ON THE GENERALIZED NYQUIST STABILITY CRITERION

by

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Abstract

This paper starts with a straightforward self-contained treatment of the generalized Nyquist stability criterion based on the eigenloci of the open-loop transfer function matrix. The derivation is straightforward and does not invoke algebraic function theory, branch cuts and Riemann surfaces. A similar criterion is derived for the distributed case where the open-loop transfer function matrices belong to the recently proposed algebra $\mathcal{B}(\sigma_o)^{m\times m}$ [9,14].

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I. Introduction

The Nyquist stability criterion has been generalized in several ways for the multi-input multi-output case (see e.g. [1] to [7]). The concept of plotting eigenloci of the open-loop transfer function matrix is particularly attractive because it allows one to check the closed-loop stability for a family of gain parameters by inspection. A rigorous theory of such test was first given in [6] where the rational transfer function case was treated in detail and much use was made of algebraic function theory, branch cuts, etc. [6]. Later MacFarlane et al. (see [7]) used Riemann surfaces to discuss this stability test.

In this paper we present first a rigorous, straighforward and self-contained derivation of the stability test based on eigenloci without using either branch-cuts or Riemann surfaces; only standard facts of analytic function theory are used (e.g. as in Chap IX of [8]). Second, we derive the stability test for distributed systems; more precisely, for those plants whose transfer function matrices belong to the recently proposed algebra $\mathcal{B}(\sigma_o)^{mxm}$ ([9,14]).

Section II derives the generalized Nyquist stability criterion for the lumped case. Section III develops the counterpart of section II for the distributed case. To our knowledge this is the first treatment of a Nyquist test based on eigenloci which considers unstable distributed open-loop transfer functions. The recent summary [17] considers L_2-stable transfer functions.

Preliminaries

Let $\mathbb{R}$ ($\mathbb{C}$) denote the field of real (complex, resp.) numbers. Let $\bar{\mathbb{C}}$ denote $\mathbb{C} \cup \{\infty\}$. Let $\mathbb{R}_+$ denote the non-negative real line. Let $\mathbb{C}_{\sigma+}$ ($\mathring{\mathbb{C}}_{\sigma-}$) denote the closed-half complex plane $\{s \in \mathbb{C} | \text{Re } s \geq \sigma\}$ (open-half complex plane $\{s \in \mathbb{C} | \text{Re } s < \sigma\}$). $\mathbb{C}_{o+}$, $\mathring{\mathbb{C}}_{o-}$ are abbreviated as $\mathbb{C}_+$, $\mathring{\mathbb{C}}_-$, respectively. Let $\mathbb{R}[s]$ ($\mathbb{R}(s)$) be the set of all polynomials (rational functions, resp.) in s with real coeffients. Let $\mathcal{Z}[\phi(s)]$ denote the set of zeros of the polynomial $\phi(s) \in \mathbb{R}[s]$. Let $\mathcal{P}[\psi(s)]$ denote the set of poles of the rational functions $\psi(s) \in \mathbb{R}(s)$. Let $G(s) \in \mathbb{R}(s)^{pxq}$; $N_r D_r^{-1}$ is said to be a right coprime factorization of G(s) iff $G(s) = N_r D_r^{-1}$ and (N_r, D_r) is right-coprime [4]. For $\sigma \in \mathbb{R}$, $L_{1,\sigma}$ denote the set $\{f(\cdot) | f(\cdot) : \mathbb{R}_+ \to \mathbb{C}, \int_0^\infty |f(t)| e^{-\sigma t} dt < \infty\}$. Let $\sigma \in \mathbb{R}$, the convolution algebra $\mathcal{A}(\sigma)$ consists of the elements of the form

$$f(t) = \begin{cases} 0 & , t < 0 \\ f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t - t_i), & t \geq 0 \end{cases}$$

where (i) $f_a(\cdot) \in L_{1,\sigma}$; (ii) $t_o = 0$ and $t_i > 0$, for i = 1,2,...; (iii) $f_i \in \mathbb{C}$ and $\delta(\cdot - t_i)$ is the Dirac delta distribution applied at t_i; (iv) $\sum_{i=0}^{\infty} |f_i| e^{-\sigma t_i} < \infty$. $\mathcal{A}(0)$ is abbreviated as $\mathcal{A}$. $f(\cdot)$ is said to belong to $\mathcal{A}_-(\sigma)$ iff there exists $\sigma_1 \in \mathbb{R}$, $\sigma_1 < \sigma$ such that $f(\cdot) \in \mathcal{A}(\sigma_1)$. Let $\hat{f}$ denote the Laplace transform of $f(\cdot)$. Let $\hat{\mathcal{A}}(\sigma)$ denote the set $\{\hat{f} | f \in \mathcal{A}(\sigma)\}$. Let $\hat{\mathcal{A}}_-^\infty(\sigma)$ denote the set $\{\hat{f} | \hat{f} \in \hat{\mathcal{A}}_-(\sigma)$, $\hat{f}$ is bounded away from zero at infinity in $\mathbb{C}_{\sigma+}\}$. Let $\mathcal{B}(\sigma)$ be the convolution algebra corresponding to the pointwise product algebra $\hat{\mathcal{B}}(\sigma) = [\hat{\mathcal{A}}_-(\sigma)][\hat{\mathcal{A}}_-^\infty(\sigma)]^{-1}$, i.e. $\hat{\mathcal{B}}(\sigma)$ is the algebra of quotients $\hat{f} = \hat{n}/\hat{d}$ with $\hat{n} \in \hat{\mathcal{A}}_-(\sigma)$ and $\hat{d} \in \hat{\mathcal{A}}_-^\infty(\sigma)$ [9,14].

II. Lumped case:

We consider the feedback system S shown in Fig. 1 where

$$\hat{G}(s) \in \mathbb{R}(s)^{mxm} \text{ is proper;} \tag{1}$$

$$k > 0. \tag{2}$$

Let $\hat{N}_r \hat{D}_r^{-1}$ be a right coprime factorization of $\hat{G}(s)$, then the closed-loop system transfer function matrix $r \mapsto y$ is given by

$$\hat{H}_{yr}(s) := k\hat{G}(s)[I + k\hat{G}(s)]^{-1} = k\hat{N}_r(s)[\hat{D}_r(s) + k\hat{N}_r(s)]^{-1} \tag{3}$$

Since $\hat{N}_r$ and $\hat{D}_r+k\hat{N}_r$ are also right coprime, $\det(\hat{D}_r+k\hat{N}_r)$ is the closed-loop characteristic polynomial. Consequently the closed-loop system is exp. stable if and only if $\mathcal{Z}[\det(\hat{D}_r+k\hat{N}_r)] \subset \mathring{\mathbb{C}}_-$ $\hat{H}_{yr}$ and $\hat{H}_{yr}$ is proper. (Since $\hat{G}$ is proper, $\hat{H}_{yr}$ is proper if and only if $\det[I+k\hat{G}(\infty)] \neq 0$; the latter condition will be guaranteed by the condition (i) of Theorems L1 and L3). By substituting $\hat{G}(s) = \hat{N}_r D_r^{-1}$, we have

$$\det[I + k\hat{G}(s)] = \frac{\det[\hat{D}_r+k\hat{N}_r]}{\det D_r(s)} \tag{4}$$

Since $\det[I + k\hat{G}(s)] \in \mathbb{R}(s)$ is analytic on $\mathbb{C}$, except at the poles of $\hat{G}(s)$, we can apply the argument principle [8,p.246-247] to $\det[I + k\hat{G}(s)]$ to determine whether $\mathcal{Z}[\det(\hat{D}_r+k\hat{N}_r)] \subset \mathring{\mathbb{C}}_-$. For this purpose, we construct the Nyquist path $\bar{N}_\infty$ as shown in Fig. 2,(1) where $\bar{N}_\infty$ includes the point $+j\infty$ and $-j\infty$ (the ε-indentations guarantee that $\hat{G}(s)$ is analytic on $\bar{N}_\infty$). The following result is well-known (see e.g. [10,p.141]).

Theorem L1 (Stability theorem based on $\det[I + k\hat{G}(s)]$: Lumped case)

Let $\bar{N}_\infty$ be defined as above. Let the feedback system S of Fig. 1 satisfy eqns. (1) and (2). The closed-loop system S is exp. stable

$\Leftrightarrow$
(i) $\det[I + k\hat{G}(s)] \neq 0, \quad \forall s \in \bar{N}_\infty$
(ii) $\det[I + k\hat{G}(s)]\big|_{s\in\bar{N}_\infty}$ encircles the origin p_+° times in the counterclockwise sense, where p_+° denotes the number of $\mathbb{C}_+$-zeros of the open-loop system characteristic polynomial, counting multiplicities. ¤

Remarks L1: (a) In the single-input, single-output case, i.e., m=1, Theorem L1 reduces to the classical Nyquist stability criterion (see e.g. [11,p.324]).

(1) The ε-indentations must be in $\mathbb{C}_-$ since for the multi-input multi-output case, the closed-loop system and the open-loop system may have some common $j\omega$-axis poles, e.g. if $\hat{G}(s)$ and k are such that $I + k\hat{G}(s) = \begin{bmatrix} \frac{s+1}{s} & 0 \\ 0 & \frac{s}{s+1} \end{bmatrix}$, then s=0 is a open-loop system pole as well as a closed-loop pole.

