SOLVING NONLINEAR INEQUALITIES IN A FINITE NUMBER OF ITERATIONS

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Solving Nonlinear Inequalities in a Finite Number of Iterations*

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ABSTRACT

This paper describes a modified Newton algorithm for solving a finite system of inequalities in a finite number of iterations.

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1. INTRODUCTION

It is well known that it is possible to solve the following set of inequalities:

$$g^j(x) \le 0, \; j = 1, 2, \ldots, m$$

in a finite number of iterations using standard feasible directions algorithms [1]. However such algorithms are rather slow. Newton's method may also be employed, and has, of course, a quadratic rate of convergence; nevertheless quadratic convergence does not imply finite convergence and examples, for which Newton's method requires an infinite number of iterations, abound. Since many engineering problems, including computer aided design problems [2, 3], require the solution or repeated solution of such inequalities, the reward for efficiency is high. We show in this paper that it is possible to modify Newton's method, preserving its desirable properties, in such a way as to obtain a solution in a finite number of iterations. An alternative approach is presented in [4].

The principle underlying the new algorithm can easily be understood by means of simple example: determine an x such that $g(x) = x^2 - 1 \le 0$. The Newton step p_i^1, if the current point is x_i, is that p of the minimum norm which solves $g(x_i) + g_x(x_i)p = x_i^2 - 1 + 2x_i p \le 0$; $p_i^1 = -(x_i^2-1)/2x_i$, and $x_{i+1} = (1/2)(x_i+1/x_i)$ if the step length is unity and $x_i > 1$. If $x_0 = 2$, the successive points generated by the Newton algorithm are 2, 1.25, 1.025, 1.0003, ... so that $x_i \to 1$ but $x_i > 1$ for all i. On the other hand a first order algorithm would generate a search direction p_i^2 by solving $\min \{g_x(x_i)p \,\big|\, |p| \le \varepsilon\}$ yielding $x_{i+1} = -\varepsilon \text{ sign } (x_i)$. If $x_0 = 2$ and $\varepsilon = 0.01$, the algorithm yields a sequence of points 2, 1.99, 1.98, 1.97, ..., which converges slowly but satisfies the inequality in a finite number (200) of iterations. Our algorithm combines the virtue of both the above algorithms by choosing the search direction p to be the sum of a Newton step and a first order step. For the simple example we are now considering the algorithm generates a search direction $p_i = p_i^1 + p_i^2$, so that $x_{i+1} = (1/2)(x_i+1/x_i) - \varepsilon \text{ sign } (x_i)$. With $\varepsilon = 0.01$, the algorithm generates the following sequence of points: 2, 1.24, 1.013, 0.990, and so achieves a feasible point in three iterations.

The algorithm presented calculates a modified Newton step p_i (a suitable generalization of the step p_i described above) and accepts this step if it satisfies certain tests. If the modified Newton step is not satisfactory, the algorithm utilizes a conventional first order step. The algorithm is described in §2 and its finite convergence is established in §3.. Some numerical examples are presented in §4.

2. THE ALGORITHM

The following notation will be employed. The feasible set is denoted by F:

$$F \triangleq \{x \in R^n | g^j(x) \le 0, \; j = 1,\ldots,m\} \tag{1}$$

The maximum value of the constraint functions is specified by $\psi(\cdot) : R^n \to R$:

$$\psi(x) \triangleq \max \{g^j(x) | j = 1,\ldots,m\} \tag{2}$$

We shall employ $\hat{\psi}(x,p)$ to denote a first order estimate of $\psi(x+p)$, so that:

$$\hat{\psi}(x,p) \triangleq \max \{g^j(x)+g_x^j(x)p \mid j=1,\ldots,m\} \tag{3}$$

Similarly $\theta(x,p)$ is a first order estimate of $\psi(x+p) - \psi(x)$, i.e.

