THE CONSTRUCTION AND EVALUATION OF FUZZY MODELS

by

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1. Introduction

This paper discusses the idea of a fuzzy model, ways in which such models may be constructed and ways in which their performance may be evaluated.

The first part of the paper is concerned with the relevance of fuzziness as a concept in systems modelling. It is argued that, because fuzzy set theory has the ability to handle linguistic information, the most natural definition of a fuzzy model is in terms of finite discrete relations.

The second part of the paper reviews some techniques for model construction. These fall into three broad categories, namely verbalisation, fuzzification and identification. Each is considered in turn and its advantages and disadvantages examined.

The next section considers ways in which the performance of a fuzzy model may be assessed. The idea of model quality is introduced and is seen to be a multi-faceted concept which includes complexity, uncertainty and accuracy.

The main body of the paper concludes with two examples. The first of these is a well studied problem in the system identification field. The second is a problem in river water quality modelling and shows quite clearly the main features of the fuzzy modelling methodology.

Research supported by Naval Electronics Systems Command Contract N00039-76-C-0022.

2. The Fuzzy Model

There are many situations in which our understanding of process behaviour is imprecise. In the steelmaking industry, for example, dynamic models of the basic oxygen furnace rely on a representation of very complex chemical and physical phenonema in terms of nonlinear ordinary differential equations. This may be adequate in a given context but no modeller of this process would argue that it is a true understanding. Similarly, in economics it is possible to construct time series models for the highly inter-connected variables which characterise the system. However, such models are 'notoriously' poor.

The limited performance of the models in these examples, and in other cases as well, may be a result of the "Principle of Incompatibility". This is an intuitive idea introduced by Zadeh[1] which states that as systems become more complex, it becomes increasingly difficult to make statements about them which are both meaningful and precise.

Fuzzy set theory, originated by Zadeh[2] and developed by many others, is a tool for handling imprecision. What it allows the modeller to do is make statements about process behaviour which, although inaccurate in the usual sense, convey an understanding of the basic characteristics.

A distinction should be made though between 'subjective' and 'objective' fuzziness. The former is typified by the steelmaking example. Here the imprecision lies in our inability to gather sufficient information from the process rather in any actual fuzziness in the laws governing its behaviour. In the economic system though it is possible to argue that not only can we not measure all the relevant variables but also that the decisions taken, and hence the laws of economics, are almost certainly governed by fuzzy reasoning.

There are many forms which a fuzzy model could take. It could simply be a static description of relationships between variables. Or it could be a forecasting model for use in economic planning. In this paper the emphasis is on dynamic systems models for use in the control engineering type of application.

One paticularly interesting form is derived from the study of fuzzy numbers (Dubois and Prade[3]). Let $\mathbb{R}$ be the set of real numbers and $\underset{\sim}{\mathbb{R}}$ be the power set, $P(\mathbb{R})$, of strongly convex and normal fuzzy sets in $\mathbb{R}$. A fuzzy system may then be defined as a function

$$f: \underset{\sim}{\mathbb{R}}^n \times \underset{\sim}{\mathbb{R}}^m \times \underset{\sim}{\mathbb{R}}^d \rightarrow \underset{\sim}{\mathbb{R}}^n$$

such that

$$X(t+1) = f(X(t), U(t), P)$$

where
$$X^T(t) = (x_1(t), \ldots, x_n(t))$$
$$U^T(t) = (u_1(t), \ldots, u_m(t))$$

are the state and input vectors of the fuzzy system at time t and

$$P^T = (p_1, \ldots , p_d)$$

is the parameter vector. The vector components x_i, u_i and p_i being elements in $\mathbb{R}$. Thus the function, f, and the parameter vector, P, define a model of the system.

Whilst there are few results using this formulation, it is hoped that concepts in conventional systems theory can be generalised to what might be called a fuzzy systems theory (deGlas[4]).

However, this approach does not explicitly utilise the linguistic properties of fuzzy set theory. One of the reasons for the success of fuzzy set theory in tackling control problems is exactly this parallel with natural language (Tong[5]). It is unfortunate, therefore, that this is lost when pursuing the algebraic notions outlined above.

An alternative is to model the system in terms of linguistic relations. In doing so, the model becomes the dual of a fuzzy logic controller and fits naturally into the theoretical framework of fuzzy control systems described in Tong[6].

Thus, in the remainder of the paper, a fuzzy model is defined as a finite set of linguistic relations, $\{r_i \; ; \; i=1, \ldots ,n\}$, which together form an algorithm, A, for determining the outputs of the process from some finite number of past inputs and past outputs. Usually, of course, the model is to be implemented on a digital computer. In which case, the output space and the input space of the system have to be discretised and will be denoted S and U respectively. The set of numerical definitions of the primary fuzzy sets used in specifying A will be denoted, F, so that the fuzzy model is now defined in terms of a quadruple, (S,U,F,A). Then using the calculus of fuzzy sets it can be rewritten in terms of a finite discrete relation, R, and a composition operator, $\circ$, such that $(S,U,F,A) \rightarrow (R, \circ)$.

3. Model Construction

There appear to be three ways in which information about a systems behaviour may be obtained. They are not mutually exclusive and indeed it is reasonable to suppose that some combination of all three would be the most effective way of constructing a 'good' model.

3.1 Verbalisation

This is the process of recording verbal descriptions and formalising them as fuzzy relations. The descriptions may be given by process operators, plant managers or process technologists. In fact, anyone who has some 'feel' for the process could be considered a possible source of information.

The idea of asking operators to describe their actions is not new and has been well studied by workers in the field of industrial psychology (Bainbridge[7]). There are, however, some severe problems with this approach, principally methodological. It is, for example, difficult to ensure that in observing the operator all significant responses have been recorded (Umbers[8]). Similarly, there is no guarantee that the operators verbal responses are accurate descriptions of his mental model.

Nonetheless, this approach has been successfully used in the study of the Activated Sludge sewage treatment process (Beck, Latten and Tong[9]). In this work the plant manager was given a carefully constructed questionnaire in which he was asked to describe the behaviour of the process under certain hypothetical operating conditions. From his answers it was possible to construct a fuzzy control algorithm which embodied several important feature of the plant's dynamic behaviour.

Much work needs to be done to perfect this technique. At present, there is no systematic way of approaching a given verbal modelling problem although some obvious guidelines may be drawn up. Difficulties, apart from those already mentioned, include selecting the primary sets and the appropriate discretisations.

