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# GEOMETRIC PROPERTIES OF DYNAMIC NONLINEAR NETWORKS: TRANSVERSALITY, WELL-POSEDNESS AND EVENTUAL PASSIVITY

by

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# GEOMETRIC PROPERTIES OF DYNAMIC NONLINEAR NETWORKS: TRANSVERSALITY, WELL-POSEDNESS AND EVENTUAL PASSIVITY†

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## Abstract

This paper discusses several general properties of dynamic nonlinear networks from a geometric point of view. One of the main advantages of a geometric approach is that it is coordinate-free, i.e., the results obtained by a geometric method do not depend on the particular choices of a tree, a loop matrix, state variables, etc.

Firstly, it is shown that the transversality between resistor constitutive relations and the Kirchhoff space is a sufficient condition for the configuration space to be well defined. Secondly, the concept of well-posedness is shown to be important for the dynamics to be well defined on the configuration space. It is also clarified that transversality and well-posedness are two distinct mechanisms which are responsible for the non-existence of state equations. Perturbation results are given which guarantee transversality and/or well-posedness. Finally, several other perturbation results are given which guarantee the eventual strict passivity of dynamic nonlinear networks.

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## I. Introduction

This paper discusses several general properties of dynamic nonlinear networks from a geometric point of view. One of the main advantages of a geometric approach is that it is coordinate-free, i.e., the results obtained by a geometric method do not depend on particular choices of a tree, a loop matrix, state variables, etc. This approach allows us to resolve and clarify a number of subtle paradoxes and perplexing questions which lie at the very foundation of nonlinear circuit theory. In particular, several basic questions involving the formulation of state equations for nonlinear networks are hereby resolved in a rigorous manner. Among other things, we have clearly identified two mathematically distinct mechanisms which are responsible for the non-existence of state equations; one involving the concept of transversality, while the other involving the property of well-posedness. Under the assumptions that all capacitors are voltage-controlled and all inductors are current-controlled, we have also shown that the capacitor voltages and inductor currents are a good choice of variables to describe the dynamics in the sense that if we cannot describe the dynamics in terms of capacitor voltages and inductor currents, then, there is no choice of variables in the network in terms of which the dynamics is well defined. Conversely, if the dynamics can be described with respect to some set of variables in a network, then it can be described by capacitor voltages and inductor currents also. Our geometric approach allows us to choose a convenient coordinate system and use it to derive general conclusions which hold with respect to any other coordinate system.

In Section II we show how the dynamics of nonlinear networks can be described in a coordinate-free manner. In Section III we discuss transversality of the resistor constitutive relations and the Kirchhoff space. Transversality is important in that it guarantees the configuration space to be a well-defined submanifold. We give two perturbation results which guarantee transversality. One involves element perturbations, i.e., perturbing the existing resistor constitutive relations. The other involves network perturbations, i.e., augmenting the network with capacitors and inductors. In Section IV we discuss well-posedness which is a condition for the dynamics to be well defined. We give a network perturbation technique which guarantees well-posedness. In Section V we give several perturbation results which guarantee eventual strict passivity of dynamic nonlinear networks. Eventual strict passivity is important in that it implies boundedness of both voltage and current waveforms.

General Remark For simplicity, we will usually delete the superscript "T" denoting the "transpose" of a vector or matrix whenever no confusion arises.

## II. Coordinate-Free Description of Network Dynamics

Throughout the paper, we need to use the fact that transversality, well-posedness and eventual passivity are coordinate-free properties, i.e., they are independent of the choices of a tree, a loop matrix, a cut set matrix, state variables etc. Here we explain how nonlinear networks are described in a coordinate-free manner.

Consider a nonlinear network $\mathcal{N}$ containing $n_R$ resistors, $n_C$ capacitors and $n_L$ inductors. Let $b = n_R + n_C + n_L$ and let $\underset{\sim}{v}$ and $\underset{\sim}{i}$ represent the branch voltages and currents of the network. Then $(\underset{\sim}{v},\underset{\sim}{i}) \in \mathbb{R}^b \times \mathbb{R}^b$. The following are the standing assumptions of this paper:

(a) The linear graph $\mathcal{G}$ which defines the topology of $\mathcal{N}$ is connected.