$$\theta(x,p) \triangleq \hat{\psi}(x,p) - \psi(x) \tag{4}$$

We note that $F = \{x \in R^n \mid \psi(x) \leq 0\}$. The "most active" constraint set $I(x)$ is defined by:

$$I(x) \triangleq \{j \in \{1,\ldots,m\} \mid g^j(x) = \psi(x)\} \tag{5}$$

For any x in R^n, a Newton step p^1 is *any* vector in the solution set $P^1(x)$ of the program $L^1(x)$ defined by:

$$\min_p \{\|p\|_\infty \mid g(x)+g_x(x)p \leq 0\} \tag{6}$$

where $g(x) \in R^m$ is the vector whose components are $g^j(x)$, $j = 1, \ldots, m$. For any x in R^n, any Newton step p^1 in $P^1(x)$, we require an additional step p^2, which is *any* vector in the solution set $P^2(x,p^1)$ of the program $L^2(x,p^1)$ defined (for some $\varepsilon_1 > 0$) by:

$$\begin{aligned}\theta^2(x,p^1) &\triangleq \min_p \{\theta(x,p^1+p) \mid \|p\|_\infty \leq \varepsilon_1\} \\ &= \min_p \max \{g^j(x)+g_x^j(x)p^1+g_x^j(x)p-\psi(x),\ j=1,\ldots,m \mid \|p\|_\infty \leq \varepsilon_1\}\end{aligned} \tag{7}$$

Hence, for any x in R^n, the modified Newton step generated by the algorithm is:

$$p = p^1 + p^2 \tag{8}$$

where p^1 is any vector in the solution set $P^1(x)$ and p^2 is any vector in the solution set $P^2(x,p^1)$. Note that (7) reduces to $\min \{g_x(x)(p^1+p) \mid |p| \leq \varepsilon_1\}$, yielding $p^2 = -\varepsilon_1 \operatorname{sgn}(g_x(x))$ when $m = 1$.

The Newton step may not exist or, if it does, it may be unsatisfactory; in such cases the algorithm employs a first order step p^3 which is any vector in the solution set $P^3(x)$ of the program $L^3(x)$ defined by:

$$\theta^3(x) = \min_p \{\theta(x,p) \mid \|p\|_\infty \leq 1\} \tag{9}$$

We employ a standard test on the magnitude of the Newton step (e.g. $\|p^1(x)\| \leq L$) to judge whether it is satisfactory or not. We will use $\underline{m}$ to denote the set $\{1,2,\ldots,m\}$.

We are now able to state our algorithm.

<u>Algorithm to Determine $x \in F$</u>

Parameters: $\gamma \in (0,\frac{1}{2})$, $\beta \in (0,1)$, $\varepsilon_1 > 0$, $L >> 1$.

Data: $x_0 \in R^n$.

Step 0: Set $i = 0$.

Step 1: If $\psi(x_i) \leq 0$, stop.

Step 2: If $P^1(x_i) \neq \phi$, solve $L^1(x_i)$ to obtain p^1. If $\|p^1\|_\infty \leq L$ solve $L^2(x_i,p^1)$ to obtain p^2. Set $p_i = p^1 + p^2$.
If $P^1(x_i) = \phi$ or if $\|p^1\|_\infty > L$, solve $L^3(x_i)$ to obtain p^3 and set $p_i = p^3$.

Step 3: Determine the smallest integer $k_i \geq 0$ such that:

$$\psi(x_i+\beta^{k_i}p_i) - \psi(x_i) \leq \gamma\beta^{k_i}\theta(x_i,p_i)$$

Step 4: Set $x_{i+1} = x_i + \beta^{k_i}p_i$.
Set $i = i+1$ and go to Step 1. □

It is shown later that $L^1(x)$, $L^2(x,p^1)$ and $L^3(x)$ can be transcribed into linear programs. At each iteration of the above algorithm two linear programs have to be solved, L^1 followed by *either* L^2 *or* L^3.