3.2 Fuzzification

Zadeh's "Extension Principle" [10] can be used to extend the meaning of an ordinary mathematical expression from points to fuzzy sets. In this way any exact relationship known to hold for the process can be transformed into a fuzzy relation.

Thus if 'f' is a scalar function of 'n' variables $x_1, \dots, x_n$ such that $y=f(\underline{x})$ and $\mu_i(x)$ is the membership function associated with the 'ith' variable then

$$\mu(y) = \max_{y=f(\underline{x})} \left[\min_i \left(\mu_i(x) : i=1, \dots, n \right) \right]$$

In practice it is necessary to restrict the values that $\underline{x}$ and y can take to be both finite and discrete. This presents some implementation problems which are basically concerned with the discrete representation of fuzzy sets. Suppose a fuzzy set of temperature is defined on the continuous range $[0, 100^\circ]$, then a finite discrete representation corresponds either to a set of representative points in $[0, 100^\circ]$ or a set of intervals on $[0, 100^\circ]$. The first of these may be termed a 'point - set' and the second an 'interval - set'.

Obviously, there are four combinations of $\underline{x}$ and y representation. However, only one leads to a straightforward computational problem and that is to choose $\underline{x}$ to be a point-set and y an interval-set.

Of course the different representations will give different fuzzifications of the original function, although they may be similar linguistically. Very little practical work has been done on the evaluation of the Extension Principle and it is not at all clear if the choice of fuzzy set representation is of real significance.

3.3 Identification

In many cases, industrial processes for example, it is possible to perform data logging experiments. These produce various amounts of non-fuzzy data relating to input variables and output variables. The generation of fuzzy relational descriptions of this data, and hence the process, is called 'identification'.

A technique for doing this has been reported by Tong[11]. It is essentially a method for testing fuzzy propositions about the process against the data to see if they are 'true'. Those which are then constitute valid fuzzy descriptions for incorporation in the model.

The technique has deficiencies, pricipally because it does not eliminate the need to do some kind of correlation analysis on the input and output variables. However, it remains the only published method for utilising input-output data.

What is required in this area is a theoretical investigation of the relationship between models and data, analagous to that done in conventional system identification (see Eykhoff[12] for example).

4. Model Evaluation

After the model has been constructed it must be tested and evaluated. In the conventional modelling excercise this rarely consists of more than the calculation of some accuracy measure between the model and the data. In the fuzzy case this is inadequate and a more detailed inspection of the model is required.

Usually, several models have been constructed and the modeller wishes to choose the best amongst them. The first requirement is that the models all come from the same 'class'. In the context of the model definition given in section two, this means that S,U and F must be the same. Thus the only thing that varies is A, the set of fuzzy relations.

Of primary importance is model complexity, denoted p_1. Clearly, if the models are the same in all other respects then the one with least complexity is preferred. It is natural to choose as a measure for p_1 the number of linguistic relations which make up A.

Related to complexity is the notion of accuracy. This is problematic in fuzzy models since they generate fuzzy sets as outputs, in contrast to the measured data which is non-fuzzy. Although non-fuzzy values can be generalised to fuzzy singletons, the distance measures usually proposed for fuzzy sets (Kaufmann[13a] for example) are not really appropriate. Furthermore, the model operates with discretised variables and consequently there are bounds on the accuracy that can be achieved.

One way of overcoming these difficulties is to de-fuzzify the outputs of the fuzzy model. Note that whilst this seems appropriate in the current context it is not necessarily a general solution. De-fuzzification may be achieved in several ways and there is no published evidence to suggest that any method is superior. One obvious way is simply to select the value corresponding to the peak in the fuzzy set, averaging when there are several such values. Another is to select the value which divides the area under the membership function curve in half.

Having fixed on a method, two accuracy measures seem appropriate. The first is simply the squared error, such that

$$p_2(i) = (\hat{y}(i) - y(i))^2$$

where $\hat{y}(i)$ is the de-fuzzified model output at the 'ith' data point and $y(i)$ is the 'ith' non-fuzzy observation. This accuracy measure is useful because it is the most commonly employed measure in non-fuzzy modelling. The second accuracy measure is the absolute difference between the discretised de-fuzzified model output and the discretised measurement. That is,

$$p_3(i) = \left| D\left[\hat{y}(i)\right] - D\left[y(i)\right] \right|$$

where $D[.]$ denotes a discretisation mapping and is such that $D: y \rightarrow \{ j \; ; \; j=1, \ldots , L \}$ where L is the number of discretisation points (or levels) for the output space. The value of this accuracy measure is that it does not discriminate against the inherent inaccuracy introduced by using dicretised fuzzy sets.

A third, and final, aspect of model evaluation is the notion of uncertainty. It is obviously the case that output sets with different membership functions can generate the same de-fuzzified output but, intuitively, the 'sharper' the output set the better. The notions of non-probalistic entropy developed by DeLuca and Termini[13] are of some use in this context. However, a simpler and more direct measure of uncertainty is to set

$$p_4(i) = 1 - \mu[\hat{y}(i)]$$

where $\mu[\cdot]$ is the membership function of the model output set.

Measures $p_2(i)$, $p_3(i)$ and $p_4(i)$ are for single data points only and thus for a given data set of 'N' points they become

$$p_2 = \frac{1}{N}\sum_{i=1}^{N} p_2(i) \qquad p_3 = \frac{1}{N}\sum_{i=1}^{N} p_3(i) \qquad p_4 = \frac{1}{N}\sum_{i=1}^{N} p_4(i)$$

So any model can be characterised in relation to any set of data by means of complexity, p_1, accuracy, p_2 and p_3, and uncertainty, p_4.

In general, the more complex the model the higher the accuracy and the smaller the uncertainty. However, it may not be a trivial task to impose an ordering on the models on the basis of such measures. The trade-offs between complexity and accuracy and between complexity and uncertainty may require the modeller to use some external criteria in order to select a 'best' model.

5. Application Studies

The following two examples illustrate some of the characteristics of fuzzy models and emphasise the points made above. The first example is a model of Box and Jenkins[14] gas furnace data and shows how effective the model construction techniques can be. The second example consists of two models of Beck's [15] river water quality data. The first is simple and appears satisfactory, but has some unexpected consequences. The second is much more complex but overcomes the deficiencies of the first.

5.1 The gas furnace data of Box and Jenkins

The data of Box and Jenkins is extremely well known and is often used as a standard test for identification techniques. It provides, therefore, a useful starting point for assessment of the methods proposed in section three. The data consists of 296 pairs of input-output measurements. The input is gas flow rate into the furnace, the output is the CO_2 concentration in the outlet gases and the sampling interval is nine seconds.

