REPRESENTATION AND TRANSFORMATION OF TWO-PARAMETER MARTINGLES
UNDER A CHANGE OF MEASURE

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Memorandum No. UCB/ERL M79/23

13 April 1979

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1. Introduction

Let $(\Omega,\{F_t, 0\leq t\leq 1\},P_0)$ be a probability space and let $\{X_t, 0\leq t\leq 1\}$ be a P_0 local martingale. Doleans-Dade showed that the integral equation

$$L_t = 1+\int_0^t L_{s-}dX_s$$

has a unique solution $\{L_t, 0\leq t\leq 1\}$ which is a positive local martingale. If $E_0L_1 = 1$ then L_t is a martingale and $\frac{dP}{dP_0} = L_1$ defines a new probability measure P with

$$L_t = E_0(\frac{dP}{dP_0}|F_t) \ .$$

Results concerning P_0 local martingales under P or P local martingales under P_0 have come to be known as Girsanov's theorem [3]. For example, if N_t is a continuous P_0 local martingale then $N_t-[N,X]_t$ is a P local martingale where $[N,X]_t$ is defined intrinsically as a quadratic variation process.

If $\{F_t, 0\leq t\leq 1\}$ is generated by a P_0 Wiener process W_t then there exists a P Wiener process $\tilde{W}_t$. Further, any P local martingale Z_t has the integral representation $Z_t = \int_0^t q_s d\tilde{W}_s$ where $\int_0^1 q_s^2 ds < \infty$ a.s.

Research sponsored by the U.S. Army Research Office Grant DAAG29-78-G-0186.

For processes with two-dimensional parameter the martingale concept has several extensions [1,8]. The purpose of this paper is to present some results of the Girsanov type for such processes defined on the probability space of a 2-parameter Wiener process. When possible, transformation results are stated and proved in an intrinsic (representation independent) form.

Preliminary material is presented in Section 2, while the main results are collected in Section 3. In the remaining two sections, the theorems regarding martingles and weak martingales, respectively, are proved.

2. The Stochastic Calculus and Likelihood Ratios

The basic definitions of [1] will be used in this paper, and are summarized as follows. Let $R_+ = [0,\infty) \times [0,\infty)$ denote the positive quadrant of the plane. For two points $s = (s_1,s_2)$ and $s' = (s_1',s_2')$ in R_+, $s \succ s'$ will denote the condition $s_1 \geq s_1'$ and $s_2 \geq s_2'$, $s \curlywedge s'$ will denote the condition $s_1' \geq s_1$ and $s_2 \geq s_2'$, $s \times s'$ will denote the point (s_1,s_2'), and $s \vee s'$ will denote the point $(\max(s_1,s_1'),\max(s_2,s_2'))$. $I(s \vee s')$ will denote the indicator function of the set $\{s \vee s'\}$. 0 will denote the origin in R_+ and R_z the rectangle $\{0 \prec s \prec z\}$. $R_z \otimes R_z$ is the set $\{(s,s'): s \in R_z,\ s' \in R_z,\ s \curlywedge s'\}$.

Let $\{W_z,\ z \in R_{z_0}\}$ be a standard Wiener process defined on $(\Omega,\{F_z\},P_0)$ where $F_z = \{\sigma(W_s,\ s \prec z)$ completed with respect to $P_0\}$. A stochastic process $X = \{X_z,\ z \in R_{z_0}\}$ is said to be adapted if X_z is F_z measurable for each $z \in R_{z_0}$. In the following definitions, the process X is assumed to be adapted and for each z, X_z is integrable. X is a martingale if $z' > z$ implies that $E_0(X_{z'}|F_z) = X_z$ a.s., X is an adapted 1-martingle

(2-martingale) if $\{X_{s_1s_2}, F_{s_1s_2}\}$ is a one parameter martingale in s_1 for each s_2 (in s_2 for each s_1). X is a weak martingale if $s' \succ s$ implies that $E[X_{s'}-X_{s'\times s}-X_{s\times s'}+X_s | F_s] = 0$ a.s.

A proper 1-martingale (2-martingale) is a square integrable, sample continuous process M_1 (M_2) which is an adapted 1-martingale (2-martingale) and mean square differentiable in the 2-direction (1-direction).

A process Z is a local martingale if there is a sequence Z_n of square integrable martingales such that $Z_n(z,\omega) = Z(z,\omega)$ for $z \in R_{z_0}$ and $n \geq N(\omega)$ where $N(\omega) < \infty$ a.s. A local i-martingale (proper local i-martingale, weak local martingale) is similarly defined as the limit of square integrable i-martingales (proper i-martingales, weak martingales) for $i = 1,2$. By the theorems of Wong and Zakai [see 2], all local martingales have the stochastic integral representation

$$Z_z = \int_{R_z} q(s)dW_s + \int_{R_z \otimes R_z} r(s,s')dW_s dW_{s'}, \qquad (2.1)$$

and all proper local 1-martingales (2-martingales) are given by mixed area integrals

$$\int_{R_z \otimes R_z} \alpha(s,s')ds\,dW_{s'}, \qquad \left(\int_{R_z \otimes R_z} \beta(s,s')dW_s ds'\right) \qquad (2.2)$$

where q is F_s adapted and (ds-measure) square integrable a.s., and r, α and β are $F_{s \vee s'}$ adapted and (dsds' measure) square integrable a.s.

A local semimartingale is by definition the sum of a local martingale, a proper local 1-martingale, a proper local 2-martingale, and an absolutely continuous process

$$B_z = \int_{R_z} b_s ds$$

where b_s is a.s. square integrable. Denoting Lebesgue measure by μ, a local semimartingale is conveniently written as

$$Z = W\circ r\circ W + q\circ W + \mu\circ\alpha\circ W + W\circ\beta\circ\mu + b\circ\mu \tag{2.3}$$

where r, q, α, β are a.s. square integrable and b is a.s. integrable. A local semimartingale will also be referred to as a representable process. If ψ_s is an adapted, a.s. bounded process (for example, if ψ_s is a.s. continuous) then the stochastic integral

$$\psi\circ Z(z) = \int_{R_z} q_s\psi_s dW_s + \int_{R_z\otimes R_z} \rho_{s,s'}\psi_{s\wedge s'}dW_s dW_{s'} + \int_{R_z\otimes R_z} \alpha_{s,s'}\psi_{s\wedge s'}ds dW_{s'} + \int_{R_z\otimes R_z} \beta_{s,s'}\psi_{s\wedge s'}dW_s ds' + \int_{R_z} b_s\psi_s ds$$

is again a local semimartingale.

