)_{\underset{\sim}{x}} - (\underset{\sim}{D}\underset{\sim}{i}_P)^T_{\underset{\sim}{x}}(\underset{\sim}{D}\underset{\sim}{v}_P)_{\underset{\sim}{x}}$. But this is equivalent to saying that the matrix in (136) is symmetric. ¤

Using Proposition 11 we can check reciprocity in terms of the internal resistor variables. The following can be proved in the same way as that of Proposition 14.

Proposition 15. Let N be weakly regular with dim $\Sigma = m$ and let $(\underset{\sim}{\psi}, \Sigma \cap U)$ be a local chart at $(\underset{\sim}{v}, \underset{\sim}{i})$ for Σ. Then N is reciprocal if and only if for each $(\underset{\sim}{v}, \underset{\sim}{i}) \in \Sigma$, the following m×m matrix is symmetric:

$$(D\underset{\sim}{v}_R)^T_{\underset{\sim}{x}}(D\underset{\sim}{i}_R)_{\underset{\sim}{x}} \tag{139}$$

where

$$(\underset{\sim}{v}_R(\underset{\sim}{x}),\underset{\sim}{i}_R(\underset{\sim}{x})) \triangleq \underset{\sim}{\pi}_R \circ \underset{\sim}{\psi}^{-1}(\underset{\sim}{x}) \tag{140}$$

and $\underset{\sim}{\pi}_R$ is defined by (42).

If N is normal the following holds.

<u>Corollary 7</u>. Let N be normal, i.e., Σ and $\mathcal{R}$ are globally diffeomorphic to $\mathbb{R}^n$ and $\mathcal{R}$ admits a generalized port coordinate. (See Def. 9). Then N is reciprocal if and only if the following $n\times n$ matrix is symmetric for each $\underset{\sim}{\eta}_P$:

$$[\underset{\sim}{a}_P(D\underset{\sim}{F}_P)_{\underset{\sim}{\eta}_P} + \underset{\sim}{b}_P]^T[\underset{\sim}{c}_P(D\underset{\sim}{F}_P)_{\underset{\sim}{\eta}_P} + \underset{\sim}{d}_P] \quad . \tag{141}$$

<u>Proof</u>. In this case we can choose $\underset{\sim}{x} = \underset{\sim}{\eta}_P$ as a global coordinate for Σ and $(D\underset{\sim}{v}_P)^T_{\underset{\sim}{\eta}_P} = [\underset{\sim}{a}_P(D\underset{\sim}{F})_{\underset{\sim}{\eta}_P} + \underset{\sim}{b}_P]^T$, $(D\underset{\sim}{i}_P)_{\underset{\sim}{\eta}_P} = [\underset{\sim}{c}_P(D\underset{\sim}{F}_P)_{\underset{\sim}{\eta}_P} + \underset{\sim}{d}_P]$. Hence the result follows from <u>Proposition 14</u>. ¤

<u>Corollary 8</u>. Let N be normal and let $\mathcal{R}$ admit a global hybrid representation, i.e., let $\underset{\sim}{v}_P = (\underset{\sim}{v}_a,\underset{\sim}{v}_b)$, $\underset{\sim}{i}_P = (\underset{\sim}{i}_a,\underset{\sim}{i}_b)$, $\underset{\sim}{\eta}_P = (\underset{\sim}{v}_a,\underset{\sim}{i}_b)$, $\underset{\sim}{\xi}_P = (\underset{\sim}{i}_a,\underset{\sim}{v}_b)$ and $\underset{\sim}{\xi}_P = \underset{\sim}{F}_P(\underset{\sim}{\eta}_P)$. Let

$$(D\underset{\sim}{F}_P)_{\underset{\sim}{\eta}_P} = \begin{bmatrix} \underset{\sim}{H}_{11} & \underset{\sim}{H}_{12} \\ \underset{\sim}{H}_{21} & \underset{\sim}{H}_{22} \end{bmatrix}_{\underset{\sim}{\eta}_P} \tag{142}$$

where the matrix partition corresponds to those of $\underset{\sim}{\xi}_P$ and $\underset{\sim}{\eta}_P$. Then N is reciprocal if and only if, for each $\underset{\sim}{\eta}_P$, the following hold:

$$(\underset{\sim}{H}_{11})_{\underset{\sim}{\eta}_P} = (\underset{\sim}{H}_{11})^T_{\underset{\sim}{\eta}_P}, \quad (\underset{\sim}{H}_{22})_{\underset{\sim}{\eta}_P} = (\underset{\sim}{H}_{22})^T_{\underset{\sim}{\eta}_P}, \quad (\underset{\sim}{H}_{12})_{\underset{\sim}{\eta}_P} = -(\underset{\sim}{H}_{21})^T_{\underset{\sim}{\eta}_P}.$$

In particular, if $\mathcal{R}$ is globally voltage controlled or globally current controlled, then N is reciprocal if and only if $(D\underset{\sim}{F}_P)_{\underset{\sim}{\eta}_P}$ is symmetric for all $\underset{\sim}{\eta}_P \in \mathbb{R}^n$.

<u>Proof</u>. In this case

$$\underset{\sim}{a}_P = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} \end{bmatrix}, \quad \underset{\sim}{b}_P = \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix}, \quad \underset{\sim}{c}_P = \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix}, \quad \underset{\sim}{d}_P = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} \end{bmatrix}$$

and the matrix of (140) becomes

$$\begin{bmatrix} \underset{\sim}{H}_{11} & \underset{\sim}{H}_{12}+\underset{\sim}{H}^T_{21} \\ 0 & \underset{\sim}{H}_{22} \end{bmatrix}_{\underset{\sim}{\eta}_P}.$$

The result follows from the symmetry of this matrix. ¤

If Λ is represented by (13) we can derive a sufficient condition for reciprocity in terms of Λ_R instead of Σ.

__Proposition 16__. Let N be weakly regular and let Λ be described by (13). If Λ_R satisfies

$$\underset{\sim}{\iota}_R^*\left(\sum_{k=1}^{n_R} di_{R_k} \wedge dv_{R_k}\right) = \underset{\sim}{0} \tag{143}$$

where

$$\underset{\sim}{\iota}_R : \Lambda_R \to \mathbb{R}^{n_R} \times \mathbb{R}^{n_R} \tag{144}$$

is the inclusion map, then N is reciprocal.

__Proof__. Let $\underset{\sim}{\iota}_1$ and $\underset{\sim}{\iota}_2$ be inclusion maps defined by

$$\Sigma \xrightarrow{\underset{\sim}{\iota}_2} \Lambda \xrightarrow{\underset{\sim}{\iota}_1} \mathbb{R}^b \times \mathbb{R}^b \ . \tag{145}$$

Then

$$\underset{\sim}{\iota} = \underset{\sim}{\iota}_1 \circ \underset{\sim}{\iota}_2 \ . \tag{146}$$

It follows from this that

$$\underset{\sim}{\iota}^*\left(\sum_{k=1}^{n_R} di_{R_k} \wedge dv_{R_k}\right) = \underset{\sim}{\iota}_2^* \underset{\sim}{\iota}_1^*\left(\sum_{k=1}^{n_R} di_{R_k} \wedge dv_{R_k}\right) \ . \tag{147}$$

We claim that

$$\underset{\sim}{\iota}_1^*\left(\sum_{k=1}^{n_R} di_{R_k} \wedge dv_{R_k}\right) = \underset{\sim}{0} \ . \tag{148}$$

To prove this we first locally parametrize Λ_R;

$$(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = (\underset{\sim}{v}_R(\underset{\sim}{\rho}), \underset{\sim}{i}_R(\underset{\sim}{\rho})) \tag{149}$$

where $\underset{\sim}{\rho}$ varies over an open subset of $\mathbb{R}^{n_R}$. Then we locally parametrize Λ by $\underset{\sim}{\sigma} \triangleq (\underset{\sim}{\rho}, \underset{\sim}{v}_P, \underset{\sim}{i}_P)$;

$$(\underset{\sim}{v}_R, \underset{\sim}{v}_P, \underset{\sim}{i}_R, \underset{\sim}{i}_P) = (\underset{\sim}{v}_R(\underset{\sim}{\rho}), \underset{\sim}{v}_P, \underset{\sim}{i}_R(\underset{\sim}{\rho}), \underset{\sim}{i}_P) \ . \tag{150}$$

Using an argument similar to the proof of __Proposition 13__ one can show that (148) holds if and only if the following matrix is symmetric:

$$\begin{aligned}(\underset{\sim}{D}\underset{\sim}{v}_R)_{\underset{\sim}{\sigma}}^T(\underset{\sim}{D}\underset{\sim}{i}_R)_{\underset{\sim}{\sigma}} &= [(\underset{\sim}{D}_{\underset{\sim}{\rho}}\underset{\sim}{v}_R)_{\underset{\sim}{\rho}} \;\; \underset{\sim}{0} \;\; \underset{\sim}{0}]^T[(\underset{\sim}{D}_{\underset{\sim}{\rho}}\underset{\sim}{i}_R)_{\underset{\sim}{\rho}} \;\; \underset{\sim}{0} \;\; \underset{\sim}{0}] \\ &= \begin{bmatrix} (\underset{\sim}{D}_{\underset{\sim}{\rho}}\underset{\sim}{v}_R)_{\underset{\sim}{\rho}}^T(\underset{\sim}{D}_{\underset{\sim}{\rho}}\underset{\sim}{i}_R)_{\underset{\sim}{\rho}} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \ .\end{aligned} \tag{151}$$

Similarly, (143) holds if and only if the following matrix is symmetric:

$$(\underset{\sim}{D}_{\underset{\sim}{\rho}}\underset{\sim}{v}_R)_{\underset{\sim}{\rho}}^T(\underset{\sim}{D}_{\underset{\sim}{\rho}}\underset{\sim}{i}_R)_{\underset{\sim}{\rho}} \ . \tag{152}$$

By assumption, (152) is symmetric and hence (151) is symmetric. This implies (148). But since [12]

$$\underset{\sim}{\iota}_2^*(\underset{\sim}{0}) = \underset{\sim}{0} \ , \tag{153}$$

(153) and (148) imply (131). ¤

If Λ_R admits a generalized port coordinate then we have the following:

<u>Corollary 9</u>. Let Λ_R admit a generalized port coordinate. Then Λ_R satisfies (143) if and only if for each $\underset{\sim}{\eta} \in \mathbb{R}^{n_R}$, the following matrix is symmetric: $[\underset{\sim}{a}(\underset{\sim\sim}{DF})_{\underset{\sim}{\eta}}+\underset{\sim}{b}]^T[\underset{\sim}{c}(\underset{\sim\sim}{DF})_{\underset{\sim}{\eta}}+\underset{\sim}{d}]$. In particular, if Λ_R is globally voltage controlled or globally current controlled, then it is reciprocal if and only if $(\underset{\sim\sim}{DF})_{\underset{\sim}{\eta}}$ is symmetric for all $\underset{\sim}{\eta}$.

Condition (143) is sufficient for N to be reciprocal but not necessary as the following example shows.

<u>Example 20</u>. Consider the 1-port of Fig. 1(a) where Λ_R is given by

$$v_{R_1} - g_{R_1}(i_{R_1}, i_{R_2}) = 0, \quad v_{R_2} - g_{R_2}(i_{R_1}, i_{R_2}) = 0 \ . \tag{154}$$

Since Λ_R is globally current controlled, it satisfies (143) if and only if the following matrix is symmetric for all (i_{R_1}, i_{R_2}):

$$\begin{bmatrix} D_{i_{R_1}} g_{R_1} & D_{i_{R_2}} g_{R_1} \\ D_{i_{R_1}} g_{R_2} & D_{i_{R_2}} g_{R_2} \end{bmatrix}_{(i_{R_1}, i_{R_2})} \ . \tag{155}$$

Hence if the matrix of (155) is not symmetric at some point, then (143) does not hold. Next, it is easy to show that $\Lambda \,\bar{\pitchfork}\, K$ for any C^1 functions g_{R_1} and g_{R_2}. Therefore Σ is a 1-dimensional submanifold. Since i_P serves as a coordinate for Σ, and since $i_{R_1} = i_{R_2} = i_P$, we have

$$D_{i_P}\underset{\sim}{v}_R = \begin{bmatrix} D_{i_{R_1}} g_{R_1} + D_{i_{R_2}} g_{R_2} \\ D_{i_{R_1}} g_{R_1} + D_{i_{R_2}} g_{R_2} \end{bmatrix}, \quad D_{i_P}\underset{\sim}{i}_R = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ .$$

This implies that $(D_{i_P}\underset{\sim}{v}_R)^T_{i_P}(D_{i_P}\underset{\sim}{i}_R)_{i_P}$ is a scalar and hence symmetric. Therefore N is reciprocal for any C^1 functions g_{R_1} and g_{R_2} whereas (143) is not satisfied if the matrix of (155) is not symmetric. Finally, if $\underset{\sim}{\beta} = \underset{\sim}{\iota}^*(\sum_{k=1}^{2} v_{R_k} di_{R_k})$ then, in terms of the coordinate i_P, we have $\underset{\sim}{\beta} = \left(g_{R_1}(i_P,i_P)+g_{R_2}(i_P,i_P)\right)di_P$. Hence there is a potential function; $\underset{\sim}{\beta} = dP(i_P)$ where $P(i_P) = \int_{i_{P_0}}^{i_P}\left(g_{P_1}(i_P,i_P)+g_{R_2}(i_P,i_P)\right)di_P$ and i_{P_0} is arbitrary.