Using a combination of the techniques described earlier, the fuzzy model shown in Figure 1 was obtained. The table, which in this case is 'complete' since all the cells are filled, is often called a 'transition table' and should be interpreted as follows.

The model is a relation between gas flow four sampling intervals ago, CO_2 concentration one sampling interval ago and current CO_2 concentration. That is, $y(t) = f(y(t-1), u(t-4))$. The row indices are mnemonics for primary fuzzy sets of gas flow. The column indices are mnemonics for primary fuzzy sets of CO_2 concentration as are the entries in the table. Thus each entry forms an 'elementary' relation of the form

$$\text{IF } \{u(t-4) \text{ is NQ1} \;\&\; y(t-1) \text{ is C3}\} \text{ THEN } \{y(t) \text{ is C4}\}$$

where & corresponds to conjunction.

Interpretation of implication is, of course, a subject of debate but in this example, and in the following one, a cartesian product form is used (ie. $A \Rightarrow B \triangleq A \times B$). The reason for this is simply that it is unreasonable for the modeller to infer anything at all if proposition A is not true. Thus the transition table can be thought of as a tabulation of a fuzzy function.

To test the model it is necessary to generate one-step-ahead, OSA, predictions from the data. This is straightforward. The procedure is to take the non-fuzzy measurements corresponding to y(t-1) and u(t-4), construct the fuzzy singleton $\underset{\sim}{y(t-1)\&u(t-4)}$, calculate $\underset{\sim}{\hat{y}(t)}=\underset{\sim}{y(t-1)\&u(t-4)} \circ R$ and then de-fuzzify to give $\hat{y}(t)$.

The gas furnace model performs very well indeed. The OSA predictions are shown in Figure 2 (continuous line) together with the measured output data (crosses). The complexity is equal to the number of compound relations (ie. relations which fill more than one cell in the table) and here is equal to 19. The other measures are

$$p_2 = 0.469 \qquad p_3 = 0.558 \qquad p_4 = 0.220$$

so that the discretised de-fuzzified model output coincides with the discretised measured output about half the time. The average value of the membership function at the output value is 0.780.

For comparison, the OSA predictions of Box and Jenkins deterministic model are shown in Figure 3. Their model has the form

$$y(t) = - \frac{0.53 + 0.37B + 0.51B^2}{1.00 - 0.57B - 0.01B^2} \cdot u(t-3)$$

where B is the backward time shift operator, and gives an equivalent p_2 measure of 0.202.

The most interesting thing about the fuzzy model is that it fits the last section of the data better than Box and Jenkins model. The data is known to exhibit some non-stationarity over the last forty samples or so (Young et al[16]) and it's clear that the inherent non-linearity of the fuzzy model gives it the ability to compensate for this. Inspection of Figure 1 shows that the transition table has a broadly 'monotonic' form except for a few cells. Analysis of the model shows that those cells marked with '*' are only in operation during the last portion of the data and thus account for the improved performance.

5.2 The river water quality data of Beck

The river can be thought of as a five-input two-output process, see Figure 4, with biochemical oxygen demand, BOD, and dissolved oxygen, DO, used as measures of water quality. The data consists of 81 consecutive daily sampled values of upstream BOD, upstream DO, volumetric flow rate, river water temperature, hours of sunlight incident on the river during each day, the downstream BOD and the downstream DO.

A preliminary analysis of the data using only identification suggests that downstream DO is a function only of itself and downstream BOD. It also suggests that downstream BOD is a function of itself and upstream BOD. The model developed is shown in Figure 5 and is essentially two single-input single-output processes in series.

The OSA predictions are shown in Figure 6 and the quality measures are

DO:	$p_1 = 9$	$p_2 = 0.4204$	$p_3 = 1.2468$	$p_4 = 0.2727$
BOD:	$p_1 = 8$	$p_2 = 0.5127$	$p_3 = 1.1606$	$p_4 = 0.3182$

For comparison, Beck's model gives the output shown in Figure 7. His differential equation model has equivalent accuracy measure, p_2, of DO: 0.4313 and BOD: 1.0364.

On the whole, the fuzzy model does rather well. But inspection of the transition tables in Figure 5 shows that essentially the model is "next output equals current output". Thus if the model is run as a pure predictor (ie. using defuzzified model outputs instead of measured outputs) the results are as given in Figure 8. The quality measures are now

DO:	$p_1 = 9$	$p_2 = 1.1648$	$p_3 = 1.8276$	$p_4 = 0.3039$
BOD:	$p_1 = 8$	$p_2 = 0.9782$	$p_3 = 1.6646$	$p_4 = 0.3169$

and it's obvious that the model is really a crude approximation to the river's dynamic behaviour.

The reasons for this become clear after a study of Beck's model. This shows that sunlight, or rather a weighted moving average of sunlight, plays an important role. In particular, it is largely responsible for the peaks which occur in both the downstream DO and Downstream BOD at about forty days.

A second model was developed accordingly. It is much more complicated, having a structure such that downstream DO is a function of itself, downstream BOD, upstream BOD and sunlight, and such that downstream BOD is a function of itself, upstream BOD and sunlight. The model may not be conveniently described by a transition table, but consists of 20 rules describing downstream DO behaviour and 24 rules describing downstream BOD behaviour.

The OSA predictions for this second model are shown in Figure 9. The quality measures are

DO: $p_1 = 20$ $p_2 = 0.6155$ $p_3 = 1.2756$ $p_4 = 0.5455$

BOD: $p_1 = 24$ $p_2 = 0.5912$ $p_3 = 1.2306$ $p_4 = 0.4727$

Thus despite the large increase in complexity, both accuracy and uncertainty are worse! However, using the model in a purely predictive mode gives the output shown in Figure 10 and the quality measures

DO: $p_1 = 20$ $p_2 = 0.9306$ $p_3 = 1.7148$ $p_4 = 0.5922$

BOD: $p_1 = 24$ $p_2 = 0.9420$ $p_3 = 1.5526$ $p_4 = 0.5182$

So, whilst these are still poor, the second model is a better predictor than the first.

Summarising these results, it is clear that neither of the fuzzy models is as good as Beck's model. However, it would be surprising if they were. What they do show, though, is an ability to capture the underlying behaviour of the river. It is also the case that neither model is 'best' in any absolute sense. It depends on the proposed application. Thus the first model might be useful in controller design whilst the second might serve as a qualitative description for non-specialist personnel.