(b) Technically, Theorem L1 follows directly from the argument principle together with the fact that $\det[I + k\hat{G}(s)]$ tends to a constant as $|s| \to \infty$.

A great virtue of the classical Nyquist criterion is that it checks the closed-loop stability for any $k > 0$ by inspection of the Nyquist diagram of $\hat{G}(s)$, whereas Theorem L1 requires a plot of $\det[I + k\hat{G}(s)]$ for each $k > 0$ of interest which is very inconvenient. Our objective is to derive straightforwardly a generalization of the classical Nyquist stability criterion for multi-input, multi-output systems which allows us to test the closed-loop system stability for any $k > 0$ by inspection.

Now, for each $s \in \bar{N}_\infty$, $\hat{G}(s)$ is a well-defined $\mathbb{C}^{m\times m}$ matrix; thus for each $s \in \bar{N}_\infty$, we have m (not necessarily distinct) eigenvalues of $\hat{G}(s)$, $\lambda_1(s), \lambda_2(s), \ldots, \lambda_m(s)$, which satisfy

$$\det[\lambda_i(s)I - \hat{G}(s)] = 0, \quad i = 1,2,\ldots,m \tag{5}$$

Note that

$$\det[\lambda I - \hat{G}(s)] = \lambda^m + g_1(s)\lambda^{m-1} + \ldots + g_m(s)$$

where the $g_i(s)$'s are proper rational functions. Hence we can write

$$\det[\lambda I - \hat{G}(s)] = \frac{1}{\beta_o(s)}[\beta_o(s)\lambda^m + \beta_1(s)\lambda^{m-1} + \ldots + \beta_m(s)]$$

$$=: \frac{1}{\beta_o(s)}\beta(\lambda,s) \tag{6}$$

where $\beta_i(s) \in \mathbb{R}[s]$ for $i = 1,2,\ldots,m$; $\beta_o(s)$ is the monic least common denominator polynomial of $\{g_1(s), g_2(s), \ldots, g_m(s)\}$. Since $\mathcal{Z}[\beta_o(s)] \subset \mathcal{P}[\hat{G}(s)]$ and since $\bar{N}_\infty$ does not contain any pole of $\hat{G}(s)$,

$$\beta_o(s) \neq 0, \quad \forall s \in \bar{N}_\infty \tag{7}$$

Thus, for all $s \in \bar{N}_\infty$, $\lambda_i(s)$ is an eigenvalue of $\hat{G}(s)$ iff $\lambda_i(s)$ is a zero of the polynomial $\lambda \mapsto \beta(\lambda,s)$. Now, for any labelling, $(\lambda_i(s))_{i=1,2,\ldots,m}$, of the eigenvalues of $\hat{G}(s)$, we have

$$\det[I + k\hat{G}(s)] = \prod_{i=1}^{m} (1+k\lambda_i(s)), \qquad \forall s \in \bar{N}_\infty \tag{8}$$

Note that, for each $s \in \bar{N}_\infty$

$$\begin{aligned} \measuredangle \det[I + k\hat{G}(s)] &= \measuredangle \prod_{i=1}^{m} (1+k\lambda_i(s)) \\ &= \sum_{i=1}^{m} \measuredangle (1+k\lambda_i(s)) \\ &= \sum_{i=1}^{m} \measuredangle \left(\frac{1}{k} + \lambda_i(s)\right) \qquad \text{(by assumption (2), } k > 0\text{)} \end{aligned} \tag{9}$$

Now, as s travels along $\bar{N}_\infty$, we can label the eigenvalues of $\hat{G}(s)$ such that, for each i, $s \mapsto \lambda_i(s)$ is a <u>continuous</u> function; hence we obtain m <u>continuous</u> eigenloci $\lambda_i(\bar{N}_\infty)$ (this is possible because eigenvalues of a matrix are <u>continuous</u> functions of the elements of the matrix, see e.g. [12,p.45]). Thus, at first sight, one might expect to count the number of encirclements of the point 0 by $\det[I + k\hat{G}(s)]\big|_{s\in\bar{N}_\infty}$ by summing the number of encirclements of the point $-\frac{1}{k}$ by the $\lambda_i(\bar{N}_\infty)$'s. Unfortunately, this does not work because some curves $\lambda_i(\bar{N}_\infty)$ may not form a <u>closed</u> path as the example 1 below shows.

<u>Example 1</u>:

Consider $\hat{G}(s) = \begin{bmatrix} 0 & 1 \\ \frac{s-1}{s+1} & 0 \end{bmatrix} \in \mathbb{R}(s)^{2\times 2}$

Then $\det[\lambda I - \hat{G}(s)] = \lambda^2 - \frac{s-1}{s+1}$. The eigenloci $\lambda_1(\bar{N}_\infty)$, $\lambda_2(\bar{N}_\infty)$ are shown in Fig. 3, where $\bar{N}_\infty$ is the $j\omega$-axis, $\omega \in [-\infty,+\infty]$. ⌑

It turns out that (a) since the set of eigenvalues of $\hat{G}(-j\infty)$ is equal to the set of eigenvalues of $\hat{G}(+j\infty)$ counting multiplicities, we can always form an indexed family of closed paths $(\gamma_j^*)_{j=1,2,\ldots,p}$ from the set of eigenloci $\lambda_i(\bar{N}_\infty)$, $i = 1,2,\ldots,m$ (see (Lemma L2, part (a) below); (b) since the paths γ_j^*'s are closed paths, it is legitimate to define the number of encirclements of some point in $\mathbb{C}$ by γ_j^*; and (c) for any choice of $(\gamma_j^*)_{j=1,2,\ldots,p}$, the sum of the number of encirclements of $-\frac{1}{k}$ by the γ_j^*'s is equal to the number of encirclements of the origin by $\det[I + k\hat{G}(s)]\big|_{s\in\bar{N}_\infty}$. Thus we can test the closed-loop stability for any $k > 0$ by inspection of the number of encirclements of $-\frac{1}{k}$ by an indexed family of closed paths (obtained from the eigenloci $\lambda_i(\bar{N}_\infty)$'s).

Notation

Let $\gamma(\cdot) : [\alpha,\beta] \to \mathbb{C}$ be a closed path, i.e. $\gamma(\cdot)$ is continuous and $\gamma(\alpha) = \gamma(\beta)$ [8,p.217]. Let the point $a \notin \gamma$, where $\gamma := \gamma([\alpha,\beta])$. Then $C(a;\gamma)$ denotes the number of counterclockwise encirclements of $a \in \mathbb{C}$ by γ, i.e. [8,p.223]

$$C(a;\gamma) = \frac{1}{2\pi j}\int_\gamma \frac{dz}{z-a} \tag{10}$$

Note that strictly speaking, the integral in eqn. (10) requires $\gamma(\cdot)$ to be a circuit [8,p.223], i.e. $\gamma(\cdot)$ has to be a closed path and differentiable almost everywhere. However, Eilenberg's method specifies a standard procedure to treat the case where $\gamma(\cdot)$ is not necessarily differentiable almost everywhere, [8,p.251].

Generalized Nyquist Stability Criterion

For technical reasons, we parametrize the Nyquist part $\bar{N}_\infty$ shown in Fig. 2 by $t \in [\alpha_o,\alpha_{q+1}]$, a compact interval. More precisely, we define a continuous function $\bar{N}_\infty(\cdot) : [\alpha_o,\alpha_{q+1}] \to \bar{\mathbb{C}}$ so that[2] $\bar{N}_\infty([\alpha_o,\alpha_{q+1}]) = \bar{N}_\infty$, $\bar{N}_\infty(\alpha_o) = -j\infty$, $\bar{N}_\infty(\alpha_{q+1}) = +j\infty$. For convenience, we shall choose the parametrization $\bar{N}_\infty(\cdot)$ to be a strictly increasing function; thus as t increases from α_o to α_{q+1}, the point $\bar{N}_\infty(t)$ moves upward along $\bar{N}_\infty$ from $-j\infty$ to $+j\infty$ (e.g. first use the ordinate ω of $\bar{N}_\infty$ as a parameter and then choose $t = \tan^{-1}\omega$). Note that there are only a finite number of points in $\mathbb{C}$, thus in $\bar{N}_\infty-\{-j\infty,+j\infty\}$, where $\hat{G}(s)$ has multiple eigenvalues.[3] Consequently, the ε-indentations of $\bar{N}_\infty$ in Fig. 2 can be chosen so small that $\forall s \in \bar{N}_\infty-\{-j\infty,+j\infty\}$, $\hat{G}(s)$ has multiple eigenvalues only at a finite number of points on the $j\omega$-axis, $\omega \in (-\infty,+\infty)$. Let $jb_1,jb_2,\ldots,jb_q$ denote such points, i.e. $\hat{G}(jb_i)$, $i = 1,2,\ldots,q$, has multiple eigenvalues. Let $\alpha_1,\alpha_2,\ldots,\alpha_q \in \mathbb{R}$ be such that $\alpha_o < \alpha_1 < \ldots < \alpha_q < \alpha_{q+1}$ and that $\bar{N}_\infty(\alpha_i) = jb_i$, $i = 1,2,\ldots,q$. Let $I_i(\cdot) = [\alpha_{i-1},\alpha_i] \to \bar{\mathbb{C}}$, $i = 1,2,\ldots,q+1$, be such that $I_i(t) = \bar{N}_\infty(t)$, $\forall t \in [\alpha_{i-1},\alpha_i]$, i.e. $\bar{N}_\infty(\cdot)$ is the juxtaposition of $I_i(\cdot)$, $i = 1,2,\ldots,q+1$ [8,p.217]. The images of the $I_i(\cdot)$'s are shown in Fig. 4. Let $\ell \in \{1,2,\ldots,q+1\}$; for each $s \in I_\ell$ which is not an endpoint of I_ℓ, $\hat{G}(s)$ has m distinct eigenvalues $\lambda_i(s)$, $i = 1,2,\ldots,m$. Since $s \mapsto \hat{G}(s)$ is continuous on $\bar{N}_\infty$ and since the eigenvalues of a matrix are continuous functions of the elements of the matrix [12,p.45], we can uniquely define m continuous oriented eigenloci $\gamma_{i\ell}$, $i = 1,2,\ldots,m$ by

$$\gamma_{i\ell}(t) := \lambda_i(I_\ell(t)), \quad t \in [\alpha_{\ell-1},\alpha_\ell] \tag{11}$$

The Lemma L2 below shows that from $(\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$, we can construct an indexed family of closed paths whose image will be called the

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(2) We use $f(\cdot)$ to denote a function and f to denote the image of its domain under the map $f(\cdot)$.