3. CONVERGENCE ANALYSIS

The following assumptions are made:

H1: The function $g(\cdot) : R^n \to R^m$ is continuously differentiable.

H2: The set $\{\nabla g^j(x) | j \in I(x)\}$ is positive linearly independent for all x such that $\psi(x) \geq 0$. □

We note that $\overline{F^c}$ (the closure of the complement of F) satisfies:

$$\overline{F^c} = \{x \in R^n | \psi(x) \geq 0\} \tag{10}$$

This is easily proven. The boundary of F, $\delta(F)$, is clearly a subset of $\{x \in R^n | \psi(x) = 0\}$. Moreover, by H2, if $\psi(x) = 0$, then $\nabla g^j(x) \neq 0$ for all $j \in I(x)$ so that $x \in \delta(F)$. Hence:

$$\delta(F) = \{x \in R^n | \psi(x) = 0\} \tag{11}$$

Since $\delta(F) = \delta(F^c)$, it follows that $\overline{F^c} = \{x \in R^n | \psi(x) \geq 0\}$. Let M denote the set of points at which L^1 has a solution, i.e.

$$M \triangleq \{x \in \overline{F^c} | P^1(x) \neq \phi,\ \|p\|_\infty \leq L \text{ for all } p \in P^1(x)\} \tag{12}$$

We note that if there exists a $p \in P^1(x)$ such that $\|p\|_\infty \leq L$, then $\|p\|_\infty \leq L$ for all $p \in P^1(x)$. Let N denote the complement of M in $\overline{F^c}$ (i.e. $\overline{F^c} = M \cup N$). For all $x \in \overline{F^c}$ (i.e. all x such that $\psi(x) \geq 0$) let $P(x)$ denote the set of possible search directions generated in Step 2 of the algorithm, i.e. $P(x) \triangleq \{p \in R^n | p = p^1 + p^2$ where $p^1 \in P^1(x)$ and $p^2 \in P^2(x,p^1)$ if $x \in M$, and $p \in P^3(x)$ if $x \in N\}$. Also, for all

$x \in \overline{F^c}$, all $p \in P(x)$, let $k(x,p)$ be the smallest integer $k \geq 0$ such that $\psi(x+\beta^k p) - \psi(x) \leq \gamma\beta^k\theta(x,p)$. Finally, for all $x \in \overline{F^c}$ let $A(x)$ be defined by:

$$A(x) \triangleq \{x+\beta^{k(x,p)}p | p \in P(x)\} \tag{13}$$

i.e., given an initial point x, A(x) is the set of possible points generated by Steps 2 and 3 of our algorithm. It is shown in the proof of Theorem 3 that A(x) is well defined. Hence our algorithm, if the stop condition in Step 1 is removed, has the structure of the following model:

<u>Algorithm Model</u>

Data: $x_0 \in \overline{F^c}$.

Step 0: Set $i = 0$.

Step 1: Compute any $x_{i+1} \in A(x_i)$.

Step 2: Set $i = i+1$ and go to Step 1. □

The purpose of our algorithm is to generate a point in F. If our algorithm is such that ψ is reduced at any point in $\overline{F^c}$, and possesses certain continuity properties, then any accumulation point generated by our algorithm cannot be in $\overline{F^c}$, and hence must lie in F^0, the interior of F. A direct consequence of this is that if the algorithm (without a stopping condition) generates an infinite sequence $\{x_i\}$ which has accumulation points, then there exists a finite integer j such that $x_j \in F$. These comments are made precise in the following result.

