A SUPERLINEARLY CONVERGENT ALGORITHM FOR CONSTRAINED OPTIMIZATION PROBLEMS

by

D. Q. Mayne and E. Polak

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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D.Q. Mayne[†] and E. Polak[††]

ABSTRACT

This paper presents an algorithm which is globally convergent and whose rate of convergence is superlinear. The superlinear rate of convergence is achieved by using a search arc which solves a quadratic approximation to the original program, and global convergence is obtained by using an exact penalty function to determine step length. The algorithm incorporates a rule for choosing the penalty parameter, and employs a search arc rather than a search direction to avoid truncation of the step length near solution points.

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† Department of Computing and Control, Imperial College of Science and Technology, London SW7 2BZ

†† Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, California 94720

1. INTRODUCTION

There exist many algorithms, such as the constrained Newton method of Levitan and Polyak [1], Wilson's method [2] and Robinson's method [3], for solving constrained optimization problems, which have a superlinear rate of convergence but are only locally convergent. A major difficulty in globally stabilizing these algorithms is the fact that they generate sequences which are not necessarily feasible, making comparison of successive points difficult. A way of overcoming this difficulty was supplied by Han [4] and later, but independently, by Mayne and Maratos [5] and Maratos [6], who suggested using an exact penalty function to order possibly non-feasible points, thereby permitting a suitable choice of step-length to be made. The computational results of Powell [7] and of Mayne and Maratos [5,6] show that this class of algorithms, in which the search direction is determined by a suitable approximation to the original problem and step length determined by (approximately) minimizing an exact penalty function, has considerable promise. These algorithms differ conceptually, even though there may exist many similarities, from the pioneering algorithms of Conn [8] and Conn and Pietrzkyowski [9] which directly generate descent directions for the (non-differentiable) exact penalty function. In the former, the search direction is determined first and the penalty function parameter c is then chosen to ensure that the resultant search direction is a descent direction for the exact penalty function; in the latter the penalty function parameter c is first chosen (to ensure equivalence of the original and exact penalty function problems) and then a descent direction for the resultant exact penalty function is chosen.

Although the new procedure brings advantages, it also causes several difficulties which have been listed by Han [4]. The first and

most obvious of these is the choice of the penalty function parameter c. Much of the literature [4,8,9] assumes that a suitable value for c is available a priori. Powell [7] gives a heuristic rule for iteratively choosing c; since global convergence is not established, cycling may occur. In contrast to this, the algorithm presented here incorporates a procedure for iteratively choosing c in such a way that global convergence, to points satisfying Kuhn-Tucker conditions of optimality, is ensured. The procedure is based on similar procedures successfully employed in [5], [6] and [10], and on a model devised by Polak [11] for iteratively choosing a parameter.

The second problem concerns the minimization of the exact penalty function along search directions. Han [4] (like Conn [8] and Conn and Pietrzykowski [9]) employs exact minimization but emphasizes the desirability of approximate procedures, like those of Armijo and Goldstein, which, however, cannot be directly employed because of the non-differentiability of the exact penalty function. Our algorithm incorporates an Armijo type procedure for choosing step length; the procedure is shown to have the necessary properties required for convergence and is based on that employed in [5] and [6]. It is interesting to note that Powell, although he supplies no theoretical justification, has employed a similar procedure [7].

A third problem is that of step length as a solution point is approached. Unless the step length is asymptotically unity, superlinear convergence is not achieved. Han [4] therefore suggests that the step length procedure described above be discarded near solution points but there is no known automatic procedure for doing this. Powell [7] does not discuss this point. Maratos [6], however, has shown that the problem cannot be ignored in that there exist problems for which the step length procedure produces step lengths of less than unity, no matter how close the current iterate is to a solution point. Maratos [6] has therefore proposed replacing the search direction by a search arc, yielding asymptotic step lengths of unity (for the equality constrained minimization problem). We employ an appropriate modification of this procedure.

A final problem concerns the suitability of the search direction (if it exists) yielded by the quadratic approximation to the original problem. An analagous problem arises in employing Newton's method for unconstrained optimization problems; search directions so generated may not be descent directions. A common method to achieve global convergence replaces the Newton direction by a descent direction whenever the former is not suitable. After investigating several alternatives we concluded that an appropriate generalization of this technique, to make it suitable for constrained optimization, was to be preferred. Accordingly we developed a relatively simple first order method for minimizing the (non-differentiable) exact penalty function. This sub-algorithm is employed whenever the search direction, yielded by the quadratic approximation, does not satisfy certain tests.

In the next section we motivate and describe the main algorithm. In §3 we describe and establish convergence of the first order sub-algorithm and in §4 we establish global convergence properties of the main algorithm. Finally, in §5, we show that the main algorithm has a superlinear convergence rate.

2. THE ALGORITHM

We consider the following nonlinear program:

$$\min \{f(x) \mid g(x) \leq 0,\ h(x) = 0\} \tag{2.1}$$

where $f : R^n \to R$, $g : R^n \to R^m$ and $h : R^n \to R^{m_e}$. The quadratic approximation QP(x,H) to this problem is defined to be the program

$$\min \{f_x(x)p + \tfrac{1}{2}p^T Hp \mid p \in \hat{F}(x)\} \tag{2.2}$$

$$\hat{F}(x) \triangleq \{p \in R^n \mid g(x)+g_x(x)p \leq 0,\ h(x)+h_x(x)p = 0\} \tag{2.3}$$

If $p \in \hat{F}(x)$, then x+p satisfies the constraints to first order. Thus the set $x + \hat{F}(x)$ is a first order approximation to the feasible set F, defined by:

$$F \triangleq \{x \in R^n \mid g(x) \leq 0,\ h(x) = 0\} \tag{2.4}$$

H is an approximation to $L_{xx}(x,\lambda,\mu)$ where the Lagrangian $L : R^n \times R^m \times R^{m_e} \to R$ is defined by:

$$L(x,\lambda,\mu) \triangleq f(x) + \lambda^T g(x) + \mu^T h(x) \tag{2.5}$$

If p solves (2.2) (with suitable values for λ and μ) then x+p approximately solves (2.1). Solving (2.2) also yields multipliers which may be employed to calculate the next approximate to L_{xx}. To compare non-feasible points we introduce an exact penalty function $\gamma : R^n \times R \to R$ defined by:

$$\gamma(x,c) = f(x) + c\psi(x) \tag{2.6}$$

where $\psi : R^n \to R$ is defined by:

$$\psi(x) = \max \{0;\ g^j(x),\ j \in \underline{m};\ |h^j(x)|,\ j \in \underline{m}_e\} \tag{2.7}$$

$\underline{m}$ denotes the set $\{1,2,\ldots,m\}$ and $\underline{m}_e$ the set $\{1,2,\ldots,m_e\}$. Other exact penalty functions such as those employed by Han [4], Maratos and Mayne [5] and Powell [7] may be employed. We note that x is feasible ($x \in F$) if and only if $\psi(x) = 0$.

We wish to know whether a search direction, determined by solving QP(x,H), is a descent direction for $\gamma(x,c)$. To do this we define the following first order estimates: $\hat{\psi} : R^n \times R^n \to R$, is defined by:

$$\hat{\psi}(x,p) \triangleq \max \{0;\ g^j(x)+g_x^j(x)p,\ j \in \underline{m};\ |h^j(x)+h_x^j(x)p|,\ j \in \underline{m}_e\} \tag{2.8}$$

$\hat{\gamma} : R^n \times R^n \times R \to R$, is defined by:

$$\hat{\gamma}(x,p,c) \triangleq f(x) + f_x(x)p + c\hat{\psi}(x,p) \tag{2.9}$$

These estimates are obtained by replacing f, g and h, in the definitions of ψ and γ, by their first order estimates. Provided that f, g, h are continuously differentiable, it is easily shown (see the appendix) that $\hat{\psi}$ and $\hat{\gamma}$ are first order estimates of, respectively, ψ and γ, i.e. that, for all x, p and c:

$$|\psi(x+p)-\hat{\psi}(x,p)| = o(||p||) \tag{2.10}$$

and:

$$|\gamma(x+p,c)-\hat{\gamma}(x,p,c)| = o(||p||) \tag{2.11}$$

For our purpose the (convex) estimates $\hat{\psi}$ and $\hat{\gamma}$ are more useful than the (linear) directional derivatives. p is said to be a descent direction for $\gamma(x,c)$ if:

$$\hat{\gamma}(x,p,c) < \gamma(x,c) \tag{2.12}$$

It follows from (2.11) that, if p is a descent direction for $\gamma(x,c)$, then there exists an $\alpha_1 > 0$ such that $\gamma(x+\alpha p,c) < \gamma(x,c)$ for all $\alpha \in (0,\alpha_1]$.

It will be convenient to introduce the function $\theta : R^n \times R^n \times R \to R$ defined by:

$$\theta(x,p,c) = \hat{\gamma}(x,p,c) - \gamma(x,c) \tag{2.13}$$

$\theta(x,p,c)$ is a first order estimate of $\gamma(x+p,c) - \gamma(x,c)$.

That a solution p to QP(x,H) can be a descent direction for $\gamma(x,c)$, provided c is large enough, is now established.

<u>Proposition 1</u>

Let f, g, h be continuously differentiable and let $\{p,\lambda,\mu\}$ be a Kuhn-Tucker triple for QP(x,H). Then:

$$\theta(x,p,c) \le -p^T Hp - [c - \sum_{j=1}^{m} \lambda^j - \sum_{j=1}^{m_e} |\mu^j|]\psi(x) \tag{2.14}$$

<u>Proof</u>

A Kuhn-Tucker triple $\{p,\lambda,\mu\}$ for QP(x,H) satisfies the following well known conditions:

$$\nabla f(x) + Hp + g_x^T(x)\lambda + h_x^T(x)\mu = 0 \tag{2.15}$$

$$h(x) + h_x(x)p = 0 \tag{2.16}$$

$$g(x) + g_x(x)p \le 0 \tag{2.17}$$

$$\lambda \ge 0 \tag{2.18}$$

$$\lambda^T(g(x)+g_x(x)p) = 0 \tag{2.19}$$

It follows from (2.9) that:

$$\theta(x,p,c) \triangleq \hat{\gamma}(x,p,c) - \gamma(x,c)$$

$$= f_x(x)p + c(\hat{\psi}(x,p)-\psi(x)) \tag{2.20}$$

From (2.8), (2.16) and (2.17) we have:

$$\hat{\psi}(x,p) = 0 \qquad (2.21)$$

Hence, for p a solution of $Q(x,H)$, $\theta(x,p,c) = f_x(x)p - c\psi(x)$.

From (2.15):

$$f_x(x)p = p^T\nabla f(x)$$

$$= -p^T Hp - \lambda^T g_x(x)p - \mu^T h_x(x)p \qquad (2.22)$$

From (2.16):

$$\mu^T h_x(x)p = -\mu^T h(x) \qquad (2.23)$$

From (2.19)

$$\lambda^T g_x(x)p = -\lambda^T g(x) \qquad (2.24)$$

Hence, from (2.20)-(2.24):

$$\theta(x,p,c) = -p^T Hp + \lambda^T g(x) + \mu^T h(x) - c\psi(x) \qquad (2.25)$$

An upper bound for the right hand side of (2.25) can be obtained by noting that, since $\lambda \geq 0$, $\lambda^j g^j(x) \leq \lambda^j \psi(x)$ for all $j \in \underline{m}$ and that $\mu^j h^j(x) \leq |\mu^j|\ |h^j(x)| \leq |\mu^j|\psi(x)$ for all $j \in \underline{m}_e$. Hence:

$$\theta(x,p,c) \leq -p^T Hp - [c - \sum_{j=1}^{m}\lambda^j - \sum_{j=1}^{m_e}|\mu^j|]\psi(x) \qquad (2.26)$$

which is the desired result. □

An obvious corollary is that, if $\{p,\lambda,\mu\}$ is a Kuhn-Tucker triple for $Q(x,H)$, then p is a descent direction for $\gamma(x,c)$ if $p^T Hp > 0$ (e.g. if H is positive definite) and $c > \sum_{j=1}^{m}\lambda^j + \sum_{j=1}^{m_e}|\mu^j|$. Consequently the following algorithm is worth considering:

Data: $x_0 \in R^n$, $H_0 \in R^{n\times n}$, $\beta \in (0,1)$

Step 0: Set $i = 0$

Step 1: Compute the Kuhn-Tucker triple $\{p_i,\lambda_i,\mu_i\}$ for $QP(x_i,H_i)$

Step 2: Choose $c_i > \sum_{j=1}^{m}\lambda_i^j + \sum_{j=1}^{m_e}|\mu_i^j|$

Step 3: Choose k_i, the first integer k in the sequence 0, 1, 2, ... satisfying:

$$\gamma(x_i+\beta^{k_i}p_i,c_i) - \gamma(x_i,c_i) \leq \beta^{k_i}\theta(x_i,p_i,c_i)/8$$

Step 4: Set $x_{i+1} = x_i + \beta^{k_i}p_i$

Update H_i to H_{i+1}

Set $i = i+1$

Go to Step 1

This algorithm is appealing in its simplicity. Powell [7] has studied the rate of convergence of a similar algorithm, *under the assumption* that it produces convergent sequences. Note that, since $\hat{\gamma}(x,\alpha p,c)$ is convex in α (for all x, p, c):

$$\theta(x,\alpha p,c) = \hat{\gamma}(x,\alpha p,c) - \gamma(x,c)$$

$$\leq \alpha[\hat{\gamma}(x,p,c)-\gamma(x,c)]$$

$$= \alpha\theta(x,p,c) \qquad (2.27)$$

so that Step 3 is an Armijo type step, with $\beta^{k_i}\theta(x_i,p_i,c_i)$ replacing the usual linear estimate of the change in the cost function as x changes from x_i to $x_i+\beta^{k_i}p_i$. However global convergence has not been (and possibly will not be) established for this algorithm for the following reasons:

(a) A solution for $QP(x_i,H_i)$ may not exist, either because the problem is infeasible ($\hat{F}(x_i) = \phi$) or H_i is not positive definite.