6. Conclusions

Fuzzy models can obviously be made to work, and work very well indeed. However its important to define those situations in which a fuzzy model is appropriate.

The big advantage of a fuzzy model is that it is relatively simple to construct and is in itself quite a simple stucture. It does not require the modeller to have a deep mathematical insight but relys more on intuition and experience of the process. Its greatest value must be, therefore, in those areas where such qualitative process knowledge is predominant and essential for understanding.

It seems likely that in 'hard' technological areas, where precision is often an over-riding consideration, fuzzy models would be of most value as devices for assessing approximate behaviour rather than as tools for detailed engineering design. However, in 'softer' areas, such as water quality control, where the goals are usually less clearly defined, fuzzy models may be useful in a wider range of tasks.

This paper has not been concerned with the use of fuzzy models in socio-economic systems, although the work of Wenstop[17] and Kickert[18] gives indications of the difficulties. It does seem that for the ideas discussed in the preceding sections to be generalised to such systems a considerable amount of work needs to be done. In particular, the relationship between data set size and model complexity needs to be examined, as does the the concept of 'model purpose'. Perhaps a combination of fuzzy and non-fuzzy models would allow broad generalisations to be made about systems without sacrificing accuracy where this is required?

The overall conclusion must be that, whilst fuzzy models can be successfully constructed, the overall concept needs a considerably more detailed investigation before its true worth can be evaluated.

Acknowledgement

The author would like to acknowledge the financial support provided by a NATO/SRC Postdoctoral Research Fellowship.

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y(t-1)

u(t-4)	C1	C2	C3	C4	C5	C6
NQ3	C4	C4	C5	C5	C6	C6
NQ2	C3	C3	C4	C4	C5	C6
NQ1	C3	C3	C4	C4	C5	C5
ZEQ	C2	C3	C4	C4	C4	C6*
PQ1	C2	C2	C3	C4	C5	C6*
PQ2	C1	C2	C2	C2	C4*	C4*
PQ3	C1	C1	C2	C2	C3	C3

y(t)