(3) The polynomial $\lambda \mapsto \beta(\lambda,s)$ in eqn. (6) has multiple zeros for some $\zeta \in \mathbb{C}$ iff the discriminant $\theta(s)$ of $\beta(\lambda,s)$ is zero at ζ [13,p.248-250]. Since $\theta(s)$ is a polynomial in s, there are only a finite number of such ζ's.

generalized Nyquist diagram of $\hat{G}(s)$.

Lemma L2 (Generalized Nyquist diagram: Lumped case)

Given a proper $\hat{G}(s) \in \mathbb{R}(s)^{m\times m}$ and the associated eigenloci $(\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$ defined by eqn. (11), then

(a) the members of $(\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$ can be juxtaposed to form an indexed family of closed paths $(\gamma_j^*)_{j=1,2,\ldots,p}$ for some $1 \leq p \leq m$;

(b) let $\Delta(\cdot) : [\alpha_o,\alpha_{q+1}] \to \mathbb{C}$ be defined by

$$\Delta(t) := \prod_{i=1}^{m} (1+k\gamma_{i\ell}(t)), \ \forall t \in [\alpha_{\ell-1},\alpha_\ell], \ \ell = 1,2,\ldots,q+1 , \tag{12}$$

then

(i) $\Delta(\cdot)$ is a closed path, i.e. $\Delta(\cdot)$ is continuous on $[\alpha_o,\alpha_{q+1}]$ and $\Delta(\alpha_o) = \Delta(\alpha_{q+1})$;

(ii) $0 \notin \Delta([\alpha_o,\alpha_{q+1}])$ iff $-\frac{1}{k} \notin (\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$;

(iii) $C(0;\Delta) = \sum_{j=1}^{p} C(-\frac{1}{k};\gamma_j^*)$ for any indexed family $(\gamma_j^*)_{j=1,2,\ldots,p}$ obtained in part (a).

Proof of Lemma L2: see Appendix

Now we are ready to present the generalized Nyquist stability criterion for the lumped case.

Theorem L3 (Generalized Nyquist Stability Criterion: Lumped Case)

Consider the feedback system S shown in Fig. 1 and described by eqns. (1), (2). Associated with $\hat{G}(s) \in \mathbb{R}(s)^{m\times m}$ are the eigenloci $(\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$ defined by eqn. (11). Then the closed-loop system S is exp. stable

$$\Leftrightarrow \begin{cases} \text{(i)} & -\frac{1}{k} \notin (\gamma_j^*)_{j=1,2,\ldots,p} ; \\ \text{(ii)} & \sum_{j=1}^{p} C(-\frac{1}{k};\gamma_j^*) = p_+^o , \end{cases}$$

where p_+^o denotes the number of $\mathbb{C}_+$-zeros of the open-loop system characteristic polynomial counting multiplicities and $(\gamma_j^*)_{j=1,2,\ldots,p}$ denote <u>any</u> indexed family of closed paths formed from $(\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$.

<u>Remarks L3</u>: (a) Theorem L3 generalizes the well-known classical Nyquist stability criterion to the multi-input, multi-output case. Note Theorem L3 requires us to check the number of encirclements of $-\frac{1}{k}$ by a finite <u>family of closed</u> paths $(\gamma_j^*)_{j=1,2,\ldots,p}$ which is formed from the eigenloci of $\hat{G}(s)$ as s moves up along the Nyquist path $\bar{N}_\infty$ of Fig. 2.

(b) Theorem L3 uses the identity $\det[I + k\hat{G}(s)] = \prod_{i=1}^{m} [1 + k\lambda_i(s)]$, (which holds for all s where $\hat{G}(\cdot)$ is well defined) and the fact that we can label the $\lambda_i(s)$'s such that $s \mapsto \lambda_i(s)$ are <u>continuous</u> functions on $\bar{N}_\infty$. Hence the net change in phase of $\det[I + k\hat{G}(j\omega)]$ along $\bar{N}_\infty$ is simply the <u>sum</u> of the changes in phase of $[\lambda_i(s) - (-1/k)]$ along $\bar{N}_\infty$. This observation does <u>not</u> require that each $\lambda_i(s)$ be defined as an analytic function in any part of $\mathbb{C}_+$: it simply requires that we compute the λ_i's for each $s \in \bar{N}_\infty$. Note that in our treatment, the argument principle is only used in Theorem L1 and is only applied to $\det[I + k\hat{G}(s)]$, thus we need not mention branch cuts and/or Riemann surfaces and/or the extended argument principle [16,17].

III. <u>Distributed case</u>

We consider the feedback system S shown in Fig. 1, where

$$\hat{G}(s) \in \hat{\mathcal{B}}(\sigma_o)^{m\times m} \text{ for some } \sigma_o < 0 \tag{13}$$

$$k > 0 \tag{14}$$

Note that any proper rational function is an element of $\mathcal{B}(\sigma)$ for any $\sigma \in \mathbb{R}$. Thus the present formulation includes the lumped case in sec. II as a special case. Let $(\hat{\mathcal{N}}_r, \hat{\mathcal{D}}_r)$ be a σ_o-right-representation of $\hat{G}(s)$ [9,14]; equivalently:

(i) $\hat{G}(s) = \hat{\mathcal{N}}_r(s)\hat{\mathcal{D}}_r(s)^{-1}$ with $\hat{\mathcal{N}}_r \in \hat{\mathcal{A}}_-(\sigma_o)^{mxm}$, $\hat{\mathcal{D}}_r \in \hat{\mathcal{A}}_-(\sigma_o)^{mxm}$; (15)

(ii) there exist $\hat{\mathcal{U}}_r \in \hat{\mathcal{A}}_-(\sigma_o)^{mxm}$, $\hat{\mathcal{V}}_r \in \hat{\mathcal{A}}_-(\sigma_o)^{mxm}$ such that

$$\hat{\mathcal{U}}_r \hat{\mathcal{N}}_r + \hat{\mathcal{V}}_r \hat{\mathcal{D}}_r = I \quad ; \tag{16}$$

(iii) det $\hat{\mathcal{D}}_r(s)$ is analytic and bounded away from zero at ∞ in C_{σ_o+}. (17)

Without loss of generality [14], $\hat{\mathcal{D}}_r(s)$ may be taken to be a proper rational function matrix with all its poles in $\mathring{\mathbb{C}}_{\sigma_o-}$, hence by eqn. (17),

$$\lim_{|s|\to\infty} \det \hat{\mathcal{D}}_r(s) = \text{constant} \neq 0 \tag{18}$$

Note that the closed-loop system transfer function matrix $r \mapsto y$ is given by

$$\hat{H}_{yr} = k\hat{G}(s)[I + k\hat{G}(s)]^{-1} = k\,\hat{\mathcal{N}}_r[\hat{\mathcal{D}}_r + k\hat{\mathcal{N}}_r]^{-1} \tag{19}$$

Since $(\hat{\mathcal{N}}_r, \hat{\mathcal{D}}_r + k\hat{\mathcal{N}}_r)$ is also σ_o-right-coprime, $\det[\hat{\mathcal{D}}_r + k\hat{\mathcal{N}}_r]$ is the closed-loop characteristic function [14]. Consequently, the closed-loop system is $\mathcal{A}$-stable iff $\inf_{\text{Re } s \geq 0} |\det[\hat{\mathcal{D}}_r + k\hat{\mathcal{N}}_r]| > 0$. By substituting $\hat{G} = \mathcal{N}_r \mathcal{D}_r^{-1}$, we have

$$\det[I + k\hat{G}(s)] = \frac{\det[\hat{\mathcal{D}}_r(s) + k\hat{\mathcal{N}}_r(s)]}{\det \hat{\mathcal{D}}_r(s)} \tag{20}$$

Since $\det[I + k\hat{G}(s)]$ is analytic on $\mathbb{C}_{\sigma_o+}$ except for a finite number of poles in $\mathbb{C}_{\sigma_o+}$, (these poles occur at zeros of det $\hat{\mathcal{D}}_r(s)$), we can apply the argument principle [8,p.246-247] to $\det[I + k\hat{G}(s)]$ to determine whether $\inf_{\mathrm{Re}\, s \geq 0} |\det[\hat{\mathcal{D}}_r + k\hat{\mathcal{N}}_r]| > 0$. For this purpose, we construct a Nyquist path $\bar{N}_\infty$ as in the lumped case (see Fig. 2). Note that (a) the ε-indentations are again taken to the <u>left</u> of the jω-axis for the same reason as stated in the lumped case; (b) by assumptions (13) and (15)∿(17), $\hat{G}(s)$ is <u>analytic</u> on $\mathbb{C}_{\sigma_o+}$ except for a <u>finite</u> number of poles in $\mathbb{C}_{\sigma_o+}$ (these poles are the zeros of $\det \hat{\mathcal{D}}_r(s)$). Since $\sigma_o < 0$, the ε-indentations can be taken to the left of the jω-axis poles of $\hat{G}(s)$; furthermore, we choose the indentation radius $\varepsilon \leq |\sigma_o|$ so that $\hat{G}(s)$ is <u>analytic on $\bar{N}_\infty$</u>.