Simple connectedness of Σ cannot be relaxed as the following example shows.

Example 21. Consider the 2-port of Fig. 23 where

$$\Lambda_R = \left\{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)\,\middle|\, i_{R_1}+\frac{v_{R_2}}{v_{R_1}^2+v_{R_2}^2}=0,\ i_{R_2}-\frac{v_{R_1}}{v_{R_1}^2+v_{R_2}^2}=0,\ v_{R_1}^2+v_{R_2}^2\neq 0\right\}. \tag{156}$$

This is a 2-dimensional submanifold of $\mathbb{R}^2\times\mathbb{R}^2$ which is not simply connected. It is easy to show that $\Lambda \pitchfork K$. The space Σ is essentially the same as Λ_R and is parametrized by $\underset{\sim}{x} = (v_{P_1},v_{P_2})$. One can easily show that $(D\underset{\sim}{i}_P)_{\underset{\sim}{x}}$ is symmetric and therefore N is reciprocal. Let

$$\underset{\sim}{\beta} \triangleq \sum_{k=1}^{2} i_{P_k}dv_{P_k} = \frac{v_{P_2}}{v_{P_1}^2+v_{P_2}^2}dv_{P_1} - \frac{v_{P_1}}{v_{P_1}^2+v_{P_2}^2}dv_{P_2}\ .$$

Then $d\underset{\sim}{\beta} = \sum_{k=1}^{2} di_{P_k}\wedge dv_{P_k} = \underset{\sim}{0}$. But it is known [12] that there is no function P satisfying $\underset{\sim}{\beta} = dP$.

Corollary 10. Let Λ_R be uncoupled, i.e.,

$$\Lambda_R = \{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)\,|\,(v_{R_k},i_{R_k})\in\Lambda_{R_k},\ k=1,\dots,n_R\}$$

where Λ_{R_k} is a 1-dimensional submanifold, $k = 1,\dots,n_R$. Then (143) holds and hence N is reciprocal.

Proof. Each $di_{R_k}\wedge dv_{R_k}$ is a 2-form on a 1-dimensional submanifold Λ_{R_k}. The only possible 2-form on a 1-dimensional submanifold is $\underset{\sim}{0}$ [12]. This implies (143). ⌑

Remark. Definition of reciprocity in terms of a 2-form was first given by Brayton [13] and used by Matsumoto [11] to give a necessary and sufficient condition for the dynamics to be a gradient dynamical system.

We next define "anti-reciprocity" which sometimes plays important roles in the theory of n-ports [14,17]. In order to define anti-reciprocity in a coordinate-free manner, we introduce <u>symmetric product</u> of two symmetric tensors instead of exterior product of two forms. A formal definition is given in APPENDIX 3. Here we will give an example showing the operation of symmetric product. Let $y_1 = f(x_1,x_2)$, $y_2 = g(x_1,x_2)$ and let $\underset{\sim}{\eta} \triangleq dy_1 = D_{x_1} f\, dx_1 + D_{x_2} f\, dx_2$ and $\underset{\sim}{\xi} \triangleq dy_2 = D_{x_1} g\, dx_1 + D_{x_2} g\, dx_2$ be associated 1-forms. Then the <u>symmetric product</u> of $\underset{\sim}{\eta}$ and $\underset{\sim}{\xi}$ is defined by the following formula:

$$\underset{\sim}{\eta} \circledcirc \underset{\sim}{\xi} = (D_{x_1} f)(D_{x_1} g) dx_1 \circledcirc dx_1 + (D_{x_2} f)(D_{x_2} g) dx_2 \circledcirc dx_2 + \big((D_{x_1} f)(D_{x_2} g) + (D_{x_2} f)(D_{x_1} g)\big) dx_1 \circledcirc dx_2 \;. \tag{157}$$

In particular

$$dx_1 \circledcirc dx_2 = dx_2 \circledcirc dx_1 \;. \tag{158}$$

The set of symmetric tensors is closed under the operation $\circledcirc$ of symmetric product [15]. Since any 1-form is trivially symmetric, we used 1-forms in (157). Of course higher order forms are not closed under the symmetric product operation. Conversely, for higher order symmetric tensors, exterior derivative is not well defined.

<u>Definition 13</u>. A weakly regular n-port N is said to be <u>anti-reciprocal</u> if

$$\underset{\sim}{\imath}^* \left(\sum_{k=1}^{n} di_{P_k} \circledcirc dv_{P_k} \right) = \underset{\sim}{0} \;. \tag{159}$$

This definition is, of course, coordinate-free. We will give next a method for checking anti-reciprocity in terms of local coordinates.

<u>Proposition 17</u>. Let N be weakly regular with dim $\Sigma = m$ and let $(\underset{\sim}{\psi}, \Sigma \cap U)$ be a local chart at $(\underset{\sim}{v},\underset{\sim}{i})$ for Σ. Then N is anti-reciprocal if and only if for each $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$, the matrix of (136) is <u>skew symmetric</u>.

<u>Proof</u>. In terms of the coordinate $\underset{\sim}{\psi}$, the left hand side of (159) can be recast with the help of (157) as follows:

$$(\underset{\sim}{\imath} \circ \underset{\sim}{\psi}^{-1})^* \left(\sum_{k=1}^{n} di_{P_k} \circledcirc dv_{P_k} \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{m} \frac{\partial i_{P_k}}{\partial x_j}(\underset{\sim}{x})\, dx_j \right) \circledcirc \left(\sum_{\ell=1}^{m} \frac{\partial v_{P_k}}{\partial x_\ell}(\underset{\sim}{x})\, dx_\ell \right)$$

$$= \sum_{k=1}^{n} \sum_{j,\ell=1}^{m} \frac{\partial i_{P_k}}{\partial x_j}(\underset{\sim}{x}) \frac{\partial v_{P_k}}{\partial x_\ell}(\underset{\sim}{x})\ dx_j \circledcirc dx_\ell$$

$$= \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{\partial i_{P_k}}{\partial x_j}(\underset{\sim}{x}) \frac{\partial v_{P_k}}{\partial x_j}(\underset{\sim}{x})\ dx_j \circledcirc dx_j$$

$$+ \sum_{j<\ell}^{m} \sum_{k=1}^{n} \left(\frac{\partial i_{P_k}}{\partial x_j}(\underset{\sim}{x}) \frac{\partial v_{P_k}}{\partial x_\ell}(\underset{\sim}{x}) + \frac{\partial i_{P_k}}{\partial x_\ell}(\underset{\sim}{x}) \frac{\partial v_{P_k}}{\partial x_j}(\underset{\sim}{x}) \right) dx_j \circledcirc dx_\ell . \tag{160}$$

In order for (160) to vanish, each coefficient must vanish:

$$\sum_{k=1}^{n} \frac{\partial i_{P_k}}{\partial x_j}(\underset{\sim}{x}) \frac{\partial v_{P_k}}{\partial x_j}(\underset{\sim}{x}) = 0, \quad j = 1,\ldots,m$$

$$\sum_{k=1}^{n} \left(\frac{\partial i_{P_k}}{\partial x_j}(\underset{\sim}{x}) \frac{\partial v_{P_k}}{\partial x_\ell}(\underset{\sim}{x}) + \frac{\partial i_{P_k}}{\partial x_\ell}(\underset{\sim}{x}) \frac{\partial v_{P_k}}{\partial x_j}(\underset{\sim}{x}) \right) = 0, \quad 1 \le j < \ell \le m.$$

These conditions are equivalent to saying that the matrix of (136) is skew symmetric. ¤

Results corresponding to Propositions 12, 14-16 and Corollaries 7-10 hold also for anti-reciprocity, by merely replacing the word "symmetry" with "skew symmetry" and the symbol $\wedge$ with $\circledcirc$.

Acknowledgment. The authors are indebted to Prof. M. W. Hirsch of the University of California for discussions on the structural stability result.

APPENDIX 1

Let M_i be m_i-dimensional C^1 submanifold of $\mathbb{R}^n$, $i = 1,2$. For each $\underset{\sim}{x}_0 \in M_1 \cap M_2$, there are neighborhoods U_i of $\underset{\sim}{x}_0$ in $\mathbb{R}^n$ and C^1 functions $\underset{\sim}{f}_i : U_i \to \mathbb{R}^{n-m_i}$ such that

$$M_i \cap U_i = \underset{\sim}{f}_i^{-1}(\underset{\sim}{0})$$

$$\operatorname{rank}(\underset{\sim}{D}\underset{\sim}{f}_i)_{\underset{\sim}{x}} = n - m_i, \quad \text{for all} \quad \underset{\sim}{x} \in U_i \cap M_i .$$

The following fact can be proved in an essentially the same manner as in the APPENDIX of [16].

__Fact A.__ $M_1 \overline{\pitchfork} M_2$ if and only if for each $\underset{\sim}{x} \in M_1 \cap M_2$

$$\operatorname{rank}\begin{bmatrix} \underset{\sim}{D}\underset{\sim}{f}_1 \\ \underset{\sim}{D}\underset{\sim}{f}_2 \end{bmatrix}_{\underset{\sim}{x}} = n - m_1 + n - m_2 . \tag{A.1}$$

APPENDIX 2

__Proof of Lemma 1.__ Let M_1 and M_2 be C^1 submanifolds of $\mathbb{R}^n$. Then every neighborhood of the inclusion map $\underset{\sim}{\iota}_{M_1} : M_1 \to \mathbb{R}^n$ contains an embedding $\underset{\sim}{G}$ such that $\underset{\sim}{G} \overline{\pitchfork} M_2$. Hence either $\underset{\sim}{G}(M_2) \cap M_2$ is empty or for $\underset{\sim}{x} \in M_1$ with $\underset{\sim}{G}(\underset{\sim}{x}) \in M_2$, we have

$$(\underset{\sim}{d}\underset{\sim}{G})_{\underset{\sim}{x}}(T_{\underset{\sim}{x}}M_1) + T_{\underset{\sim}{G}(\underset{\sim}{x})}M_2 = \mathbb{R}^n .$$

We claim that

$$(\underset{\sim}{d}\underset{\sim}{G})_{\underset{\sim}{x}}(T_{\underset{\sim}{x}}M_1) = T_{\underset{\sim}{G}(\underset{\sim}{x})}(\underset{\sim}{G}(M_1)) . \tag{A.2}$$

To see this, let $\underset{\sim}{\psi}$ be a local coordinate for M_1. Then $\underset{\sim}{G}\circ\underset{\sim}{\psi}$ is a local coordinate for M_2 since $\underset{\sim}{G}$ is an embedding. Therefore

$$T_{\underset{\sim}{G}(\underset{\sim}{x})}(\underset{\sim}{G}(M_1)) = \operatorname{Im}(\underset{\sim}{d}\underset{\sim}{G})_{\underset{\sim}{x}} \circ (\underset{\sim}{D}\underset{\sim}{\psi}^{-1})_{\underset{\sim}{\psi}(\underset{\sim}{x})} \tag{A.3}$$

and

$$T_{\underset{\sim}{x}}M_1 = \operatorname{Im}(\underset{\sim}{D}\underset{\sim}{\psi}^{-1})_{\underset{\sim}{\psi}(\underset{\sim}{x})} . \tag{A.4}$$

Equations (A.3) and (A.4) imply (A.2) and hence

$$T_{\underset{\sim}{G}(\underset{\sim}{x})}(\underset{\sim}{G}(M_1)) + T_{\underset{\sim}{G}(\underset{\sim}{x})}M_2 = \mathbb{R}^n . \tag{A.5}$$

Let $\hat{M}_1 \triangleq \underset{\sim}{G}(M_1)$. Then (A.5) implies $\hat{M}_1 \overline{\pitchfork} M_2$. ¤

__Remarks.__ 1. Note that the intersection $\underset{\sim}{G}(M_1) \cap M_2$ may be empty.

2. Perhaps, we should mention here that the term "embedding" in [3] is slightly different from the one used in [4]. The definition in [3] is the same as the

one given in this paper, whereas in [4], a map is called an embedding if it is a proper injective immersion. The former is weaker than the latter in the sense that embedding in the former sense plus the "proper" condition imply embedding in the latter sense. In fact, an embedding in the former sense is proper if and only if its image is closed as a subset of the range space. Theorem 2.4 of [3] which we used in the proof of Lemma 1 uses the weaker definition. It is easy, however, to obtain the same result with the stronger definition of embedding provided that M_1 is closed. Since we are assuming that Λ is closed, there is no confusion.