Figure 1. Complete fuzzy model of the gasfurnace

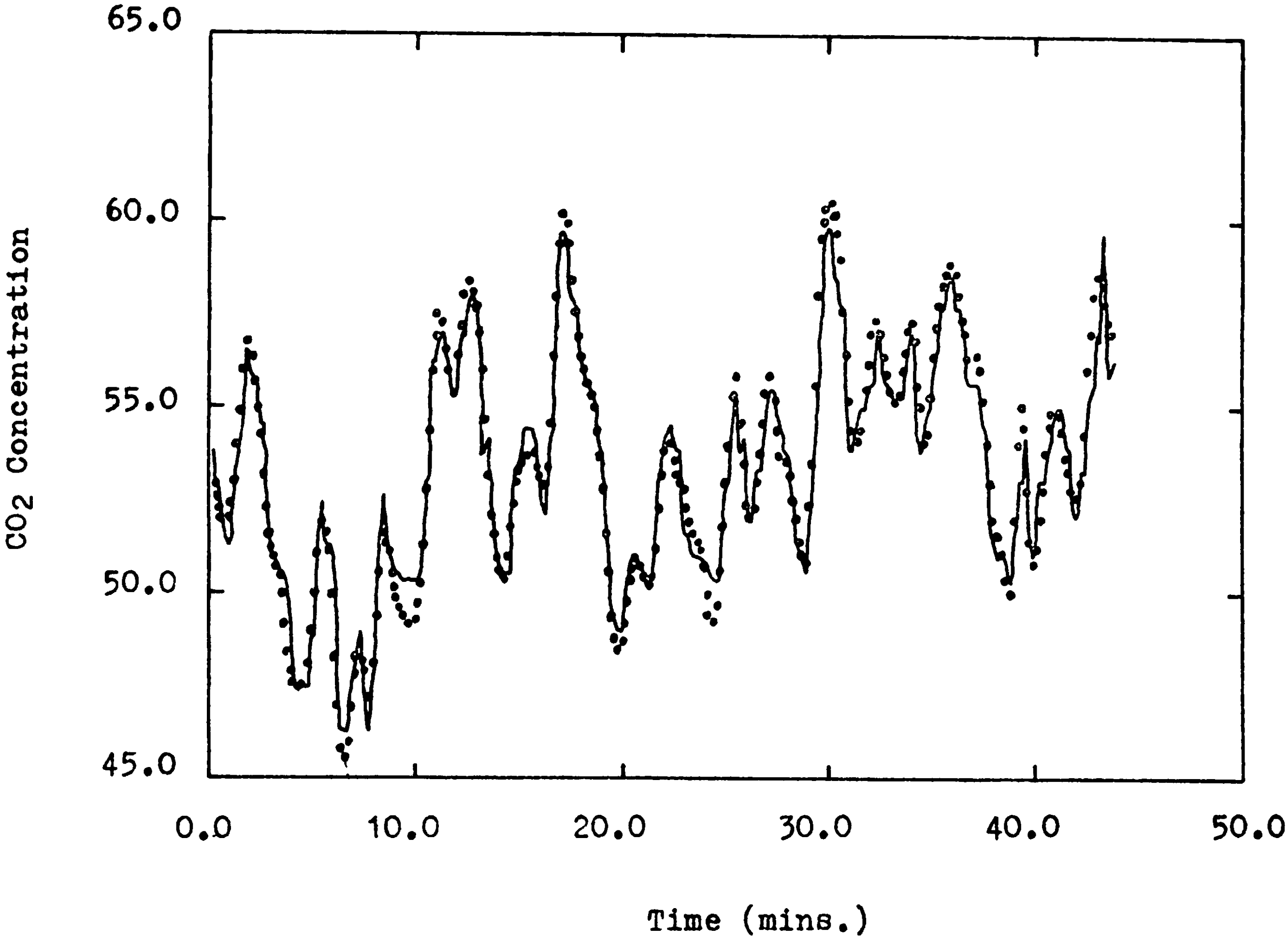

Figure 2. OSA Predictions for the complete model

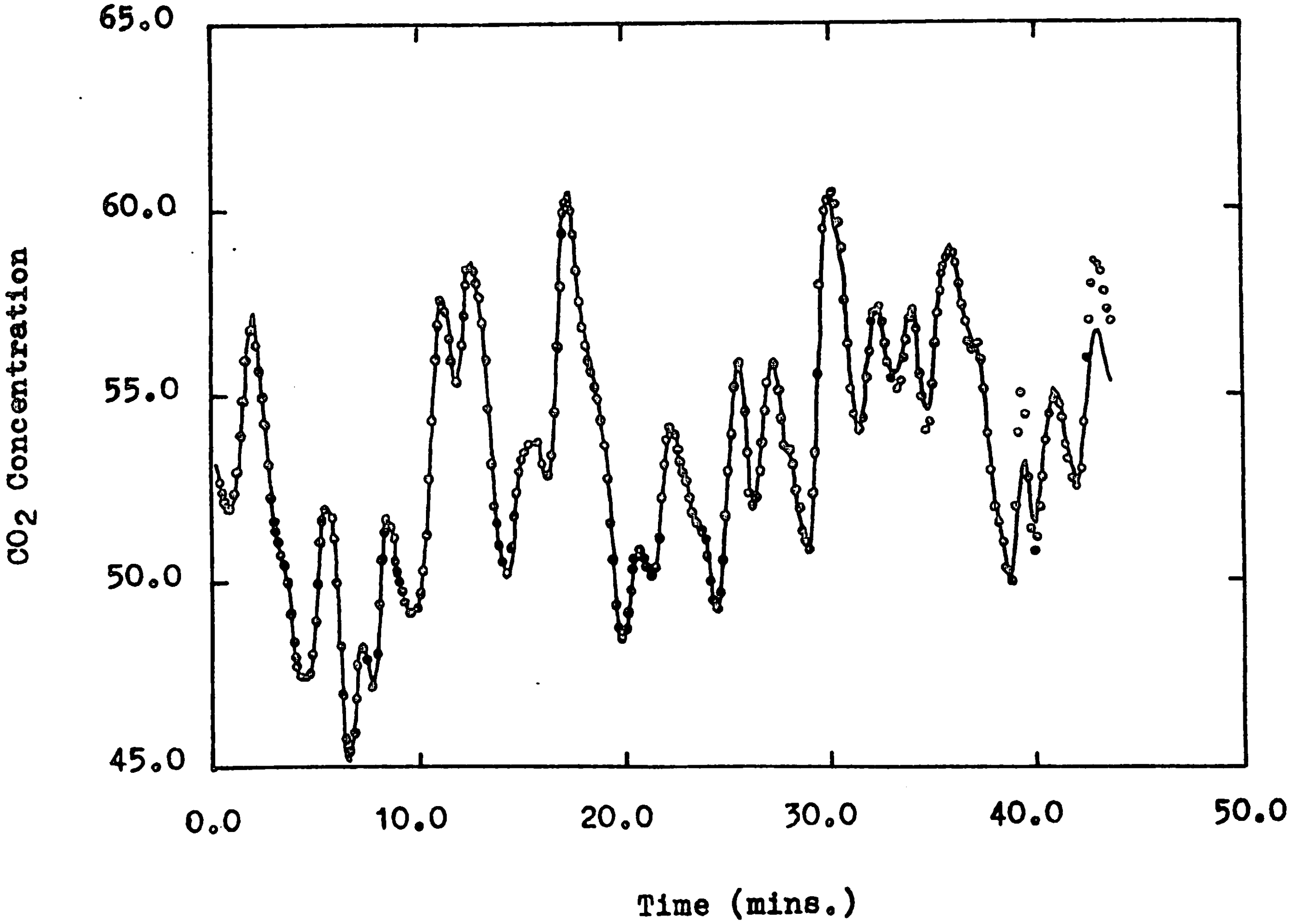

Figure 3. OSA Predictions for Box and Jenkins' model

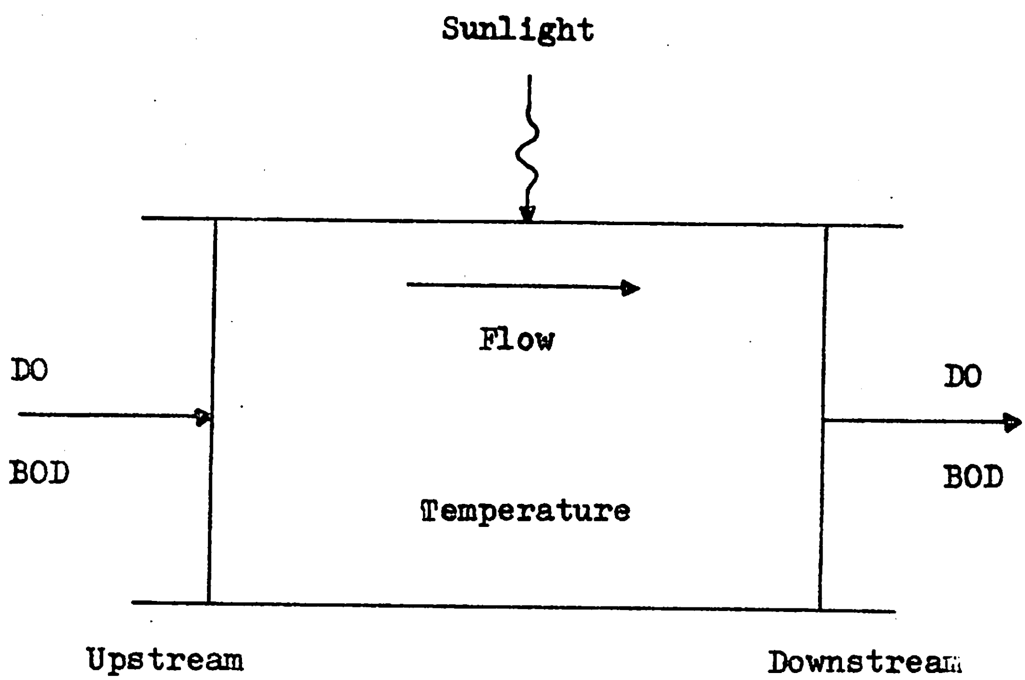

Figure 4. Schematic of the river system

Downstream DO(t-1)

Downstream BOD(t-3)	DO1	DO2	DO3	DO4	DO5
BOD1		DO2	DO3		
BOD2	DO2	DO2	DO3	DO5	DO4
BOD3	DO1	DO2	DO3	DO5	DO5
BOD4	DO2	DO2	DO3	DO4	
BOD5	DO1	DO2	DO2		

Downstream DO(t)

Downstream BOD(t-1)

Upstream BOD(t-3)	BOD1	BOD2	BOD3	BOD4	BOD5
BOD1	BOD1	BOD2			
BOD2	BOD2	BOD2	BOD3		
BOD3		BOD3	BOD3	BOD4	BOD5
BOD4			BOD3	BOD4	BOD4
BOD5		BOD2	BOD3	BOD4	BOD4