(b) Even if a solution exists, or if $\hat{F}(x_i)$ is replaced (as suggested in [7]) by a larger non-empty set $\tilde{F}(x_i)$) the Kuhn-Tucker triple $\{p_i,\lambda_i,\mu_i\}$ for $QP(x_i,H_i)$ and, consequently, $\theta(x_i,p_i,c)$ may vary discontinuously with x_i. Unless θ has some semi-continuity properties it seems unlikely that global convergence can be established.

(c) The choice of c_i in Step 2, as it stands, is heuristic, and may result in cycling. Moreover the test function should vary continuously with x_i (see [11]) and so the possible discontinuity of the multipliers may cause trouble.

(d) Step 3 may generate steps whose lengths are strictly less than unity no matter how close x_i is to a solution point $\hat{x}$.

Our response to these difficulties is as follows. If $QP(x_i,H_i)$ is infeasible, or if the search direction p_i obtained by solving $QP(x_i,H_i)$ is unsuitable, then a search direction is generated using a robust first order descent algorithm for γ. The test for c_i is modified to conform with Polak's algorithm model [11], and continuous estimates for the multipliers are employed in place of those yielded by solving $QP(x_i,H_i)$. Although p_i may consequently occasionally fail to be a descent direction for $\gamma(x_i,c_i)$, the first order algorithm will take over and no difficulty will ensue. Finally the search direction will be replaced by an arc. Near a Kuhn-Tucker point for the original problem, the solution to $QP(x_i,H_i)$ is guaranteed to possess continuity properties which ensure that p_i will be accepted for all subsequent iterations, resulting in superlinear convergence.

Our new estimates for the multipliers are modifications of those proposed by Glad and Polak [12]. The estimators $\bar{\lambda} : R^n \to R^m$ and $\bar{\mu} : R^n \to R^{m_e}$ are defined by:

$$(\bar{\lambda}(x),\bar{\mu}(x)) \triangleq \arg\min_{(\lambda,\mu)} \{||\nabla f(x)+g_x^T(x)\lambda+h_x^T(x)\mu||^2 + \lambda^T G(x)\lambda+\mu^T K(x)\mu\} \tag{2.28}$$

where G and K : $R^n \to R^{n\times n}$ are defined by:

$$G(x) \triangleq \text{diag}\,\{(\psi(x)-g^j(x))^2\} \tag{2.29}$$

$$K(x) \triangleq \text{diag}\,\{(\psi(x)-|h^j(x)|)^2\} \tag{2.30}$$

$\bar{\lambda}$ and $\bar{\mu}$ are continuous functions and if $\{\hat{x},\hat{\lambda},\hat{\mu}\}$ is a Kuhn-Tucker triple for problem 2.1, then $\hat{\lambda} = \bar{\lambda}(x)$ and $\hat{\mu} = \bar{\mu}(x)$. This will be established later. We also define the function $\bar{c} : R^n \to R$ by:

$$\bar{c}(x) \triangleq \max\,\{[\sum_{j=1}^{m} \bar{\lambda}^j(x) + \sum_{j=1}^{m_e} |\bar{\mu}^j(x)|+b],b\} \tag{2.31}$$

where $b \in (0,\infty)$ is an arbitrary, "small" constant. We also require a test function $T : R^n \to R$ defined by:

$$T(x) \triangleq \min\,\{\varepsilon,[\psi(x)+||\nabla f(x)+g_x^T(x)\bar{\lambda}(x)+h_x^T(x)\bar{\mu}(x)||^2]^2\} \tag{2.32}$$

Assume that our first order sub-algorithm generates, at each x and c, a search direction and $\bar{p}(x,c)$. Let $\hat{I}_i$ denote the set of active inequality constraints predicted by the quadratic program at iteration i, i.e.

$$\hat{I}_i \triangleq \{j|\lambda_i^j > 0\} \tag{2.33}$$

where $\{p_i,\lambda_i,\mu_i\}$ is the Kuhn-Tucker triple for $QP(x_i,H_i)$. For each i, $\tilde{p}_i$ denotes the solution, if a solution with norm less than or equal to $||p_i||$ exists, of:

$$\begin{aligned} g^j(x_i+p_i) + g_x^j(x)p &= 0,\ j \in \hat{I}_i \\ h(x_i+p_i) + h_x(x)p &= 0 \end{aligned} \tag{2.34}$$

$\tilde{p}_i$ denotes the zero vector otherwise. We can now specify our algorithm:

<u>Main Algorithm</u>

Data: $x_1 \in R^n$; $b, c_0, \delta \in (0,\infty)$; $\beta, \delta_1 \in (0,1)$, H_1

Step 0: Set $i = 1$, set $j = 0$

Step 1: If $c_{i-1} \geq \bar{c}(x_i)$, set $c_i = c_{i-1}$

If $c_{i-1} < \bar{c}(x_i)$, set $c_i = \max\{c_{i-1}+\delta, \bar{c}(x_i)\}$

Step 2: *If*: (α) Solution p_i of $QP(x_i,H_i)$ exists

(β) $||p_i|| \leq \delta_1^j$ (2.35)

(γ) $\theta(x_i,p_i,c_i) \leq -T(x_i)$ (2.36)

Then: (a) Compute $\tilde{p}_i$ (see (2.34))

(b) Compute k_i, the first integer in the sequence 0, 1, 2, ... satisfying:

$$\gamma(x_i+\beta^{k_i}p_i+\beta^{2k_i}\tilde{p}_i,c_i) - \gamma(x_i,c_i) \leq \beta^{k_i}\theta(x_i,p_i,c_i)/8 \quad (2.37)$$

(c) Set $x_{i+1} = x_i + \beta^{k_i}p_i + \beta^{2k_i}\tilde{p}_i$

(d) Update H_i to H_{i+1}

(e) Set $i = i+1$, $j = j+1$ and go to Step 1

Else: proceed to Step 3.

Step 3: (a) Compute $\bar{p}(x_i,c_i)$ (by solving (3.1))

(b) Compute the smallest integer k_i in the sequence 0, 1, 2, ... such that:

$$\gamma(x_i+\beta^{k_i}\bar{p})x_i,c_i),c_i) - \gamma(x_i,c_i) \leq \beta^{k_i}\theta(x_i,\bar{p}(x_i,c_i),c_i)/4$$

(2.38)

(c) Set $x_{i+1} = x_i + \beta^{k_i}\bar{p}(x_i,c_i)$

(d) Update H_i to H_{i+1}

(e) Set $i = i+1$ and go to Step 1 □

For simplicity in analysis, a stopping test, based on satisfaction of the Kuhn-Tucker conditions (e.g. $T(x) = 0$) has been excluded but should be incorporated in a practical algorithm. Note that i is the iteration number and j the number of times that a solution p_i of $QP(x_i,H_i)$ has been accepted as a search direction. Several possibilities exist for updating H_i, but this choice does not affect the global convergence. We now specify the first order sub-algorithm used in Step 3.

3. <u>FIRST ORDER SUB-ALGORITHM</u>

For all x, c the first order search direction is any p (p will later be shown to be unique) which solves the following program:

$$\theta'(x,c) = \min_p \{(n/2)||p||^2+\theta(x,p,c)\} \quad (3.1)$$

where $\theta(x,p,c)$ denotes $\hat{\gamma}(x,p,c) - \gamma(x,c)$. The search direction (3.1) is easily seen, using (2.20), to be equivalent to the following quadratic program:

$$\theta'(x,c) = \min_{p,\xi} \{(n/2)||p||^2+\langle\nabla f(x),p\rangle+c(\xi-\psi(x)) \,|\, g^j(x)+g_x^j(x)p \leq \xi,\ j \in \underline{m};\ |h^j(x)+h_x^j(x)p| \leq \xi,\ j \in \underline{m}_e;\ \xi \geq 0\} \quad (3.2)$$

Let

$$I(x) \triangleq \{j \in \underline{m} \,|\, g^j(x) = \psi(x)\}$$

and

$$I_e(x) \triangleq \{j \in \underline{m}_e \,|\, |h^j(x)| = \psi(x)\}$$

Let $P'(x,c)$ denote the solution set of (3.1), and let the following assumptions be satisifed:

H1.: The functions f, g and h are continuously differentiable.

H2.: For all x, the vectors $\{\nabla g^j(x),\ j \in I(x);\ \nabla h^k(x),\ k \in I_e(x)\}$ are linearly independent.

Then we have the following result.

Proposition 2

(i) For all $c \geq 0$ the function $\theta'(\cdot,c) : R^n \to R$ is continuous.

(ii) For all (x,c), $c \geq 0$, $\theta'(x,c) \leq 0$.

(iii) For all (x,c), $c \geq 0$, the solution to (3.1) is unique, i.e. $P'(x,c) = \{\bar{p}(x,c)\}$.

(iv) For all $c \geq 0$, the function $\bar{p}(\cdot,c) : R^n \to R^n$ is continuous.

Proof

(ii) is established by setting p = 0 in the right hand side of (3.1). Next we note that (3.1) is equivalent to:

$$\theta'(x,c) = \min_p \{(n/2)||p||^2+\theta(x,p,c) \mid p \in C\} \tag{3.3}$$

where C is any compact set which includes $\Gamma(x,c)$ defined by:

$$\Gamma(x,c) \triangleq \{p \mid (n/2)||p||^2+\theta(x,p,c) \leq 0\} \tag{3.4}$$

Since $\theta(x,p,c) = \langle\nabla f(x),p\rangle+c\hat{\psi}(x,p)-c\psi(x)$ (3.3) and $\hat{\psi}(x,p) \geq 0$ we see that $\Gamma(x,c) \subset \tilde{\Gamma}(x,c)$ where:

$$\tilde{\Gamma}(x,c) \triangleq \{p \mid (n/2)||p||^2+\langle\nabla f(x),p\rangle-c\psi(x) \leq 0\} \tag{3.5}$$

But $p \in \tilde{\Gamma}(x,c)$ implies that:

$$(n/2)||p||^2 \leq ||\nabla f(x)||\ ||p|| + c\psi(x) \tag{3.6}$$

and hence that:

$$||p|| \leq (1/n)||\nabla f(x)|| + [(1/n^2)||\nabla f(x)||^2 + 2(c/n)\psi(x)]^{1/2} \tag{3.7}$$

Let $N(x)$ denote the right hand side of (3.7). Clearly the map $x \mapsto N(x)$ is continuous and $\tilde{\Gamma}(x,c) \subset \{p \mid ||p|| \leq N(x)\}$. Hence, for any $\hat{x} \in R^n$, any $\hat{\varepsilon} > 0$, there exists a compact set C such that $\Gamma(x,c) \subset \tilde{\Gamma}(x,c) \subset C$ for all $x \in B(\hat{x},\varepsilon)$, and hence that $\theta'(x,c)$ is defined by (3.3) for all such x. Hence [13, p. 116] $\theta'(\cdot,c)$ is continuous and $P'(\cdot,c)$ is upper semi-continuous at $\hat{x}$. From the strict convexity of $(n/2)||\cdot||^2 + \theta(x,\cdot,c)$, $P'(x,c)$, the solution set, has a single element $\bar{p}(x,c)$. Since $P'(\cdot,c) = \{\bar{p}(\cdot,c)\}$ and is upper semi-continuous at $\hat{x}$ it follows [13] that $\bar{p}(\cdot,c)$ is continuous at $\hat{x}$, and since $\hat{x}$ is arbitrary, continuous. □

Let $A_c' : R^n \to R^n$ be defined by Step 3 of the main algorithm, i.e.

$$A_c'(x) \triangleq x + \beta^{k(x,c)}\bar{p}(x,c) \tag{3.8}$$

where $k(x,c)$ is the least integer in $\{0,1,2\ldots\}$ satisfying:

$$\gamma(x+\beta^k\bar{p}(x,c),c) - \gamma(x,c) \leq \beta^k\theta(x,\bar{p}(x,c),c)/4 \tag{3.9}$$

Let D_c be defined by:

$$D_c \triangleq \{x \mid \theta'(x,c) = 0\} \tag{3.10}$$

It follows from (3.1) that:

$$\theta'(x,c) = (n/2)||\bar{p}(x,c)||^2 + \theta(x,\bar{p}(x,c),c) \tag{3.11}$$

and hence, for all x, $c \geq 0$, that $\theta(x,\bar{p}(x,c),c) \leq \theta'(x,c) \leq 0$. Hence $\tilde{D}_c$ defined by:

$$\tilde{D}_c = \{x \mid \theta(x,\bar{p}(x,c),c) = 0\} \tag{3.12}$$

satisfies $\tilde{D}_c \subset D_c$.