Proof of Lemma 2. Let $\{f_1,\dots,f_{n_X}\}$ and $\{g_1,\dots,g_{n_Y}\}$ be bases for X and Y, respectively, and let $Z \triangleq \text{span}\{f_1,\dots,f_{n_X},g_1,\dots,g_{n_Y}\}$. Let $\dim Z \triangleq n_X+k$ and without loss of generality assume $Z = \text{span}\{f_1,\dots,f_{n_X},g_1,\dots,g_k\}$. Let $Z^\perp$ be the orthogonal complement of Z and let $\{e_1,\dots,e_{n-(n_X+k)}\}$ be its orthonormal basis. Define

$$\hat{g}_1 \triangleq g_1,\dots,\hat{g}_k \triangleq g_k$$
$$\hat{g}_{k+1} \triangleq g_{k+1}+e_1,\dots,\hat{g}_{n-n_X} \triangleq g_{n-n_X}+e_{n-(n_X+k)}$$
$$\hat{g}_{n-n_X+1} \triangleq g_{n-n_X+1},\dots,\hat{g}_{n_Y} \triangleq g_{n_Y} .$$

We first claim that

$$\text{span}\{f_1,\dots,f_{n_X},\hat{g}_1,\dots,\hat{g}_{n_Y}\} = \mathbb{R}^n \tag{A.6}$$

i.e., the vectors in the bracket of (A.6) are linearly independent. To this end let

$$\alpha_1 f_1 + \cdots + \alpha_{n_X} f_{n_X} + \beta_1\hat{g}_1 + \cdots + \beta_{n-n_X}\hat{g}_{n-n_X} = 0$$

i.e.,

$$\alpha_1 f_1 + \cdots + \alpha_{n_X} f_{n_X} + \beta_1 g_1 + \cdots + \beta_k g_k + \beta_{k+1}(g_{k+1}+e_1) + \cdots + \beta_{n-n_X}(g_{n-n_X}+e_{n-(n_X+k)}) = 0 . \tag{A.7}$$

Since $\{e_1,\dots,e_{n-(n_X+k)}\} \subset Z^\perp$ and since e_i's are orthonormal, taking inner product of (A.7) with e_1, we have

$$\beta_{k+1} = 0 . \tag{A.8}$$

Similarly, taking inner product with $e_2,\dots,e_{n-(n_X+k)}$, we have

$$\beta_{k+2} = \cdots = \beta_{n-n_X} = 0 . \tag{A.9}$$

Equations (A.7)-(A.9) imply

$$\alpha_1 f_1 + \cdots + \alpha_{n_X} f_{n_X} = 0 . \tag{A.10}$$

But since $\{f_1,\ldots,f_{n_X}\}$ was assumed to be linearly independent, (A.9) implies

$$\alpha_1 = \cdots = \alpha_{n_X} = 0 . \tag{A.11}$$

This together with (A.8) and (A.9) imply (A.6). Next let $\hat{Y} \triangleq \text{span}\{\hat{g}_1,\ldots,\hat{g}_{n_Y}\}$. If $\varepsilon > 0$ is small enough, then $\|e_1\|,\ldots,\|e_{n-(n_X+k)}\| < \varepsilon$ imply that $\dim \hat{Y} = n_Y$ because small perturbations do not destroy linear independence of vectors. Hence

$$X + \hat{Y} = \mathbb{R}^n . \tag{A.12}$$

Finally it is clear that $\hat{Y} = A(Y)$ for some matrix A and if $\varepsilon > 0$ is small enough, then $\|A-1\|$ can be arbitrarily small. Since any linear subspace contains the origin, $X \cap A(Y) \neq \emptyset$. This and (A.12) imply (102). ¤

Proof of Lemma 3. Let $\mathcal{U}(f;\varepsilon(\cdot))$ be an arbitrarily small neighborhood of f in $C^1(\mathbb{R}^n,\mathbb{R}^n)$ as defined by (99). Since $\varepsilon(x) > 0$ for all $x \in \mathbb{R}^n$, there is a neighborhood $U = \{x \in \mathbb{R}^n \mid \|x-x_0\| < \delta_0\}$ for some $\delta_0 > 0$ and there is an $\varepsilon > 0$ such that

$$\varepsilon(x) \geq \varepsilon \text{ for all } x \in U . \tag{A.13}$$

Next recall Taylor's formula;

$$f(x) - g(x) = f(x_0) - g(x_0) + \left((Df)_{x_0} - (Dg)_{x_0}\right)(x-x_0) + \|x-x_0\| R(\|x-x_0\|) \tag{A.14}$$

where the remainder term satisfies $R(\|x-x_0\|) \to 0$ as $\|x-x_0\| \to 0$. Choose $\delta_1 > 0$ in such a way that

$$\sup_{\|x-x_0\| \leq \delta_1} |R(\|x-x_0\|)| < \frac{\varepsilon}{9} . \tag{A.15}$$

Choose $\delta_2 > 0$ in such a way that

$$\sup_{\|x-x_0\| \leq \delta_2} \|(Df)_x - (Dg)_x\| < \frac{\varepsilon}{3} \tag{A.16}$$

and let $\delta \triangleq \min\{\delta_0,\delta_1,\delta_2\}$. Set

$$U_2 \triangleq \{x \in \mathbb{R}^n \mid \|x-x_0\| < \delta\} . \tag{A.17}$$

Then there is a neighborhood U_1 of x_0 and a C^1 function $\mu\colon \mathbb{R}^n \to [0,1]$ such that

$$\text{(a)} \quad \bar{U}_1 \subset U_2 \tag{A.18}$$

(b) $\mu(\underset{\sim}{x}) = \begin{cases} 1 & \text{on } U_1 \\ 0 & \text{off } U_2 \end{cases}$ (A.19)

(c) $\|(D\mu)_{\underset{\sim}{x}}\| \le \frac{2}{\delta}$ for all $\underset{\sim}{x} \in \mathbb{R}^n$. (A.20)

Such a function is called a <u>bump</u> function [3]. (See Fig. 24.) Set

$$\hat{g}(x) \triangleq \mu(x)f(x) + (1-\mu(x))g(x) \ . \tag{A.21}$$

Then

$$\hat{g} = f \text{ on } U_1 \tag{A.22}$$

and

$$\hat{g} = g \text{ off } U_2. \tag{A.23}$$

We claim that $\hat{g}$ is in $\mathcal{U}(f;\varepsilon(\cdot))$. To this end we compute the following:

$$\begin{aligned} &\|\hat{g}(x)-g(x)\| + \|(D\hat{g})_x-(Dg)_x\| \\ &\quad\le \mu(x)\|f(x)-g(x)\| + \mu(x)\|(Df)_x-(Dg)_x\| + \|(D\mu)_x\|\|f(x)-g(x)\| \\ &\quad\le \big(\mu(x) + \|(D\mu)_x\|\big)\|f(x)-g(x)\| + \mu(x)\|(Df)_x-(Dg)_x\| \ . \end{aligned} \tag{A.24}$$

Since $\mu(x) \equiv 0$ off U_2, the right hand side of (A.24) is zero off U_2. Now for $\|x-x_0\| < \delta$ we have, using (A.14)-(A.17), (A.20),

$$\begin{aligned} &\big(\mu(x) + \|(D\mu)_x\|\big)\|f(x)-g(x)\| + \mu(x)\|(Df)_x-(Dg)_x\| \\ &\quad\le (1+\tfrac{2}{\delta})\|x-x_0\|R(\|x-x_0\|) + \|(Df)_x-(Dg)_x\| \\ &\quad< (1+\tfrac{2}{\delta})\delta\tfrac{\varepsilon}{9}+\tfrac{\varepsilon}{3} = (\delta+2)\tfrac{\varepsilon}{9}+\tfrac{\varepsilon}{3} \le \tfrac{\varepsilon}{3}+\tfrac{\varepsilon}{3} < \varepsilon \ . \end{aligned}$$

It follows from this and (A.13) that $\hat{g}$ belongs to $\mathcal{U}(f;\varepsilon(\cdot))$. Hence $\hat{g}$ is close to g in the strong C^1 topology.

<u>Proof of Lemma 4</u>. The proof is similar to that of <u>Lemma 3</u>. Let $\mathcal{U}(i_d;\varepsilon(\cdot))$ be an arbitrarily small neighborhood of the identity map i_d in $C^1(\mathbb{R}^n,\mathbb{R}^n)$. Let U and $\varepsilon > 0$ be defined as in the proof of <u>Lemma 3</u>. Let A satisfy $\|A-1\| < \frac{\varepsilon}{4}$ and let $\delta \triangleq \min\{\delta_0,\frac{1}{4}\}$ where $\delta_0 > 0$ is as in the proof of <u>Lemma 3</u>. Define $U_2 \triangleq \{x \in \mathbb{R}^n \,|\, \|x\| < \delta\}$. Then there is a neighborhood U_1 of the origin and there is a C^1 function μ: $\mathbb{R}^n \to [0,1]$ satisfying (A.18)-(A.20). Let $G_1(x) \triangleq \mu(x)Ax + (1-\mu(x))x$. Then $G_1 = A$ on U_1, $G_1 = i_d$ off U_2. We claim that $G_1 \in \mathcal{U}(i_d;\varepsilon(\cdot))$. Since $\mu(x) \equiv 0$ off U_2, we need to check it only on U_1;

$$\|\underset{\sim}{G}_1(\underset{\sim}{x})-\underset{\sim}{x}\| + \|(D\underset{\sim}{G}_1)_{\underset{\sim}{x}}-\underset{\sim}{1}\| \le \mu(\underset{\sim}{x})\|\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{x}\| + \mu(\underset{\sim}{x})\|\underset{\sim}{A}-\underset{\sim}{1}\| + (D\mu)_{\underset{\sim}{x}}\|\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{x}\|$$
$$\le \left(\mu(\underset{\sim}{x})+(D\mu)_{\underset{\sim}{x}}\right)\|\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{x}\| + \mu(\underset{\sim}{x})\|\underset{\sim}{A}-\underset{\sim}{1}\|$$
$$\le (1+2\cdot 4)\|\underset{\sim}{x}\|\|\underset{\sim}{A}-\underset{\sim}{1}\| + \|\underset{\sim}{A}-\underset{\sim}{1}\|$$
$$\le 9\cdot\frac{1}{4}\cdot\frac{\varepsilon}{4}+\frac{\varepsilon}{4} = \frac{13}{16}\varepsilon < \varepsilon .$$

Hence $\underset{\sim}{G}_1 \in \mathcal{U}(i_d;\varepsilon(\cdot))$ and therefore $\underset{\sim}{G}_1$ is close to $\underset{\sim}{G}$ in the strong C^1 topology.