A process is a weak martingale if and only if it is the sum of a 1-martingale and a 2-martingale [8]. It follows that all representable local weak martingales have the representation

$$Z = W\circ r\circ W + q\circ W + \mu\circ\alpha\circ W + W\circ\beta\circ\mu \tag{2.4}$$

A local semimartingale is a one-parameter local semimartingale in each direction with the local semimartingale representations

$$Z_z = \int_{R_z} Z_{W1}(z,s')dW_{s'} + \int_{R_z} Z_{\mu 1}(z,s')ds' \tag{2.5}$$

$$Z_z = \int_{R_z} Z_{W2}(z,s)dW_s + \int_{R_z} Z_{\mu 2}(z,s)ds \tag{2.6}$$

where

$$Z_{W1}(z,s') = q_{s'} + \int_{R_z} I(s\wedge s')r_{s,s'}dW_s + \int_{R_z} I(s\wedge s')\alpha_{s,s'}ds \tag{2.7}$$

$$Z_{\mu 1}(z,s') = b_{s'} + \int_{R_z} I(s \curlywedge s')\beta_{s,s'}dW_s$$

$$Z_{W2}(z,s) = q_s + \int_{R_z} I(s \curlywedge s')r_{s,s'}dW_{s'} + \int_{R_z} I(s \curlywedge s')\beta_{s,s'}ds'$$

$$Z_{\mu 2}(z,s) = b_s + \int_{R_z} I(s \curlywedge s')\alpha_{s,s'}dW_{s'}$$

It is convenient to write (2.5) and (2.6) in the compact form

$$Z = Z_{W1} \circ W + Z_{\mu 1} \circ \mu \tag{2.8}$$

$$Z = Z_{W2} \circ W + Z_{\mu 1} \circ \mu \tag{2.9}$$

Note that if $Z = \mu \circ \alpha \circ W$ and $\bar{Z} = \mu \circ \bar{\alpha} \circ W$, then $Z \equiv \bar{Z}$ a.s. if and only if $Z_{W1}(z,s') = \dot{Z}_{W1}(z,s')$ for ($ds' \times dP$ measure) a.e. (s',ω) for each z. In this case $\bar{\alpha}$ will be called a version of α. Hence, α and β in (2.3) are uniquely determined by Z up to versions.

The composition $Y*X$ of two local semimartingales X and Y is the process defined by

$$\begin{aligned}(Y*X)_z = &\int_{R_z \otimes R_z} Y_{W2}(s \curlyvee s',s)X_{W1}(s \curlyvee s',s')dW_s dW_{s'} \\ &+ \int_{R_z \otimes R_z} Y_{\mu 2}(s \curlyvee s',s)X_{W1}(s \curlyvee s',s')ds dW_{s'} \\ &+ \int_{R_z \otimes R_z} Y_{W2}(s \curlyvee s',s)X_{\mu 1}(s \curlyvee s',s')dW_s ds' \\ &+ \int_{R_z \otimes R_z} Y_{\mu 2}(s \curlyvee s',s)X_{\mu 1}(s \curlyvee s',s')ds ds'\end{aligned} \tag{2.10}$$

Formally, $Y*X$ satisfies $\partial_1\partial_2(Y*X) = \partial_2 Y \partial_1 X$. $Y*X$ is a well defined local semimartingale if the integrands in (2.10) are a.s. square integrable. In abbreviated form, (2.10) can be expressed as

$$Y*X = W\circ Y_{W2}X_{W1}\circ W + \mu\circ Y_{\mu 2}X_{W1}\circ W + W\circ Y_{W2}X_{\mu 1}\circ\mu + \mu\circ Y_{\mu 2}X_{\mu 1}\circ\mu$$

One may define quadratic variation processes for a local semimartingale with representation (2.1) by

$$[Z,Z]_z = b^2\circ\mu + \mu\circ r^2\circ\mu \tag{2.11}$$

and, for $i = 1$ or 2,

$$\langle Z,Z\rangle_{iz} = \int_{R_z} X^2_{Wi}(z,s)ds\ . \tag{2.12}$$

The definition (2.12) is consistent with the definition of quadratic variation for one-parameter local semimartingales. Both $[Z,Z]$ and $\langle Z,Z\rangle_i$ are intrinsic to Z in the sense that they have representation free, quadratic variation interpretations [1,2]. Define $[Z,\tilde{Z}]$ and $\langle Z,\tilde{Z}\rangle_i$ for local semimartingales Z and $\tilde{Z}$ by bilinearity. If Z has representation (2.3) and

$$\tilde{Z} = W\circ\tilde{r}\circ W + \tilde{q}\circ W + \mu\circ\tilde{\alpha}\circ W + W\circ\tilde{\beta}\circ\mu + \tilde{b}\circ\mu \tag{2.13}$$

then

$$\langle Z,\tilde{Z}\rangle_1(z) = \int_{R_z} Z_{W1}(z,s')\tilde{Z}_{W1}(z,s')ds'\ , \quad z = (z_1,z_2) \tag{2.14}$$

and

$$\langle Z,\tilde{Z}\rangle_1 = [Z,\tilde{Z}] + (W\circ r + \mu\circ\alpha)\tilde{Z}_{W1}\circ\mu + (W\circ\tilde{r} + \mu\circ\tilde{\alpha})Z_{W1}\circ\mu \tag{2.15}$$

To obtain (2.15) apply the 1-parameter differential formula [3] to the integrand in (2.14) as a function of z_2 for fixed z_1 and use (2.7). Similarly,

$$\langle Z,\tilde{Z}\rangle_2 = [Z,\tilde{Z}] + \mu\circ\tilde{Z}_{W2}(r\circ W + \beta\circ\mu) + \mu\circ Z_{W2}(\tilde{r}\circ W + \tilde{\beta}\circ\mu) \tag{2.16}$$