<u>Theorem 1</u>

Suppose that for all $x \in \overline{F^c}$ there exists an ε, $\delta > 0$ such that

$$\psi(x'') - \psi(x') \leq -\delta \tag{14}$$

for all $x' \in B(x,\varepsilon) \cap \overline{F^c}$ and all $x'' \in A(x')$ where $B(x,\varepsilon) \triangleq \{z \mid \|z-x\|_\infty \le \varepsilon\}$. Then, if an infinite sequence $\{x_i\}$ generated by the algorithm model has an accumulation point x^*, this accumulation point must lie in the interior of F. $\square$

Corollary

(i) If the algorithm model generates a sequence $\{x_i\}$, then either there exists a finite integer j such that $x_j \in F$ or $\{x_i\}$ is an infinite sequence possessing no accumulation points.

(ii) If the set $\{x \mid \psi(x) \le \psi(x_0)\}$ is compact, or the sequence of $\{x_i\}$ generated by the algorithm model is bounded, there exists a finite integer j such that $x_j \in F$. $\square$

The proof of Theorem 1 is essentially the same as that of Theorem 1.3.3 in [1]. Hence, to show that our algorithm has finite convergence (if it generates a bounded sequence) we have to show that A, defined by (13) satisfies the hypothesis (see (14)) of Theorem 1. Our first step is to show that the estimate $\theta(x,p)$ of $\psi(x+p) - \psi(x)$ satisfies $\theta(x,p) < 0$ for all $x \in \overline{F^c}$, all $p \in P(x)$ *and* has certain continuity properties. For convenience we define $\hat{\theta}(\cdot), \hat{\theta}^2(\cdot) : \overline{F^c} \to \mathbb{R}$ as follows:

$$\hat{\theta}(x) \triangleq \sup \{\theta(x,p) \mid p \in P(x)\} \tag{15}$$

$$\hat{\theta}^2(x) \triangleq \sup \{\theta^2(x,p) \mid p \in P^1(x)\} \tag{16}$$

The following properties of $\hat{\theta}(\cdot)$ and $P(\cdot)$ are easily established.

Proposition 1

(a) For all $x \in M$:

$$\hat{\theta}(x) = \hat{\theta}^2(x) \le -\psi(x)$$

(b) For all $x \in N$

$$\hat{\theta}(x) = \theta^3(x)$$

(c) $\|p\|_\infty \le \max\{L+\varepsilon_1, 1\}$ for all $p \in P(x)$, all $x \in \overline{F^c}$

Proof

(a) Let $x \in M$, $p \in P(x)$. Then $p = p^1 + p^2$ where $p^1 \in P^1(x)$, $\|p^1\| \le L$, $p^2 \in P^2(x,p^1)$ so, from (7), $\theta(x,p) = \theta^2(x,p^1)$ for all $p \in P(x)$, all $p^1 \in P^1(x)$. Also, from (7), $\theta^2(x,p^1) \le \theta(x,p^1) \le -\psi(x)$ since, from (6), $\hat{\psi}(x,p^1) = 0$.

(b) Let $x \in N$. Then $p = p^3$ where $p^3 \in P^3(x)$. Hence $\theta(x,p) = \theta^3(x)$.

(c) Follows directly from the definition of $P(\cdot)$. $\square$

We next establish certain properties of $\hat{\theta}^2(\cdot)$ and $\theta^3(\cdot)$.

Proposition 2

(a) For all $x \in \overline{F^c}$, $\theta^3(x) < 0$. Also $\theta^3(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is continuous.

(b) For all $x \in M$, there exists a ρ, $\eta > 0$ such that $\hat{\theta}^2(x') \le -\eta$ for all $x' \in B(x,\rho) \cap M$.