Downstream BOD(t)

Figure 5. First fuzzy model of the river

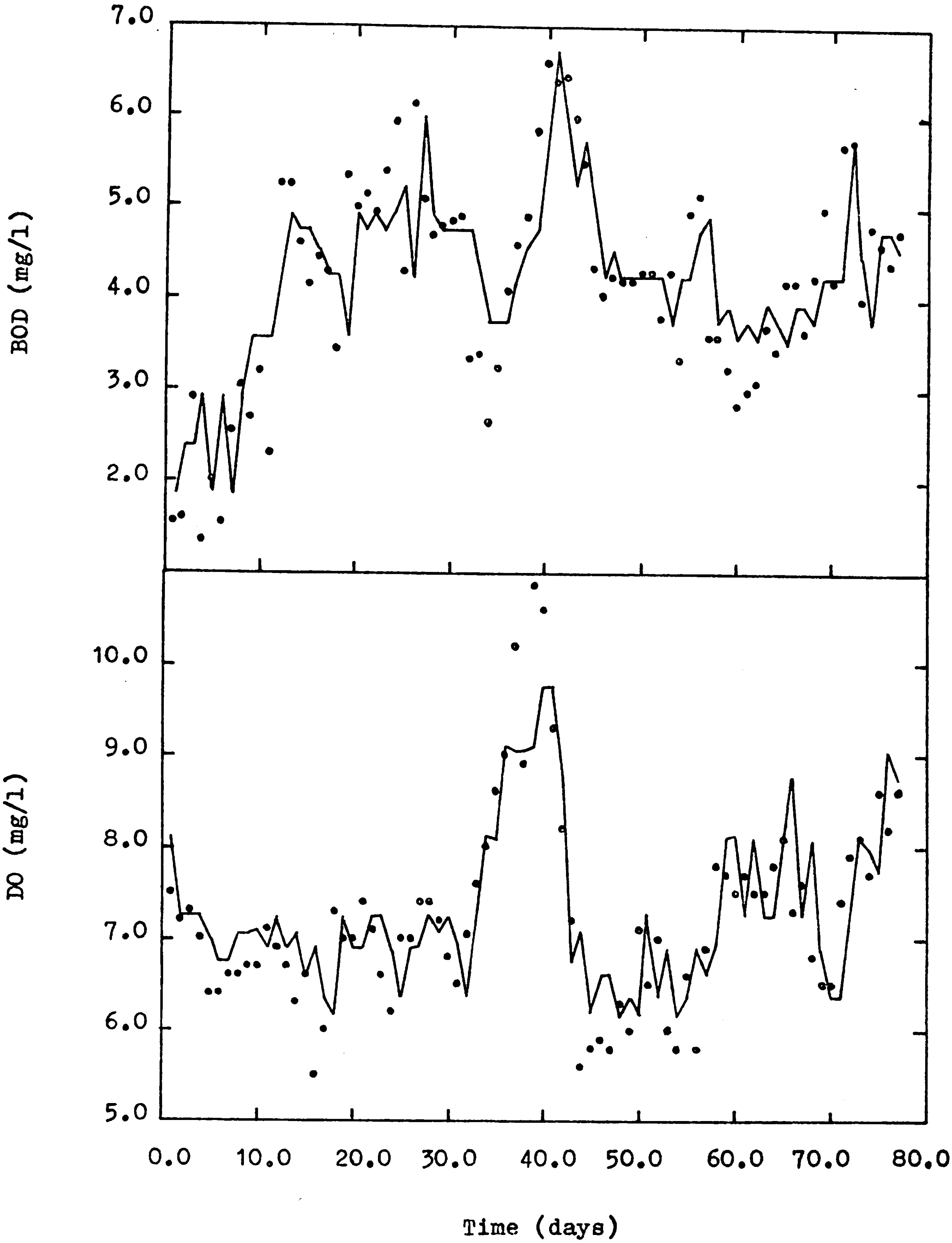

Figure 6. OSA Predictions for the first model