Suppose now $x \in D_c$ so that $\theta'(x,c) = 0$. It follows from (3.11) that $\theta(x,\bar{p}(x,c),c) = -(n/2)||\bar{p}||^2 \leq 0$. If $\theta(x,\bar{p}(x,c),c) = -\rho < 0$, then (from (2.27)) $\theta(x,\alpha\bar{p}(x,c),c) \leq -\alpha\rho$ while $(\eta/2)||\alpha\bar{p}(x,c)||^2 = \alpha^2\rho$ so that $\theta'(x,c) \leq \alpha^2\rho - \alpha\rho = \alpha\rho(1-\alpha) < 0$ for $\alpha = \frac{1}{2}$ say. This is a contradiction. Hence $\theta'(x,c) = 0 \Rightarrow \theta(x,\bar{p}(x,c),c) = 0$, i.e. $D_c \subset \tilde{D}_c$. Hence $D_c = \tilde{D}_c$.

For convenience in the sequel we denote $\theta(x,\bar{p}(x,c),c)$ by $\theta(x)$. Note that $x \mapsto \theta(x)$ is continuous. We can now establish a result which will be employed in the next section to establish global convergence of our main algorithm.

<u>Proposition 3</u>

(i) $x \in D_c$ is a necessary condition of optimality for the unconstrained program:

$$\min_x \{\gamma(x,c)\}$$

(ii) For all $x \notin D_c$, there exists an $\varepsilon_1 > 0$, $\delta_1 > 0$ such that:

$$\gamma(A_c'(x'),c) - \gamma(x',c) \leq -\delta_1 \quad \text{for all } x' \in B(x,\varepsilon_1) \tag{3.13}$$

(iii) $D_c = \tilde{D}_c$.

<u>Proof</u>

(i) Follows from (ii). Suppose $x \notin D_c$. Then $\theta(x) \triangleq \theta(x,\bar{p}(x,c),c) < 0$. Now, from (2.27):

$$\hat{\gamma}(x',\alpha\bar{p}(x',c),c) - \gamma(x',c) \leq \alpha\theta(x') \tag{3.14}$$

for all $\alpha \in [0,1]$. It is shown in the appendix that, given any compact neighbourhood N of x,

$$|\gamma(x'+\alpha\bar{p}(x',c),c) - \hat{\gamma}(x',\alpha\bar{p}(x',c),c)| \leq \alpha\phi(\alpha,x') \tag{3.15}$$

where $\phi(\alpha,x') \to 0$, uniformly in $x' \in N$, as $\alpha \to 0$. Hence there exists a positive integer k' such that $\phi(\beta^{k'},x') \leq -\theta(x)/4$ for all $x' \in N$.

Hence, from (3.14 and (3.15):

$$\gamma(x'+\beta^k\bar{p}(x',c),c) - \gamma(x',c) \leq \beta^k\ (\theta(x') - \theta(x)/4) \tag{3.16}$$

for all $k \geq k'$, for all $x' \in N$. If we now choose an $\varepsilon_1 > 0$ such that $B(x,\varepsilon_1) \subset N$ and $\theta(x') \leq \theta(x)/2$ for all $x' \in B(x,\varepsilon_1)$, we obtain:

$$\gamma(x'+\beta^k\bar{p}(x',c),c) - \gamma(x',c) \leq \beta^k\theta(x)/4 \tag{3.17}$$

for all $x' \in B(x,\varepsilon_1)$, for all $k \geq k'$. From (3.9), $k(x',c) \leq k'$ so that $\gamma(A_c(x'),c) - \gamma(x',c) \leq \beta^{k'}\theta(x)/4 \triangleq \delta$ for all $x' \in B(x,\varepsilon_1)$, thus establishing (i) and (ii). Part (iii) has been proven above. □

It follows from Theorem 1.3.3 in [14] that any accumulation point x^* of an infinite sequence $\{x_i\}$, where $x_{i+1} = A_c'(x_i)$ for all i, satisfies $x^* \in D_c$. Hence A_c' defines a first order algorithm for solving $\min_x \{\gamma(x,c)\}$.

We can now turn our attention to establishing the global convergence of the main algorithm.

4. <u>GLOBAL CONVERGENCE</u>

We again assume that H1 and H2 are satisfied. Step 3 of the main algorithm, if entered, generates a new point $x_{i+1} = A_c'(x_i)$. Since $QP(x_i,H_i)$ may have more than one solution, Step 2 of the main algorithm, may generate any point in a set, which we denote $\bar{A}_{c_i}(x_i)$ (the parameters H_i and j are omitted to simplify the notation). Thus Step 2 defines a point to set map $\bar{A}_c : R^n \to 2^{R^n}$, i.e. $\bar{A}_c(x)$ is the set of points that could be generated in Step 2 with x_i replaced by x, c_i replaced by c. Steps 2 and 3 together define a point to set map $A_c : R^n \to 2^{R^n}$ such that:

$$A_c(x) = \bar{A}_c(x) \quad \text{if } x \text{ satisfies tests in Step 2}$$

$$= \{A_c'(x)\} \quad \text{otherwise} \tag{4.1}$$

With this definition, our main algorithm has the structure of the following model:

<u>Algorithm Model</u>

Data: $x_1 \in R^n$, $c_0 > 0$

Step 0: Set $i = 1$

Step 1: If $c_{i-1} \geq \bar{c}(x_i)$ set $c_i = c_{i-1}$
If $c_{i-1} < \bar{c}(x_i)$ set $c_i = \max\{c_{i-1}+\delta, \bar{c}(x_i)\}$

Step 2: Compute any $x_{i+1} \in A_{c_i}(x_i)$
Set $i = i+1$
Go to Step 1 □

The following result, which is a slight modification of Theorem 4 in [11], gives conditions on A_c and $\bar{c}$ which, if satisfied, guarantee global convergence. D denotes the set of Kuhn-Tucker points for (2.1).

<u>Theorem 1</u>

If $\bar{c}$ and A_c have the following properties:

(i) $\bar{c} : R^n \to R$ is continuous

(ii) $x \in D_c$ and $c \geq \bar{c}(x) \Rightarrow x \in D$

(iii) Let $\{x_i\}$ be any infinite sequence such that $x_{i+1} \in A_c(x_i)$ and $c \geq \bar{c}(x_i)$ for all i; any accumulation point x^* of $\{x_i\}$ satisfies $x^* \in D_c$.

Then any sequence $\{x_i\}$ generated by the Algorithm Model has the following properties.

(a) If c_{i-1} is increased finitely often when $i = i_1, i_2, \ldots, i_J$ so that $c_i = c' \triangleq c_{i_J}$ for all $i \geq i_J$, then any accumulation point x^* of $\{x_i\}$ satisfies $x^* \in D$.

(b) If c_{i-1} is increased infinitely often when $i \in K \triangleq \{i_1, i_2, i_3, \ldots\}$, then the sequence $\{x_i\}_{i \in K}$ has no accumulation points. □

To appreciate the significance of (b) we note the following consequence of Theorem 1.

<u>Corollary</u>

If the sequence $\{x_i\}$ is bounded then c_i is increased only finitely often, and any accumulation point x^* of $\{x_i\}$ is desirable ($x^* \in D$). □

Theorem 1 is proven in the appendix.

Hence, to establish that our algorithm has the same convergence properties given in (a) and (b) of Theorem 1 (and in the Corollary) we need to establish that $\bar{c}$ satisfies conditions (i) and (ii) and A_c satisfies condition (iii). First of all we establish:

<u>Proposition 4</u>

(i) $\bar{\lambda} : R^n \to R^m$ and $\bar{\mu} : R^n \to R^{m_e}$ (defined in (2.28)) are well-defined and continuous.

(ii) If $(\hat{x}, \hat{\lambda}, \hat{\mu})$ is a Kuhn-Tucker triple for (2.1), then $\hat{\lambda} = \bar{\lambda}(\hat{x})$ and $\hat{\mu} = \bar{\mu}(\hat{x})$.

<u>Proof</u>

(i) We note that the second order term in (2.28) is positive definite in (λ, μ). For if:

$$||g_x^T(x)\lambda + h_x^T(x)\mu||^2 + \sum_{j=1}^{m} (\psi(x) - g^j(x))^2 (\lambda^j)^2 + \sum_{j=1}^{m} (\psi(x) - |h^j(x)|)^2 (\mu^j)^2 = 0 \tag{4.2}$$

then $\lambda^j = 0$ for all $j \notin I(x)$ and $\mu^j = 0$ for all $j \notin I_e(x)$. Hence:

$$||\sum_{j \in I(x)} \lambda^j \nabla g^j(x) + \sum_{j \in I_e(x)} \mu^j \nabla h^j(x)||^2 = 0 \tag{4.3}$$

But this implies, from Assumption H2, that $\lambda = 0$ and $\mu = 0$. Hence the second order term in (2.28) is positive definite. The continuity of $\bar{\lambda}$, $\bar{\mu}$ then follows from the continuity of g_x, h_x, g, h and ψ.

(ii) If $\{\hat{x},\hat{\lambda},\hat{\mu}\}$ is a Kuhn-Tucker triple for (1), then:

$$\nabla f(\hat{x}) + g_x^T(\hat{x})\hat{\lambda} + h_x^T(\hat{x})\hat{\mu} = 0 \tag{4.4}$$

$$g(\hat{x}) \le 0, \quad h(\hat{x}) = 0 \tag{4.5}$$

$$\langle\hat{\lambda},g(\hat{x})\rangle = 0, \quad \hat{\lambda} \ge 0 \tag{4.6}$$

Hence $\psi(\hat{x}) = 0$ so that $G(\hat{x}) = \text{diag}\,\{(g^j(\hat{x}))^2\}$. Since $\hat{\lambda}^j = 0$ if $g^j(\hat{x}) \ne 0$, it follows that $G(\hat{x})\hat{\lambda} = 0$. Also $H(\hat{x}) = \text{diag}\,\{(h^j(\hat{x}))^2\} = 0$. Hence $\hat{\lambda}$ and $\hat{\mu}$ satisfy:

$$g_x(\hat{x})[\nabla f(\hat{x})+g_x^T(\hat{x})\lambda+h_x^T(\hat{x})\mu] + G(\hat{x})\lambda = 0 \tag{4.7}$$

$$h_x(\hat{x})[\nabla f(\hat{x})+g_x^T(\hat{x})\lambda+h_x^T(\hat{x})\mu] + H(\hat{x})\mu = 0 \tag{4.8}$$

But, from (2.28) and Proposition 4 (i), $\bar{\lambda}(\hat{x})$, $\bar{\mu}(\hat{x})$ are the unique solutions of (4.7) and (4.8). The desired result follows. □

<u>Proposition 5</u>

$\bar{c} : R^n \to R$, defined by (2.31) and $T : R^n \to R$ defined by (2.32) are continuous.

<u>Proof</u>

This is a direct consequence of the continuity of $\bar{\lambda}$ and $\bar{\mu}$. □

<u>Proposition 6</u>

Let $c \ge \bar{c}(x)$. Then $x \in D_c \Longleftrightarrow x \in D$.