Proof

(a) Let $x \in \overline{F^c}$. It follows from (9) that $\theta^3(x) \le 0$. Since the vectors in $\{\nabla g^j(x) \mid j \in I(x)\}$ are positive linearly independent, there exists a $p \ne 0$ and a $\delta > 0$ such that $g_x^j(x)p \le -\delta$ for all $j \in I(x)$ and $g^j(x) \triangleq \psi(x) - \delta$ for all $j \notin I(x)$. Hence, for all $\alpha \in [0,1]$:

$$g^j(x) + \alpha g_x^j(x)p \le \psi(x) - \alpha\delta \quad \text{for all } j \in I(x)$$

$$g^j(x) + \alpha g_x^j(x)p \le \psi(x) - \delta + \alpha d \quad \text{for all } j \notin I(x) \tag{17}$$

where $d \triangleq \max\{g_x^j(x)p \mid j \in \underline{m}\}$. Hence:

$$\theta^3(x) \le \max\{-\alpha\delta, -\delta+\alpha d\} \tag{18}$$

for all $\alpha \in [0,1]$. Hence $\theta^3(x) < 0$.

Since $\theta^3(x) = \min\{\theta(x,p) \mid p \in S\}$, where $\theta(\cdot,\cdot)$ is continuous and S is the unit cube in $\mathbb{R}^n$, it follows that $\theta^3(\cdot)$ is continuous.

(b) Let $x \in M$. We consider two cases:

(i) $\psi(x) > 0$.

It follows from Proposition 1(a) that $\hat{\theta}^2(x') \leq -\psi(x')$ for all $x' \in M$ where $-\psi(\cdot)$ is continuous (and $-\psi(x) < 0$).

(ii) $\psi(x) = 0$

The function $\theta^2(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous. It is shown in the Appendix that $d(0,P^1(x')) \to 0$ as $x' \to x$ in M (For all $x \in \mathbb{R}^n$, $\Gamma \in 2^{\mathbb{R}^n}$, $d(x,\Gamma) \triangleq \sup\{\|x-y\|_\infty \mid y \in \Gamma\}$.) Since $\psi(x) = 0$, $P^1(x) = \{0\}$ so that $\hat{\theta}^2(x') = \sup\{\theta^2(x',p') \mid p' \in P'(x')\} \to \theta^2(x,0) = \hat{\theta}^2(x)$ as $x' \to x$ in M. But $\theta^2(x,0) = \varepsilon_1\theta^3(x) < 0$. Hence $\hat{\theta}^2(x) < 0$ and $\hat{\theta}^2(\cdot)$ is (relatively) continuous at x on M.

It follows from (i), (ii) that, for all $x \in M$, there exists a ρ, $\eta > 0$ such that $\hat{\theta}^2(x') \leq -\eta$ for all $x' \in B(x,\rho) \cap N$. □

Combining Propositions 1 and 2 we obtain:

<u>Theorem 2</u>

For all $x \in \overline{F^c} = M \cup N$ there exists a η, $\rho > 0$ such that

$$\hat{\theta}(x') \leq -\eta$$

for all $x' \in B(x,\rho) \cap \overline{F^c}$. □

We wish to show that A satisfies the hypothesis of Theorem 1. The first step is showing this is Theorem 2; the next consists in showing that $\theta(x,p)$ is a sufficiently good estimate of $\psi(x+p) - \psi(x)$, i.e. $\hat{\psi}(x,p)$ is a sufficiently good estimate of $\psi(x,p)$.

<u>Proposition 3</u>

Let N be any compact subset of $\overline{F^c}$. There exists a function $(\alpha,x) \mapsto \phi(\alpha,x)$, $\mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ such that:

$$\sup\{|\hat{\psi}(x,\alpha p) - \psi(x+\alpha p)| \mid p \in P(x)\} \leq \alpha\phi(\alpha,x)$$

for all $x \in N$, all $\alpha \in [0,1]$, and

$$\phi(\alpha,x) \to 0$$

uniformly in $x \in N$ as $\alpha \to 0$.