The differentiation formula of [7,8] for local semimartingales has been put into a representation free form [9]. Let $F : R \to R$ be a function with continuous derivatives through the fourth order, and let Z be given by (2.1). Let $F_k(x) = \frac{\partial^k}{\partial x^k} F(x)$. Then

$$\begin{aligned} F(Z) &= F(X_0) + F_1(Z)\circ Z + F_2(Z)\circ(Z*Z) \\ &+ \tfrac{1}{2}F_2(Z)\circ(\langle Z,Z\rangle_1 + \langle Z,Z\rangle_2 - [Z,Z]) \\ &+ \tfrac{1}{2}F_3(Z)\circ(Z*\langle Z,Z\rangle_1 + \langle Z,Z\rangle_2 * Z + 2[Z,Z*Z]) \\ &+ \tfrac{1}{4}F_4(Z)\circ\langle X,X\rangle_2 * \langle X,X\rangle_1 \end{aligned} \tag{2.17}$$

whenever the composition operations yield local semimartingales.

If $Z = (Z_1,\ldots,Z_n)$ is a vector of n local semimartingales and if $F\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ has continuous partial derivatives to fourth order, then (2.17) still yields the integral representation of $F(Z)$ if the terms are interpreted appropriately. For example, identify

$$\begin{aligned} F_1(Z)\circ Z &= \sum_i \frac{\partial F}{\partial z_i}\circ Z_i \\ F_2(Z)\circ(Z*Z) &= \sum_{i,j} \frac{\partial^2 F}{\partial z_i \partial z_j}\circ(Z_i * Z_j) \\ F_3(Z)\circ(Z*\langle Z,Z\rangle_1) &= \sum_{i,j,k} \frac{\partial^3 F}{\partial z_i \partial z_j \partial z_k}\circ(Z_i * \langle Z_j, Z_k\rangle_1). \end{aligned}$$

For $n = 2$ and $F(z,\tilde{z}) = z\tilde{z}$, this yields

$$Z\tilde{Z} = Z_0\tilde{Z}_0 + Z\circ\tilde{Z} + \tilde{Z}\circ Z + Z*\tilde{Z} + \tilde{Z}*Z + \langle Z,\tilde{Z}\rangle_1 + \langle Z,\tilde{Z}\rangle_2 - [Z,\tilde{Z}] \tag{2.18}$$

This generalization of the differentiation formula given in [7,8] may be proved, as in [7], by repeated application of the differential formula for one parameter martingales. (2.18) shows that $Z*\tilde{Z} + \tilde{Z}*Z$ is intrinsic to Z, $\tilde{Z}$ since all the other terms are. Thus the symmetrization of $*$ is an intrinsic operation.

Let P be a probability measure on $(\Omega,\{F_z\})$ equivalent to P_0. Then if $E_0[(\frac{dP}{dP_0})^2]$ is finite, the likelihood ratio

$$L_z = E_0[\frac{dP}{dP_0}|F_z]$$

satisfies $L = e^X$ where [6]

$$X_z = \int_{R_z} \theta_s dW_s - \frac{1}{2}\int_{R_z} \theta_s^2 ds - \frac{1}{2}\int_{R_z \otimes R_z} \rho_{s,s'}^2 ds ds' \tag{2.19}$$
$$+ \int_{R_z \otimes R_z} \rho_{s,s'}[dW_s - \tilde{u}(s' \times s, s)ds][dW_{s'} - u(s' \times s, s')ds']$$

for some functions ρ, θ, u and $\tilde{u}$. In an abbreviated form, (2.19) is expressed as

$$X = \theta \circ W - \frac{1}{2}\theta^2 \circ \mu - \frac{1}{2}\mu \circ \rho^2 \circ \mu + (W - \mu\tilde{u}) \circ \rho \circ (W - u\mu) \quad .$$

If ρ and θ are any a.s. bounded and adapted processes then u and $\tilde{u}$ are uniquely determined by

$$u(z,s') = \theta_{s'} + \int_{R_z} I(s \curlywedge s')\rho_{s,s'}[dW_s - \tilde{u}(s' \times s, s)ds] \tag{2.20}$$

$$\tilde{u}(z,s) = \theta_s + \int_{R_z} I(s \curlywedge s')\rho_{s,s'}[dW_{s'} - u(s' \times s, s')ds'] \tag{2.21}$$

and X is a local semimartingale. It then makes sense to write $L = e^X = \mathcal{E}(\rho,\theta)$. $L = e^X$ also has the representation [6]

$$e^X = e^X \circ (m + M_2 * M_1) \tag{2.22}$$

where m is the local martingale part of X and M_1 (M_2) is the local 1-martingale (2-martingale) part of X. It follows from (2.19)-(2.21) that

$$X_{W1}(z,s') = (M_1)_{W1}(z,s') = u(z,s') \tag{2.23}$$

$$X_{W2}(z,s) = (M_2)_{W2}(z,s) = \tilde{u}(z,s) \quad . \tag{2.24}$$

$L = e^X$ also has essentially one parameter representations, as expressed by

$$e^X_{W1}(z,s') = e^X_{s'\times z}u(z,s') \tag{2.25}$$

$$e^X_{W2}(z,s) = e^X_{z\times s}\tilde{u}(z,s) \tag{2.26}$$

Lemma 2.1. Let $e^X = \&(\rho,\theta)$ and suppose ρ, θ, u and $\tilde{u}$ are a.s. bounded (for example, they may be sample continuous). Let $Z = \{Z_z, z \in R_{Z_0}\}$ be a local semimartingale with the representation (2.3). Then the following identities hold and all terms are local semimartingales.