<u>Proof</u>

(i) $(x \in D_c) \Rightarrow (x \in D)$

It is shown in the appendix that program (3.1) is dual to the program:

$$\theta'(x,c) = \max_{\omega,\rho} \{c(\langle\omega,g(x)\rangle+\langle\rho,h(x)\rangle-\psi(x))-(\tfrac{1}{2}n)\,\|\nabla L\|^2 \mid$$

$$\omega \ge 0, \rho = \rho_1-\rho_2,\ \rho_1 \ge 0,\ \rho_2 \ge 0,$$

$$\sum_{j=1}^{m}\omega^j + \sum_{j=1}^{m_e}\rho_1^j + \sum_{j=1}^{m_e}\rho_2^j \le 1\} \tag{4.9}$$

where:

$$\nabla L \triangleq \nabla f(x) + c(g_x^T(x)\omega + h_x^T(x)\rho) \tag{4.10}$$

(a) Let $c \ge \bar{c}(x)$ and $x \in D_c$. Suppose also that $x \in F$. Then $\psi(x) = 0$, $g(x) \le 0$ and $h(x) = 0$. Hence:

$$\theta'(x,c) = 0$$

$$= \max_{\omega,\rho} \{c\langle\omega,g(x)\rangle-(1/2n)\,\|\nabla L\|^2 \mid$$

$$\omega \ge 0,\ \rho = \rho_1-\rho_2,\ \rho_1 \ge 0,\ \rho_2 \ge 0,$$

$$\sum_{j=1}^{m}\omega^j + \sum_{j=1}^{m_e}\rho_1^j + \sum_{j=1}^{m_e}\rho_2^j \le 1\} \tag{4.11}$$

Let $\hat{\omega}$, $\hat{\rho}$ denote a solution of (4.11). Then (4.11) implies that:

$$\hat{\omega}^j = 0 \quad \text{for all } j \notin I(x) \tag{4.12}$$

and

$$\nabla L = \nabla f(x) + cg_x^T(x)\omega + ch_x^T(x)\rho = 0 \tag{4.13}$$

Hence we have:

$$g(x) \le 0, \quad h(x) = 0 \tag{4.14}$$

$$\langle\hat{\omega},g(x)\rangle = 0 \tag{4.15}$$

$$\hat{\omega} \ge 0 \tag{4.16}$$

$$\nabla f(x) + g_x^T(x)\ c\hat{\omega} + h_x^T(x)\ c\hat{\rho} = 0 \tag{4.17}$$

i.e. $\{x, c\hat{\omega}, c\hat{\rho}\}$ is a Kuhn-Tucker triple for (2.1), i.e. $x \in D$.

(b) Again let $c \geq \bar{c}(x)$ and suppose $x \notin F$, so that $x \notin D$. Suppose, contrary to what is to be proven, that $x \in D_c$, i.e. $\theta'(x,c) = 0$. Let $I_e^1(x) \triangleq \{j | h^j(x) = \psi(x)\}$ and $I_e^2(x) \triangleq \{j | -h^j(x) = \psi(x)\}$ so that $I_e^1(x) = I_e(x) \cup I_e^2(x)$. From (4.9), if $\theta'(x,c) = 0$, then (since $g^j(x) \leq \psi(x)$, $|h^j(x)| \leq \psi(x)$, and the multipliers sum to less than unity):

$$\langle\hat{\omega}, g(x)\rangle + \langle\hat{\rho}_1, h(x)\rangle + \langle\hat{\rho}_2, -h(x)\rangle = \psi(x) \tag{4.18}$$

($\psi(x) > 0$) and:

$$\nabla L = 0 \tag{4.19}$$

From (4.18), since:

$$\sum_{j=1}^{m} \hat{\omega}^j + \sum_{j=1}^{m_e} \hat{\rho}_1^j + \sum_{j=1}^{m_e} \hat{\rho}_2^j \leq 1 \tag{4.20}$$

we must have:

$$\hat{\omega}^j = 0, \quad \text{for all } j \notin I(x) \tag{4.21}$$

$$\hat{\rho}_1^j = 0, \quad \text{for all } j \notin I_e^1(x) \tag{4.22}$$

$$\hat{\rho}_2^j = 0, \quad \text{for all } j \notin I_e^2(x) \tag{4.23}$$

and

$$\sum_{j\in I(x)} \hat{\omega}^j + \sum_{j\in I_e^1(x)} \hat{\rho}_1^j + \sum_{j\in I_e^2(x)} \hat{\rho}_2^j = 1 \tag{4.24}$$

From (4.19):

$$\nabla f(x) + c g_x^T(x)\hat{\omega} + c h_x^T(x)(\hat{\rho}_1 - \hat{\rho}_2) = 0 \tag{4.25}$$

Now $(\bar{\lambda}(x), \bar{\mu}(x))$ is the unique solution of:

$$\min_{\lambda,\mu} \{ ||\nabla f(x) + \sum_{j\in \underline{m}} \lambda^j \nabla g^j(x) + \sum_{j\in \underline{m}_e} \mu^j \nabla h^j(x)||^2 + \sum_{j\notin I(x)} (\lambda^j)^2 (\psi(x) - g^j(x))^2 + \sum_{j\notin I_e(x)} (\mu^j)^2 (\psi(x) - |h^j(x)|)^2 \tag{4.26}$$

Making use of (4.21)-(4.23) and (4.25), we see that $(\bar{\lambda}(x), \bar{\mu}(x))$ is the solution of:

$$\begin{aligned} \min_{\lambda,\mu} \{ || &\sum_{j\in I(x)} [\lambda^j - c\hat{\omega}^j]\nabla g^j(x) + \sum_{j\in I_e^1(x)} [\mu^j - c\hat{\rho}_1^j]\nabla h^j(x) \\ &+ \sum_{j\in I_e^2(x)} [\mu^j + c\hat{\rho}_2^j]\nabla h^j(x) + \sum_{j\notin I(x)} \lambda^j \nabla g^j(x) \\ &+ \sum_{j\notin I_e(x)} \mu^j \nabla h^j(x)||^2 + \sum_{j\notin I(x)} (\lambda^j)^2 (\psi(x) - g^j(x))^2 \\ &+ \sum_{j\notin I_e(x)} (\mu^j)^2 (\psi(x) - |h^j(x)|)^2 \} \end{aligned} \tag{4.27}$$

It is evident that a solution of (4.27) is:

$$\begin{aligned} \bar{\lambda}^j(x) &= c\hat{\omega}^j \geq 0, \ j \in I(x) \\ \bar{\lambda}^j(x) &= 0, \qquad j \notin I(x) \\ \mu^j(x) &= c\hat{\rho}_1^j \geq 0, \ j \in I_e^1(x) \\ \mu^j(x) &= -c\hat{\rho}_2^j \leq 0, \ j \in I_e^2(x) \\ \mu^j(x) &= 0, \qquad j \notin I_e(x) \end{aligned} \tag{4.28}$$

But this solution is unique. Hence:

$$\begin{aligned} \bar{c}(x) &= \sum_{j\in \underline{m}} \bar{\lambda}^j(x) + \sum_{j\in \underline{m}_e} |\bar{\mu}^j(x)| + b \\ &= \sum_{j\in I(x)} c\hat{\omega}^j + \sum_{j\in I_e^1(x)} c\hat{\rho}_1^j + \sum_{j\in I_e^2(x)} c\hat{\rho}_2^j + b \end{aligned} \tag{4.29}$$

Making use of (4.24) we obtain:

$$\bar{c}(x) = c + b \tag{4.30}$$

But this contradicts the fact that $c \geq \bar{c}(x)$. Hence $x \notin D_c$, thus establishing that, given $c \geq \bar{c}(x)$, $x \notin F \Rightarrow x \notin D_c$, i.e. given $c \geq c(x)$, $x \in D_c \Rightarrow x \in F$. Hence $x \in D_c \Rightarrow x \in D$.

(ii) To prove the converse suppose $x \in D$. Then $h(x) = 0$, $g(x) = 0$, $\psi(x) = 0$ and there exist multipliers λ, μ such that $\lambda \geq 0$, $\langle\lambda, g(x)\rangle = 0$ and $\nabla f(x) + g_x^T(x)\lambda + h_x^T(x)\mu = 0$. Let $c \geq \bar{c}(x)$ be arbitrary. Setting $\omega = (1/c)\lambda$, $\rho = (1/c)\mu$ yields $\nabla L = 0$ (see (4.10), $\omega > 0$, and $\sum_{j\in\underline{m}} \omega^j + \sum_{j\in\underline{m}_e} \rho^j \leq 1$ (see (2.31) and Proposition 4(ii)). Hence substituting this ω, ρ into (4.9) yields $\theta'(x,c) \geq 0$, which implies that $\theta'(x,c) = 0$ or, equivalently, $x \in D_c$. □

Propositions 5 and 6 establish that $\bar{c}$ has the properties required by Theorem 1. We now turn our attention to A_c, defined in (4.1).

Proposition 7

For all $x \notin D_c$, where $c \geq \bar{c}(x)$, there exists an $\epsilon > 0$, $\delta > 0$ such that:

$$\gamma(x'',c) - \gamma(x',c) \leq -\delta$$

for all $x' \in B(x,\epsilon)$, for all $x'' \in A_c(x')$. □

Proposition 7 is proven in the appendix.

A direct consequence of Proposition 7 is:

Proposition 8

Any accumulation point x^* of an infinite sequence $\{x_i\}$, where $x_{i+1} \in A_c(x_i)$ and $c \geq \bar{c}(x_i)$, for all i, satisfies $x^* \in D_c$.

Proof

It follows from the continuity of $\bar{c}(\cdot)$ that $c \geq \bar{c}(x^*)$. The desired result then follows from Proposition 7 and Theorem (1.3.3) of [14]. □

We can now establish the convergence properties of our main algorithm.

Theorem 2

Suppose the main algorithm generates an infinite sequence $\{x_i\}$.

(i) If the algorithm increases c_{i-1} finitely often, then any accumulation point x^* of $\{x_i\}$ satisfies $x^* \in D$.

(ii) If the algorithm increases c_{i-1} infinitely often (so that $c_{i-1} \to \infty$) at $i \in K \triangleq \{i_1, i_2, i_3, \ldots\}$ then the sub sequence $\{x_i\}_{i\in K}$ has no accumulation points.

(iii) If $\{x_i\}$ is bounded, then c_{i-1} is increased only finitely often (and every accumulation point x^* of $\{x_i\}$ satisfies $x^* \in D$).

Proof

(i), (ii), (iii) follow from Propositions 5, 6 and 8, which show that $\bar{c}$ and A_c satisfy the conditions of Theorem 1. □

5. RATE OF CONVERGENCE

Although the main algorithm generates sequences whose accumulation points are desirable, the purpose of Step 2 is to ensure superlinear convergence. Hence we have to show that eventually the tests in Step 2 are always satisfied, that the step length is eventually always unity and that H_i is updated suitably to ensure this rate of convergence. First of all we have to strengthen our assumptions; our new assumptions include those of Robinson [15]. H1 is replaced by H1A and we add two new assumptions H3 and H4 to obtain:

H1A: f, g and h are three times continuously differentiable.

H3: At each Kuhn-Tucker triple $(\hat{x},\hat{\lambda},\hat{\mu})$ for (2.1) the second order sufficiency conditions hold with strict complementary slackness, i.e. $\hat{\lambda}^j > 0$ for all $j \in I(\hat{x})$ and $L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})$ is positive definite on the subspace $\{p \mid g_x(\hat{x})p = 0,\ h_x(\hat{x})p = 0\}$.

H4: The sequence $\{x_i\}$ (generated by the main algorithm) is bounded.

In the sequel H1A, H2, H3 and H4 are assumed to hold. From Theorem 2.1 of [15] we deduce that Kuhn-Tucker points ($\hat{x}$ is said to be a Kuhn-Tucker point of $\{\hat{x},\hat{\lambda},\hat{\mu}\}$ is a Kuhn-Tucker triple) are isolated i.e. there exists a neighbourhood of each Kuhn-Tucker point which contains no other Kuhn-Tucker points.

For simplicity we adopt the secant method for updating H_i. Thus Steps 2(d) and 3(d) in the main algorithm become:

<u>Updating Step</u>: Replace column (i mod n) of H_i by:

$$(1/\Delta_i)[\nabla_x L(x_{i+1}+\Delta_i e_i,\bar{\lambda}(x_{i+1}),\bar{\mu}(x_{i+1}))-\nabla_x L(x_{i+1},\bar{\lambda}(x_{i+1}),\bar{\mu}(x_{i+1}))] \tag{5.1}$$

where:

$$\Delta_i \triangleq \min\{||x_{i+1}-x_i||,||\bar{\lambda}(x_{i+1})-\bar{\lambda}(x_i)||,||\bar{\mu}(x_{i+1})-\bar{\mu}(x_i)||,\varepsilon\} \tag{5.2}$$

e_i, $i = 1, \ldots, n$, denotes the ith basis vector and $\varepsilon > 0$ is a small number based on the word length of the computer.

It is not essential to update, as above, H_i at every iteration; updating may be done every kth iteration, for example.

We require, for superlinear convergence, that $H_i \to L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})$, where $(\hat{x},\hat{\lambda},\hat{\mu})$ is a Kuhn-Tucker triple, as $i \to \infty$ (see Theorem 3). If the secant updating procedure, specified in (5.1), is employed then, since $\bar{\lambda}$ and $\bar{\mu}$ are continuous, a sufficient condition for the convergence of H_i to $L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})$ is the convergence of x_i to $\hat{x}$ as $i \to \infty$. We now establish the latter.

<u>Proposition 9</u>

Let $\{x_i\}$ be a bounded infinite sequence generated by the main algorithm. Then $\|x_{i+1}-x_i\| \to 0$ and $x_i \to \hat{x}$, where $\hat{x} \in D$ ($\hat{x}$ is a Kuhn-Tucker point), as $i \to \infty$.