<u>Proof</u>

It follows from Proposition 1(c) that $P(x) \subset \Gamma \triangleq \{p \in \mathbb{R}^n \mid \|p\|_\infty \leq \max\{L+\varepsilon_1, 1\}\}$ for all x in $\overline{F^c}$. Hence:

$$e(\alpha,x) \triangleq \sup\{|\hat{\psi}(x,\alpha p) - \psi(x+\alpha p)| \mid p \in P(x)\}$$

$$\leq \sup\{|\hat{\psi}(x,\alpha p) - \psi(x+\alpha p)| \mid p \in \Gamma\} \qquad (13)$$

Now:

$$\max(A,B) - \max(C,D) \leq \max(A-C, B-D)$$

and:

$$\max(C,D) - \max(A,B) \leq \max(C-A, D-B)$$

so that:

$$|\max(A,B) - \max(C,D)| \leq \max\{|A-C|, |B-D|\}$$

Hence:

$$|\hat{\psi}(x,\alpha p)-\psi(x+\alpha p)| \leq \max \{|\hat{g}^j(x,\alpha p)-g^j(x+\alpha p)|, j \in \underline{m}\} \qquad (20)$$

where:

$$\hat{g}^j(x,p) \triangleq g^j(x) + g^j_x(x)p, \quad j = 1, \ldots, m \qquad (21)$$

Because $g(\cdot)$ is continuously differentiable:

$$|\hat{\psi}(x,\alpha p)-\psi(x+\alpha p)| \leq [\max \{|g^j_x(x)-g^j_x(\xi^j)|, j \in \underline{m}\}]\alpha\|p\|_2 \qquad (22)$$

where, for all $j \in \underline{m}$, ξ^j lies in the line segment $[x,x+\alpha p]$. As $\alpha \to 0$, $|g^j_x(x)-g^j_x(\xi^j)| \to 0$, uniformly in $x \in N$, $p \in \Gamma$ (since $g_x(\cdot)$ is continuous). Since there exists a $d < \infty$ such that $\|p\|_2 \leq d$ for all $p \in \Gamma$, we see that $\phi(\cdot)$, defined by:

$$\phi(\alpha,x) \triangleq d \max_p \max_\xi \max_j \{|g^j_x(x)-g^j_x(\xi)| \,\big|\, j \in \underline{m}, \xi \in [x,x+\alpha p], p \in \Gamma\} \qquad (23)$$

satisfies Proposition 3, i.e.:

$$\epsilon(\alpha,x) \leq \alpha\phi(\alpha,x) \qquad (24)$$

for all $x \in N$, all $\alpha \in [0,1]$ and $\phi(\alpha,x) \to 0$, uniformly in $x \in N$, as $\alpha \to 0$. □

We can now employ Theorem 2 and Proposition 3 to establish an important property of A, as defined in (13).

<u>Theorem 3</u>

Let $\{x_i\}$ be a sequence generated by the algorithm. Then either there exists a finite integer j such that $x_j \in F$, or the sequence $\{x_i\}$ is infinite and has no accumulation points. If $\{x_i\}$ is bounded then there exists a finite integer j such that $x_j \in F$.

<u>Proof</u>

It follows from Theorem 1 that all we have to be is to show

that for $x \in \overline{F^c}$ there exists an ϵ, $\delta > 0$ such that $\psi(x'') - \psi(x') \leq -\delta$ for all $x' \in B(x,\epsilon) \cap \overline{F^c}$, all $x'' \in A(x')$. It is easily checked that the map $\alpha \to \hat{\psi}(x,\alpha p)$ is convex (for all x, p) so that:

$$\hat{\psi}(x,\alpha p) - \psi(x+\alpha p) \leq \alpha\theta(x,p) \qquad (25)$$

for all $x \in F^c$, all $p \in P(x)$, all $\alpha \in [0,1]$. From (25) and Proposition 3, for any compact neighbourhood $N \subset \overline{F^c}$ of x:

$$\psi(x'+\alpha p) - \psi(x') = \psi(x'+\alpha p) - \hat{\psi}(x',\alpha p) + \hat{\psi}(x',\alpha p) - \psi(x')$$