$$Z*e^X = e^X\circ(Z*M_1) \tag{2.27}$$

$$e^X*Z = e^X\circ(M_2*Z) \tag{2.28}$$

$$\langle X,Z\rangle_1 = [X,Z] + (W-\mu\tilde{u})\circ\rho Z_{W1}\circ\mu + W\circ ru\circ\mu + \mu\circ\alpha u\circ\mu \tag{2.29}$$

$$\langle X,Z\rangle_2 = [X,Z] + \mu\circ Z_{W2}\rho\circ(W-u\mu) + \mu\circ\tilde{u}r\circ W + \mu\circ\tilde{u}\alpha\circ\mu \tag{2.30}$$

$$\langle e^X,Z\rangle_1 = e^X\circ\{\langle X,Z\rangle_1 + M_2*\langle X,Z\rangle_1 + [X*Z,X] + [X*X,Z]\} \tag{2.31}$$

$$\langle e^X,Z\rangle_2 = e^X\circ\{\langle X,Z\rangle_2 + \langle X,Z\rangle_2*M_1 + [Z*X,X] + [X*X,Z]\} \tag{2.32}$$

$$\begin{aligned} e^X Z = Z\circ e^X + e^X\circ\{&Z + M_2*(Z+\langle Z,X\rangle_1) + (Z+\langle Z,X\rangle_2)*M_1 \\ &+ \langle X,Z\rangle_1 + \langle X,Z\rangle_2 + [X,X*Z] + [X,Z*X] - [Z,X-X*X]\} \end{aligned} \tag{2.33}$$

$$\begin{aligned} e^{-X} = 1 + e^{-X}\circ\{&-X*\langle X,X\rangle_1 - \langle X,X\rangle_2*X + \langle X,X\rangle_1 + \langle X,X\rangle_2 - [X,X] \\ &- m + M_2*M_1\} \end{aligned} \tag{2.34}$$

$$\begin{aligned} e^{-X}Z = Z\circ e^{-X} + e^{-X}\circ\{&Z - \langle Z,X\rangle_1 - \langle Z,X\rangle_2 + [Z,X] - [X,X*Z] \\ &- [X,Z*X] - (M_2-\langle X,X\rangle_2)*(Z-\langle X,Z\rangle_1) \\ &- (Z-\langle X,Z\rangle_2)*(M_1-\langle X,X\rangle_1)\} \end{aligned} \tag{2.35}$$

Proof. Z appears at most once in each of the terms in (2.27)-(2.35) so that each term has a local semimartingale representation with integrands which are a.s. square integrable (each integrand being the product of a.s.

bounded processes and at most one a.s. square integrable process). (2.27) and (2.28) follow from (2.23)-(2.26). (2.29) and (2.30) follow from (2.15), (2.16), (2.23) and (2.24). (2.31) follows by applying the differentiation rule for one parameter local semimartingales to the integrand in

$$\langle e^X, Z\rangle_1 = e^X_{W1} Z_{W1} \circ \mu = \int_{R_z} e^X(s'\times z)u(z,s')Z_{W1}(z,s')ds'$$

as a function of z_2 for s' fixed. (2.32) follows similarly. (2.33) is obtained by applying the differentiation formula to $F(e^X, Z) = e^X Z$ and using (2.27), (2.28), (2.31) and (2.32). (2.34) is obtained by applying the differentiation formula to $F(X) = e^{-X}$ and (2.35) follows by applying the differentiation formula to $F(e^{-X}, Z) = Ze^{-X}$.

Local martingales, local i-martingales, and local weak martingales may be defined under the law P exactly as they were for law P_0. It follows that a process Z is a P local martingale (a P local i-martingale, $i = 1$ or 2, a P local weak martingale) if and only if LZ is a P_0 local martingale (P_0 local i-martingale, P_0 local weak martingale).

The notion of local semimartingale will always refer to representation with respect to the process $\{W_z\}$ under the law P_0. Under the conditions of Lemma 2.1, a process $Z = e^{-X}(e^X Z)$ is representable if and only if $e^X Z$ is representable.

3. Compensation and Representation Theorems

The main results of this paper, which are summarized in this section, describe martingales, weak martingales and i-martingales under the change of measure $P_0 \to P$ described in Section 2. There are two types of results; one type concerns compensation (or transformation) of P_0 martingales,

P_0 i-martingales and P_0 weak martingales to obtain P martingales, P i-martingales, and P weak martingales (and vice versa). The other type of result concerns the integral representation of P weak martingales and P martingales (i.e., the counterpart of (2.1) and (2.4) when P_0 is replaced by P).

Throughout the remainder of this paper, assume that $P \sim P_0$, $\frac{dP}{dP_0} = \mathcal{E}(\rho,\theta) = \exp(X)$ and that ρ, θ, u and $\tilde{u}$ in (2.19) are a.s. bounded.

Theorem 1 (i-Martingale Compensation). Let Z and N be local semimartingales, and let i = 1 or 2.

(a) If Z is a P_0 local i-martingale, then (and only then) $Z - \langle X,Z\rangle_i$ is a representable P local i-martingale.

(b) If N is a P local i-martingale, then (and only then) $N + \langle X,N\rangle_i$ is a P_0 local i-martingale.

If Z is a P_0 local i-martingale, then $\langle X,Z\rangle_i$ is the unique local semimartingale with $(\langle X,Z\rangle_i)_{Wi} = 0$ such that $Z - \langle X,Z\rangle_i$ is a P local i-martingale.

Remark. Theorem 1 follows easily from the theorem on transformation of one parameter local martingales. It is also easily proved using the identities in Lemma 2.1.

It will be convenient to define some operators on the linear space of local semimartingales. If Y is a local semimartingale, let

$$T(Y) = \langle X,Y\rangle_1 + \langle X,Y\rangle_2$$
$$V(Y) = [X, X*Y + Y*X] - [Y, X - X*X]$$
$$R = (I+T)^{-1} = \sum_{n=0}^{\infty} (-1)^n T^n$$
$$\Gamma = (I-V)R$$

I is the identity operator. It follows easily from Lemma 2.1 that T and V are well defined and that $V^2 = TV = 0$. Hence (as will be justified by Lemma 4.1),

$$\Gamma^{-1} = ((I-V)(I+T)^{-1})^{-1} = (I+T)(I-V)^{-1}$$
$$= (I+T)(I+V) = I + T + V$$

These operators are intrinsic since they are defined in terms of $\langle\ \rangle_i$, [ ], and the symmetrization of *.

<u>Theorem 2</u> (Martingale Compensation). The operators T, V, R, Γ and Γ^{-1} are well defined, linear, and intrinsically defined on the space of local semimartingales. Γ is invertible with inverse Γ^{-1}. Γ maps the subspace consisting of P_0 local martingales onto the space of P local martingales.