<u>Proof</u>

(a) Any accumulation point $\hat{x}$ of $\{x_i\}$ is a Kuhn-Tucker point (Theorem 2). Let B be a compact ball in R^n containing $\{x_i\}$. Then, since the Kuhn-Tucker points are *isolated*, B contains a finite number of Kuhn-Tucker points. (Suppose, contrary to this assumption, that $\{z_j\} \triangleq \{(x_j,\lambda_j,\mu_j)\}$ is an infinite sequence of Kuhn-Tucker triples (for problem (2.1)) in B such that $x_j \to x$ as $j \to \infty$. From Proposition 4, $\lambda_j = \bar{\lambda}(x_j)$ and $\mu_j = \bar{\mu}(x_j)$ so that $z_j \to z = (x,\bar{\lambda}(x),\bar{\mu}(x))$ as $j \to \infty$. Since $\nabla f(x_j) + g_x^T(x_j)\lambda_j + h_x^T(x_j)\mu_j = 0$, $\langle\lambda_j,g(x_j)\rangle = 0$ and $\lambda_j \geq 0$ for all j, it follows that $\nabla f(x) + g_x^T(x)\bar{\lambda}(x) + h_x^T(x)\bar{\mu}(x) = 0$, $\langle\bar{\lambda}(x),g(x)\rangle = 0$ and $\bar{\lambda}(x) \geq 0$. Hence x is a Kuhn-Tucker point which is *not* isolated, a contradiction.) Hence the set $\{\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_k\} \subset B$ of accumulation points of $\{x_i\}$ is finite, and each element of this set is a Kuhn-Tucker point.

(b) For all $\varepsilon > 0$, there exists a positive integer i_0, such that $x_i \in \bigcup_{j\in\underline{k}} B(\hat{x}_j,\varepsilon)$ for all $i \geq i_0$. (Theorem 2.)

(c) There exists an $i_1 \geq i_0$ such that $c_i = \hat{c}$ for all $i \geq i_1$. (Theorem 2.) Hence $\hat{x}_j \in D_{\hat{c}}$ (Proposition 8) for all $j \in \underline{k}$. Since $D_{\hat{c}} = \tilde{D}_{\hat{c}}$ it follows that $\theta'(\hat{x}_j,\hat{c}) = \theta(\hat{x}_j,\bar{p}(\hat{x}_j,\hat{c}),\hat{c}) = 0$ so that (from (3.11)), $\bar{p}(\hat{x}_j,\hat{c}) = 0$ for all $j \in \underline{k}$. Also $\bar{p}(\cdot,\hat{c})$ is continuous (Proposition 2).

(d) Let $K_1 \subset \{0,1,2,...\}$ denote the subsequence in which Step 2 is entered and K_2 the complement of K_1 (the subsequence in which Step 3 is entered).

(e) From (c), for all $\delta > 0$ there exists an $\varepsilon > 0$ such that $x_i \in \bigcup_{i=1}^{k} B(\hat{x}_j,\varepsilon)$ implies that $||\bar{p}(x_i,\hat{c})|| \leq \delta$.

(f) Combining (b) and (e) we see that given any $\delta > 0$, there exists an i_1 such that $||x_{i+1}-x_i|| \leq ||\bar{p}(x_i,\hat{c})|| \leq \delta$, for all $i > i_1$, $i \in K_2$. This implies that $||x_{i+1}-x_i|| \to 0$ as $i \to \infty$, $i \in K_2$.

(g) For all $i \in K_1$, $||x_{i+1}-x_i|| \leq ||p_i|| + ||\tilde{p}_i||$.

(h) $||p_i|| \to 0$ and $||\tilde{p}_i|| \to 0$ as $i \to \infty$, $i \in K_1$ because of the test in Step 2 and the definition of $\tilde{p}_i$.

(i) Combining (g) and (h) yields $||x_{i+1}-x_i|| \to 0$ as $i \to \infty$, $i \in K_1$.

(j) Combining (f) and (i) yields $||x_{i+1}-x_i|| \to 0$ as $i \to \infty$.

(k) Let $d \triangleq \min \{||\hat{x}_{j_1}-\hat{x}_{j_2}|| \mid j_1,j_2 \in \underline{k}\}$ and let $\varepsilon \triangleq d/4$. Then there exists an $i_2 \geq i_1$ such that:

(i) $x_i \in \bigcup_{j\in\underline{k}} B(\hat{x}_j,\varepsilon)$

(ii) $||x_{i+1}-x_i|| \leq \varepsilon$

for all $i \geq i_2$. Hence, if $x_{i_2} \in B(\hat{x},\varepsilon)$ where $\hat{x} \subset \{\hat{x}_1,...,\hat{x}_k\}$, then $x_i \in B(\hat{x},\varepsilon)$ for all $i \geq i_2$. Hence $x_i \to \hat{x}$ as $t \to \infty$. □

It follows from [16] and [17] that $H_i \to L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})$ if $(x_i,\bar{\lambda}(x_i),\bar{\mu}(x_i)) \to (\hat{x},\hat{\lambda},\hat{\mu})$. Hence, from Theorem 2, Proposition 9 and the continuity of $\bar{\lambda}$, $\bar{\mu}$, we have:

Proposition 10

If $\{x_i\}$ is a bounded infinite sequence generated by the main algorithm, then $x_i \to \hat{x} \in D$ and $H_i \to L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})$ as $i \to \infty$, where $(\hat{x},\hat{\lambda},\hat{\mu}) \triangleq \{\hat{x},\bar{\lambda}(\hat{x}),\bar{\mu}(\hat{x})\}$ is a Kuhn-Tucker triple for Problem (2.1). □

We establish next the convergence of the multipliers λ_i and μ_i, recalling that, for each i, $\{p_i,\lambda_i,\mu_i\}$ is a Kuhn-Tucker triple for problem $QP(x_i,H_i)$.

Proposition 11

Let $\{x_i\}$ be a bounded infinite sequence generated by the main algorithm. For each i let $\{p_i,\lambda_i,\mu_i\}$ denote the Kuhn-Tucker triple for $QP(x_i,H_i)$.

(a) $(x_i,\lambda_i,\mu_i) \to (\hat{x},\hat{\lambda},\hat{\mu})$ as $i \to \infty$, where $\{\hat{x},\hat{\lambda},\hat{\mu}\}$ is a Kuhn-Tucker triple for Problem (2.1).

(b) There exists an integer i_0 such that $QP(x_i,H_i)$ has a unique solution p_i for all $i \geq i_0$ (so that the test in Step 2(a) of the main algorithm is satisfied for $i \geq i_0$). Also $p_i \to 0$ as $i \to \infty$.

(c) There exists an integer i_1 such that $\hat{I}_i = I(\hat{x})$ for all $i \geq i_1$.

Proof

We make use of Theorem (2.1) in [15], with $q \triangleq (x_k,H_k)$, $\bar{q} \triangleq (\hat{x},\hat{H})$ with $H = L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})$ and:

$$\Theta(x,q) \triangleq f_x(x_k)(x-x_k) + (½)(x-x_k)^T H_k(x-x_k)$$

$$g(x,q) \triangleq g(x_k) + g_x(x_k)(x-x_k)$$

$$h(x,q) \triangleq h(x_k) + h_x(x_k)(x-x_k)$$

Problem 1(q) is defined to be:

$$\min \{\Theta(x,q) \mid g(x,q) \leq 0,\ h(x,q) = 0\}$$

This is seen to be identical to our $QP(x_k,H_k)$ with p in $QP(x_k,H_k)$ replaced by $x - x_k$. Problem $1(\bar{q})$ is:

$$\min \{f_x(\hat{x})(x-\hat{x})+½(x-\hat{x})^T\hat{H}(x-\hat{x}) \mid g(x)+g_x(\hat{x})(x-\hat{x}) \leq 0$$
$$h(x)+h_x(\hat{x})(x-\hat{x}) = 0\}$$

Clearly $\{\hat{x},\hat{\lambda},\hat{\mu}\}$ is also a Kuhn-Tucker triple for Problem 1(q) at which the second order sufficiency conditions are satisfied with strict complementarity slackness and linear independence of the active constraints. Hence we satisfy all of the conditions of Robinson's theorem, (2.1) so that there exists, in a suitably small neighbourhood of $(\hat{x},\hat{H})$, a continuous function $Z : (x_k,H_k) \mapsto Z(x_k,H_k) = x_k+p_k,\lambda_k,\mu_k)$ such that $Z(\hat{x},\hat{H}) = (\hat{x},\hat{\lambda},\hat{\mu})$. In this neighbourhood $\{x_k+p_k,\lambda_k,\mu_k\}$ is the unique Kuhn-Tucker triple for Problem 1(q). Since $x_k \to \hat{x}$ and $H_k \to \hat{H}$ (by Proposition 10), (a) and (b) are proven. That (c) is true follows from the facts that $\lambda_i \to \hat{\lambda}$ as $i \to \infty$, that $\hat{I}_i = \{j \in \underline{m} | \lambda_i^j > 0\}$ and that (from strict complementarity) $I(\hat{x}) = \{j \in \underline{m} | \hat{\lambda}^j > 0\}$. □

We now proceed to establish that the tests in Step 2 of the main algorithm are always satisfied. Since we are dealing with a convergent sequence $\{x_i\}$ we adopt, for convenience, the following conventions. "A is true" should be interpreted as "there exists an integer i_1 such that A is true for all $i \geq i_1$". Following Powell [18], "$a_i \sim b_i$" should be interpreted as "there exists an integer i_2 and positive constants d_1, d_2 such that $(a_i/b_i) \in [d_1,d_2]$ for all $i \geq i_2$". Similarly "$a_i \approx b_i$" means that "there exists an integer i_3 and a positive constant d_3 such that $a_i \leq d_3 b_i$ for all $i \geq i_3$".

Let $R(x_i)$ denote the matrix whose rows are $g_x^j(x_i)$, $j \in \hat{I}_i$ and $h_x^j(x_i)$, $j \in \underline{m}_e$. Similarly $r(x_i)$ denotes the vector whose components are $g^j(x_i)$, $j \in \hat{I}_i$, and $h^j(x_i)$, $j \in \underline{m}$, ordered in the same way as are the rows of $R(x_i)$. $r(x_i) \in R^{\tilde{m}+m_e}$, where $\tilde{m}$ is the cardinality of $\hat{I}_i$. Let $Y_i(\hat{Y})$ denote a matrix whose columns are orthogonal basis vectors for the null space of $R(x_i)$ $(R(\hat{x}))$ so that $||Y_i|| = 1$ $(||\hat{Y}|| = 1)$ and $R(x_i)Y_i = 0$, for all i $(R(\hat{x})\hat{Y}=0)$. Let $P_i \triangleq Y_iY_i^T$ for all i and $\hat{P} \triangleq \hat{Y}\hat{Y}^T$. We note that P_iy is the orthogonal projection of y onto the null space of $R(x_i)$.

<u>Proposition 12</u>

(a) $y^TH_iy \sim ||y||^2$ for all y such that (i) $R(\hat{x})y = 0$ or (ii) $R(x_i)y = 0$.

(b) $R(x_i)$ has maximum rank.

(c) $||p_i|| \sim ||r(x_i)|| + ||P_i\nabla f(x_i)||$.

<u>Proof</u>

(a) (i) Follows from the convergence of H_i to $\hat{H}$ and the fact that $y^THy \sim ||y||^2$ for all y in the null space of $R(\hat{x})$ (by virtue of H3).

(b) Follows from H2 (with $x = \hat{x}$) and the continuity of g_x^j, $j \in \underline{m}$, h_x^j, $j \in \underline{m}_e$, Proposition 11(c) and the fact that $I_e(\hat{x}) = \underline{m}_e$.

(c) Proven by Powell in [18]. □

We now consider the test in Step 2(γ) of the main algorithm.