$$\leq \alpha\Theta(x',p) + \alpha\phi(\alpha,x') \qquad (26)$$

for all $x' \in N$, all $p \in P(x')$, all $\alpha \in [0,1]$. From Theorem 2 there exists ρ, $\eta > 0$ such that $\Theta(x',p) \leq \hat{\Theta}(x') \leq -\eta$ for all $x' \in B(x,\rho) \cap \overline{F^c}$, all $p \in P(x')$. Choose N so that $N \subset B(x,\rho) \cap \overline{F^c}$ and an integer $k' < \infty$ such that $\phi(\alpha,x') \leq (1-\gamma)\eta$ for all $x' \in N$, all $\alpha \in [0,\beta^{k'}]$. Then, since $\eta \leq -\Theta(x',p)$:

$$\psi(x'+\alpha p) - \psi(x') \leq \alpha[\Theta(x',p)+(1-\gamma)\eta]$$

$$< \alpha\gamma\Theta(x',p) \qquad (27)$$

for all $x' \in N$, all $p \in P(x')$ all $\alpha \in [0,\beta^{k'}]$. This shown that the algorithm is well defined ($k(x,p) \leq K' < \infty$ for all $p \in P(x)$) and that:

$$\psi(x'') - \psi(x') \leq -\beta^{k'}\gamma\eta \qquad (28)$$

for all $x' \in N$, all $x'' \in A(x')$. Choosing $\epsilon > 0$ so that $B(x,\epsilon) \cap \overline{F^c} \subset N$ and setting $\delta = \beta^{k'}\gamma\eta$ yields the desired result. □

4. NUMERICAL EXAMPLES

Firstly we note that the programs $L^1(x)$, $L^2(x,p^1)$ and $L^3(x)$ are all equivalent to linear programs. Thus $L^1(x)$ is equivalent to:

minimize σ subject to

(a) $p^i \leq \sigma$, $i = 1, \ldots, n$

(b) $-p^i \leq \sigma$, $i = 1, \ldots, n$

(c) $g^j(x) + g_x^j(x)p \leq 0$, $j = 1, \ldots, m$.

$L^2(x,q)$ is equivalent to:

minimize σ subject to

(a) $p^i \leq \varepsilon$, $i = 1, \ldots, n$

(b) $-p^i \leq \varepsilon$, $i = 1, \ldots, n$

(c) $g^j(x) + g_x^j(x)q + g_x^j(x)p \leq \sigma$, $j = 1, \ldots, m$.

$L^3(x)$ is equivalent to:

minimize σ subject to

(a) $p^i \leq 1$, $i = 1, \ldots, n$

(b) $-p^i \leq 1$, $i = 1, \ldots, n$

(c) $g^j(x) + g_x^j(x)p \leq \sigma$, $j = 1, \ldots, m$.

In the following examples the values of the parameters employed were $\beta = 0.1$, $\gamma = 0.1$, $\varepsilon = 0.1$, $L = 100.0$.

Example 1

The feasible set consists of infinitely many discs of radius $\pi/2$ centred at $((2m-\frac{1}{2})\pi, 2n\pi)$ for all integer values of m, n. The set is defined by:

$$\sin x_1 \leq 0$$

$$-\cos x_2 \leq 0$$

With $x_0 = (1,2)$, a feasible point (-3,0416,1.4708) was located in one step (a (modified) Newton step).

Example 2

The feasible set consists of a pair of discs, each of radius $\pi/2$, centred at $(-\pi/2,0)$ and $(3\pi/2,0)$, and is defined by:

$$\sin x_1 \leq 0$$

$$-\cos x_2 \leq 0$$

$$x_1 - 3\pi \leq 0$$

$$x_2 - \pi/2 \leq 2$$

$$-x_1 - \pi \leq 0$$

$$-x_2 - \pi/2 \leq 0$$

With $x_0 = (0,75)$, a feasible point (-3.0416,1.4708) was located in four iterations. The sequence of points generated were (1,74), (1.73), (1,72), (-3.0416,1.4708), the first three steps being first order, and the final one a (modified) Newton step.