Proof of (i) of Theorem 2. In order to prove that Σ and $\hat{\Sigma}$ are diffeomorphic we first define a family of C^1 maps $\underset{\sim}{G}_t: \Lambda \to \mathbb{R}^b\times\mathbb{R}^b$ by

$$\underset{\sim}{G}_t(\underset{\sim}{v},\underset{\sim}{i}) \triangleq (1-t)\underset{\sim}{\imath}_\Lambda(\underset{\sim}{v},\underset{\sim}{i}) + t\underset{\sim}{G}(\underset{\sim}{v},\underset{\sim}{i})$$

where $t \in \mathbb{R}$ and G is obtained in the proof of the first half of (i). Define the map $\underset{\sim}{Y}: \Lambda \times \mathbb{R} \to \mathbb{R}^b \times \mathbb{R}^b \times \mathbb{R}$ by

$$\underset{\sim}{Y}(\underset{\sim}{v},\underset{\sim}{i},t) \triangleq \left(\underset{\sim}{G}_t(\underset{\sim}{v},\underset{\sim}{i}),t\right).$$

Since $\underset{\sim}{G}_t(\underset{\sim}{v},\underset{\sim}{i}) - \underset{\sim}{\imath}_\Lambda(\underset{\sim}{v},\underset{\sim}{i}) = t\left(\underset{\sim}{G}(\underset{\sim}{v},\underset{\sim}{i}) - \underset{\sim}{\imath}_\Lambda(\underset{\sim}{v},\underset{\sim}{i})\right)$ and since $\underset{\sim}{Y}(\underset{\sim}{v},\underset{\sim}{i},t) - \left(\underset{\sim}{\imath}_\Lambda(\underset{\sim}{v},\underset{\sim}{i}),t\right) = \left(\underset{\sim}{G}_t(\underset{\sim}{v},\underset{\sim}{i})-\underset{\sim}{\imath}_\Lambda(\underset{\sim}{v},\underset{\sim}{i}),0\right)$, we see that if $\underset{\sim}{G}$ is close enough to $\underset{\sim}{\imath}_\Lambda$ in the strong C^1 topology, then there is a number $\varepsilon > 0$ such that $\underset{\sim}{G}_t$ is an embedding transversal to K for all $t\in I \triangleq(-\varepsilon,1+\varepsilon)$, and $\underset{\sim}{Z} \triangleq Y|(\Lambda\times I)$ is an embedding transversal to $K \times I$. By the same reasoning as that of the proof of Theorem 1 we see that $M \triangleq \underset{\sim}{Z}(\Lambda\times I) \cap (K\times I)$ is an $(n+1)$-dimensional submanifold. Now, let

$$M_t \triangleq \underset{\sim}{Z}(\Lambda\times\{t\})\cap(K\times\{t\}).$$

Then $M = \bigcup_{t\in I} M_t$ and M_t is naturally identified with $\underset{\sim}{G}_t(\Lambda)\cap K$. Hence $\Sigma = M_0$ and $\hat{\Sigma} = M_1$, where $\Sigma = \Lambda\cap K$ and $\hat{\Sigma} = \hat{\Lambda}\cap K$. We next construct a diffeomorphism between M_0 and M_1. To this end let $(\underset{\sim}{0},\frac{\partial}{\partial t}) \in T_{(\underset{\sim}{v},\underset{\sim}{i},t)}(\Lambda\times I)$ be the vector field[9] on $\Lambda\times I$ along the t-axis. Let $\underset{\sim}{G}_t(\underset{\sim}{v},\underset{\sim}{i}) \triangleq \underset{\sim}{x}$. Then, since $T_{(\underset{\sim}{x},t)}M = T_{(\underset{\sim}{x},t)}M_t \oplus T_tI$, we can decompose the vector $(d\underset{\sim}{Z})_{(\underset{\sim}{v},\underset{\sim}{i},t)}((\underset{\sim}{0},\frac{\partial}{\partial t}))$ as $(d\underset{\sim}{Z})_{(\underset{\sim}{v},\underset{\sim}{i},t)} = \underset{\sim}{X}_{1(\underset{\sim}{x},t)} \oplus (\frac{\partial}{\partial t})_t$ where $\underset{\sim}{X}_{1(\underset{\sim}{x},t)} \in T_{(\underset{\sim}{x},t)}M_t$. Next, let $\underset{\sim}{P}_{(\underset{\sim}{x},t)}: T_{(\underset{\sim}{x},t)}\underset{\sim}{Z}(\Lambda\times\{t\}) \to T_{(\underset{\sim}{x},t)}M_t$ be the orthogonal projection and set

$$\underset{\sim}{X}_{(\underset{\sim}{x},t)} \triangleq \underset{\sim}{P}_{(\underset{\sim}{x},t)}\underset{\sim}{X}_{1(\underset{\sim}{x},t)} + (\frac{\partial}{\partial t})_t . \tag{A.25}$$

---

[9] A vector field $\underset{\sim}{X}$ on a manifold M is a function such that for each $\underset{\sim}{x} \in M$, the value $\underset{\sim}{X}_{\underset{\sim}{x}}$ at $\underset{\sim}{x}$ belongs to $T_{\underset{\sim}{x}}M$. The vector field $\partial/\partial t$ has the property that $(\frac{\partial}{\partial t})_t = \tilde{1}$ for all t.

We first show that $\underset{\sim}{X}$ is a C^1 vector field on M. For $(\underset{\sim}{x},t) \in M$ choose a local coordinate $\underset{\sim}{\psi}_Z : \underset{\sim}{Z}(\Lambda\times I) \cap U \to \mathbb{R}^{2b-n_R+1}$ for $\underset{\sim}{Z}(\Lambda\times I)$ in such a manner that $\underset{\sim}{\psi}_Z(\underset{\sim}{x}) = (y_1,\ldots,y_{n+1},0,\ldots,0)$. Without loss of generality assume

that the basis $(D\underset{\sim}{\psi}_Z^{-1})(\frac{\partial}{\partial y_k})$, $k = 1,\ldots,2b-n_R+1$, for $T_{(\underset{\sim}{x},t)}\underset{\sim}{Z}(\Lambda\times I)$ is orthonormal. It is clear that with respect to this basis, the projection $\underset{\sim}{P}_{(\underset{\sim}{x},t)}$ is represented by the following $(2b-n_R+1) \times (2b-n_R+1)$ matrix:

$$\begin{array}{r c} n+1 \{ & \left[\begin{array}{cc} \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{array}\right] \\ b \{ & \\ & \underbrace{\quad}_{n+1} \; \underbrace{\quad}_{b} \end{array} .$$

This is true for all $\underset{\sim}{x} \in \underset{\sim}{Z}(\Lambda\times I) \cap U$. Hence $\underset{\sim}{P}_{(\underset{\sim}{x},t)}\underset{\sim}{X}_{1(\underset{\sim}{x},t)}$ is C^1. Clearly $(\frac{\partial}{\partial t})_t$ is C^1 and therefore $\underset{\sim}{X}$ defined by (A.25) is C^1. We next show that if $\underset{\sim}{G}$ is close enough to $\underset{\sim}{1}_\Lambda$ in the strong C^1 topology, then for each $(\underset{\sim}{x}_0,0) \in M_0$, the trajectory $\underset{\sim}{\phi}(\underset{\sim}{x}_0,t)$ generated by $\underset{\sim}{X}$ is defined for all $t \in [0,1]$. To prove this suppose that the maximal interval of existence of $\underset{\sim}{\phi}(\underset{\sim}{x}_0,t)$ is $[0,\beta)$, where $0 < \beta < 1$, i.e., either $\underset{\sim}{\phi}(\underset{\sim}{x}_0,t) \to \underset{\sim}{y} \notin M$ or $\|\underset{\sim}{\phi}(\underset{\sim}{x}_0,t)\| \to \infty$ as $t \to \beta$. We first claim that the first case is impossible. Since $\underset{\sim}{Z}$ is an embedding and since Λ is assumed to be closed, $\underset{\sim}{Z}(\Lambda\times I)$ is a closed subset of $\mathbb{R}^b \times \mathbb{R}^b \times I$. Therefore $M = \underset{\sim}{Z}(\Lambda\times I) \cap (K\times I)$ is a closed subset of $\mathbb{R}^b \times \mathbb{R}^b \times I$. Hence, if $\underset{\sim}{\phi}(\underset{\sim}{x}_0,t) \to \underset{\sim}{y}$ as $t \to \beta$, then we must have $\underset{\sim}{y} \in M$. In order to prove that the second case is also impossible notice that if $\underset{\sim}{G}$ is close enough to $\underset{\sim}{1}_\Lambda$ in the strong C^1 topology, then $\underset{\sim}{dZ}$ is close to $\underset{\sim}{d}(\underset{\sim}{1}_\Lambda,i_d)$ and hence $\underset{\sim}{X}_1$ is small. Since $\underset{\sim}{P}_{(\underset{\sim}{x},t)}$ is a projection, $\underset{\sim}{P}_{(\underset{\sim}{x},t)}\underset{\sim}{X}_{1(\underset{\sim}{x},t)}$ is also small. Now let $\underset{\sim}{Q} : \mathbb{R}^b \times \mathbb{R}^b \times \underset{\sim}{I} \to I$ be the orthogonal projection. It follows from (A.25) that $\underset{\sim}{Q}\,\underset{\sim}{X}_{(\underset{\sim}{x},t)} = (\frac{\partial}{\partial t})_t$. Hence $\underset{\sim}{X}$ generates a solution $\underset{\sim}{\phi}(\underset{\sim}{x}_0,t)$ on M such that $\underset{\sim}{Q}\,\underset{\sim}{\phi}(\underset{\sim}{x}_0,t) = t$. Next, since $\underset{\sim}{\phi}(\underset{\sim}{x}_0,t)$ is a curve in $\mathbb{R}^b \times \mathbb{R}^b \times I$, we can write $\underset{\sim}{\phi}(\underset{\sim}{x}_0,t) = \left(\phi^1(\underset{\sim}{x}_0,t),\ldots,\phi^{2b}(\underset{\sim}{x}_0,t),t\right)$. Similarly, we have $\underset{\sim}{X}_{(\underset{\sim}{x},t)} = \left(X^1_{(\underset{\sim}{x},t)},\ldots,X^{2b}_{(\underset{\sim}{x},t)},1\right)$. It follows from the above argument that there is a number $L > 0$ such that $|X^k_{(\underset{\sim}{x},t)}| \le L$ for all $t \in [0,\beta)$. Integrating $X^k_{(\underset{\sim}{x},t)}$ with respect to t, we have $\phi^k(\underset{\sim}{x}_0,t)-\phi^k(\underset{\sim}{x}_0,0) = \int_0^t X^k\left(\underset{\sim}{\phi}(\underset{\sim}{x}_0,s),s\right) ds$. Hence $|\phi^k(\underset{\sim}{x}_0,t)-\phi^k(\underset{\sim}{x}_0,0)| \le L\beta$ for all $t \in [0,\beta)$. Therefore, the second case is also impossible. Since $\underset{\sim}{\phi}(\underset{\sim}{x}_0,t)$ is well defined for $t \in [0,1]$, we have $\underset{\sim}{Q}\,\underset{\sim}{\phi}(\underset{\sim}{x}_0,1) = 1$. It follows from a property of solution of differential equations [3, p. 150] that $\underset{\sim}{\phi}(M_0,1) = M_1$ and the map $\underset{\sim}{\phi}(\cdot,1): M_0 \to M_1$

is a diffeomorphism. Therefore Σ and $\hat{\Sigma}$ are diffeomorphic. ¤

APPENDIX 3

Let $X \triangleq \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{p \text{ times}}$ and let $\underset{\sim}{f}$ be a p-linear function on X, i.e., for any $(\underset{\sim}{x}^1, \ldots, \underset{\sim}{x}^{j-1}, \underset{\sim}{x}^{j+1}, \ldots, \underset{\sim}{x}^p) \in \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{(p-1) \text{ times}}$, the function $\underset{\sim}{f}(\underset{\sim}{x}^1, \ldots, \underset{\sim}{x}^{j-1}, \cdot, \underset{\sim}{x}^{j+1}, \ldots, \underset{\sim}{x}^p)$: $\mathbb{R}^n \to \mathbb{R}$ is linear for all $j = 1, \ldots, p$. The set of all p-linear functions on X is a vector space X^*. Let $\underset{\sim}{f}$ be a p-linear function on X and let $\underset{\sim}{g}$ be a q-linear function on $Y \triangleq \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{q \text{ times}}$. Then one can naturally define a product of $\underset{\sim}{f}$ and $\underset{\sim}{g}$ by the following formula:

$$\underset{\sim}{f}(\underset{\sim}{x}^1, \ldots, \underset{\sim}{x}^p)\underset{\sim}{g}(\underset{\sim}{y}^1, \ldots, \underset{\sim}{y}^q) .$$

This is called the <u>tensor product</u> of $\underset{\sim}{f}$ and $\underset{\sim}{g}$ and denoted by $\underset{\sim}{f} \otimes \underset{\sim}{g}$. Let G_p be the set of all permutations of $(1, \ldots, p)$ and let $\sigma \in G_p$. For any $\underset{\sim}{f} \in X^*$ and any $(\underset{\sim}{x}^1, \ldots, \underset{\sim}{x}^p) \in X$, define

$$(\sigma \underset{\sim}{f})(\underset{\sim}{x}^1, \ldots, \underset{\sim}{x}^p) \triangleq \underset{\sim}{f}(\underset{\sim}{x}^{\sigma(1)}, \ldots, \underset{\sim}{x}^{\sigma(p)}) .$$

If $\sigma \underset{\sim}{f} = \underset{\sim}{f}$ for all $\sigma \in G_p$, then $\underset{\sim}{f}$ is called a <u>symmetric p-tensor</u>. The set of all symmetric p-tensors X_S^* is a linear subspace of X^*. Let

$$\underset{\sim}{S}_p \underset{\sim}{f} \triangleq \sum_{\sigma \in G_p} \sigma \underset{\sim}{f} .$$

For any symmetric p-tensor $\underset{\sim}{f}$ and symmetric q-tensor $\underset{\sim}{g}$ define

$$\underset{\sim}{f} \circledcirc \underset{\sim}{g} \triangleq \frac{1}{(p+q)!} \underset{\sim}{S}_{p+q}(\underset{\sim}{f} \otimes \underset{\sim}{g}) .$$