If N(Z) is a P_0 (P) local martingale, then $\Gamma(N)$ ($\Gamma^{-1}(Z)$) is the <u>unique</u> P (P_0) local martingale such that $N-\Gamma(N)$ ($Z-\Gamma^{-1}(Z)$) has no P_0 local weak martingale component.

<u>Theorem 3</u> (Weak Martingale Compensation). If N is a P_0 local weak martingale then

$$N - \langle N,X\rangle_1 - \langle N,X\rangle_2 + [N,X+X*X] \tag{3.1}$$

and

$$\tilde{N} = N - \mu\circ\alpha u\circ\mu - \mu\circ\tilde{u}\beta\circ\mu - [N,X+X*X] \tag{3.2}$$

are representable P local weak martingales. The process in (3.1) is determined from N and X by intrinsic operations. $\tilde{N}$ is the unique representable P local weak martingale such that $N-\tilde{N}$ is an absolutely continuous process.

If M is a P_0 local martingale, then $M-[M,X+X*X]$ is a representable P local weak martingale.

Remark. It would be desirable to find an expression for $\tilde{N}$ in Theorem 3 which is intrinsically determined by N and X. The last part of the theorem shows that this is possible if N is a P_0 local martingale (rather than just a P_0 weak local martingale).

Theorem 4 (Weak Martingale Representation). All local semimartingales which are P local weak martingales may be represented as

$$q\circ(W-\theta\mu) + (W-\mu\tilde{u})\circ r\circ(W-u\mu) - \mu\circ r\rho\circ\mu + \mu\circ a\circ(W-u\mu) \qquad (3.3)$$
$$+ (W-\mu\tilde{u})\circ b\circ\mu$$

where q, r, a and b are a.s. square integrable. Hence, ρ, θ, u and $\tilde{u}$ have the interpretation (E denotes expectation under law P)

$$\theta(s)ds = E[dW_s|F_s]$$
$$u(z,s')ds' = E[dW_{s'}|F_{s'\times z}]$$
$$\tilde{u}(z,s)ds = E[dW_s|F_{z\times s}]$$
$$\rho(s,s')dsds' = E[(dW_s-\tilde{u}(z,s)ds)(dW_{s'}-u(z,s')ds')|F_{s\vee s'}] \quad .$$

Remark. Since the σ-fields F_z are generated by a P_0 Wiener process, if $z'\curlywedge z$ then F_z and $F_{z'}$ are conditionally independent given $F_{z\times z'}$ (using probability measure P_0). However, when P_0 is replaced by P, the conditional independence is lost unless ρ is identically zero. Indeed, this is suggested by the interpretation given in Theorem 4 of ρ as a conditional correlation. As a result, the σ-fields $\{F_z\}$ cannot be generated by a process which is Wiener on (Ω, F_z, P).

A corollary to the representation (3.3) is that the class of representable P local weak martingales is stable under stochastic integration. Unless ρ is identically zero, the classes of P local martingales and representable P local i-martingales are not stable under the operation of stochastic integration. Hence, there does not exist a counterpart to the representation (3.3) for P local martingales.

If ρ is identically zero, then $\tilde{W} = W - \theta \circ \mu$ is a Wiener process on (Ω, F_z, P_0) [6]. If $\tilde{W}$ generates the same σ-fields as W (i.e., "innovations equivalence" holds for W under P), then P local martingales may be expressed as $q \circ \tilde{W} + \tilde{W} \circ r \circ \tilde{W}$ by the Wong-Zakai representation theorem. However, innovations equivalence is not necessary.

Theorem 5 (Martingale Representation when $\rho = 0$). If ρ is identically zero, then any P martingale may be represented as $q \circ \tilde{W} + \tilde{W} \circ r \circ \tilde{W}$ where $\tilde{W} = W - \theta \circ \mu$.

4. Martingale Results

Theorems 2 and 5 will be proved in this section. The operators T and V are well defined maps of local semimartingales by Lemma 2.1. The following lemma shows that R, $\Gamma = (I-V)R$ and Γ^{-1} are also well defined.

Lemma 4.1. Let $T(Y) = \langle X,Y \rangle_1 + \langle X,Y \rangle_2$ for local semimartingales Y. For each such Y, the series

$$\sum_{n=0}^{\infty} (-1)^n T^n(Y) \tag{4.1}$$

converges pointwise in probability to a local semimartingale $R(Y)$. Moreover $(I+T)R = R(I+T) = I$.

Proof. By Lemma 2.1 and induction on n, $T^n Y$ is a semimartingale for all $n \geq 0$. Let

$$T^n(Y) = q^{(n)} \circ W + W \circ r^{(n)} \circ W + \mu \circ f_1^{(n)} \circ W + W \circ f_2^{(n)} \circ \mu + b^{(n)}$$

be the semimartingale representation. By (2.29) and (2.30),

$$\begin{aligned} T(Y) &= W \circ (\rho Y_{W1} + r^{(0)} \tilde{u}) \circ \mu + \mu \circ (Y_{W2} \rho + u r^{(0)}) \circ W + 2[X,Y] \\ &\quad + \mu \circ (f_2^{(0)} u - \tilde{u} \rho Y_{W1} + \tilde{u} f_1^{(0)} - Y_{W2} \rho u) \circ \mu \end{aligned}$$

Hence $q^{(1)} = r^{(1)} = 0$, and moreover, $q^{(n)} = r^{(n)} = 0$ for all $n \geq 1$. Also

$$f_1^{(1)} = Y_{W2} \rho + u r^{(0)} , \quad f_2^{(1)} = \rho Y_{W1} + r^{(0)} \tilde{u}$$

We claim that $f_1^{(1)}(s,s')$ is a.s. square integrable. Indeed, since u and ρ are a.s. bounded and $r^{(0)}$ is a.s. square integrable, it suffices to show that Y_{W2} is a.s. square integrable. Recall that

$$Y_{W2}(s,s') = \int_{R_{s'} \times s} I(s \curlywedge t') r^{(0)}(s,t') dW_{t'} + \int_{R_{s'} \times s} I(s \curlywedge t') f_2^{(0)}(s,t') dt' \tag{4.2}$$