Example 3

The feasible set is defined by:

$$(0.999)^2 \leq (x_1^2+x_2^2) \leq 1$$

With $x_0 = (0,15)$ a feasible step was located in four iterations, all steps being of the (modified) Newton type.

Example 4

The feasible set is narrow crescent defined by:

$$(x_1-\tfrac{1}{2})^2 + (x_2-1)^2 \leq 0.25$$

$$-(x_1-\tfrac{1}{2})^2 - (x_1-1.1)^2 \leq 0.26$$

$$x_2 - 1 \leq 0$$

Starting from an initial point (0.5,-6.0) a feasible point is located a five iterations, all of the (modified) Newton type.

Example 6

This problem is based on one suggested by Powell. The vector coordinates $(x_1,0)$, (x_2,x_3) of two vertices of a triangle (the third being (0,0)) and the centres (x_4,x_5), (x_6,x_7) of two discs of unit radius. A point x is feasible if the triangle has a specified area a and the two discs lie inside the triangle and do not overlap each other. These constraints can be specified as:

$$x_1 \geq 0$$

$$x_3 \geq 0$$

$$(x_4-x_6)^2 + (x_5-x_7)^2 - 4 \geq 0$$

$$x_i - 1 \geq 0, \quad i - 5,\ 7$$

$$\frac{x_3x_6-x_2x_{i+1}}{(x_2^2+x_3^2)^{\frac{1}{2}}} - 1 \geq 0, \quad i = 4,\ 6$$

$$\frac{(x_2-x_1)x_{i+1}+(x_1-x_i)x_3}{[x_3^2+(x_2-x_1)^2]^{\frac{1}{2}}} - 1 \geq 0, \quad i = 4,\ 6$$

$$a - \tfrac{1}{2}x_1x_3 \geq 0$$

With x_0 = ((3),(0,2),(-1.5,1-5),(5,0)) and a = 12, a feasible point ((5.8381),(0.4448,4.1062),(1.250,2.3237),(2.8434,1.0505)) was achieved in six iterations. The minimum value for a has been obtained by Powell as 11.6569.

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APPENDIX

Proposition A1

Let $x \in M$ satisfy $\psi(x) = 0$. Then $d(0,P^1(x')) \triangleq \sup\{\|p\|_\infty \mid p \in P^1(x')\} \to 0$ as $x' \to x$ in M.

Proof

For all $x' \in M$ let $\Gamma(x')$ be defined by:

$$\Gamma(x') \triangleq \{p \in R^n \mid g(x') + g_x(x')p \leq 0\}$$

Clearly $P^1(x') \subset \Gamma(x')$. Since $\psi(x) = 0$, it follows that $P^1(x) = \{0\}$ so that $0 \in \Gamma(x) \neq \phi$. Hence $\mathrm{dist}\,(0,\Gamma(x)) \triangleq \inf\,\{\|p\|_\infty \mid p \in \Gamma(x)\} = 0$. Since $\{\nabla g^j(x),\ j \in I(x)\}$ are positive linearly independent, there exists a $p \in R^n$ such that $g^j(x) + g^j_x(x)p < 0$, $j = 1, \ldots, m$; it follows from the continuity of $g(\cdot)$ and $g_x(\cdot)$ and [2, Theorem 4.2] that $\mathrm{dist}\,(0,\Gamma(x')) \to 0$ as $x' \to x$ in M, for all $x \in \delta(F)$. But $\mathrm{dist}\,(0,\Gamma(x')) = \inf\,\{\|p\|_\infty \mid p \in \Gamma(x')\} = \|p\|_\infty$ for any p in $P^1(x')$ by virtue of the definition of $P^1(\cdot)$. Hence $d(0,P^1(x')) \to 0$ as $x' \to x$ in M. □