It can be shown that $\underset{\sim}{f} \circledcirc \underset{\sim}{g}$ is, again, a symmetric (p+q)-tensor, i.e., the set of symmetric tensors is closed under the operation of $\circledcirc$ which we call <u>symmetric product</u>. We can define symmetric tensors on manifolds. Let M be a manifold. Then $\underset{\sim}{\omega}$ is said to be a <u>symmetric p-tensor field</u> on M if for each $\underset{\sim}{z} \in M$, the value $\underset{\sim}{\omega}_{\underset{\sim}{z}}$ of $\underset{\sim}{\omega}$ at $\underset{\sim}{z}$ is a symmetric p-tensor on $\underbrace{T_{\underset{\sim}{z}} M \times \cdots \times T_{\underset{\sim}{z}} M}_{p \text{ times}}$. Let M_1 and M_2 be two manifolds and let $\underset{\sim}{F}: M_1 \to M_2$ be C^1. Let $\underset{\sim}{\omega}$ be a symmetric p-tensor field on M_2. Then $\underset{\sim}{F}$ naturally pulls $\underset{\sim}{\omega}$ back to M_1 and induces a symmetric p-tensor field on M_1;

$$(\underset{\sim}{F}^* \underset{\sim}{\omega})_{\underset{\sim}{z}}(\underset{\sim}{x}^1, \ldots, \underset{\sim}{x}^p) \triangleq \underset{\sim}{\omega}_{\underset{\sim}{F}(\underset{\sim}{z})}\left((\underset{\sim}{dF})_{\underset{\sim}{z}}(\underset{\sim}{x}^1), \ldots, (\underset{\sim}{dF})_{\underset{\sim}{z}}(\underset{\sim}{x}^p)\right)$$

where $\underset{\sim}{z} \in M_1$ and $(\underset{\sim}{x}^1, \ldots, \underset{\sim}{x}^p) \in \underbrace{T_{\underset{\sim}{z}} M_1 \times \cdots \times T_{\underset{\sim}{z}} M_1}_{p \text{ times}}$. The map $\underset{\sim}{F}^*$ is called the induced map of $\underset{\sim}{F}$. Next, for $\sigma \in G_p$ let

$$\varepsilon_\sigma \triangleq \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

If $\underset{\sim}{f} \in X^*$ satisfies $\sigma\underset{\sim}{f} = \varepsilon_\sigma \underset{\sim}{f}$ for all $\sigma \in G_p$, then $\underset{\sim}{f}$ is called an alternating p-tensor. A differential p-form is simply an alternating p-tensor field on a manifold. Hence formally we should have used different notation from di_{P_k} and dv_{P_k} in (159). But any 1-tensor field is trivially alternating and symmetric. Hence there is no inconsistency in defining symmetric product of two 1-forms. For more on this subject, see [15].

References

[1] H. J. Carlin and A. B. Giordano, Network Theory, Prentice-Hall, Englewood Cliffs, N. J., 1964.

[2] M. Peixoto, "On structural stability," Ann. of Math., vol. 69, pp. 199-222, 1959.

[3] M. W. Hirsch, Differential Topology, Springer-Verlag, New York, 1976.

[4] V. Guillemin and A. Pollack, Differential Topology, Prentice-Hall, Englewood Cliffs, N. J., 1974.

[5] H. Flanders, "Infinite networks: I-Resistive networks," IEEE Trans. Circuit Theory, vol. CT-18, pp. 326-331, May 1971.

[6] T. Matsumoto, L. O. Chua and A. Makino, "On the implications of capacitor-only cut sets and inductor-only loops in nonlinear networks," IEEE Trans. Circuits and Systems, to appear.

[7] C. A. Desoer and F. F. Wu, "Trajectories of RLC networks: Geometric approach," IEEE Trans. Circuit Theory, vol. CT-19, pp. 562-570, Dec. 1972.

[8] L. O. Chua and Y. F. Lam, "A theory of algebraic n-ports," IEEE Trans. Circuit Theory, vol. CT-20, pp. 370-382, July 1973.

[9] F. A. Ficken, Linear Transformations and Matrices, Prentice-Hall, Englewood Cliffs, N. J., 1967.

[10] S. Smale, "On the mathematical foundations of electrical circuit theory," J. Differential Geometry, vol. 17, pp. 193-210, 1972.

[11] T. Matsumoto, "On the dynamics of electrical networks," J. Differential Equations, vol. 21, pp. 179-196, May 1976.

[12] W. H. Fleming, Functions of Several Variables, Addison-Wesley, Reading, Mass., 1967.

[13] R. K. Brayton, "Nonlinear reciprocal networks," in Mathematical Aspects of Electrical Network Analysis, Amer. Math. Soc., Providence, R. I., 1971.

[14] J. L. Wyatt and L. O. Chua, "A theory of non-energic n-ports," Int. J. Cir. Theor. Appl., vol. 5, pp. 181-208, 1977.

[15] Y. Matsushima, Differential Manifolds, Marcel Dekker, New York, 1972.

[16] T. Matsumoto, "On several geometric aspects of nonlinear networks," J. Franklin Institute, vol. 301, pp. 203-225, Jan. 1976.

[17] L. O. Chua, "Nonlinear Circuit Theory," Proc. of the 1978 European Conference on Circuit Theory and Design, vol. II, Guest Lectures, Lausanne, Switzerland, Sept. 4-8, pp. 65-172, 1978.

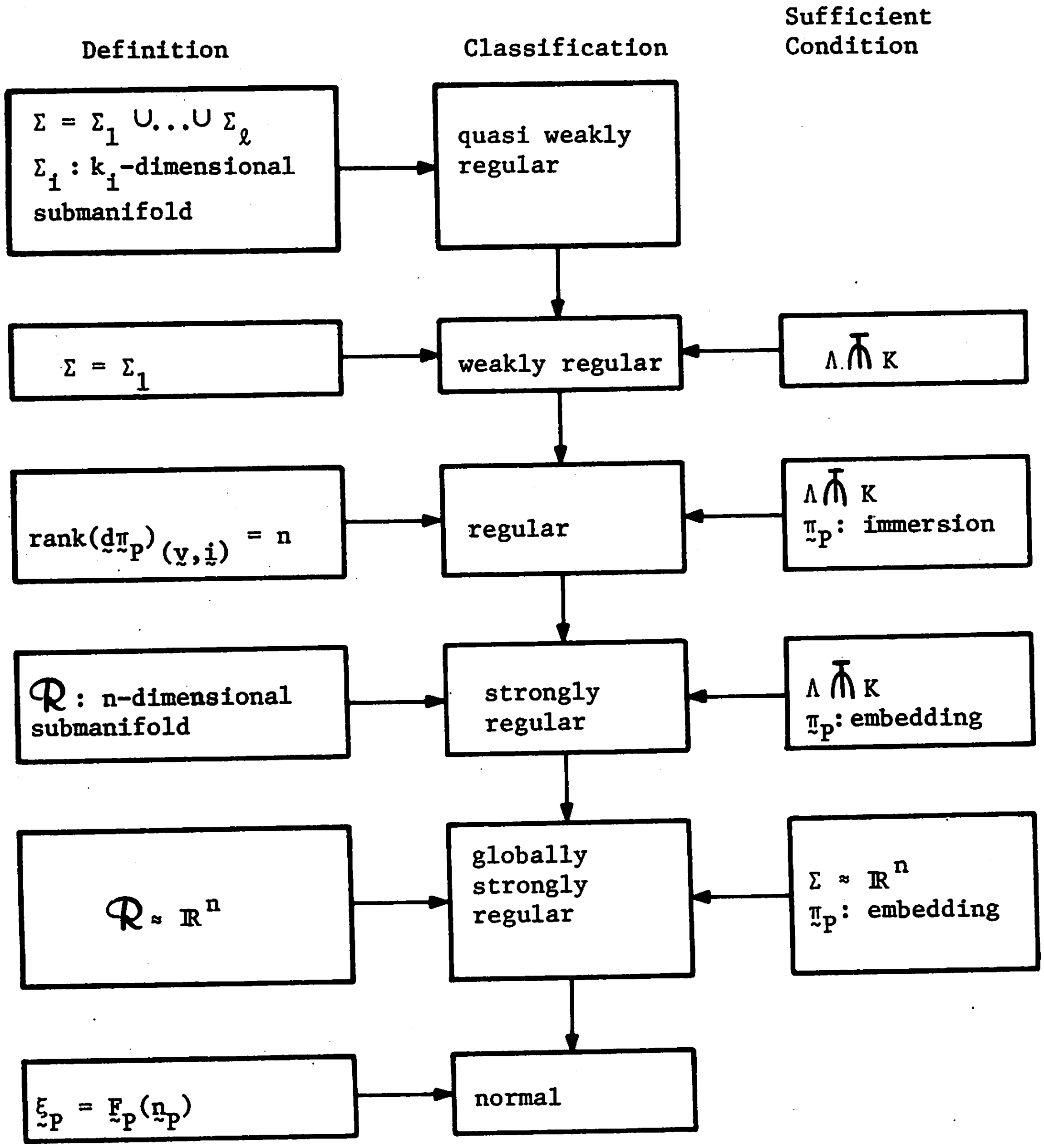


Table 1. A summary of definitions, classifications, and sufficient conditions of different categories of resistive n-ports.

Figure Captions

Fig. 1 Examples demonstrating resistors with simple constitutive relations when interconnected could give rise to complicated constitutive relations in the composite 1-port.

(a) The circuit diagram.

(b) Constitutive relations of R_1, R_2 and the composite 1-port.

Fig. 2 A 1-port (norator) whose constitutive relation consists of the entire plane ($\mathbb{R}^2$).

(a) A norator circuit realization.

(b) Constitutive relation of the norator.

Fig. 3 A 1-port (nullator) whose constitutive relation consists of a single point.

(a) A nullator circuit realization.

(b) Constitutive relation of the nullator.

Fig. 4 A 1-port whose constitutive relation consists of all points within a bounded region in $\mathbb{R}^2$.

(a) The circuit diagram.

(b) Constitutive relation of R_1.

(c) Constitutive relation of R_2.

(d) Constitutive relation of the composite 1-port.

Fig. 5 A 1-port whose constitutive relation consists of a parametrizable curve in $\mathbb{R}^2$.

(a) The circuit diagram.

(b) Constitutive relation of the composite 1-port.

Fig. 6 A 1-port whose constitutive relation consists of all points belonging to the intersection between the constitutive relations of the internal resistors.

(a) The circuit diagram.

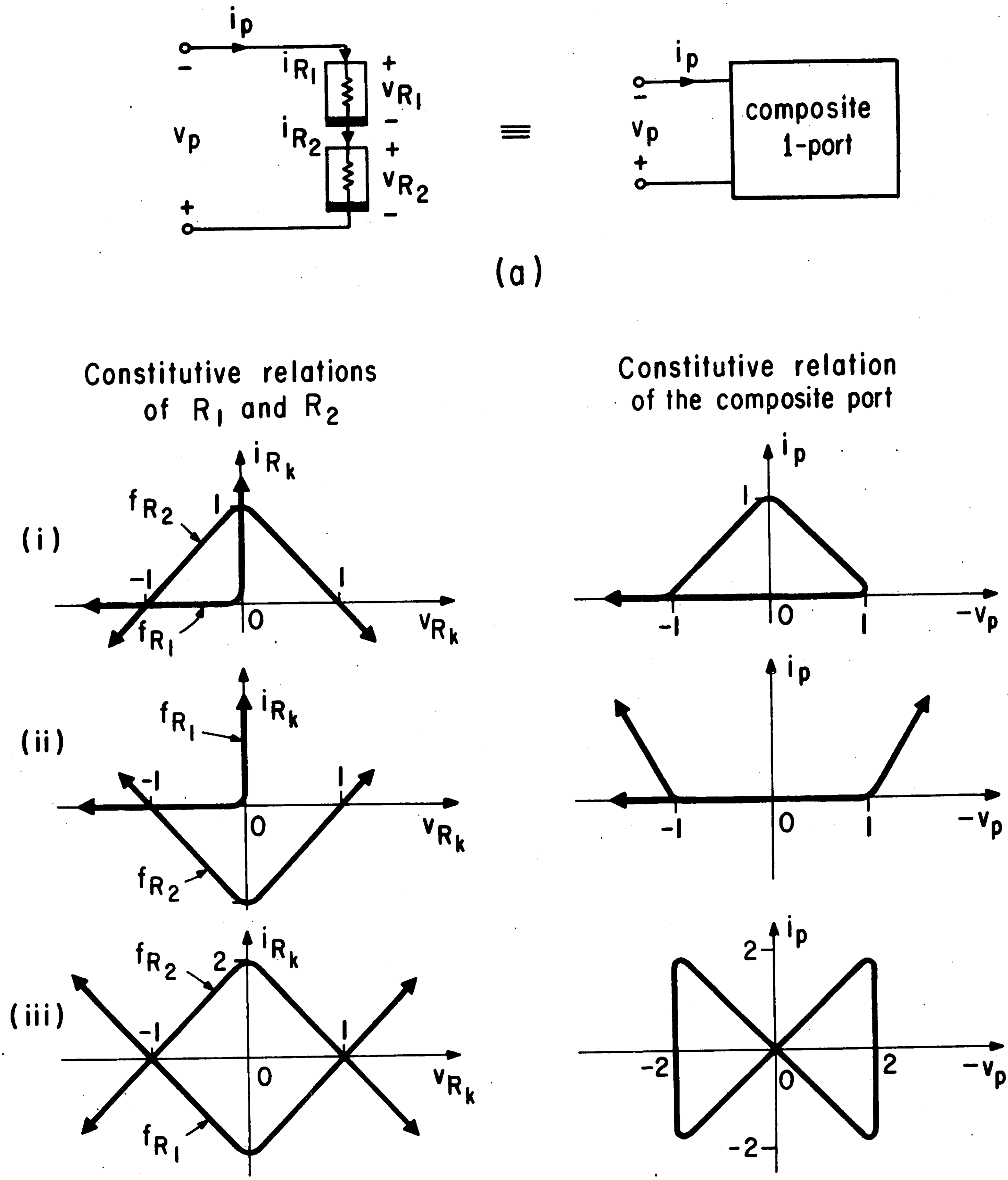
(b) Constitutive relation of R_1.