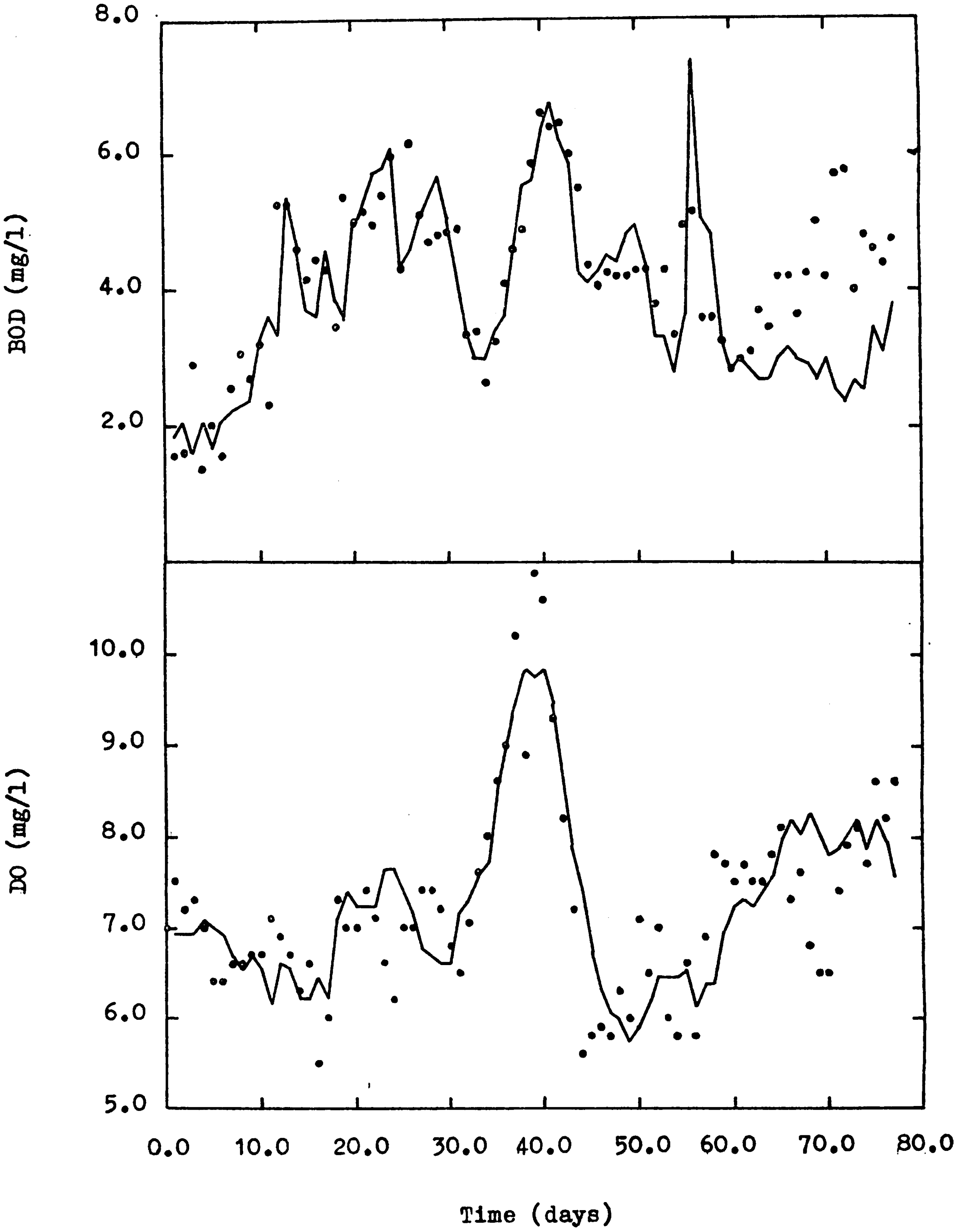

Figure 7. Response of Beck's model

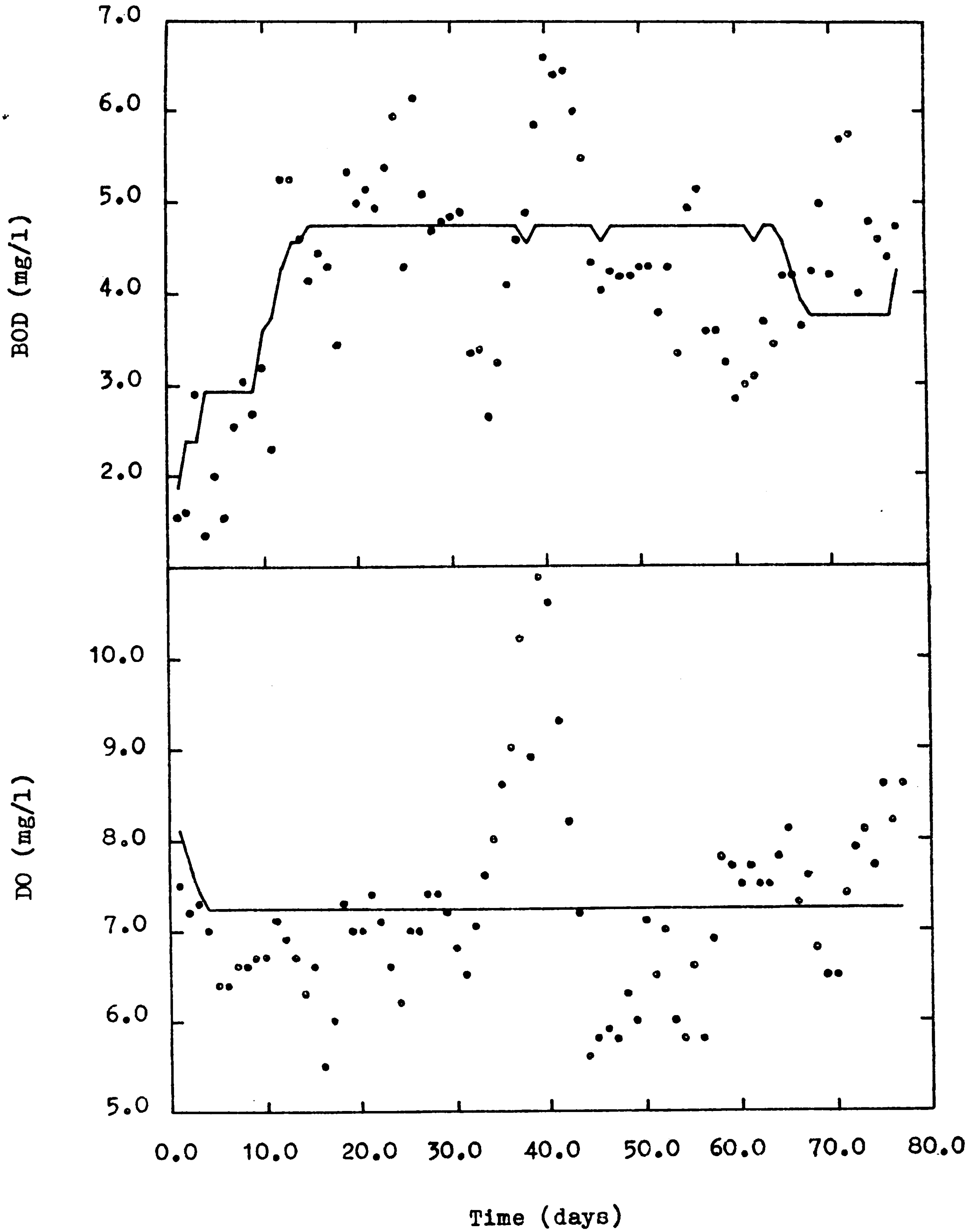

Figure 8. Pure predictions for the first model