(b) $\mathcal{N}$ is time invariant.

(c) The resistor constitutive relations are characterized by

$$(\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda \subset \mathbb{R}^b \times \mathbb{R}^b \tag{1}$$

where $\Lambda$ is a $2b-n_R$ dimensional $C^2$ submanifold.

(d) Capacitors and inductors are characterized by

$$\underset{\sim}{C}(\underset{\sim}{v}_C)\frac{d\underset{\sim}{v}_C}{dt} = \underset{\sim}{i}_C \tag{2}$$

and

$$\underset{\sim}{L}(\underset{\sim}{i}_L)\frac{d\underset{\sim}{i}_L}{dt} = \underset{\sim}{v}_L \tag{3}$$

respectively, where $\underset{\sim}{v}_C$ and $\underset{\sim}{v}_L$ denote capacitor and inductor voltages, respectively, and $\underset{\sim}{i}_C$ and $\underset{\sim}{i}_L$ are capacitor and inductor currents, respectively, $\underset{\sim}{C}(\underset{\sim}{v}_C)$ and $\underset{\sim}{L}(\underset{\sim}{i}_L)$ are incremental capacitance and incremental inductance matrices, respectively, and they are symmetric, positive definite and $C^1$.

(e) There are no capacitor-only loops and no inductor-only cut sets.

Remarks 1. There is no loss of generality in assuming (a) since disconnected subgraphs can be hinged together. Connectedness is necessary for a tree to exist.

2. Most of the results of this paper can be easily generalized to include the time-varying case under appropriate conditions. We make this assumption simply to avoid introducing complicated notations.

3. Under assumption (c) resistors can be coupled to each other and they need not be voltage or current controlled. Even couplings among $(\underset{\sim}{v}_R,\underset{\sim}{i}_R)$ and $(\underset{\sim}{v}_C,\underset{\sim}{v}_L,\underset{\sim}{i}_C,\underset{\sim}{i}_L)$ are allowed. This includes virtually all modes of representation,

including the hybrid and transmission representations. In particular, a broad class of nonlinear dependent sources are allowed in this formulation. We regard independent sources as uncoupled resistors. All multiterminal elements are represented as coupled 2-terminal elements. We need the $C^2$ property of $\Lambda$ rather than $C^1$ because we would like to define a $C^1$ vector field on the configuration space. (See Section IV.)

4. Under the present formulation, capacitors can be coupled to each other. Similarly, inductors can be coupled to each other.

5. Assumption (e) was introduced only for simplicity. This involves no loss of generality in view of the results of Chua and Green [1] and Sangiovanni-Vincentelli and Wang [2].

Now let K denote the Kirchhoff space [3], i.e., the set of all $(\underset{\sim}{v},\underset{\sim}{i})$ satisfying KVL and KCL. It is known that K is independent of a particular choice of a tree, a loop matrix, a cut set matrix etc. Since $(\underset{\sim}{v},\underset{\sim}{i})$ must satisfy the resistor constitutive relations and the Kirchhoff laws simultaneously, the operating points are restricted to within the following subset:

$$\Sigma \triangleq \Lambda \cap K. \tag{4}$$

The set $\Sigma$ is called the configuration space of $\mathcal{N}$ since this is where the dynamics takes place. In order to describe the dynamics in a coordinate-free manner, consider the following 1-form on $\mathbb{R}^b \times \mathbb{R}^b$ [4,5]:

$$\underset{\sim}{\eta} \triangleq \sum_{k=1}^{n_R} v_{R_k} di_{R_k} + d\left(\sum_{k=1}^{n_C} v_{C_k} i_{C_k}\right) \tag{5}$$

and the following symmetric 2-tensor on $\mathbb{R}^{n_C} \times \mathbb{R}^{n_L}$:

$$\underset{\sim}{G} \triangleq \sum_{m,n=1}^{n_C} C_{mn}(\underset{\sim}{v}_C)\, dv_{C_m} \otimes dv_{C_n} - \sum_{m,n=1}^{n_L} L_{mn}(\underset{\sim}{i}_L)\, di_{L_m} \otimes di_{L_n} \tag{6}$$

where $C_{mn}(\underset{\sim}{v}_C)$ $\left(\text{resp., } L_{mn}(\underset{\sim}{i}_L)\right)$ is the (m,n)-component of $\underset{\sim}{C}(\underset{\sim}{v}_C)$ $\left(\text{resp., } \underset{\sim}{L}(\underset{\sim}{i}_L)\right)$ and $v_{R_k}$ (resp., $i_{R_k}$) is the voltage (resp., current) of the k-th resistor.