(c) Constitutive relation of R_2.

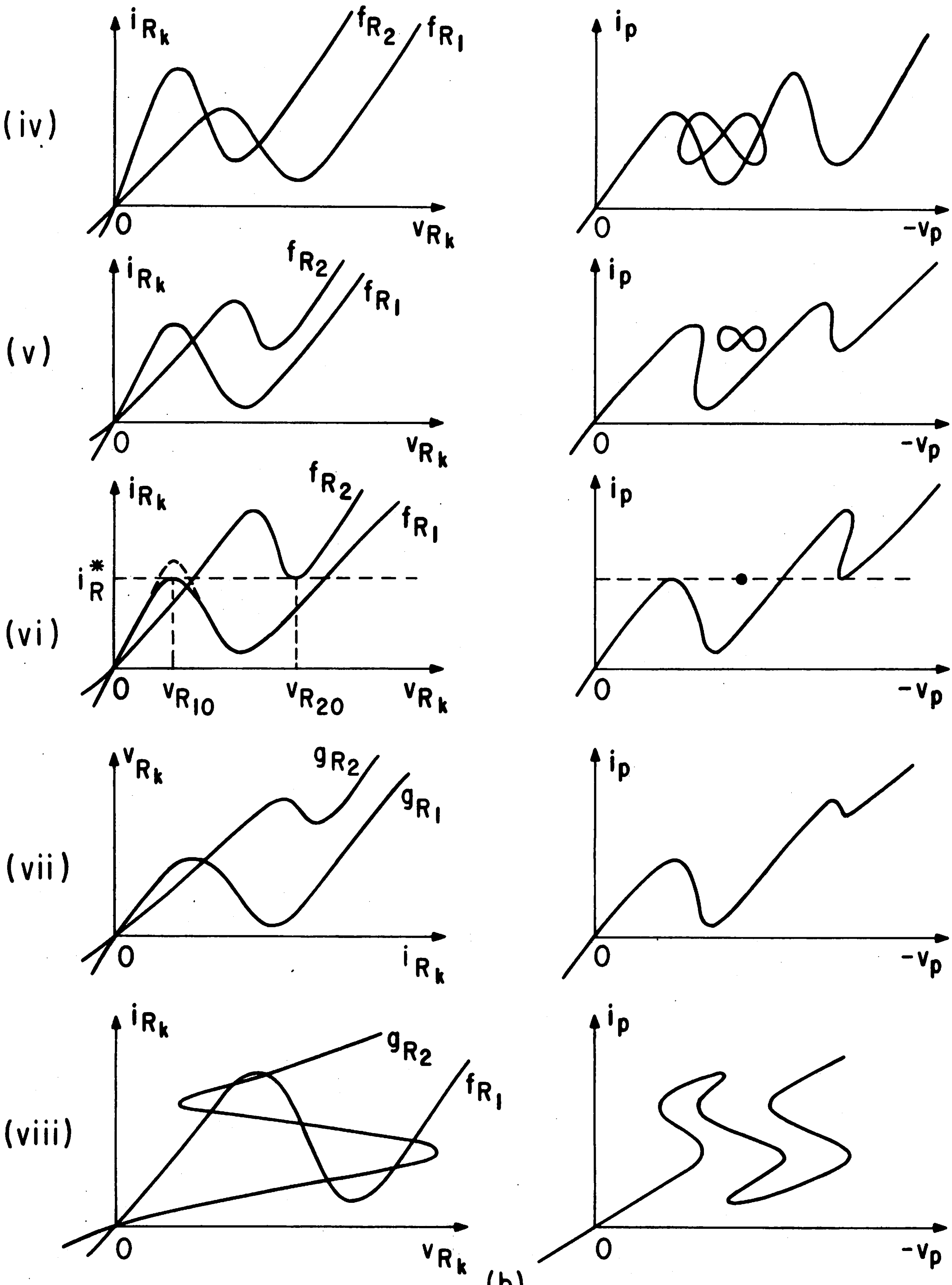
(d) Constitutive relation of the composite 1-port.

Fig. 7 A 1-dimensional submanifold of $\mathbb{R}^2$.

Fig. 8 A commutative diagram showing the relationship between the derivative $(d\underset{\sim}{F})_{\underset{\sim}{x}}$ of $\underset{\sim}{F}$ at $\underset{\sim}{x}$ and its equivalent expression via local coordinates.



ip
iR1
vR1
iR2
vR2
vp
composite 1-port
(a)
Constitutive relations of R1 and R2
Constitutive relation of the composite port
(i)
(ii)
(iii)
fR1
fR2
iRk
vRk
ip
−vp



(b)