Now we have the following stability theorem which is the counterpart of Theorem L1 in the distributed case.

<u>Theorem D1</u>: (Stability theorem based on $\det[I + k\hat{G}(s)]$: distributed case)

Let $\bar{N}_\infty$ be defined as above. Let the feedback system S of Fig. 1 satisfy eqns. (13)∿(17) . Assume that

$$\hat{G}(s) \text{ tends to a constant matrix as } |s| \to \infty \text{ in } \mathbb{C}_+ \tag{21}$$

U.t.c. the closed-loop system S is $\mathcal{A}(\sigma)$-stable, for some $\sigma < 0$

$$\Leftrightarrow \begin{cases} \text{(i)} \quad \det[I + k\hat{G}(s)] \neq 0, \quad \forall s \in \bar{N}_\infty; & (22) \\ \text{(ii)} \quad \det[I + k\hat{G}(s)]\Big|_{s\in\bar{N}_\infty} \text{ encircles the origin} & (23) \end{cases}$$

p_+^o times in the counterclockwise sense, where p_+^o denotes the number of $\mathbb{C}_+$-zeros of the open-loop system characteristic function $\det \hat{\mathcal{D}}_r(s)$, counting multiplicities.

<u>Proof of Theorem D1</u>: see Appendix

<u>Remarks D1</u>: (a) Assumption (21) means that the almost periodic asymptotic behavior of $\hat{G}(j\omega)$ reduces to a constant matrix. This constant matrix is the zero matrix in many cases.

(b) Theorem D1 guarantees $\mathcal{A}(\sigma)$-stability, for some $\sigma < 0$; this implies exp. stability : more precisely; in view of (21), $\hat{H}_{yr}(s)$ tends to a constant as $|s| \to \infty$ in $\mathbb{C}_+$, hence $H_{yr}(t) = H_o\delta(t) + H_1(t)$ where $H_o \in \mathbb{R}^{m\times m}$ and $H_1(\cdot) \in L_{1,\sigma}$. Consequently, the response $y(\cdot)$ of the feedback system S to any step input $r(t) = 1(t)v$, where $v \in \mathbb{R}^m$, will tend to $\hat{H}_{yr}(0)v$ exponentially since the error $\tilde{y}(t) := y(t) - \hat{H}_{yr}(0)v$ satisfies that for some $\rho < \infty$, $\|\tilde{y}(t)\| \leq \rho\|v\|e^{-\sigma t}$. Thus $\hat{H}_{yr}$ can be said to be exp. stable.

To obtain, for the distributed case, a generalized Nyquist stability criterion similar to Theorem L3, we note that the discriminant of the polynomial $\lambda \mapsto \det[\lambda I - \hat{G}(s)]$ is a polynomial in the elements of $\hat{G}(s)$ [13,p.248-250], hence

(i) it is an analytic function of s in $\mathbb{C}_{\sigma_o+} - \mathcal{P}[\det \hat{\mathcal{D}}_r(s)]$; (24)

(ii) it can only have a finite number of zeros in any compact subset of $\mathbb{C}_{\sigma_o+}$ (25)

Thus we can choose the radii of the ε-indentations so that for $s \in \bar{N}_\infty$, $\hat{G}(s)$ has multiple eigenvalues only for some isolated points on the $j\omega$-axis, where $\omega \in [-\infty,+\infty]$. Suppose that there is only a finite number of values of ω such that $\hat{G}(j\omega)$ has multiple eigenvalues. Then the construction of the generalized Nyquist diagram of Lemma L2 still holds and we are right back to the previous case. However, as shown in the example 2 below, it may happen that, for some $\hat{G}(s) \in \hat{\mathcal{B}}(\sigma_o)^{m\times m}$, $\hat{G}(s)$ has multiple eigenvalues for an infinite number of points on the $j\omega$-axis. Consequently, a naive generalization of Lemma L2 requires a construction of an infinite digraph (see the proof of Lemma L2).

Example 2

Consider
$$\hat{G}(s) = \begin{bmatrix} \dfrac{10^3}{(s+10)^3} & \hat{g}_{12}(s) \\ \dfrac{8^3\cdot 4}{(s+8)^3(s+4)} & \dfrac{10^3}{(s+10)^3} \end{bmatrix} \qquad (26)$$

where $\hat{g}_{12}(s) = \mathcal{L}[g_{12}(t)]$ with

$$g_{12}(t) = \begin{cases} 0, & t \notin [0,1] \\ t(1-t), & t \in [0,1] \end{cases}$$

Note that $\hat{G}(s) \in \hat{\mathcal{A}}_-(\sigma_o)^{2\times 2}$ for any $\sigma_o > -4$, thus $\hat{G}(s)$ satisfies assumption (13). The discriminant of the polynomial $\lambda \mapsto \det[\lambda I - \hat{G}(s)]$ is, for this case, equal to $\frac{8^3 \cdot 4}{(s+8)^3(s+4)^3} \cdot \hat{g}_{12}(s)$. By direct calculation using tables [15, formulas 440.11 and 440.12], we obtain

$$\hat{g}_{12}(j\omega) = \frac{e^{-j\frac{\omega}{2}}}{2} \; \frac{\sin(\frac{\omega}{2}) - \frac{\omega}{2}\cos(\frac{\omega}{2})}{(\frac{\omega}{2})^3} \tag{27}$$

Thus the discriminant has an infinite number of $j\omega$-axis zeros, i.e. $\hat{G}(s)$ has multiple eigenvalues for an infinite number of points on the $j\omega$-axis. ¤

To overcome this difficulty, we note the following. Let λ_i^∞, $i = 1,2,\ldots,\mu_o$ be the distinct eigenvalues of $\hat{G}(+j\infty)$ with multiplicities m_i, respectively. Then by continuity, for any given $\varepsilon > 0$, there exists $\Omega > 0$ such that for $i = 1,2,\ldots,\mu_o$, $\hat{G}(j\omega)$ has m_i eigenvalues inside the closed disc $\bar{D}(\lambda_i^\infty,\varepsilon)$, $\forall\ |\omega| > \Omega$. Now by (25), $\hat{G}(j\omega)$ has multiple eigenvalues for only a finite number of points within the compact interval $[-j\Omega,+j\Omega]$. Therefore, by assigning a set V_o of μ_o nodes which now represent the closed discs $\bar{D}(\lambda_i^\infty,\varepsilon)$, $i = 1,2,\ldots,\mu_o$, we obtain a finite digraph $\mathcal{G}$ by the construction stated in the proof of Lemma L2 and we are right back to the previous case. Now following the procedure stated in the proof of Lemma L2, we identify, in the digraph $\mathcal{G}$, a finite indexed family of closed paths Γ_i, $i = 1,2,\ldots,p$, for some $p \le m$. The corresponding eigenloci will then form a finite indexed family of closed paths

$(\gamma_i^*)_{i=1,2,\ldots,p}$ provided that additional m_i directed paths are joined in each disc $\bar{D}(\lambda_i^\infty,\varepsilon)$, $i = 1,2,\ldots,\mu_o$ from the eigenvalues of $\hat{G}(+j\Omega)$ to those of $\hat{G}(-j\Omega)$. These $(\gamma_i^*)_{i=1,2,\ldots,p}$ constitute the required generalized Nyquist diagram and the generalized Nyquist stability criterion for the distributed case is stated in the Theorem D3 below.

Theorem D3 (Generalized Nyquist Stability Criterion: distributed case)

Consider the feedback system S shown in Fig. 1 and described by eqns. (13)∿(17) and (21). Consider the finite indexed family of closed paths $(\gamma_j^*)_{j=1,2,\ldots,p}$ and the closed discs $\bar{D}(\lambda_i^\infty,\varepsilon)$, $i = 1,2,\ldots,\mu_o$ as indicated above. Let $k > 0$ be such that $-\frac{1}{k} \notin \bar{D}(\lambda_i^\infty;\varepsilon)$ for $i = 1,2,\ldots,\mu_o$.