<u>Proposition 13</u>

$$\theta(x_i,p_i,c_i) \approx -M(x_i)$$

where

$$M(x_i) \triangleq ||r(x_i)|| + ||P_i\nabla f(x_i)||^2$$

<u>Proof</u>

$$p_i = p_i^1 + p_i^2 \tag{5.3}$$

where p_i^1 is the minimum norm solution of:

$$R(x_i)p + r(x_i) = 0 \tag{5.4}$$

and p_i^2 is the unique solution of:

$$\min_p \{<\nabla f(x_i)+H_ip_i^1,p> + \tfrac{1}{2}p^TH_ip | R(x_i)p=0\} \tag{5.5}$$

Clearly p_i^1 satisfies (5.4), is unique, and is orthogonal to the null space of $R(x_i)$. Also p_i^2 lies in the null space of $R(x_i)$ and is thus orthogonal to p_i^1. Hence, from (5.3), the fact that $\hat{\psi}(x_i,p_i) = 0$, and the definition of θ we obtain:

$$\theta(x_i,p_i,c_i) = \langle\nabla f(x_i),p_i^1\rangle - c_i\psi(x_i) + \langle\nabla f(x_i),p_i^2\rangle \tag{5.6}$$

From (5.4), p_i^1 lies in the range space of $R^T(x_i)$ and satisfies:

$$p_i^1 = R^T(x_i)\eta_i^1 \tag{5.7}$$

where η_i is the unique solution of

$$R(x_i)R^T(x_i)\eta_i^1 + r(x_i) = 0 \tag{5.8}$$

We define, for all i, a vector $\tilde{\omega}_i = (\tilde{\lambda}_i^{j_1},\ldots,\tilde{\lambda}_i^{j_{\tilde{m}}},\tilde{\mu}_i^1,\ldots,\tilde{\mu}_i^{m_e}) \in R^{\tilde{m}+m_e}$, (where $j_k \in I(\hat{x})$, $k = 1, \ldots, \tilde{m}$) of multipliers as follows: $\tilde{\omega}_i$ is the unique minimizer of $||R^T(x_i)\omega+\nabla f(x_i)||$ on $R^{\tilde{m}+m_e}$. Hence:

$$R(x_i)[R^T(x_i)\tilde{\omega}_i+\nabla f(x_i)] = 0 \tag{5.9}$$

and (from the continuity of R and ∇f) $\tilde{\omega}_i \to \hat{\omega}$ where $\hat{\omega}$ satisfies:

$$R^T(\hat{x})\hat{\omega} + \nabla f(\hat{x}) = 0 \tag{5.10}$$

and is the vector of non-trivial Kuhn-Tucker multipliers for (2.1) at $\hat{x}$, so that $\hat{\omega}^k = \hat{\lambda}^{j_k} > 0$, $k = 1, \ldots, \tilde{m}$. Also, because $\tilde{\omega}_i$ is the least square solution of (5.9), the projection of $\nabla f(x_i)$ onto the null space of $R(x_i)$ is:

$$P_i\nabla f(x_i) = \nabla f(x_i) + R^T(x_i)\tilde{\omega}_i \tag{5.11}$$

Clearly $\hat{P}\nabla\hat{f}(x) = 0$.

From (5.7), (5.8) and (5.9):

$$\begin{aligned}\langle\nabla f(x_i),p_i^1\rangle &= \langle\nabla f(x_i),R^T(x_i)\eta_i^1\rangle \\ &= \langle R(x_i)\nabla f(x_i),\eta_i^1\rangle \\ &= -\langle R(x_i)R^T(x_i)\tilde{\omega}_i,\eta_i^1\rangle \\ &= -\langle\tilde{\omega}_i,R(x_i)R^T(x_i)\eta_i^1\rangle \\ &= \tilde{\omega}_i^T r(x_i)\end{aligned} \tag{5.12}$$

Let $\tilde{\psi}(x) \triangleq \min\{0;r^j(x),j \in \underline{\tilde{m}}\} \leq 0$. Since $\psi(x_i) = \max\{0;r^j(x_i),j \in \underline{\tilde{m}};|r^j(x_i)|,j=\tilde{m}+1,\ldots,\tilde{m}+m_e\}$, we see that:

$$||r(x_i)||_\infty = \max\{\psi(x_i),-\tilde{\psi}(x_i)\} \tag{5.13}$$

From (5.12) and the positivity of $\tilde{\lambda}_i^j$, $j \in I(\hat{x})$:

$$\begin{aligned}\langle\nabla f(x_i),p_i^1\rangle &= \sum_{j\in I(\hat{x})}\tilde{\lambda}_i^j g^j(x_i) + \sum_{j\in m_e}\tilde{\mu}_i^j h^j(x_i) \\ &\leq [\sum_{j\in I(\hat{x})}\tilde{\lambda}_i^j + \sum_{j\in\underline{m}_e}|\tilde{\mu}_i^j|]\psi(x_i) + \rho_i\tilde{\psi}(x_i)\end{aligned} \tag{5.14}$$

where

$$\rho_i \triangleq \min_j\{\tilde{\lambda}_i^j|g^j(x_i)=\tilde{\psi}(x_i)\}$$

Hence:

$$\langle\nabla f(x_i),p_i^1\rangle - c_i\psi(x_i) \leq -[c_i - \sum_{j\in I(\hat{x})}\tilde{\lambda}_i^j - \sum_{j\in\underline{m}_e}|\tilde{\mu}_i^j|]\psi(x_i) + \rho_i\tilde{\psi}(x_i) \tag{5.15}$$

From the facts that:

$$c_i - \sum_{j\in I(\hat{x})}\bar{\lambda}^j(x_i) - \sum_{j\in\underline{m}_e}|\bar{\mu}(x_i)| \geq b \tag{5.16}$$

that $\tilde{\lambda}_i^j \to \bar{\lambda}^j(x_i)$, $j \in I(\hat{x})$, $\tilde{\mu}_i^j \to \bar{\mu}(x_i)$, $j \in \underline{m}_e$, and $\rho_i \to \rho > 0$ we see from (5.13) and (5.15) that:

$$\langle \nabla f(x_i), p_i^1 \rangle - c_i \psi(x_i) \approx -||r(x_i)||_\infty \sim -||r(x_i)|| \quad (5.17)$$

Now p_i^2 lies in the null space of $R(x_i)$ (is orthogonal to p_i^1) and therefore satisfies:

$$p_i^2 = Y_i n_i^2 \quad (5.18)$$

where n_i^2 is the unique solution of:

$$Y_i^T \nabla f(x_i) + Y_i^T H_i p_i^1 + (Y_i^T H_i Y_i) n = 0 \quad (5.19)$$

Hence:

$$n_i^2 = -(Y_i^T H_i Y_i)^{-1} [Y_i^T \nabla f(x_i) + Y_i^T H_i p_i^1] \quad (5.20)$$

and:

$$\langle \nabla f(x_i), p_i^2 \rangle = (n_i^2)^T Y_i^T \nabla f(x_i)$$

$$= -[Y_i^T \nabla f(x_i) + Y_i^T H_i p_i^1]^T [Y_i^T H_i Y_i]^{-1} [Y_i^T \nabla f(x_i)] \quad (5.21)$$

Using Proposition 12(a), the fact that $||Y_i^T \nabla f(x_i)|| = ||P_i \nabla f(x_i)||$ and the fact that (from (5.8)) $p_i^1 \sim ||r(x_i)||$ we obtain:

$$\langle \nabla f(x_i), p_i^2 \rangle \approx -||P_i \nabla f(x_i)||^2 - ||P_i \nabla f(x_i)||\ ||r(x_i)|| \quad (5.22)$$

Combining (5.6), (5.17) and (5.22) and making use of the fact that $||r(x_i)|| \to 0$ and $||P_i \nabla f(x_i)|| \to 0$ yields:

$$\theta(x_i, p_i, c_i) \approx -||r(x_i)|| - ||P_i \nabla f(x_i)||^2 \quad (5.23)$$

which is what we wished to prove. □

<u>Proposition 14</u>

The test:

$$\theta(x_i, p_i, c_i) \leq -T(x_i)$$

in Step 2(c) of the algorithm is always satisfied (for $i \geq i_1$, say).

<u>Proof</u>

$$P_i \nabla f(x_i) = \nabla f(x_i) + g_x^T(x_i)\tilde{\lambda}_i + h_x^T(x_i)\tilde{\mu}_i \quad (5.24)$$

where $\tilde{\lambda}_i^T \triangleq (\tilde{\lambda}_i^1, \ldots, \tilde{\lambda}_i^m)$, $\tilde{\mu}_i^T = (\tilde{\mu}_i^1, \ldots, \tilde{\mu}_i^{m_e})$ and $\tilde{\lambda}_i^j$, $j \in I(\hat{x})$ and $\tilde{\mu}_i^j$, $j \in \underline{m}_e$ are defined by (5.9) and $\tilde{\lambda}_i^j \triangleq 0$ for $j \notin I(\hat{x})$. Since as shown in the proof of Proposition (13), $\tilde{\lambda}_i \to \bar{\lambda}(x_i)$ and $\tilde{\mu}_i \to \bar{\mu}(x_i)$ we see from Proposition 13 that:

$$\theta(x_i, p_i, c_i) \approx -||r(x_i)|| - ||\nabla f(x_i) + g_x^T(x_i)\bar{\lambda}(x_i) + h_x^T(x_i)\bar{\mu}(x_i)||^2$$

But: (5.25)

$$\psi(x_i) \leq ||r(x_i)||_\infty$$

so:

$$-||r(x_i)|| \approx -\psi(x_i)$$

yielding:

$$\theta(x_i, p_i, c_i) \approx -(T(x_i))^{\frac{1}{2}} \leq -T(x_i) \quad (5.26)$$

thus proving the proposition. □

We now consider the step length constructed in Step 2(b) of the main algorithm.

<u>Proposition 15</u>

The step length generated in Step 2(b) of the main algorithm is unity (for all i greater than some integer).

<u>Proof</u>

Let $\tilde{r}(x_i)$ be a vector whose components are $g^j(x_i+p_i)$, $j \in I(\hat{x})$ and $h^j(x_i+p_i)$, $j \in \underline{m}$, ordered in the same way as the components of r. Since $g^j(x_i) + g_x^j(x_i)p_i = 0$, $j \in I(\hat{x})$, and $h^j(x_i) + h_x^j(x_i)p_i = 0$, $j \in \underline{m}_e$, we obtain:

$$\|\tilde{r}(x_i)\| \approx \|p_i\|^2 \qquad (5.27)$$

From the definition of $\tilde{p}_i$, $\tilde{p}_i$ is the least norm solution of:

$$R(x_i)p + \tilde{r}(x_i) = 0 \qquad (5.28)$$

and hence satisfies:

$$||\tilde{p}_i|| \approx ||\tilde{r}(x_i)|| \approx ||p_i||^2 \qquad (5.29)$$

so that the test $||\tilde{p}_i|| \le ||p_i||$ in the definition of $\tilde{p}_i$ is indeed satisfied.

Next we note that:

$$g^j(x_i+p_i+\tilde{p}_i) = g^j(x_i+p_i) + g^j_x(x_i+p_i)\tilde{p}_i + (\tfrac{1}{2})\tilde{p}_i^T g^j_{xx}(\xi_i^j)\tilde{p}_i \qquad (5.30)$$

for all $j \in I(\hat{x})$, where $\xi_i^j \in [x_i+p_i, x_i+p_i+\tilde{p}_i]$. Since $g^j(x_i+p_i) + g^j_x(x_i)\tilde{p}_i = 0$, $j \in I(\hat{x})$, $||g^j_x(x_i+p_i)-g^j_x(x_i)|| \approx ||p_i||$, and $||\tilde{p}_i|| \approx ||p_i||^2$ we see that, for all $j \in I(\hat{x})$:

$$|g^j(x_i+p_i+\tilde{p}_i)| \approx ||p_i||^3 \qquad (5.31)$$

A similar result holds for $h^j(x_i+p_i+\tilde{p}_i)$, $j \in \underline{m}_e$, so that:

$$\psi(x_i+p_i+\tilde{p}_i) \approx ||p_i||^3 \qquad (5.32)$$

Also, using (5.29):

$$|f(x_i+p_i+p_i)-f(x_i)-\langle\nabla f(x_i),p_i+\tilde{p}_i\rangle-\tfrac{1}{2}\langle p_i,f_{xx}(x_i)p_i\rangle| \approx ||p_i||^3 \qquad (5.33)$$

Hence, from (5.32) and (5.33):

$$\gamma(x_i+p_i+\tilde{p}_i,c_i) - \gamma(x_i,c_i) = \phi(x_i,c_i) + e_i \qquad (5.34)$$

where:

$$\phi(x_i,c_i) \triangleq \langle\nabla f(x_i),p_i+\tilde{p}_i\rangle + \tfrac{1}{2}\langle p_i,f_{xx}(x_i)p_i\rangle - c_i\psi(x_i)$$

$$= \theta(x_i,p_i,c_i) + \langle\nabla f(x_i),\tilde{p}_i\rangle + \tfrac{1}{2}\langle p_i,f_{xx}(x_i)p_i\rangle \qquad (5.35)$$

and

$$|e_i| \approx ||p_i||^3 \qquad (5.36)$$

The step length is unity ($k_i = 0$) if:

$$\epsilon_i \triangleq \gamma(x_i+p_i+\tilde{p}_i,c_i) - \gamma(x_i,c_i) - \theta(x_i,p_i,c_i)/8 \le 0 \qquad (5.37)$$