By the Schwartz inequality,

$$\int_{R_z \otimes R_z} ds ds' \left(\int\!\!\int_{R_{s'} \times s} I(s \curlywedge t') f_2^{(0)}(s,t') dt' \right)^2 \tag{4.3}$$

$$\leq \left(\int_{R_z} ds' \right) \int_{R_z \otimes R_z} f_2^{(0)}(s,t')^2 ds dt' < \infty \text{ a.s.}$$

The stopping time constructed in [5] may be used to provide a sequence of $ds ds' P(d\omega)$ square integrable functions r_n such that $r_n(s,s',\omega) = r^{(0)}(s,s',\omega)$

for all s, s' for $n \geq N(\omega)$ where $N(\omega) < \infty$ a.s. It follows that for $n \geq N(\omega)$,

$$\int_{R_{s'\times s}} I(s\wedge t')r_n(s,t')dW_{t'} = \int_{R_{s'\times s}} I(s\wedge t')r^{(0)}(s,t')dW_{t'} \text{ for } s\wedge s' \quad (4.4)$$

Since

$$E_0\Big[\int_{R_z} dsI(s\wedge s')\Big(\int_{R_{s'\times s}} I(s\wedge t')r_n(s,t')dW_{t'}\Big)^2\Big] < \int_{R_z\otimes R_z} E[r_n(s,t')^2]\, dsdt' < \infty$$

it follows that for all n,

$$\int_{R_z\otimes R_z} dsds'\Big(\int_{R_{s'\times s}} I(s\wedge t')r_n(s,t')dW_{t'}\Big)^2 < \infty \quad \text{a.s.}$$

The right hand side of (4.4) is thus also dsds' square integrable a.s. This plus (4.3) implies that Y_{W2} and hence also $f_1^{(1)}$ is (dsds' measure) square integrable a.s. A similar argument shows that $f_2^{(1)}$ is a.s. square integrable.

Now, applying (2.29) and (2.30) to compute $T^n(Y) = T(T^{n-1}(Y))$ and using the fact that $r^{(n)} = q^{(n)} = 0$ for $n \geq 1$ yields that, for $n \geq 2$,

$$f_1^{(n)}(s,s') = \rho(s,s')\int_{R_{s'\times s}} I(s\wedge r')f_2^{(n-1)}(s,r')dr'$$

$$f_2^{(n)}(s,s') = \rho(s,s')\int_{R_{s'\times s}} I(s'\wedge r)f_1^{(n-1)}(s',r)dr$$

$$b_z^{(n)} = \int_{R_z\otimes R_z} dsds'\{(f_2^{(n-1)}(s,s') - f_1^{(n)}(s,s'))u(s'\times s,s') + (f_1^{(n-1)}(s,s') - f_2^{(n)}(s,s'))\tilde{u}(s'\times s,s)\} \quad (4.5)$$

Since $f_1^{(1)}$ and $f_2^{(1)}$ are a.s. square integrable and ρ is a.s. bounded, it follows easily that $f_1^{(3)}$ and $f_2^{(3)}$ are a.s. bounded. A standard iteration argument then shows that $\sum_{n=0}^{\infty} f_i^{(n)}$ converges uniformly a.s. to an a.s. square integrable function $f_i(s,s')$ for $i = 1,2$. The stopping time provided in [5], Lemma 7 of [4], and the uniform convergence a.s. to f_i imply that the proper P_0 local i-martingale term of $\sum_{n=1}^{k} T^n(Y)$ converges pointwise in probability to the proper P_0 local i-martingale $\mu \circ f_1 \circ W$ (if $i = 1$) or $W \circ f_2 \circ \mu$ (if $i = 2$). The uniform convergence of $\sum_n f_i^{(n)}$ and (4.5) imply the uniform convergence of $\sum_{n=0}^{\infty} b^{(n)}$ a.s. to an absolutely continuous process b. Thus the series (4.1) does indeed converge to a local semimartingale $R(Y)$.

Finally, note that

$$\begin{aligned} R(I+T)(Y) &= (\lim_{k\to\infty} \sum_{n=0}^{k} (-1)^n T^n)(I+T)(Y) \\ &= Y + \lim_{k\to\infty} T^{(k+1)}(Y) = Y \end{aligned}$$

where the limit is pointwise in probability. Similarly, $(I+T)R = I$. The proof of Lemma 4.1 is complete. ∎

Lemma 4.2. If Z is a P local martingale, then

$$\Gamma^{-1}(Z) = Z + \langle X,Z\rangle_1 + \langle X,Z\rangle_2 + [X, X*Z + Z*X] - [Z, X - X*X] \tag{4.6}$$

is the unique P_0 local martingale such that $Z - \Gamma^{-1}(Z)$ has no P_0 local martingale part.

Proof. Let Z be a P-local martingale. Then $e^X Z$ is a P_0 local martingale and hence a local semimartingale. Then by Lemma 2.1, $Z = e^{-X}(e^X Z)$ is also a local semimartingale. Hence, (2.33) applies and using (4.6) yields

$$e^X Z = e^X \circ \Gamma^{-1}(z) + e^X \circ \{M_2 * (Z + \langle Z,X\rangle_1) + (Z + \langle Z,X\rangle_2) * M_1\} \tag{4.7}$$

By Theorem 3.2, $Z+\langle Z,X\rangle_i$ is a P_0 local i-martingale for $i = 1,2$ so that

$$M_2*(Z+\langle Z,X\rangle_1) \quad \text{and} \quad (Z+\langle X,Z\rangle_2)*M_1$$

are P_0 local martingales. Since the class of P_0 local martingales is stable under stochastic integration $\Gamma^{-1}(Z)$ is also a P_0 local martingale. The uniqueness assertion in Lemma 4.2 follows from the uniqueness of local semimartingale representation. ∎

The complement of Lemma 4.2 will be considered next, completing the proof of Theorem 2. Let N be a P_0 local martingale. Let n_i, $i = 1,2$ be proper P_0 local i-martingales, and let b be an absolutely continuous process. Let Z be the local semimartingale

$$Z = N-n_1-n_2-b \tag{4.8}$$

Then Z is a P local martingale if and only if it is a representable P local i-martingale for $i = 1,2$. Hence, by Theorem 1, Z is a P local martingale if and only if

$$n_2+b = \langle N-n_1,X\rangle_1 \tag{4.9}$$

$$n_1+b = \langle N-n_2,X\rangle_2 \tag{4.10}$$

<u>Proposition 4.3</u>. (i) There exist uniquely determined n_1, n_2 and b satisfying (4.9) and (4.10). Hence there exists a unique P local martingale $\hat{N} = N-n_1-n_2-b$ such that $\hat{N}-N$ has no P local martingale component.