Remark. A simple explanation of 1-forms is given in [3]. A symmetric 2-tensor $\underset{\sim}{G}$ on $\mathbb{R}^2$ is a collection of functions: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given at each point $(x_1,x_2) \in \mathbb{R}^2$ by

$$\underset{\sim}{G}(x_1,x_2) = \sum_{m,n=1}^{2} f_{mn}(x_1,x_2)\, dx_m \otimes dx_n$$

where $f_{mn}$ are real-valued functions, $f_{mn} = f_{nm}$ and

$$dx_1 \otimes dx_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad dx_1 \otimes dx_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$dx_2 \otimes dx_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad dx_2 \otimes dx_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Consequently,

$$\underset{\sim}{G}_{(x_1,x_2)}\left([1\ 0]^T, [1\ 0]^T\right) = f_{11}(x_1,x_2)$$

etc., so that $\underset{\sim}{G}$ can be thought of as the matrix-valued function $[f_{mn}]$. One needs to be careful, however, in defining 2-tensors on a general manifold since manifolds generally are nonlinear.

Let $\underset{\sim}{\pi}_{CL} : \mathbb{R}^b \times \mathbb{R}^b \to \mathbb{R}^{n_C} \times \mathbb{R}^{n_L}$ be the projection map defined by

$$\underset{\sim}{\pi}_{CL}(\underset{\sim}{v},\underset{\sim}{i}) = (\underset{\sim}{v}_C,\underset{\sim}{i}_L) \tag{7}$$

and let $\underset{\sim}{\iota} : \Sigma \to \mathbb{R}^b \times \mathbb{R}^b$ be the inclusion map defined by

$$\underset{\sim}{\iota}(\underset{\sim}{v},\underset{\sim}{i}) = (\underset{\sim}{v},\underset{\sim}{i}). \tag{8}$$

Set

$$\underset{\sim}{\pi} \triangleq \underset{\sim}{\pi}_{CL} \circ \underset{\sim}{\iota} \ . \tag{9}$$

Assume that $\Sigma$ is a submanifold[1] and let $\underset{\sim}{\pi}^*$ and $\underset{\sim}{\iota}^*$ be the induced maps of $\underset{\sim}{\pi}$ and $\underset{\sim}{\iota}$, respectively [3]. It is shown in [4,5] that the vector field[2] $\underset{\sim}{X}_{(\underset{\sim}{v},\underset{\sim}{i})}$ at $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$ is given by the following formula:

[1] Although $\Lambda$ and K are perfectly well defined submanifolds, their intersection may not be a submanifold [3].

[2] A vector field $\underset{\sim}{X}$ on a manifold $\Sigma$ is a function such that the value $\underset{\sim}{X}_{(\underset{\sim}{v},\underset{\sim}{i})}$ at $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$ belongs to $T_{(\underset{\sim}{v},\underset{\sim}{i})}\Sigma$, the tangent space of $\Sigma$ at $(\underset{\sim}{v},\underset{\sim}{i})$. The vector field $\underset{\sim}{X}$ naturally generates a flow $\underset{\sim}{\phi}(t)$ such that $d\underset{\sim}{\phi}(t)/dt = \underset{\sim}{X}_{\underset{\sim}{\phi}(t)}$.

$$\pi^*G_{(v,i)}\left(X_{(v,i)},\xi\right) = \omega_{(v,i)}(\xi), \quad \text{for all } \xi \in T_{(v,i)}\Sigma \tag{10}$$

where

$$\omega \triangleq \iota^*\eta. \tag{11}$$

If $(v_C, i_L)$ serves as a global coordinate for $\Sigma$, then

$$X_{(v,i)} = \left(\frac{dv_C}{dt}, \frac{di_L}{dt}\right) \tag{12}$$

and (10) is reduced to

$$\begin{bmatrix} C(v_C) & 0 \\ 0 & -L(i_L) \end{bmatrix} \frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = F(v_C, i_L). \tag{13}$$

where $F$ is determined by $(i_C, -v_L)^T = F(v_C, i_L)$.