Fig. 1

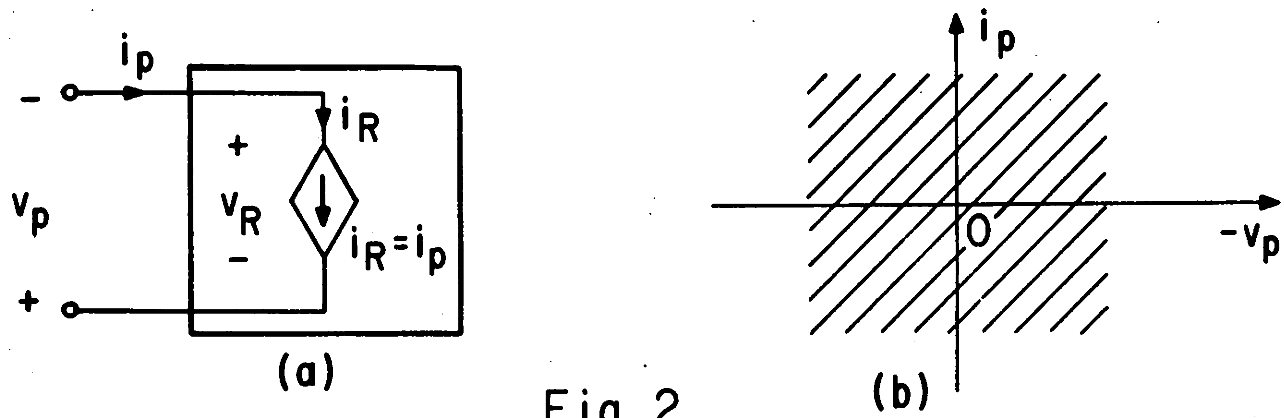


Fig.2

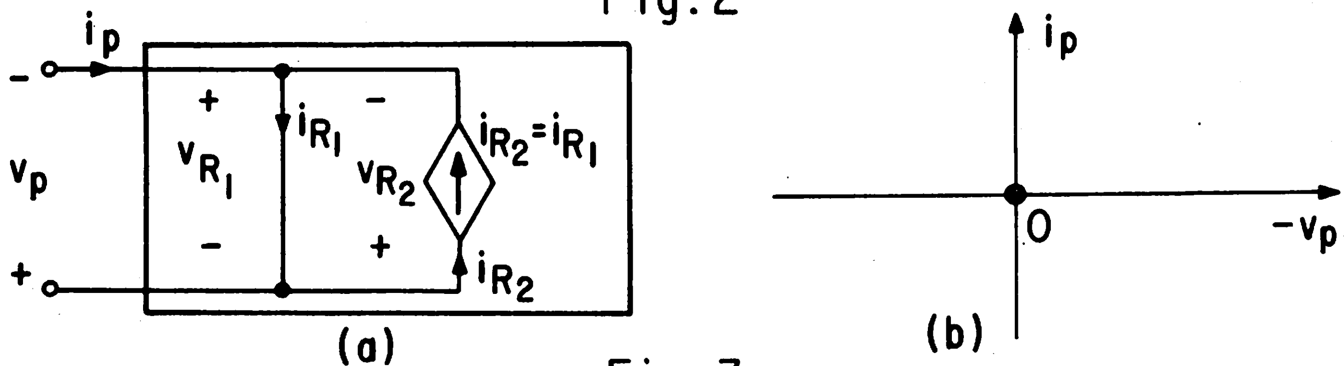


Fig.3

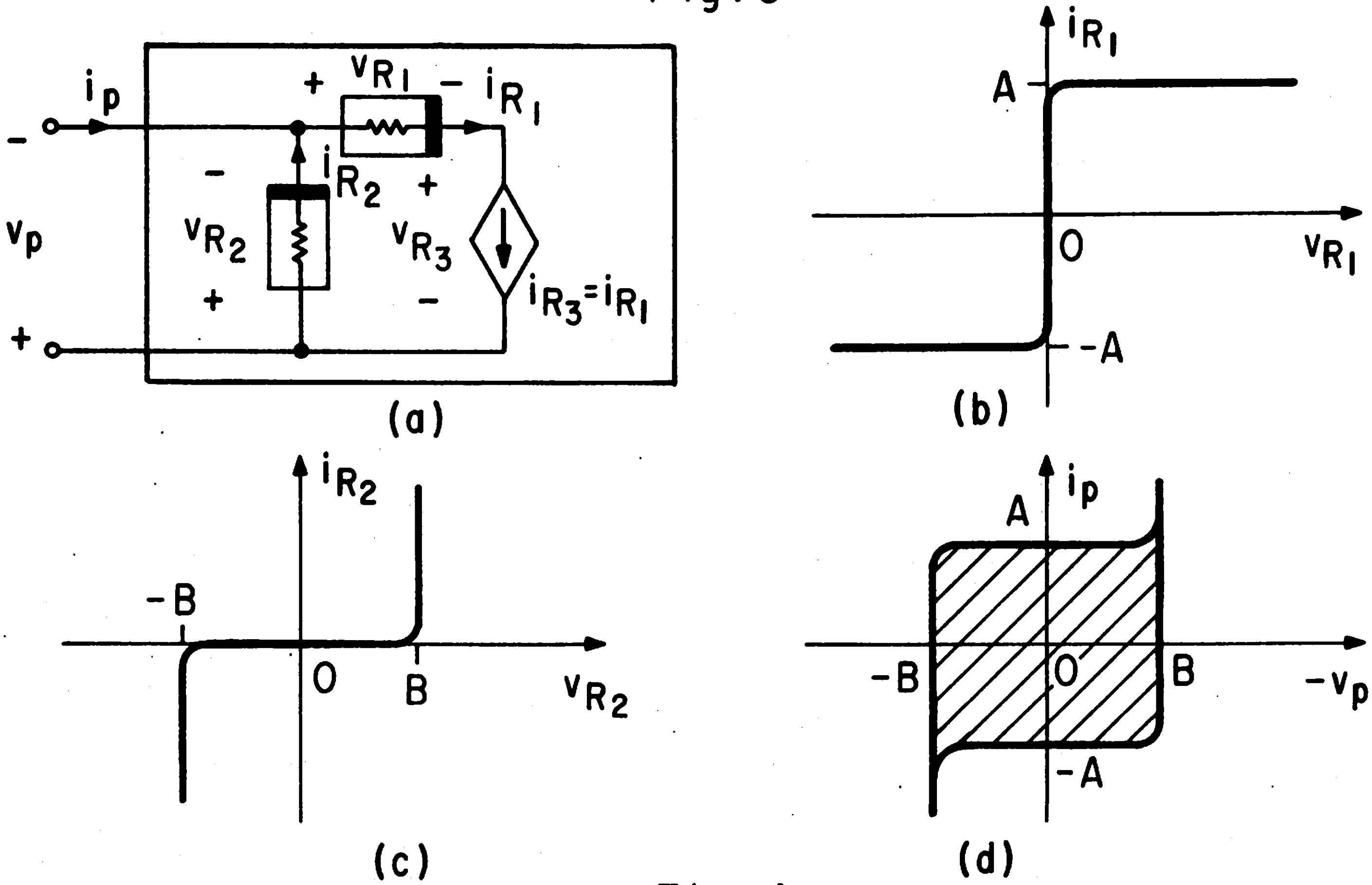


Fig.4

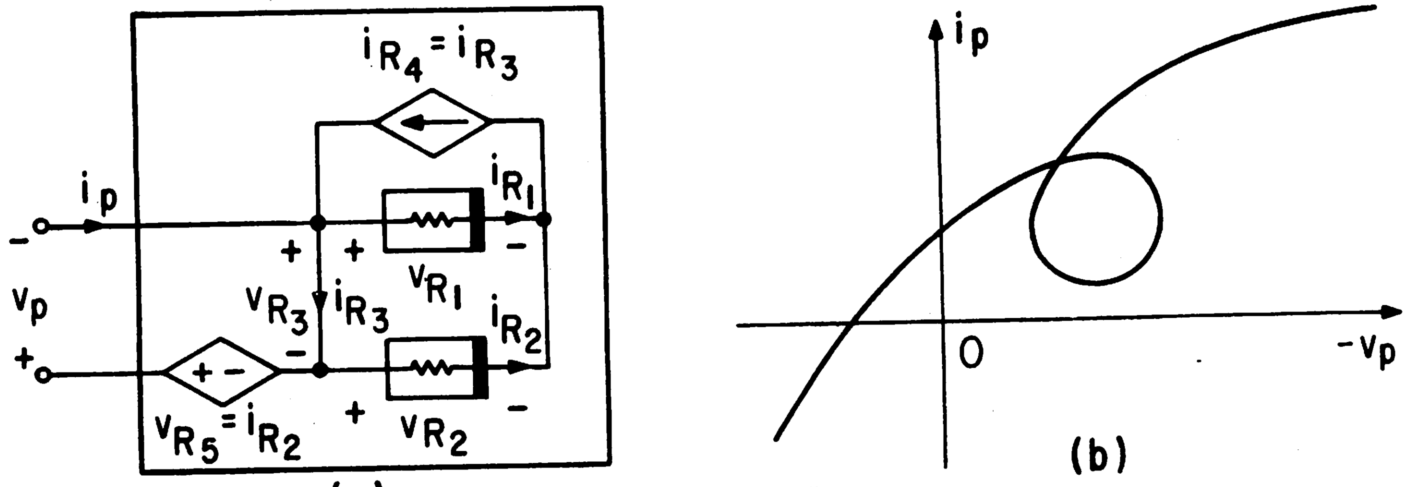


Fig. 5

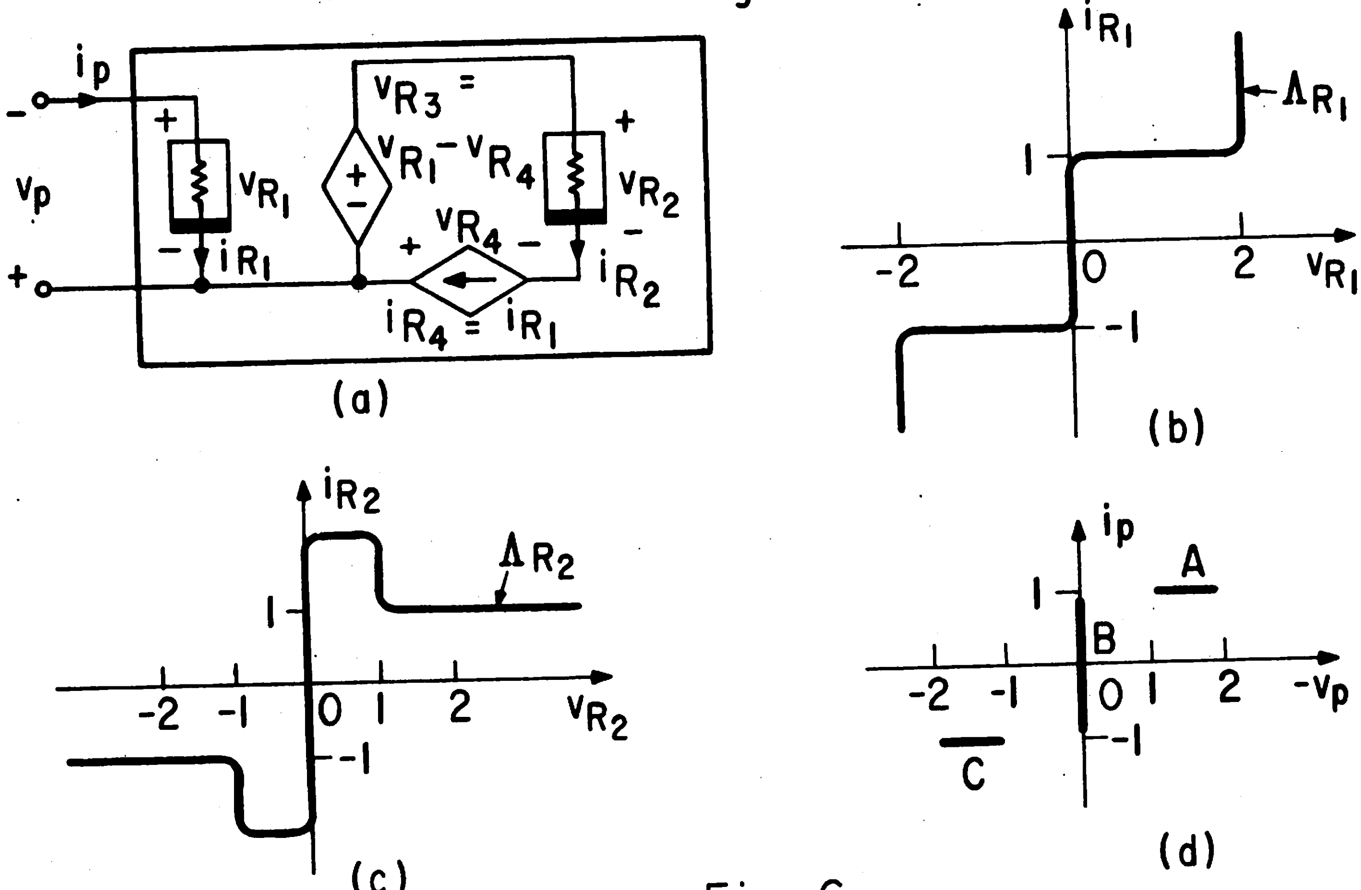


Fig. 6

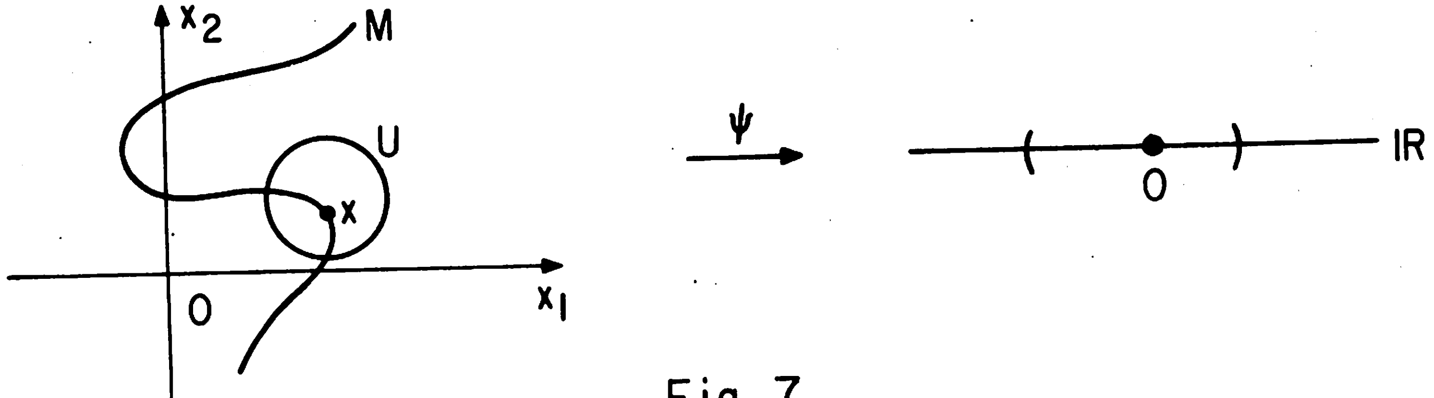


Fig. 7

$$
\begin{tikzcd}
T_{x} M_1 \arrow[r, "(dF)_{x}"] \arrow[d, "(D\psi)_{x}"'] & T_{F(x)} M_2 \\
\mathbb{R}^{m_1} \arrow[r, "(D\phi\circ F\circ\psi^{-1})_{\psi(x)}"'] & \mathbb{R}^{m_2} \arrow[u, "(D\phi^{-1})_{\phi\circ F(x)}"']
\end{tikzcd}
$$

Fig. 8

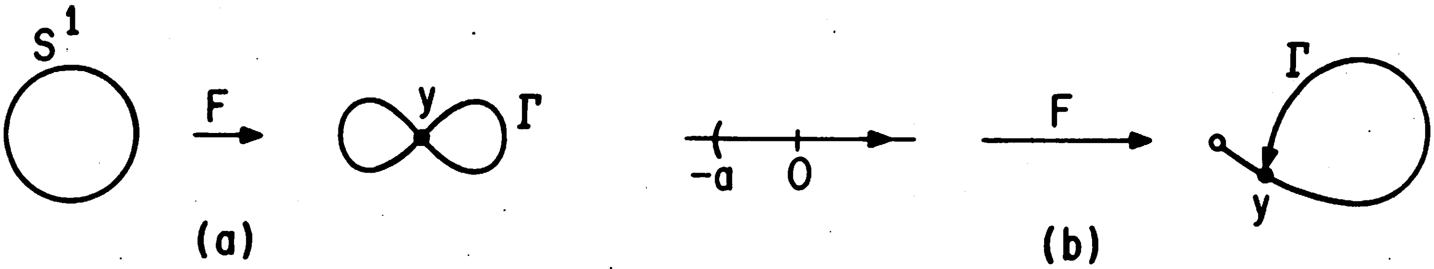


Fig. 9

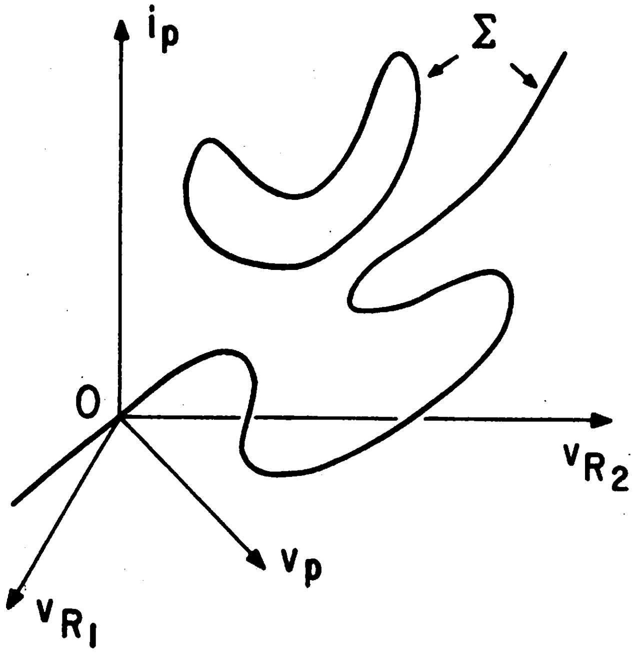


Fig. 10

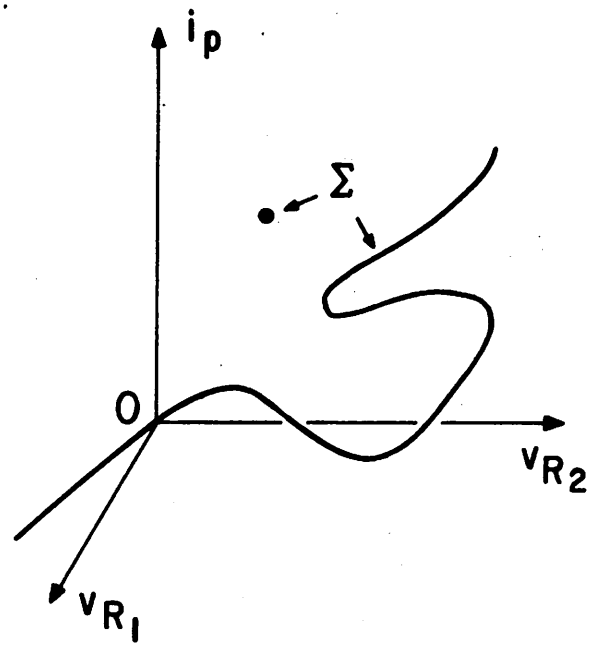
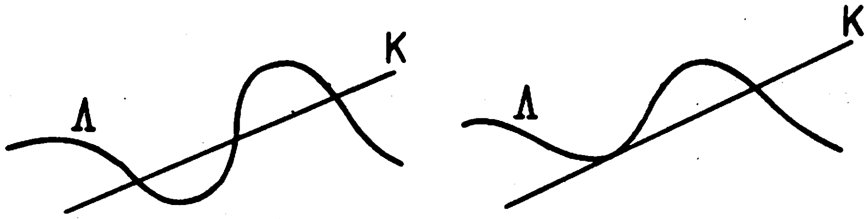