(ii) There exist unique local semimartingales m_1 and m_2 such that

$$m_1 = \langle N-m_2,X\rangle_2 \tag{4.11}$$

$$m_2 = \langle N-m_1,X\rangle_1 \tag{4.12}$$

If b is the absolutely continuous process

$$b = [N, X - X*X] - [X, X*(N-m_1) + (N-m_2)*X] \tag{4.13}$$

then $n_i = m_i - b$, $i = 1,2$ are proper P_0 local i-martingales satisfying (4.9) and (4.10).

(iii) $\hat{N} = \Gamma(N)$.

Proof. Part (i) follows immediately from part (ii) -- the uniqueness assertion in (i) follows from the fact that if n_1, n_2 and b satisfy (4.9) and (4.10), then $m_1 = n_1 + b$, $m_2 = n_2 + b$ and b satisfy (4.11)-(4.13) which have a unique solution by (ii).

If m_1 and m_2 are local semimartingales satisfying (4.11) and (4.12), then $(m_1)_{W2} = (m_2)_{W1} = 0$ so that m_1 and m_2 must have the integral representation

$$m_1 = \mu \circ f_1 \circ W + b_1 \tag{4.14}$$

$$m_2 = W \circ f_2 \circ \mu + b_2 \tag{4.15}$$

Let $N = W \circ r \circ W + q \circ W$ be the integral representation for N. As usual, f_1, f_2, r and q are square integrable a.s. and b_1 and b_2 are absolutely continuous.

Using (2.29) and (2.30) and equating proper local i-martingale terms yields that (4.11) and (4.12) are equivalent to the four equations

$$\mu \circ f_1 \circ W = \mu \circ \{\tilde{u} r + (N-m_2)_{W2} \rho\} \circ W \tag{4.16}$$

$$W \circ f_2 \circ \mu = W \circ \{r u + \rho (N-m_1)_{W1}\} \circ \mu \tag{4.17}$$

$$b_1 = [N,X] - \mu \circ \{(N-m_2)_{W2} \rho u + \tilde{u} f_2\} \circ \mu \tag{4.18}$$

$$b_2 = [N,X] - \mu \circ \{\tilde{u} \rho (N-m_1)_{W1} + f_1 u\} \circ \mu \tag{4.19}$$

Now (4.16) and (4.17) are true if and only if there are versions of f_1 and f_2 so that, for $s \curlywedge s'$,

$$f_1(s,s') = \{\tilde{u}(s,s')r(s,s') + \rho(s,s')\left(q(s) + \int_{R_{s'\times s}} I(s\curlywedge t')r(s,t')dW_{t'}\right)\}$$
$$- \rho(s,s')\int_{R_{s'\times s}} I(s\curlywedge t')f_2(s,t')dt' \tag{4.20}$$

$$f_2(s,s') = \{r(s,s')u(s,s') + \rho(s,s')\left(q(s') + \int_{R_{s'\times s}} I(t\curlywedge s')r(t,s')dW_t\right)\}$$
$$- \rho(s,s')\int_{R_{s'\times s}} I(s'\curlywedge t)f_1(s',t)dt \tag{4.21}$$

By an argument used already in the proof of Lemma 4.1, the quantities in brackets in (4.20) and (4.21) are a.s. square integrable. A standard Picard iteration argument then shows that there exist unique, a.s. square integrable solutions to (4.20) and (4.21). The convergence is uniform a.s. Hence, (4.16) and (4.17) have a unique (up to versions) solution f_1, f_2. Substitution of (4.20) and (4.21) into (4.19) and (4.18) respectively yields that $b_1 \equiv b_2 \equiv b$ a.s. where b is given by (4.13). This proves part (ii).

To prove $\hat{N} = \Gamma(N)$ note first that (4.11) and (4.12) imply that

$$m_1 + m_2 = T(N) - T(m_1 + m_2)$$

or

$$m_1 + m_2 = (I+T)^{-1}T(N) \tag{4.22}$$

Using (4.22), (4.13) becomes

$$b = -V(N - m_1 - m_2) = -V(I - (I+T)^{-1}T)(N) = -V(I+T)^{-1}(N) \tag{4.23}$$

Combining (4.22) and (4.23) yields that

$$\hat{N} = N - m_1 - m_2 + b = (I - (I+T)^{-1}T - V(I+T)^{-1})(N)$$
$$= (I-V)(I+T)^{-1}(N) = \Gamma(N) .$$

The proof of Proposition 4.3 and hence also of Theorem 2 is complete. ■■

Corollary to Theorem 2. If Z is a P local martingale with no P_0 local martingale part (i.e. $[Z,Z] = 0$), then Z is identically zero a.s.

Proof. Let 0 be the identically zero process. Then Z and $\Gamma(0)$ are each P local martingales with the same P_0 local martingale part (namely, 0) so $Z = \Gamma(0) = 0$. ■

Proof of Theorem 5. If ρ is identically zero then $X = \theta \circ W - \frac{1}{2}\theta^2 \circ \mu$ by (2.19). It follows that $T^n = 0$ for $n \geq 3$ so the series defining R is finite. The result is (using (2.29) and (2.30)),

$$\begin{aligned} R(N) &= (I+T)^{-1}N \\ &= N - \langle N,X\rangle_1 - \langle N,X\rangle_2 + \langle X,\langle N,X\rangle_1\rangle_2 + \langle X,\langle N,X\rangle_2\rangle_1 \\ &= N - \langle N,X\rangle_1 - \langle N,X\rangle_2 + 2[N,X*X] \end{aligned}$$

and

$$\begin{aligned} VR(N) &= [X,R(N)*X + X*R(N)] - [R(N),X - X*X] \\ &= 0 - [N,X - X*X] \end{aligned}$$