Now, because $\{p_i,\lambda_i,\mu_i\}$ is a Kuhn-Tucker triple for $QP(x_i,H_i)$, we have:

$$\nabla f(x_i) = -H_ip_i - g_x^T(x_i)\lambda_i - h_x^T(x_i)\mu_i \qquad (5.38)$$

so that:

$$\langle\nabla f(x_i),\tilde{p}_i\rangle = -\langle p_i,H_i\tilde{p}_i\rangle - \lambda_i^T g_x(x_i)\tilde{p}_i - \mu_i^T h_x(x_i)\tilde{p}_i \qquad (5.39)$$

Clearly, using (5.29) and the convergence of H_i:

$$|\langle p_i,H_i\tilde{p}_i\rangle| \approx ||p_i||^3 \qquad (5.40)$$

Also, since $\lambda_i^j > 0$, $j \in I(\hat{x})$, $\lambda_i^j = 0$, $j \notin I(\hat{x})$, and:

$$g_x^j(x_i)\tilde{p}_i = -g^j(x+p_i), \quad j \in I(\hat{x}) \qquad (5.41)$$

we obtain (see (2.34)):

$$\lambda_i^T g_x(x)\tilde{p}_i = -\lambda_i^T g(x_i+p_i) \qquad (5.42)$$

Similarly:

$$\mu_i^T h_x(x_i)\tilde{p}_i = -\mu_i^T h(x_i+p_i) \qquad (5.43)$$

so that, from (5.36), (5.39), (5.40), (5.42) and (5.43):

$$\varepsilon_i = \phi(x_i,c_i) - \theta(x_i,p_i,c_i)/8 + O(||p_i||^3)$$

$$= (7/8)\theta(x_i,p_i,c_i) + \lambda_i^T g(x_i+p_i)$$

$$+ \mu_i^T h(x_i+p_i) + (1/2)\langle p_i, f_{xx}(x_i)p_i\rangle$$

$$+ O(||p_i||^3) \qquad (5.44)$$

Since (from (2.14)) $\theta(x_i,p_i,c_i) \leq -\langle p_i,H_ip_i\rangle - b\psi(x_i)$, if we expand $g(x_i+p_i)$ and $h(x_i+p_i)$ to third order, recalling that $g(x_i) + g_x(x_i)p_i = 0$ etc. we obtain:

$$\varepsilon_i \leq (3/8)\theta(x_i,p_i,c_i) - (b/2)\psi(x_i)$$

$$+ (1/2)\langle p_i,(L_{xx}(x_i,\lambda_i,\mu_i)-H_i)p_i\rangle + O(||p_i||^3) \qquad (5.45)$$

Using Proposition (13) and the convergence of $[L_{xx}(x_i,\lambda_i,\mu_i)-H_i]$ to zero, we get:

$$\varepsilon_i \approx -[||r(x_i)||+||P_i\nabla f(x_i)||^2] + o(||p_i||^2) \qquad (5.46)$$

From Proposition 12(c):

$$||p_i||^2 \approx ||r(x_i)||^2 + 2||r(x_i)||\ ||P_i\nabla f(x_i)|| + ||P_i\nabla f(x_i)||^2$$

$$\approx ||r(x_i)|| + ||P_i\nabla f(x_i)||^2 \qquad (5.47)$$

Since $||r(x_i)|| + ||p_i\nabla f(x_i)||^2 \to 0$ it is obvious from (5.46) and (5.47) that

$$\varepsilon_i \leq 0 \qquad (5.48)$$

(for all i larger than some integer), so that the step length is unity. □

We are now in a position to establish our main result.

Theorem 3

If $\{x_i\}$ is a bounded infinite sequence generated by the main algorithm, then $x_i \to \hat{x} \in D$ superlinearly.

Proof

(We continue to use our previous convention "A_i is true" means "there exists an integer i_1 such that A_i is true for all $i \geq i_1$.) We have established:

(a) $QP(x_i,H_i)$ has a solution p_i which is unique (satisfying the test in Step 2(α)).

(b) $\theta(x_i,p_i,c_i) \leq -T(x_i)$ (satisfying the test in Step 2(γ))

(c) $\gamma(x_i+p_i+\tilde{p}_i,c_i) - \gamma(x_i,c_i) \leq \theta(x_i,p_i,c_i)/8$ (ensuring step length of unity in Step 2).

(d) $||\tilde{p}_i|| \approx ||p_i||^2$, ensuring $||\tilde{p}_i|| \leq ||p_i||$.

To ensure that Step 2 is executed we have to show satisfaction of the test $||p_i|| \leq \delta^j$ in Step 2(β). This depends on the rate of convergence.

We recall that $\{p_i,\lambda_i,\mu_i\}$ is the Kuhn-Tucker triple for $QP(x_i,H_i)$. Let x_i' denote $x_i + p_i$. Let z_i denote $(x_i,\lambda_{i-1},\mu_{i-1})$ and z_i' denote (x_i',λ_i,μ_i). From Proposition (11), $z_i \to \hat{z} = (\hat{x},\hat{\lambda},\hat{\mu})$. We note that $x_{i+1} = x_i' + \tilde{p}_i = x_i + p_i + \tilde{p}_i$ if the test $||p_i|| \leq \delta^j$ is satisfied.

For all z let $q(z) \in R^{m+m_e+1}$ be defined by:

$$q(z) = \begin{bmatrix} \nabla_x L(z) \\ \lambda^1 g^1(z) \\ \vdots \\ \lambda^m g^m(z) \\ h(z) \end{bmatrix} \qquad (5.49)$$

Clearly $q(z) = 0$ if and only if z is a Kuhn-Tucker triple for Problem (2.1). Under our assumption (see [19], p. 13) we have:

$$||q(z_i')|| \le M_1||z_i'-z_i||^2 + ||L_{xx}(z_i)-H_i|| \; ||(x_i'-x_i)|| \quad (5.50)$$

and (see [19], Lemma 3.7):

$$||z_{i+1}'-z_{i+1}|| \le M_2||q(z_{i+1})|| \quad (5.51)$$

for some M_1, M_2. Since $||x_i'-x_i|| \le ||z_i'-z_i||$ and $||L_{xx}(z_i)-H_i|| \to 0$ it follows that:

$$||q(z_i')|| \le \beta_i||z_i'-z_i|| \quad (5.52)$$

where $\beta_i \to 0$. Also $z_i' \to \hat{z}$ and $\tilde{p}_i \to 0$ so that, for some M_3,

$$||q(z_{i+1})|| \le ||q(z_i')|| + M_3||\tilde{p}_i|| \quad (5.53)$$

But $||\tilde{p}_i|| \approx ||p_i||^2 \le ||z_i'-z_i||^2$ so that:

$$||q(z_{i+1})|| \le \beta_i'||z_i'-z_i|| \quad (5.54)$$

where $\beta_i' \to 0$. From (5.51) and (5.54):

$$||p_{i+1}|| \le ||z_{i+1}'-z_{i+1}|| \le \beta_i'M_2||z_i'-z_i|| \quad (5.55)$$

Since both $x_i' \to \hat{x}$ and $x_i \to \hat{x}$, $z_i' - z_i \to 0$, and there exists an i such that:

$$||p_i|| \le ||z_i'-z_i|| \le \delta_1^j \quad (5.56)$$

and such that $\beta_i'M_2 \le \delta_1$, so that:

$$||p_{i+1}|| \le \delta_1^{j+1} \quad (5.57)$$

so that the test in Step 2(β) is satisfied in this and, hence, in all subsequent iterations. Since $x_{i+1} = x_i' + \tilde{p}_i$ it follows that:

$$||z_{i+1}-z_i'|| = ||\tilde{p}_i|| \approx ||p_i||^2 \le ||z_i'-z_i||^2$$

But:

$$||z_i'-z_i|| \le ||z_{i+1}-z_i|| + ||\tilde{p}_i|| \quad (5.58)$$

It follows from (5.58), the fact that $||\tilde{p}_i|| \approx ||z_i'-z_i||^2$ and (5.54) that:

$$||q(z_{i+1})|| \le \beta_i''||z_{i+1}-z_i|| \quad (5.59)$$

where $\beta_i'' \to 0$. From (5.51):

$$||z_{i+1}'-z_{i+1}|| \le M_2||q(z_{i+1})||$$
$$\le \beta_i'''||z_{i+1}-z_i|| \quad (5.60)$$

where $\beta_i''' \to 0$. Since:

$$||z_{i+2}-z_{i+1}|| \le ||z_{i+1}'-z_{i+1}|| + ||\tilde{p}_{i+1}|| \quad (5.61)$$

where $||\tilde{p}_{i+1}|| \approx ||p_{i+1}||^2 \le ||z_{i+1}'-z_{i+1}||^2$ it follows finally that:

$$||z_{i+2}-z_{i+1}|| \le \bar{\beta}_i||z_{i+1}-z_i|| \quad (5.62)$$

where $\bar{\beta}_i \to 0$, which guarantees a superlinear convergence rate. □

6. CONCLUSION

We have described an algorithm which is globally convergent and has a superlinear rate of convergence. It belongs to the family of algorithms, proposed by Han [4], in which a search direction (in our case, a search arc) is obtained by solving a quadratic approximation to the original problem and step length is determined using an exact penalty function. The algorithm has several features not present in earlier algorithms - an automatic rule for choosing the exact penalty function parameter, global convergence (without requiring uniform bounds of the form $y^T H_i y > \varepsilon ||y||^2$, say), an asymptotic step length of unity (despite the use of an exact penalty function) and an Armijo type rule for the step length.

There are, however, several features of the algorithm which could be modified, and might yield improved efficiency. One of these features is the rule for choosing c. Our rule for choosing c ensures that $c_i \to \infty$ as $i \to \infty$ *if* c_{i-1} is increased indefinitely often (i.e. if the test $c_{i-1} \geq \bar{c}(x_i)$ fails indefinitely often. As a consequence we know that if $\{x_i\}$ is bounded, c_i increases only finitely often and all accumulation points of $\{x_i\}$ are Kuhn-Tucker points for (2.1). Other rules which have been proposed, including those which allow c to *decrease*, are heuristic and may result in cycling. However it is true, as Powell [7] emphasises, that a lower value of c may be required in the final stages of convergence than in earlier stages. This feature could be incorporated within the framework of the current algorithm by allowing c to decrease a *finite* number of times. A rule worth trying would be to set $c_i = \bar{c}(x_i) + \delta$ if $T(x_i) \leq \varepsilon_i << 1$ *and* if this option for decreasing c had not been previously used.

A second feature is the choice of the rule for updating H_i. We have chosen the secant method. This enables us to establish that $H_i \to L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})$, and this, in turn, is sufficient for super-linear convergence. However, as Powell [18] points out and the analysis of this paper supports, we only require the convergence of a projection of H_i to a projection of $L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})$ (in fact the convergence $Y_i^T H_i Y_i$ to $\hat{Y}^T L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})/\hat{v}$. Powell [18] exploits the

freedom to obtain an updating rule for H_i which ensures that H_i remains positive definite. While we do not require, in our algorithm, that H_i remains positive definite, ensuring that it does may result in Step 2 being used more frequently. Whether this is advantageous is difficult to say; first order methods may be more efficient in the early iterations. However the secant method does require more work so it may be more efficient to replace it by an updating method which does not require special perturbations but which does ensure that $Y_i^T H_i Y_i \to \hat{Y}^T L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})\hat{Y}$.

Two further points are worth mentioning. Our algorithm converges (but not, of course, superlinearly) even if $\hat{Y}^T L_{xx}(\hat{x},\hat{\lambda},\hat{\mu})\hat{Y}$ is *not* positive definite. Secondly, in constructing H_i we have employed $\bar{\lambda}(x_i)$ and $\bar{\mu}(x_i)$. Since λ_i and μ_i also converge, respectively to $\hat{\lambda}$ and $\hat{\mu}$ it would be possible to employ the latter. Convergence rate is not affected, so which choice is better is not apparent.

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APPENDIX

Proposition A1

Let H1 be satisfied. Then, for all $x \in R^n$, all $c > 0$:

(a) $$|\psi(x+p)-\hat{\psi}(x,p)| = o(||p||)$$

(b) $$|\gamma(x+p,c)-\hat{\gamma}(x,c)| = o(||p||)$$

where $o(\alpha)/\alpha \to 0$ as $\alpha \to 0$.

Proof

(a) We note that:

$$\max(A,B) - \max(C,D) \leq \max(A-C,B-D)$$

and

$$\max(C,D) - \max(A,B) \leq \max(C-A,D-B)$$

so that:

$$|\max(A,B) - \max(C,D)| \leq \max(|A-C|,|B-D|)$$

Hence:

$$|\psi(x+p)-\hat{\psi}(x,p)| \leq \max\{0;|g^j(x+p)-\hat{g}^j(x,p)|, j \in \underline{m}; |h^j(x+p)-\hat{h}(x,p)|, j \in \underline{m}_e\}$$

where $\hat{g}(x,p) \triangleq g(x) + g_x(x)p$ and:

$$\hat{h}(x,p) \triangleq h(x) + h_x(x)p$$

since:

$$|g^j(x+p)-\hat{g}^j(x,p)| = o(||p||), \quad j \in \underline{m}$$

$$|h^j(x+p)-\hat{h}^j(x,p)| = o(||p||), \quad j \in \underline{m}_e$$

it follows that:

$$|\psi(x+p)-\hat{\psi}(x,p)| = o(||p||).$$

(b) Is similarly proven. □

Next we prove Theorem 1 in §3.