U.t.c. the closed-loop system S is $\mathcal{A}(\sigma)$-stable, for some $\sigma < 0$

$$\Leftrightarrow \begin{cases} \text{(i)} \quad -\frac{1}{k} \notin (\gamma_j^*)_{j=1,2,\ldots,p} \; ; \\ \text{(ii)} \quad \sum_{j=1}^{p} C(-\frac{1}{k};\ \gamma_j^*) = p_+^o \end{cases}$$

where p_+^o denotes the $\mathbb{C}_+$-zeros of the open-loop characteristic function det $\hat{\mathcal{D}}_r(s)$ counting multiplicities. ¤

Remark D3: Note that Theorem D3 does not assert anything when $-1/k$ belongs to one of the discs $\bar{D}(\lambda_i^\infty,\varepsilon)$. This represents no loss in practice, since if $-1/k$ belongs to such a disc, then $\det[I + k\hat{G}(j\omega)] = O(\varepsilon)$ at high frequencies and clearly the system would then be very sensitive to small changes in k or in $\hat{G}$ at those frequencies; indeed for such k's, the transfer function matrix, from r to e, $\hat{H}_{er}(j\omega) = O(1/\varepsilon)$ at those frequencies.

Example 3:

Consider the feedback system S shown in Fig. 1 with the open-loop transfer function matrix $\hat{G}(s)$ described by eqn. (26) of example 2. The generalized Nyquist diagram of $\hat{G}(s)$ is shown in Fig. 5. Note that $\hat{G}(j\omega)$ has multiple eigenvalues at $\omega \simeq \pm 2.9868, \pm 15.44, \ldots$, etc. (which are the solutions of $\sin \frac{\omega}{2} = \frac{\omega}{2} \cos \frac{\omega}{2}$, see eqn. (27) of example 2). The behavior of the eigenloci $\lambda_1(\cdot)$, $\lambda_2(\cdot)$ is further magnified in Fig. 6: a simple local analysis confirms that, at $\omega \simeq 8.9868$, the eigenloci $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ make a ninety degree turn.

Referring to the generalized Nyquist diagram of $\hat{G}(s)$ shown in Fig. 5, we know that, by Theorem D3 above, the closed-loop system is $\mathcal{A}(\sigma)$-stable (for some $\sigma < 0$) for $k \in (0,6]$ since the open-loop system has no $\mathbb{C}_+$-poles. $\square$

IV. Conclusion

The generalized Nyquist stability criterion based on the eigenloci of the open-loop transfer function matrix is derived for the lumped as well as the distributed case. First, a stability theorem based on $\det[I + k\hat{G}(s)]$ is presented (see Theorem L1 and Theorem D1). Construction of the generalized Nyquist diagram is shown in Lemma L2. The generalized Nyquist stability criterion is then presented in Theorem L3 (Theorem D3) which shows that the stability condition of Theorem L1 (Theorem D1, resp.) can be checked via the encirclement condition of the critical point by the closed paths formed by the eigenloci of the open-loop transfer function matrix.

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APPENDIX

Proof of Lemma L2:

(a) To show that the members of $(\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$ can be juxtaposed to form an indexed family of closed paths $(\gamma_j^*)_{j=1,2,\ldots,p}$ for some $1 \leq p \leq m$, we construct a digraph $\mathcal{G}$ associated with $(\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$. To each point $jb_\ell \in \bar{N}_\infty$ where $\hat{G}(jb_\ell)$ has multiple eigenvalues $\ell = 1,2,\ldots,q$, we assign a set V_ℓ of μ_ℓ nodes where μ_ℓ = number of distinct eigenvalues of $\hat{G}(jb_\ell)$; thus each node of V_ℓ represents a distinct eigenvalue of $\hat{G}(jb_\ell)$. We further assign a set V_o of μ_o nodes to represent the distinct eigenvalues of $\hat{G}(-j\infty)$ as well as that of $\hat{G}(+j\infty)$, (note that $\hat{G}(+j\infty) = \hat{G}(-j\infty)$, hence they have the same set of eigenvalues counting multiplicites). Now for $\ell = 1,2,\ldots,q$, we assign m directed branches from $V_{\ell-1}$ to V_ℓ, each such branch represents one of the m continuous eigenloci $\gamma_{i\ell}$, $i = 1,2,\ldots,m$. Similarly, we assign m directed branches from V_q to V_o, each such branch represents one of the m continuous eigenloci $\gamma_{i,q+1}$, $i = 1,2,\ldots,m$. A typical digraph corresponding to a 4x4 $\hat{G}(s)$ is shown in Fig. A1. Note that in this example, $\hat{G}(+j\infty)$ and $\hat{G}(-j\infty)$ has four distinct eigenvalues and $\hat{G}(j0)$ has two distinct eigenvalues of multiplicities 2, respectively; hence q=1 and we have only two set of nodes, V_o and V_1. Note that, by construction, the number of the directed branches entering (or leaving) any node of $\mathcal{G}$ is equal to the multiplicity of the corresponding eigenvalue. Hence, the digraph $\mathcal{G}$ has the following property:

for any node of $\mathcal{G}$, the number of branches entering that node is equal to the number of branches leaving that node. (A1)

We now construct a family of closed paths by suitable juxtaposition of the members of $(\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$. We select an arbitrary node of $\mathcal{G}$. We move along <u>any</u> directed branch to some other node. As we repeat this process, by property (A1) of $\mathcal{G}$ and the fact that $\mathcal{G}$ has a <u>finite</u> number of nodes, we eventually reach an already traversed node; thus this process identifies a <u>simple closed</u> path Γ_1 in the digraph $\mathcal{G}$. Corresponding to each branch of Γ_1 is one member of the family $(\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$. By the construction of the digraph $\mathcal{G}$, it follows that the eigenloci $\gamma_{i\ell}$'s corresponding to branches of Γ_1 form a not-necessarily-simple <u>closed</u> path which we call γ_1^*. Now we delete from $\mathcal{G}$ the branches associated with Γ_1. Note that the remaining digraph still has the property (A1). Now if the remaining digraph contains no branch, there is only one closed path Γ_1 in $\mathcal{G}$ and the eigenloci $(\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}}$ form a single closed path γ_1^* in $\mathbb{C}$ and we are done. Otherwise, we select an arbitrary node of $\mathcal{G}$ which has at least one outgoing branch and repeat the above procedure until all branches of $\mathcal{G}$ are exhausted. The result is a <u>finite</u> indexed family of <u>closed</u> paths $(\gamma_j^*)_{j=1,2,\ldots,p}$ in the complex plane $\mathbb{C}$, where the integer $p \in \{1,2,\ldots,m\}$.

(b)

(i) Since $\gamma_{i\ell}(\cdot)$, $i = 1,2,\ldots,m$; $\ell = 1,2,\ldots,q+1$ are continuous on $[\alpha_{\ell-1},\alpha_\ell]$ and $\prod_{i=1}^{m} (1+k\gamma_{i\ell}(\alpha_\ell)) = \prod_{i=1}^{m} (1+k\gamma_{i,\ell+1}(\alpha_\ell))$, $\Delta(\cdot)$ is continuous on $[\alpha_o,\alpha_{q+1}]$. Furthermore, $\Delta(\alpha_o) = \prod_{i=1}^{m} (1+k\gamma_{i1}(\alpha_o)) = \prod_{i=1}^{m} (1+k\lambda_i^{-\infty}) = \prod_{i=1}^{m} (1+k\lambda_i^{+\infty}) = \prod_{i=1}^{m} (1+k\gamma_{i,q+1}(\alpha_{q+1})) = \Delta(\alpha_{q+1})$ where $\lambda_i^{-\infty}$, $\lambda_i^{+\infty}$ denote the eigenvalues of $\hat{G}(-j\infty)$ and $\hat{G}(+j\infty)$ respectively. Hence $\Delta(\cdot)$ is a closed path.

(ii) follows from eqn. (11).

(iii) $2\pi j \sum_{n=1}^{p} C(-\frac{1}{k};\gamma_n^*) := \sum_{n=1}^{p} \int_{\gamma_n^*} \frac{dz}{z+\frac{1}{k}}$

$$= \sum_{i=1}^{m} \sum_{\ell=1}^{q+1} \int_{\gamma_{i\ell}} \frac{dz}{z+\frac{1}{k}}$$

$$= \sum_{i=1}^{m} \sum_{\ell=1}^{q+1} \int_{\alpha_{\ell-1}}^{\alpha_\ell} \frac{d(\gamma_{i\ell}(t)+\frac{1}{k})}{\gamma_{i\ell}(t)+\frac{1}{k}} \qquad \text{(A2)}$$

Note that we can relabel the $\gamma_{i\ell}$'s into $\gamma_{\beta\ell}$'s so that for each $\beta \in \{1,2,\ldots,m\}$, the juxtaposition of $\gamma_{\beta\ell}$, $\ell = 1,2,\ldots,q+1$ form a continuous eigenlocus, say γ_β. Thus eqn. (A2) becomes

$$2\pi j \sum_{n=1}^{p} C(-\frac{1}{k};\gamma_n^*) = \sum_{\beta=1}^{m} \int_{\alpha_o}^{\alpha_{q+1}} \frac{d(\gamma_\beta(t)+\frac{1}{k})}{\gamma_\beta(t)+\frac{1}{k}}$$

$$= \sum_{\beta=1}^{m} \int_{\alpha_o}^{\alpha_{q+1}} d[\ell n(\gamma_\beta(t)+\frac{1}{k})]$$

$$= \int_{\alpha_o}^{\alpha_{q+1}} d[\ell n \prod_{\beta=1}^{m} (\gamma_\beta(t)+\frac{1}{k})]$$

$$= \int_{\alpha_o}^{\alpha_{q+1}} d[\ell n\, \Delta(t)]$$ (by (12) and the fact that $\prod_{i=1}^{m} (\gamma_{i\ell}(t)+\frac{1}{k}) = \prod_{\beta=1}^{m} (\gamma_{\beta\ell}+\frac{1}{k})$, $\forall \ell = 1,2,\ldots,q+1$)

$$= \int_{\alpha_o}^{\alpha_{q+1}} \frac{d(\Delta(t))}{\Delta(t)}$$

$$= \int_{\Delta} \frac{dz}{z}$$

$$= 2\pi j\ \ C(0;\Delta)$$

Q.E.D.