So for any P_0 local martingale $N = W\circ r\circ W + q\circ W$,

$$\begin{aligned} \Gamma(N) &= N - \langle N,X\rangle_1 - \langle N,X\rangle_2 + [N,X + X*X] \\ &= \tilde{W}\circ r\circ\tilde{W} + q\circ\tilde{W} \end{aligned} \tag{4.24}$$

where $\tilde{W} = W - \theta\circ\mu$. Since Γ maps onto the set of all P local martingales, any P local martingale has the representation (4.24). ■

5. Weak Martingale Results

Theorems 3 and 4 will be proved in this section. Suppose that M is a P_0 local martingale. Then trivially

$$e^X(M-[M,X+X*X]) = (e^XM-[e^X,M]) - (e^X[M,X+X*X]-[e^X,M]) \quad . \qquad (5.1)$$

Since M and e^X are both P_0 local martingales, $e^XM-[e^X,M]$ is a P_0 local weak martingale. (This may be proved by using integral representations, but is an "intrinsic" fact [1].) By (2.22),

$$[e^X,M] = e^X\circ[M,m+M_2*M_1] = e^X\circ[M,X+X*X] \; . \qquad (5.2)$$

Using the differential formula (2.18) and substituting in (5.2) yields

$$e^X[M,X+X*X]-[e^X,M] = [M,X+X*X]\circ e^X + e^X*[M,X+X*X]+[M,X+X*X]*e^X$$

which is a P_0 local weak martingale. In view of (5.1), $e^X(M-[M,X+X*X])$ is a P_0 local weak martingale so that $M-[M,X+X*X]$ is a P local weak martingale.

Now, let N be a representable P_0 local weak martingale. Then N may be expressed as

$$N = M+\mu\circ\alpha\circ W+W\circ\beta\circ\mu = M+n_1+n_2 \qquad (5.3)$$

where M is a P_0 local martingale, and $n_1 = \mu\circ\alpha\circ W$ $(n_2 = W\circ\beta\circ\mu)$ is a proper P_0 local 1-martingale (proper P_0 local 2-martingale). By Theorem 1 and the result regarding M above, the following processes are all representable P_0 local weak martingales:

$$n_1-\langle n_1,X\rangle_1 = n_1 - \langle n_1,X\rangle_1 - \langle n_1,X\rangle_2 + [n_1,X+X*X]$$

$$n_2-\langle n_2,X\rangle_2 = n_2 - \langle n_2,X\rangle_1-\langle n_2,X\rangle_2 + [n_2,X+X*X]$$

$$\{M-\langle M,X\rangle_1\} + \{M-\langle M,X\rangle_2\} - \{M-[M,X+X*X]\}$$

$$= M - \langle M,X\rangle_1 - \langle M,X\rangle_2 + [M,X+X*X]$$

The sum of these processes is

$$N - \langle N,X\rangle_1 - \langle N,X\rangle_2 + [N,X+X*X] \tag{5.4}$$

which is thus also a P_0 local weak martingale as advertised.

The proofs that $M-[M,X+X*X]$ and (5.4) are P local weak martingales have been intrinsic (essentially representation independent) and are thus likely to remain valid in a more general setting.

Since $[n_1,X+X*X] = 0$ it follows that

$$\begin{aligned} &N-[N,X+X*X]-\mu\circ\alpha u\circ\mu-\mu\circ\tilde{u}\beta\circ\mu \\ &\quad = (M-[M,X+X*X]) + (n_1-\mu\circ\alpha u\circ\mu) + (n_2-\mu\circ\tilde{u}\beta\circ\mu) \end{aligned} \tag{5.5}$$

The first term on the right has been shown to be a P local weak martingale. It will be shown that the other two terms are also. Let $B_1 = \mu\circ\alpha u\circ\mu$ and apply (2.33) to get

$$e^X n_1 = n_1\circ e^X + e^X\circ\{n_1+M_2*(n_1+\langle X,n_1\rangle_1)+\langle X,n_1\rangle_1+[X*n_1,X]\}$$

and

$$e^X B_1 = B_1\circ e^X + e^X\circ\{B_1+M_2*B_1+B_1*M_1\}$$

Hence

$$e^X(n_1-B_1) = K + e^X\circ\{\langle X,n_1\rangle_1+[X*n_1,X]-B_1\}$$

where K is a representable P_0 local weak martingale. Applying (2.29) shows that

$$\langle X,n_1\rangle_1 + [X*n_1,X] - B_1 = W\circ\rho(n_1)_{W1}\circ\mu$$

is a representable P_0 local weak martingale, so that $e^X(n_1-B_1)$ is one also. Therefore $n_1 - \mu\circ\alpha u\circ\mu$ (and similarly $n_2 - \mu\circ\tilde{u}\beta\circ\mu$) is a P local weak martingale. Therefore, each side of (5.5) is a P local weak martingale.

Finally, to prove the uniqueness assertion in Theorem 3, it suffices to show that if B is an absolutely continuous process and also a P local weak martingale, then B is identically zero a.s. Since B is absolutely continuous, (2.33) yields

$$e^X B = B\circ e^X + e^X\circ\{B + M_2 * B + B * M_1\}$$

so that

$$B = e^{-X}\circ\{e^X B - B\circ e^X\} + M_2 * B + B * M_1$$

which shows that B is also a P_0 local weak martingale. Hence B is identically zero a.s. The proof of Theorem 3 is complete. ∎

<u>Proof of Theorem 4.</u> Given q, r, a and b as in (3.3), let $\alpha = a - \tilde{u}r$, $\beta = b - ru$ and $N = q\circ W + W\circ r\circ W + \mu\circ\alpha\circ W + W\circ\beta\circ\mu$. Then $\tilde{N}$ in (3.2) is equal to (3.3) which is thus a representable P local weak martingale.

Conversely, any representable P local weak martingale Z may be written $Z = N - B$ where N is a representable P_0 local weak martingale and B is a bounded variation process. Let $N = W\circ r\circ W + q\circ W + \mu\circ\alpha\circ W + W\circ\beta\circ\mu$ be the semimartingale representation of N. By the uniqueness assertion of Theorem 3, Z must equal $\tilde{N}$ of (3.2), which is equal to (3.3) with $a = \alpha + \tilde{u}r$ and $b = \beta + ru$. ∎

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