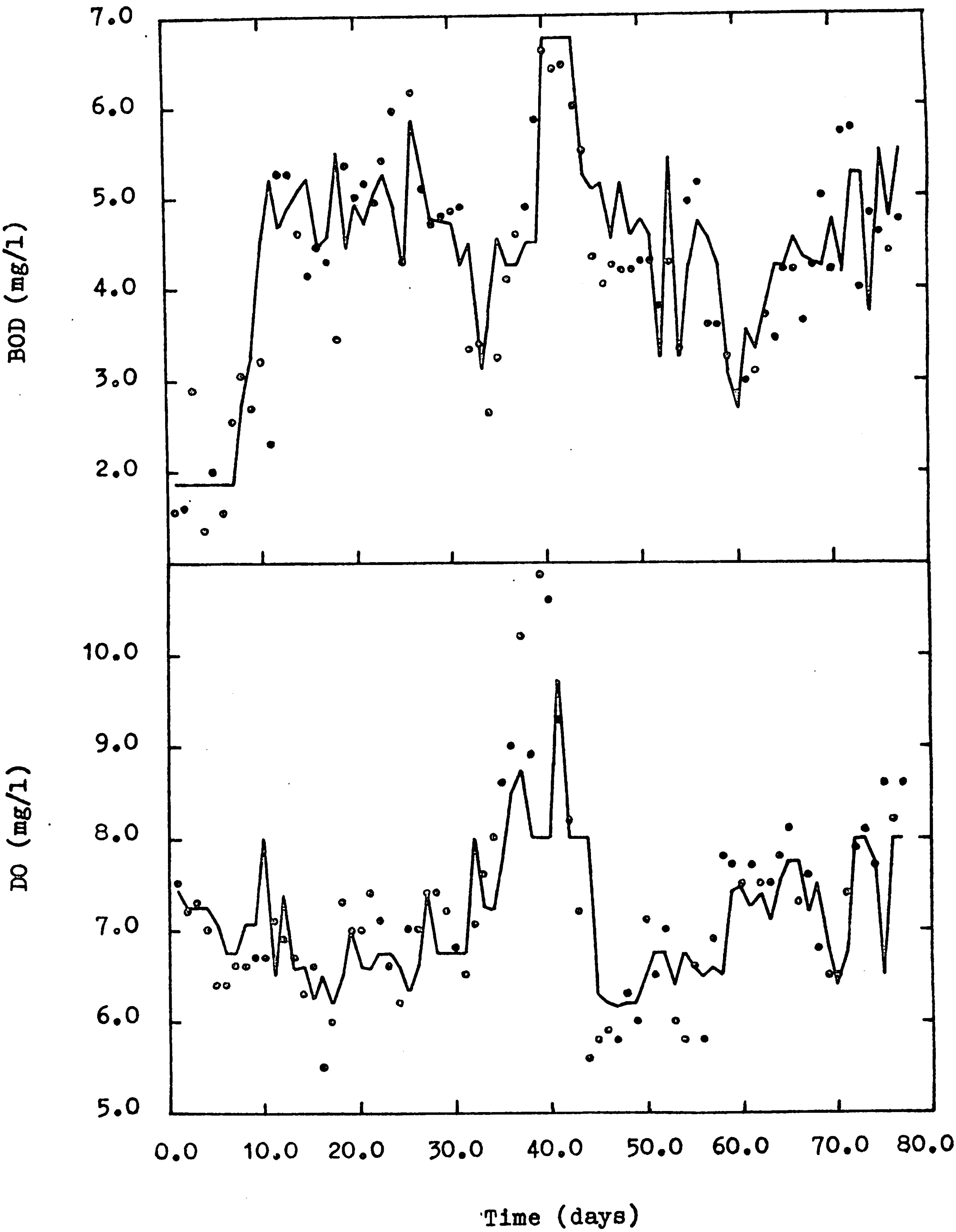

Figure 9. OSA Predictions for the second model

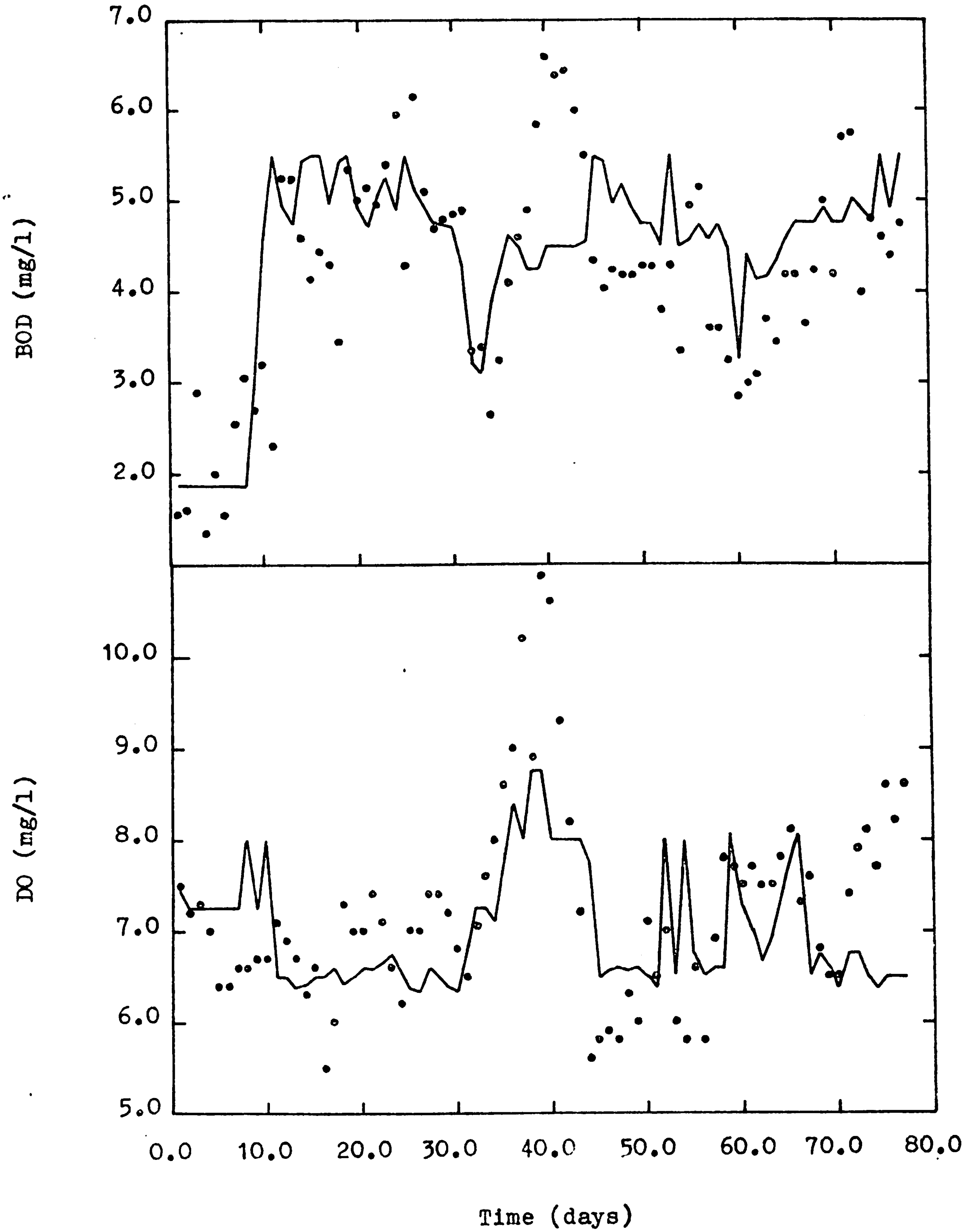

Figure 10. Pure predictions for the second model