Proof of Theorem L3:

The closed-loop system is exp. stable

$$\Leftrightarrow \begin{cases} \text{(i)} & \det[I + k\hat{G}(s)] \neq 0, \quad \forall s \in \bar{N}_\infty; \\ \text{(ii)} & C(0; \det[I + k\hat{G}(s)]\big|_{s\in\bar{N}_\infty}) = p_o^+ \end{cases} \qquad \text{(by Theorem L1)}$$

$$\Leftrightarrow \begin{cases} \text{(i)} & \det[I + k\hat{G}(I_\ell(t))] \neq 0, \quad \forall t \in [\alpha_{\ell-1},\alpha_\ell], \ \forall \ell = 1,2,\ldots,q+1 \\ \text{(ii)} & C(0; \det[I + k\hat{G}(I_\ell(t))]\Big|_{\substack{t\in[\alpha_{\ell-1},\alpha_\ell] \\ \ell=1,2,\ldots,q+1}}) = p_o^+ \end{cases} \qquad \text{(parameterization of } \bar{N}_\infty)$$

$$\Leftrightarrow \begin{cases} \text{(i)} & \prod_{i=1}^{m} [1 + k\lambda_i(I_\ell(t))] \neq 0, \quad \forall t \in [\alpha_{\ell-1},\alpha_\ell], \ \ell = 1,2,\ldots,q+1 \\ \text{(ii)} & C(0; \prod_{i=1}^{m} [1 + k\lambda_i(I_\ell(t))]\Big|_{\substack{t\in[\alpha_{\ell-1},\alpha_\ell] \\ \ell=1,2,\ldots,q+1}}) = p_o^+ \end{cases}$$

since the $\lambda_i(s)$'s are the eigenvalues of $\hat{G}(s)$, the $[1 + k\lambda_i(s)]$'s are the eigenvalues of $I + k\hat{G}(s)$.

$$\Leftrightarrow \begin{cases} \text{(i)} & \prod_{i=1}^{m} (1 + k\gamma_{i\ell}(t)) \neq 0, \quad \forall t \in [\gamma_{\ell-1},\alpha_\ell], \ \forall i = 1,2,\ldots,m, \ \forall \ell = 1,2,\ldots,q+1 \\ \text{(ii)} & C(0; \prod_{i=1}^{m} (1 + k\gamma_{i\ell}(t))\Big|_{\substack{t\in[\alpha_{\ell-1},\alpha_\ell] \\ \ell=1,2,\ldots,q+1}}) = p_o^+ \end{cases} \qquad \text{(by definition of } \gamma_{i\ell})$$

$$\Leftrightarrow \begin{cases} \text{(i)} & -\frac{1}{k} \notin (\gamma_{i\ell})_{\substack{i=1,2,\ldots,m \\ \ell=1,2,\ldots,q+1}} \\ \text{(ii)} & C(-\frac{1}{k};\Delta) = p_o^+ \end{cases} \qquad \text{(by eqn. (12))}$$

$$\Leftrightarrow \begin{cases} \text{(i)} & -\frac{1}{k} \notin (\gamma_j^*)_{j=1,2,\ldots,p} \\ \text{(ii)} & \sum_{j=1}^{p} C(-\frac{1}{k};\gamma_j^*) = p_o^+ \end{cases} \qquad \text{(by Lemma L2)}$$

Q.E.D.

Proof of Theorem D1:

We will show, in two steps, that under the assumptions (13)∿(17) and (21), the closed-loop transfer function $\hat{H}_{yr} \in \hat{\mathcal{A}}(\sigma)$ for some $\sigma < 0$ (thus the closed-loop system is exp. stable) iff (22) and (23) hold.

Claim 1: As a consequence of (20) and of assumption (21), conditions (22) and (23) are equivalent to

$$\inf_{\text{Re } s \geq 0} |\hat{\chi}(s)| > 0 \quad \text{and} \quad \inf_{s \in \bar{N}_\infty} |\hat{\chi}(s)| > 0 \tag{A3}$$

where $\hat{\chi}(s) := \det[\hat{\mathcal{D}}_r(s) + k\hat{\mathcal{N}}_r(s)]$.

Note that by (20), $\hat{\chi}(s)$ is meromorphic in $\mathbb{C}_{\sigma_o+}$ and is analytic on $\bar{N}_\infty$; furthermore, by (18) and (21), $\hat{\chi}(s)$ tends to a constant, say $\hat{\chi}(\infty)$, as $|s| \to \infty$ in $\mathbb{C}_+$. Thus the argument principle together with (20) show that (22) and (23) are equivalent to (A3).

Claim 2: (A3) is equivalent to

$$\inf_{\text{Re } s \geq \sigma} |\hat{\chi}(s)| > 0, \quad \text{for some } \sigma < 0 \tag{A4}$$

It is clear that (A4) implies (A3). Thus we only have to show that (A3) implies (A4). To see this, we note that assumption (15) and the closure of the algebra $\hat{\mathcal{A}}_-(\sigma_o)$ under addition and multiplication imply that $\hat{\chi}(s) \in \hat{\mathcal{A}}_-(\sigma_o)$; thus there is some $\sigma_1 < \sigma_o$ such that $\hat{\chi}(s) \in \hat{\mathcal{A}}(\sigma_1)$. Therefore $\chi(t)e^{-\sigma_1 t} \in \mathcal{A}$, where $\chi(t) := \mathcal{L}^{-1}[\hat{\chi}(s)]$. This implies that for some $\sigma_2 \in (\sigma_1, \sigma_o)$, $t\chi(t)e^{-\sigma_2 t} \in \mathcal{A}$, i.e. $\mathcal{L}[t\,\chi(t)] = \hat{\chi}'(s) \in \hat{\mathcal{A}}(\sigma_2) \subset \hat{\mathcal{A}}(\sigma_o)$. Thus $\hat{\chi}'(s)$ is analytic and bounded in $\mathbb{C}_{\sigma_o+}$. This together with (A3) imply that there is a $\sigma < 0$ such that (A4) holds.

Thus conditions (22) and (23) are equivalent to (A4) which, by (19) and the σ_o-right-coprimeness of $(\hat{\mathcal{N}}_r, \hat{\mathcal{D}}_r + k\hat{\mathcal{N}}_r)$, is equivalent to $\hat{H}_{yr} \in \hat{\mathcal{A}}(\sigma)^{m\times m}$, for some $\sigma < 0$. Q.E.D.

Footnotes

(1) The ε-indentations must be in $\mathbb{C}_-$ since for the multi-input multi-output case, the closed-loop system and the open-loop system may have some common $j\omega$-axis poles, e.g. if $\hat{G}(s)$ and k are such that $I + k\hat{G}(s)$ diag[(s+1)/s,s/(s+1)], then s=0 is a open-loop system poles as well as a closed-loop pole.

(2) We use $f(\cdot)$ to denote a function and f to denote the image of its domain under the map $f(\cdot)$.

(3) The polynomial $\lambda \mapsto \beta(\lambda,s)$ in eqn. (6) has multiple zeros for some $\zeta \in \mathbb{C}$ iff the discriminant $\theta(s)$ of $\beta(\lambda,s)$ is zero at ζ [13,p.248-250]. $\theta(s)$ is a polynomial in s, there are only a finite number of such ζ's.

Figure Captions

Fig. 1: Feedback system S under consideration.

Fig. 2: The Nyquist path $\bar{N}_\infty$: "x" denotes the $j\omega$-axis poles of $\hat{G}(s)$.

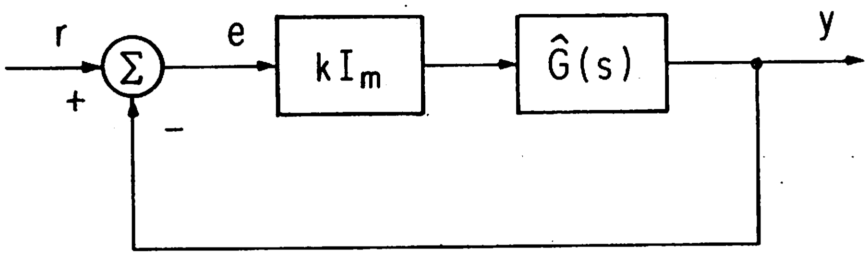
Fig. 3: An example which shows that each eigenlocus of the open loop transfer function matrix may not form a closed path.

Fig. 4: The Nyquist path $\bar{N}_\infty$ shown in Fig. 2 is considered as the juxtaposition of the paths $I_i(\cdot)$, $i = 1,2,\ldots,q+1$. jb_i, $i = 1,2,\ldots,q$, are the points on the $j\omega$-axis such that $\hat{G}(jb_i)$ has multiple eigenvalues.

Fig. 5: The generalized Nyquist diagram of $\hat{G}(s)$ considered in example 3 ($\hat{G}(s)$ is specified by (26)).

Fig. 6: Blow up of the eigenloci $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ in the neighborhood of $\omega \simeq 8.9868$ r/s where $\hat{G}(j\omega)$ has a multiple eigenvalue.