Fig. 11



(a) (b)

Fig. 12

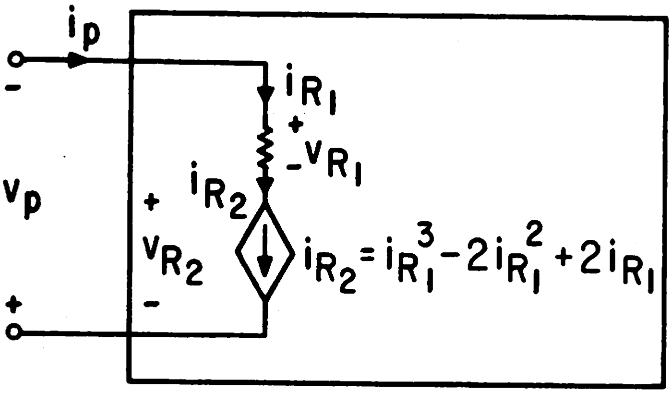


Fig. 13

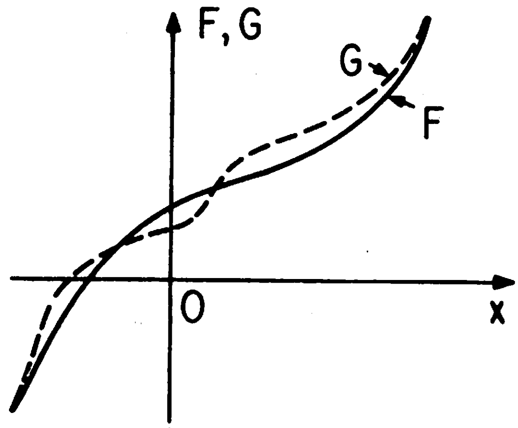


Fig. 14

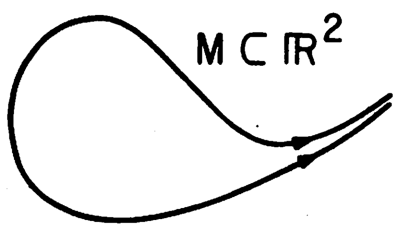


Fig. 15

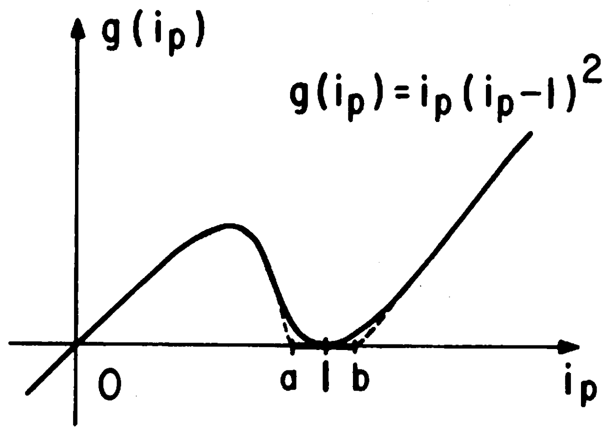


Fig. 16

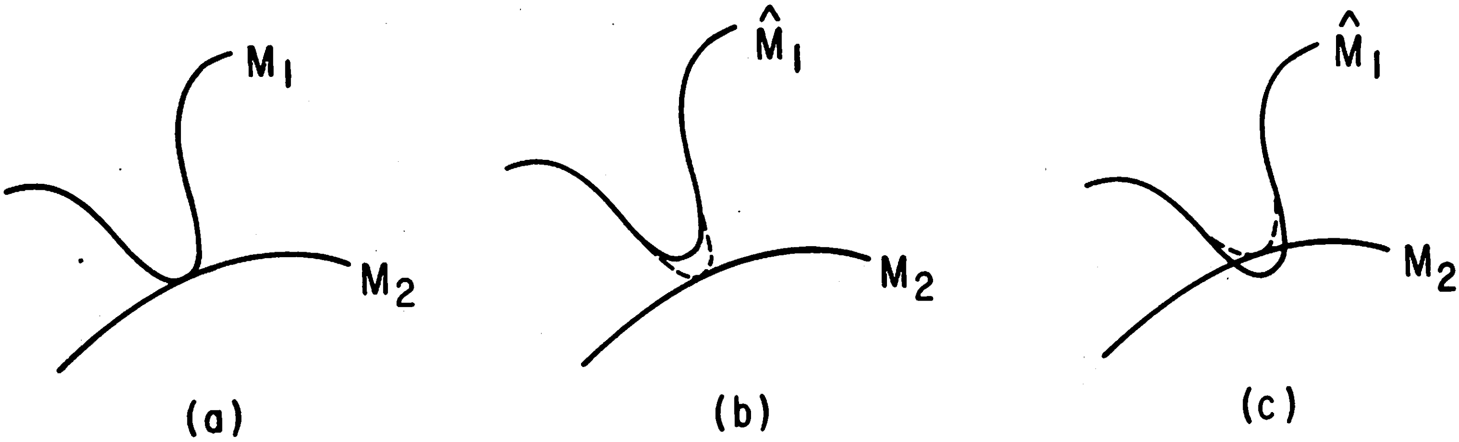


Fig. 17

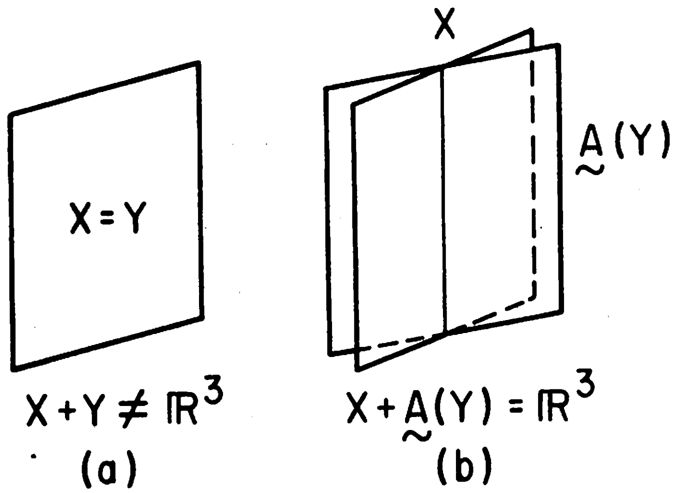


Fig. 18

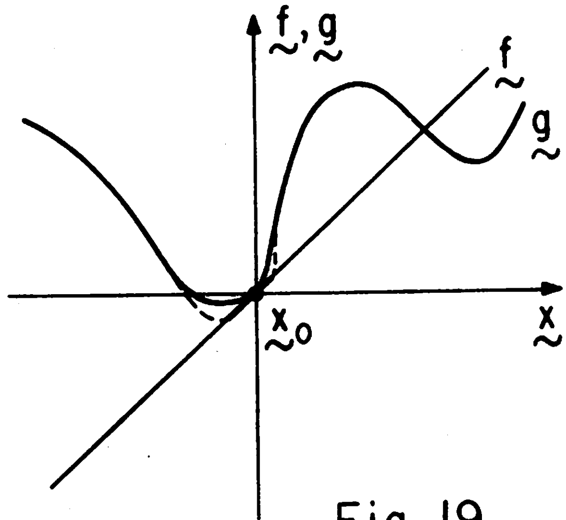


Fig. 19

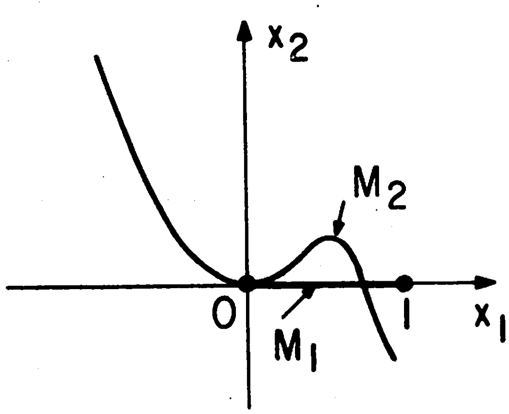


Fig. 20

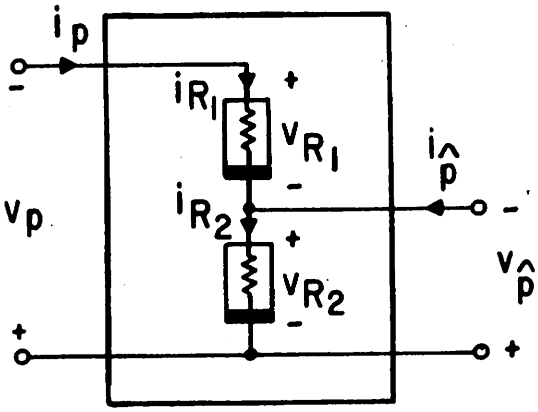


Fig. 21

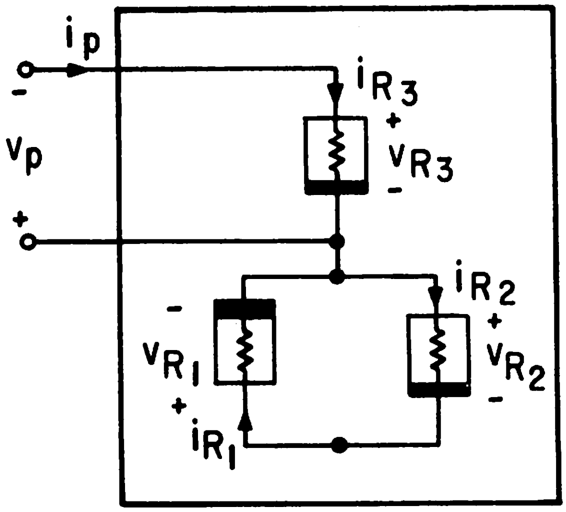


Fig. 22

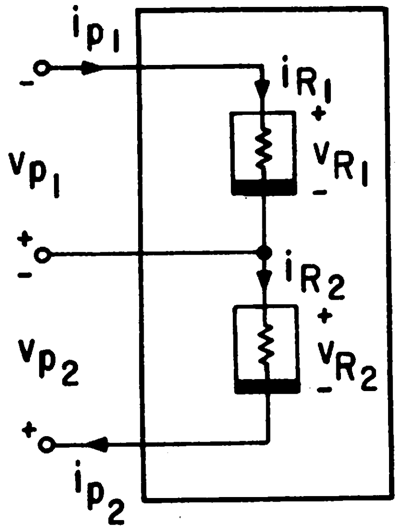


Fig. 23

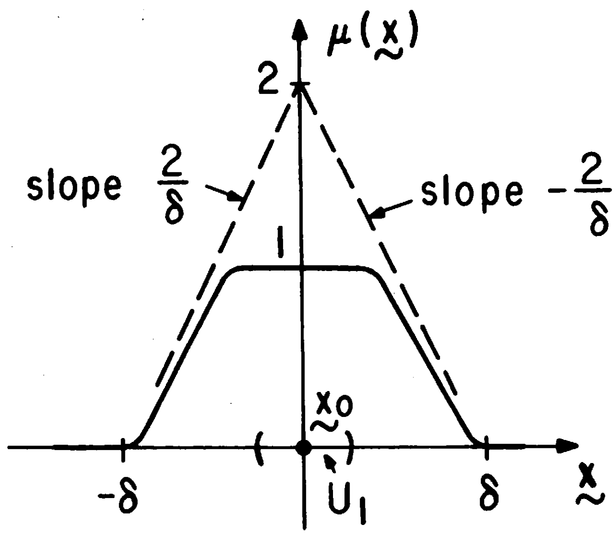


Fig. 24