Proof of Theorem 1

(a) Since c_{i-1} is increased finitely often, the sequence $\{x_i\}_{i_J}^{\infty}$ is generated by $A_{c'}$, where $c' = c_{i_J}$. From hypothesis (iii), any accumulation point x^* of $\{x_i\}$ satisfies $x^* \in D_{c'}$. From Step 1, $c_i = c' \geq \bar{c}(x_i)$ for all $i \geq i_J$. Since $\bar{c}$ is continuous, we must have $c' \geq \bar{c}(x^*)$. Hence from hypothesis (ii), $x^* \in D$.

(b) Suppose $x_i \to x^*$ as $i \to \infty$, $i \in \bar{K}$ where $\bar{K} \subset K \triangleq \{i_1, i_2, i_3, \ldots\}$ so that $c_i \to \infty$ as $i \to \infty$. Let $\varepsilon^* > 0$ and let $c^* \triangleq \max\{\bar{c}(x) \mid x \in B(x^*, \varepsilon^*)\}$. Then there exists an $i_1 < \infty$ such that:

(a) $x_i \in B(x^*, \varepsilon^*)$, for all $i \in \bar{K}$, $i \geq i_1$

(b) $c_{i-1} \geq c^* \geq \bar{c}(x_i)$, for all $i \in \bar{K}$, $i \geq i_1$

But (b) contradicts the fact (from Step 1) that $c_{i-1} < \bar{c}(x_i)$ for all $i \in K$. Hence the sequence $\{x_{i_j}\}_{j=0}^{\infty}$ has no accumulation points. □

Proposition A2

The program:

$$\theta^1(x,c) = \min_p \{(\eta/2)||p||^2 + \theta(x,p,c)\}$$

is dual to the program:

$$\theta^1(x,c) = \max_{\omega,\rho} \{c[\langle\omega, g(x)\rangle + \langle\rho, h(x)\rangle - \psi] - (\eta/2)||\nabla L||^2$$

$$\mid \omega \geq 0, \rho = \rho_1 - \rho_2, \rho_1 \geq 0, \rho_2 \geq 0, \sum_{j=1}^{m} \omega^j + \sum_{j=1}^{m_e} \rho_1^j + \sum_{j=1}^{m_e} \rho_2^j \leq 1\}$$

where:

$$\nabla L \triangleq \nabla f(x) + c(g_x^T(x)\omega + h_x^T(x)\rho)$$

Proof

$$\theta^1(x,c) \triangleq \min_p \left\{(\eta/2)||p||^2 + \langle\nabla f, p\rangle + c \max \left\{ \begin{array}{l} 0-\psi; \\ g^j + g_x^j p - \psi; j \in \underline{m}; \\ |h^j + h_x^j p| - \psi; j \in \underline{m}_e \end{array} \right\} \right\}$$

where the argument x is omitted. Hence:

$$\theta^1(x,c) \triangleq \min_p \max_{\mu,\rho_1,\rho_2} \Big\{(\eta/2)||p||^2 + \langle\nabla f, p\rangle$$

$$+c \left[\begin{array}{l} -\mu^0\psi + \sum_1^m \mu^j (g^j + g_x^j p - \psi) \\ + \sum_1^{m_e} \rho_1^j [h^j + h_x^j p - \psi] + \sum_1^{m_e} \rho_2^j [-h^j - h_x^j p - \psi] \end{array} \right] \left| \begin{array}{l} \mu^j \geq 0, \rho_i^j \geq 0, \\ \rho_2^j \geq 0, \sum_0^m \mu^j \\ + \sum_1^{m_e} \rho_1^j + \sum_1^{m_e} \rho_2^j = 1 \end{array} \right\}$$

$$= \max_{\mu,\rho_1,\rho_2} \min_p \{ \text{"} \}$$

Substituting the minimizing value of p, i.e. setting $p = -(1/\eta)\nabla L$, yields:

$$\theta^1(x,c) = \max_{\mu,\rho_1,\rho_2} \{(1/2\eta)||\nabla L||^2 - (1/\eta)\langle\nabla f, \nabla L\rangle - (c/\eta)\langle g_x^T\mu + h_x^T\rho, \nabla L\rangle$$

$$+ c\langle\mu, g\rangle + c\langle\rho, h\rangle - c\psi \mid \mu \geq 0, \rho_1 \geq 0, \rho_2 \geq 0,$$

$$\sum_1^m \mu^j + \sum_1^{m_e} (\rho_1^j + \rho_2^j) \leq 1\}$$ □

Proposition A3

Suppose H_1 is satisfied. Let x be any point in R^n and let N be a compact neighbourhood of x. Then:

(a) There exists a function $(\alpha, x') \mapsto \phi(\alpha, x')$ such that:

$$|\gamma(x' + \alpha\bar{p}(x',c), c) - \hat{\gamma}(x', \alpha\bar{p}(x',c), c)| \leq \alpha\phi(\alpha, x')$$

and $\phi(\alpha,x') \to 0$, uniformly in $x' \in N$, as $\alpha \to 0$.

(b) There exists a function $(\alpha,x',p',\tilde{p}) \to \tilde{\phi}(\alpha,x',p',\tilde{p})$ such that:

$$|\gamma(x'+\alpha p'+\alpha^2\tilde{p},c) - \hat{\gamma}(x',\alpha p'+\alpha^2\tilde{p},c)| \le \alpha\tilde{\phi}(\alpha,x',p',\tilde{p})$$

and $\phi(\alpha,x',p',\tilde{p}) \to 0$, uniformly in $(x',p',\tilde{p}) \in N \times B(0,1) \times B(0,1)$, as $\alpha \to 0$.

Proof

We prove (b), the proof of (a) being simpler. Let:

$$\begin{aligned} e(\alpha,x',p',\tilde{p}) &\triangleq |\gamma(x'+\alpha p'+\alpha^2\tilde{p},c)-\hat{\gamma}(x',\alpha p',+\alpha^2\tilde{p},c)| \\ &\le |f(x'+\alpha p'+\alpha^2\tilde{p})-\hat{f}(x',\alpha p'+\alpha^2\tilde{p})| \\ &\quad + c\max\{|g^j(x'+\alpha p'+\alpha^2\tilde{p})-\hat{g}^j(x'\alpha p'+\alpha^2\tilde{p})|, j \in \underline{m}; \\ &\quad |h^j(x'+\alpha p'+\alpha^2\tilde{p})-\hat{h}^j(x',\alpha p'+\alpha^2\tilde{p})|, j \in \underline{m}_e\} \\ &\le [|f_x(w_\alpha)-f_x(x')| + c\max\{|g_x^j(y_\alpha^j)-g_x^j(x')|, j \in \underline{m}; \\ &\quad |h_x^j(z_\alpha^j)-h_x^j(x')|, j \in \underline{m}_e\}][\|p'+\alpha\tilde{p}\|]\alpha \end{aligned}$$

where w_α, y_α^j, $j \in \underline{m}$ and z_α^j, $j \in \underline{m}_e$ all lie on the line segment $[x',x'+\alpha p'+\alpha^2\tilde{p}]$. There exists a compact set N' such that the segment $[x',x'+\alpha p'+\alpha^2\tilde{p}]$ lies in N' for all $x' \in N$, $\alpha \in [0,1]$ and p', $\tilde{p} \in B(0,1)$. f_x, g_x^j, $j \in \underline{m}$ and h_x^j, $j \in \underline{m}_e$ are all continuous, and, therefore, uniformly continuous, on N'. Hence:

$$e(\alpha,x',p',\tilde{p}) \le \phi(\alpha,x',p',\tilde{p})\alpha$$

where $\phi(\alpha,x',p',\tilde{p}) \to 0$, uniformly in $(x',p',\tilde{p}) \in N \times B(0,1) \times B(0,1)$, as $\alpha \to 0$. □

Proof of Proposition 7

Let $x \in M$ be interpreted as "x satisfies tests (α), (β) and (γ) in Step 2". Clearly the set M depends on the values of H and j, but the subsequent analysis is independent of these values. From Proposition 3, if $x \notin D_c$, there exists an ε_1, $\delta_1 > 0$ such that $\gamma(x'',c) - \gamma(x',c) \le -\delta_1$, for all $x' \in B(x,\varepsilon_1) \cap M^c$, for all $x'' \in A_c(x')$, where M^c is the complement of M in R^n. Hence we need only consider $x' \in M$, so that $x'' \in A_c(x')$ will be generated by Step 2. For any $x' \in M$, let p' denote the solution of the quadratic program and $\tilde{p}$ the solution of (2.43) (with x_i replaced by x'). By construction both p' and $\tilde{p}$ lie in the compact set $B(0,1)$. Now:

$$\begin{aligned} \hat{\gamma}(x',\alpha p'+\alpha^2\tilde{p},c) - \gamma(x',c) &= \alpha\langle\nabla f(x'),p\rangle + \alpha^2\langle\nabla f(x'),\tilde{p}\rangle \\ &\quad + c\max\{0; g^j(x')+\alpha g_x^j(x')p \\ &\quad +\alpha^2 g_x^j(x')\tilde{p}, j \in \underline{m}; \\ &\quad |h^j(x')+\alpha h_x^j(x')p+\alpha^2 h_x^j(x')\tilde{p}| j \in \underline{m}_e\} \\ &\quad -c\psi(x') \\ &\le \alpha\langle\nabla f(x'),p\rangle + c(\hat{\psi}(x',\alpha p')-\psi(x')) \\ &\quad + \alpha^2[\langle\nabla f(x'),\tilde{p}\rangle + c\max\{|g_x^j(x')\tilde{p}|, \\ &\quad j \in \underline{m}; |h_x^j(x')\tilde{p}|, j \in \underline{m}_e\}] \end{aligned}$$

Hence, for any compact neighbourhood N of x, there exists a constant d_1 such that:

$$\hat{\gamma}(x',\alpha p'+\alpha^2\tilde{p},c) - \gamma(x',c) \le \alpha[\theta(x',p',c)+d_1\alpha]$$

for all $x' \in N \cap M$. It is shown in Proposition A3 that:

$$|\gamma(x'+\alpha p'+\alpha^2\tilde{p},c) - \hat{\gamma}(x',\alpha p'+\alpha^2\tilde{p},c)| \leq \alpha\phi(\alpha,x',p',\tilde{p})$$

where $\phi(\alpha,x',p',\tilde{p}) \to 0$, uniformly in $(x',p',\tilde{p}) \in N \times B(0,1) \times B(0,1)$, as $\alpha \to 0$. Hence there exists an integer k' such that $\phi(\beta^{k'},x',p',\tilde{p}) + d_1\beta^{k'} \leq T(x')/2$† for all $(x',p',\tilde{p}) \in N \times B(0,1) \times B(0,1)$. Since $T(x') \leq -\theta'(x',p',c)$ we obtain:

$$\gamma(x'+\beta^{k'}p'+\beta^{2k'}\tilde{p},c) - \gamma(x',c) \leq \beta^{k'}[\theta(x',p',c)+T(x')/2]$$

$$\leq \beta^{k'}\theta(x',p',c)/2$$

so that, for each $x' \in N \cap M$, a step length greater than or equal to $\beta^{k'}$ is obtained in Step 2. Hence:

$$\gamma(x'',c) - \gamma(x'c) \leq \beta^{k'}\theta(x',p',c)/2$$

for all $x' \in N \cap M$, all $x'' \in A_c(x')$. Now there exists an $\bar{\varepsilon}$ such that $B(x,\bar{\varepsilon}) \subset N$ and $T(x') \geq T(x)/2$ for all $x' \in B(x,\bar{\varepsilon})$. Hence:

$$\gamma(x'',c) - \gamma(x',c) \leq -\beta^{k'}T(x')/2$$

$$\leq -\beta^{k'}T(x)/4$$

for all $x' \in B(x,\bar{\varepsilon}) \cap M$, all $x'' \in A_c(x')$. Combining this with the fact that there exists an $\varepsilon_1 > 0$, $\delta_1 > 0$ such that $\gamma(x'',c) - \gamma(x',c) \leq -\delta_1$ for all $x' \in B(x,\varepsilon_1) \cap M^c$, all $x'' \in A_c(x')$, yields the desired result, viz that for all $x \in D_c$, there exists an $\varepsilon > 0$, $\delta > 0$ such that $\gamma(x'',c) - \gamma(x',c) \leq -\delta$ for all $x' \in B(x,c)$. □

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† Since $c \geq \bar{c}(x)$ and $x \notin D_c$ then (by Proposition 6), $x \notin D$ so that $T(x) > 0$.