Fig. A1: A typical digraph $\mathcal{G}$ corresponding to a 4x4 $\hat{G}(s)$: $\hat{G}(+j\infty) = \hat{G}(-j\infty)$ has four distinct eigenvalues (represented by V_o) and $\hat{G}(j0)$ has two distinct eigenvalues with multiplicities 2, respectively (represented by V_1).



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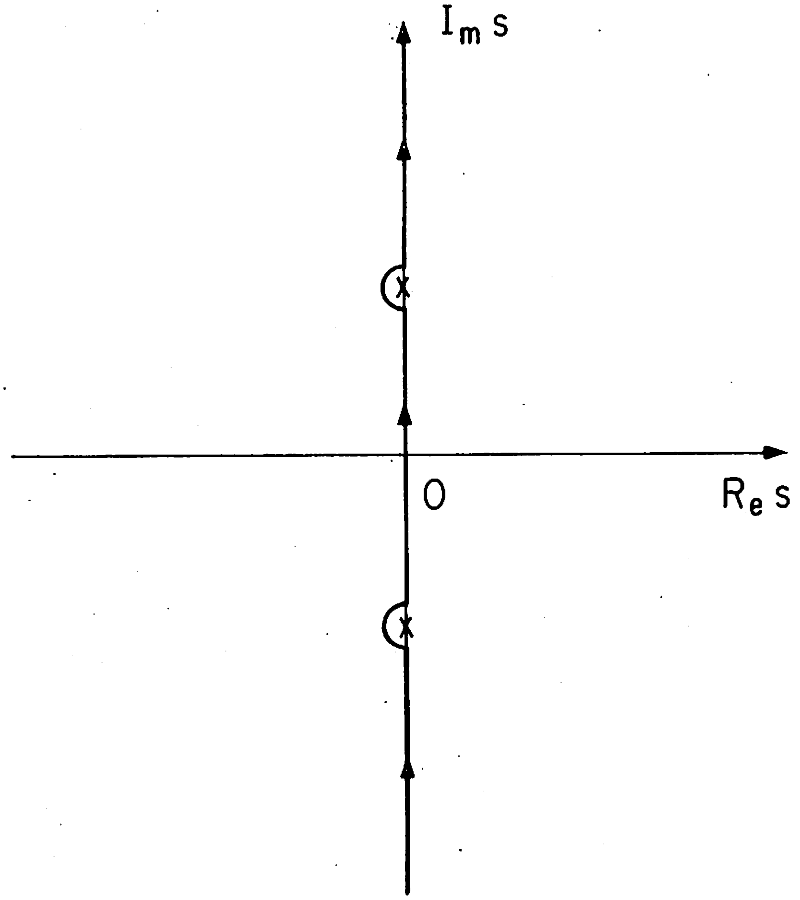

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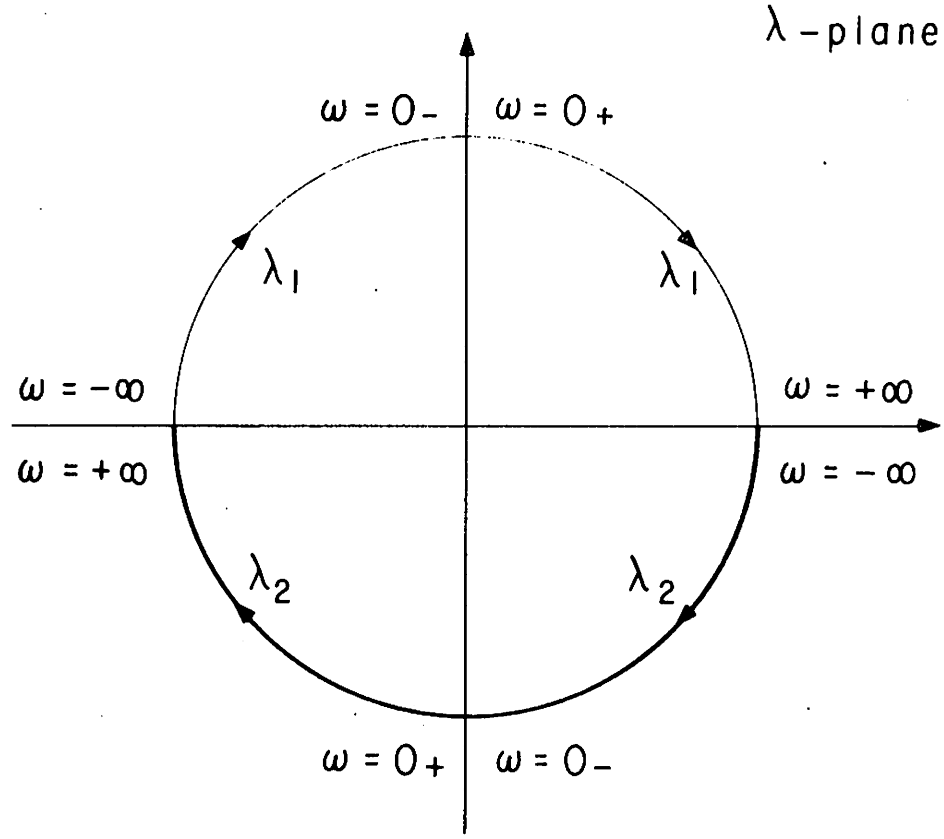
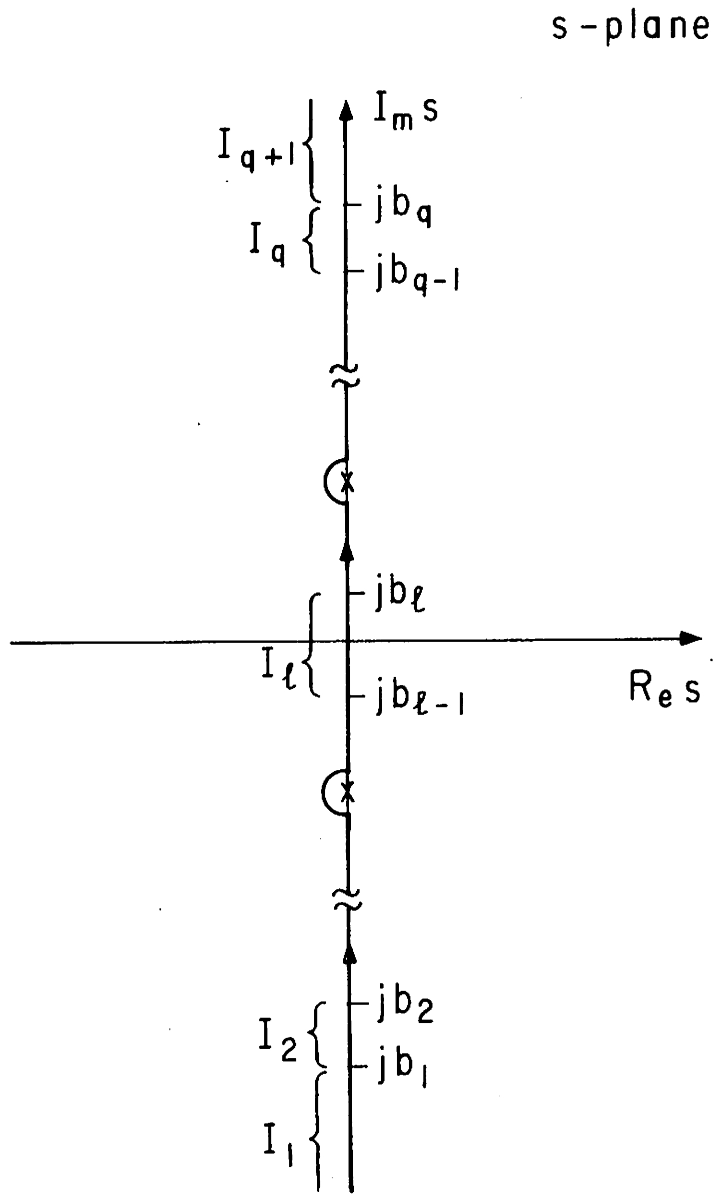
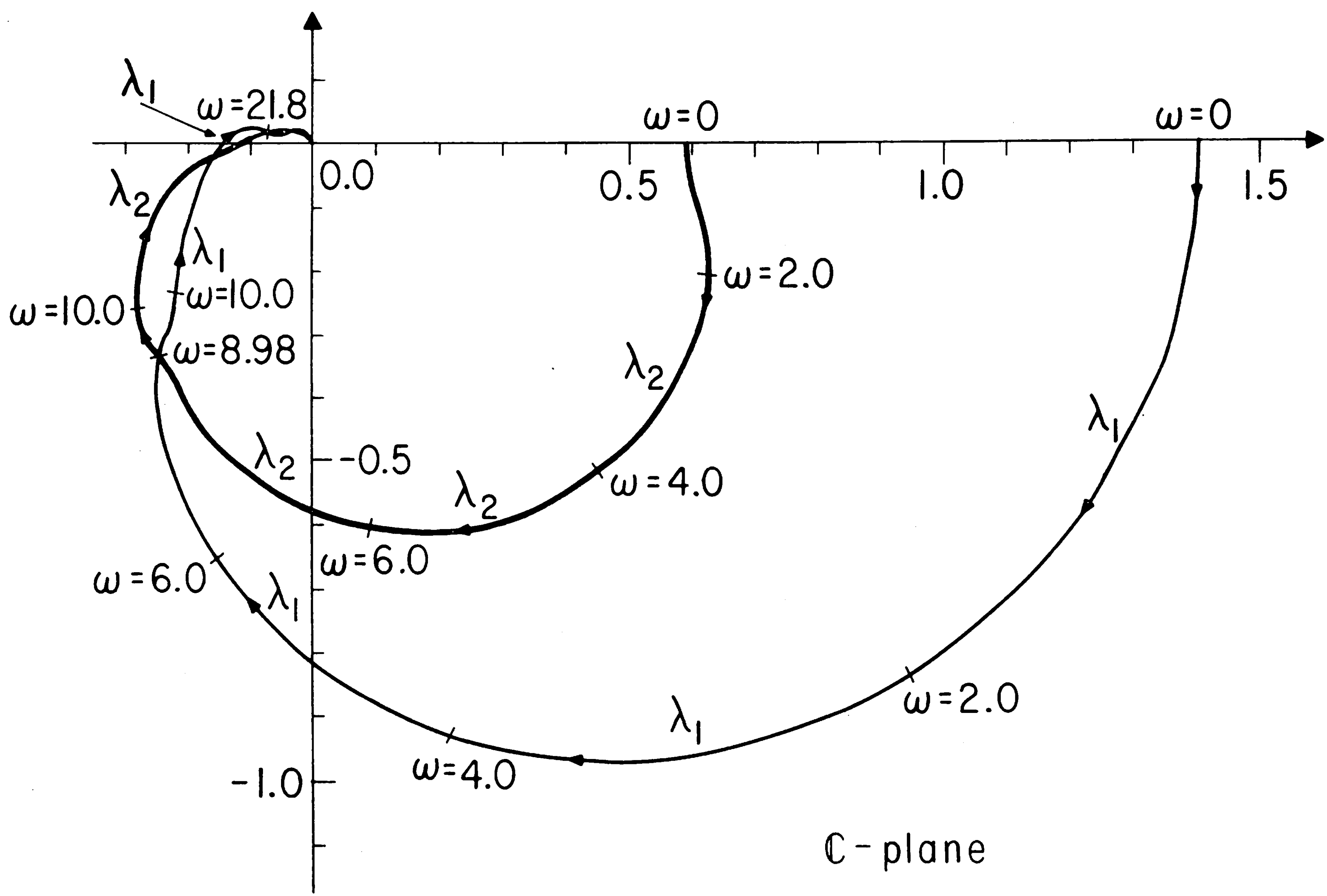
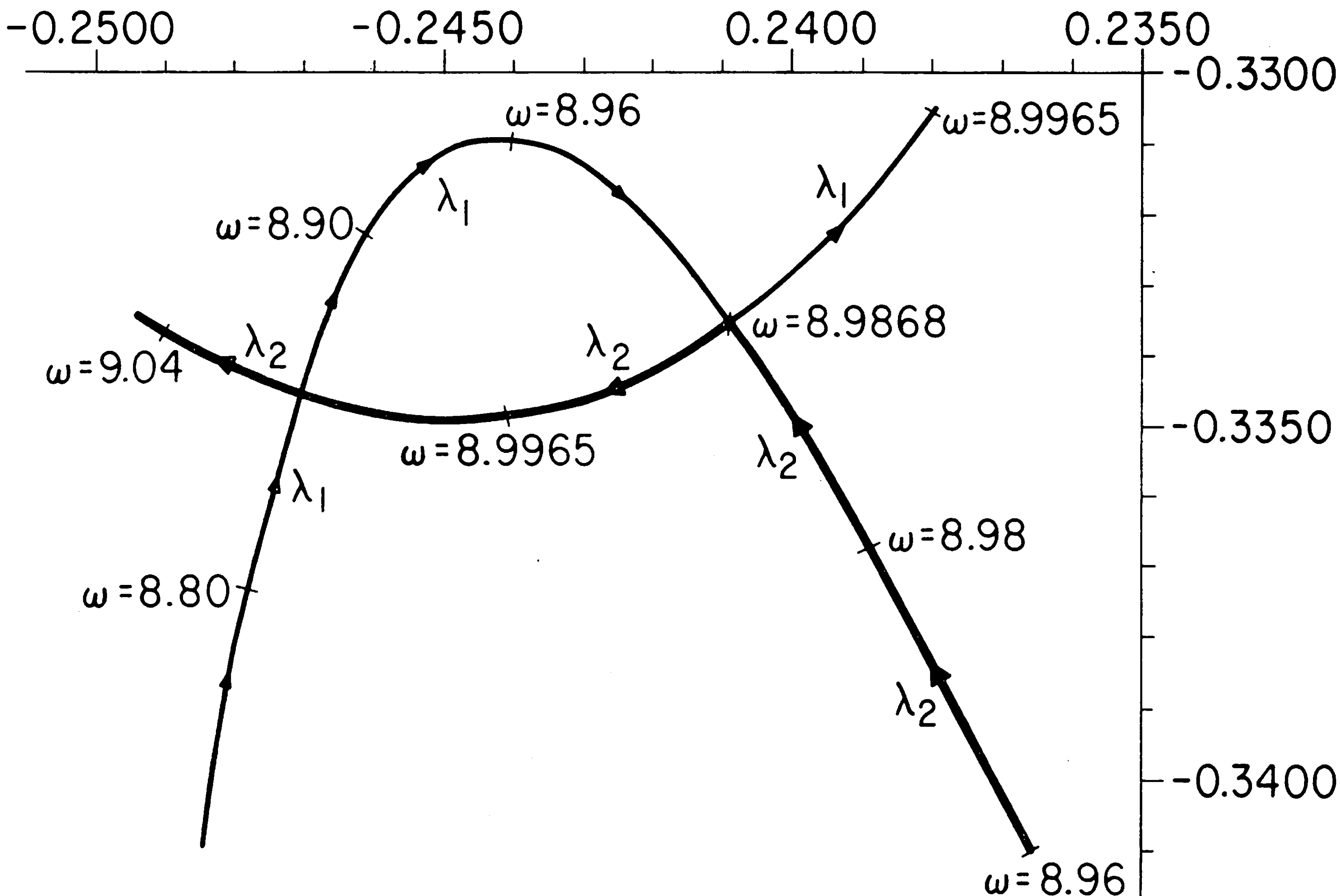

λ-plane
ω = 0-
ω = 0+
λ1
λ1
ω = -∞
ω = +∞
ω = +∞
ω = -∞
λ2
λ2
ω = 0+
ω = 0-



s-plane
Im s
I q+1
jbq
Iq
jbq-1
jbℓ
Iℓ
jbℓ-1
Re s
jb2
I2
jb1
I1

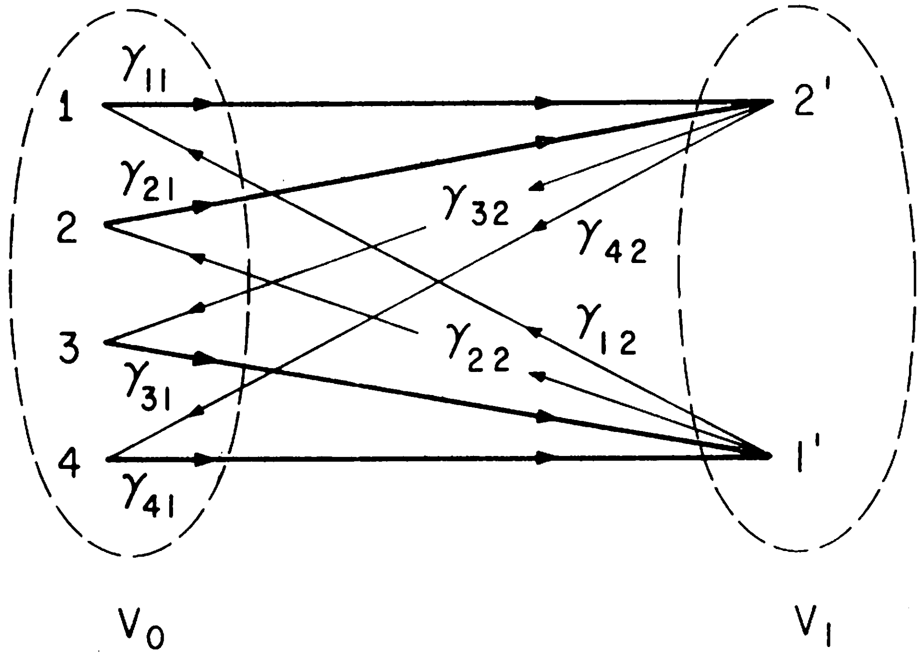


ℂ-plane
0.0
0.5
1.0
1.5
-0.5
-1.0
ω=0
ω=0
ω=2.0
ω=2.0
ω=4.0
ω=4.0
ω=6.0
ω=6.0
ω=8.98
ω=10.0
ω=10.0
ω=21.8
λ1
λ2



-0.2500
-0.2450
0.2400
0.2350
-0.3300
-0.3350
-0.3400
ω=8.96
ω=8.9965
λ1
λ1
ω=8.90
ω=8.9868
λ2
λ2
ω=9.04
ω=8.9965
λ2
λ1
ω=8.98
ω=8.80
λ2
ω=8.96

ℂ - plane



γ_{11}
γ_{21}
γ_{32}
γ_{42}
γ_{12}
γ_{22}
γ_{31}
γ_{41}
1
2
3
4
2'
1'
V_0
V_1