## III. Transversality

In order for $\pi^*G$ and $\omega$ to be well defined, the configuration space $\Sigma$ must be a submanifold. Even if $\Lambda$ and K are perfectly well defined submanifolds there is no reason to expect their intersection $\Sigma$ to be also a submanifold. A sufficient condition for $\Sigma$ to be a submanifold is the transversality [3] of $\Lambda$ and K, which is abbreviated by $\Lambda \pitchfork K$. It is shown in [5] that if $\Lambda \pitchfork K$, then $\Sigma$ is an $(n_C+n_L)$-dimensional submanifold. This is true for any $C^r$ submanifolds, $r \geq 1$. We first give a method of checking transversality of $\Lambda$ and K. To this end let $\mathcal{T}$ be a tree for $\mathcal{G}$ and let $\mathcal{L}$ be its associated cotree. Let $v$ and $i$ be partitioned accordingly;

$$v = (v_{\mathcal{L}} \vdots v_{\mathcal{T}}), \quad i = (i_{\mathcal{L}} \vdots i_{\mathcal{T}}). \tag{14}$$

Let $B$ be the fundamental loop matrix associated with $\mathcal{T}$. Then

$$B = [1 \vdots B_{\mathcal{T}}]. \tag{15}$$

Since $\Lambda$ is a $C^2$ submanifold of dimensional $2b-n_R$, for each point $(v_0, i_0) \in \Lambda$, there is a neighborhood $U \subset \mathbb{R}^b \times \mathbb{R}^b$ of this point and there is a $C^2$ function $f : U \to \mathbb{R}^{n_R}$ such that

$$\Lambda \cap U = f^{-1}(0) \tag{16}$$

and

$$\text{rank}(\underset{\sim}{D}\underset{\sim}{f})_{(\underset{\sim}{v},\underset{\sim}{i})} = n_R \text{ for all } (\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda \cap U, \tag{17}$$

where $(\underset{\sim}{D}\underset{\sim}{f})_{(\underset{\sim}{v},\underset{\sim}{i})}$ is the derivative of $\underset{\sim}{f}$ at $(\underset{\sim}{v},\underset{\sim}{i})$.

Propositions 1 and 2 and Corollaries 1 and 2 which follow have been proved in [3] for nonlinear resistive n-ports.

Proposition 1 $\Lambda \pitchfork K$ if and only if for each $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$

$$\text{rank}\, \underset{\sim}{\mathcal{F}}(\underset{\sim}{v},\underset{\sim}{i}) = n_R \tag{18}$$

where

$$\underset{\sim}{\mathcal{F}}(\underset{\sim}{v},\underset{\sim}{i}) \triangleq [D_{\underset{\sim}{v}_{\mathcal{J}}}\underset{\sim}{f} - (D_{\underset{\sim}{v}_{\mathcal{L}}}\underset{\sim}{f})\, \underset{\sim}{B}_{\mathcal{J}} \vdots D_{\underset{\sim}{i}_{\mathcal{L}}}\underset{\sim}{f} + (D_{\underset{\sim}{i}_{\mathcal{J}}}\underset{\sim}{f})\underset{\sim}{B}_{\mathcal{J}}^{T}]_{(\underset{\sim}{v},\underset{\sim}{i})} \tag{19}$$

where $D_{\underset{\sim}{v}_{\mathcal{J}}}\underset{\sim}{f}$ denotes partial derivative of $\underset{\sim}{f}$ with respect to $\underset{\sim}{v}_{\mathcal{J}}$. Other symbols have similar meanings.

We next give special cases where $\Lambda$ of (1) is given by

$$\Lambda = \{(\underset{\sim}{v},\underset{\sim}{i}) \in \mathbb{R}^b \times \mathbb{R}^b \,|\, (\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R\} \tag{20}$$

where $\Lambda_R$ is an $n_R$-dimensional $C^2$ submanifold of $\mathbb{R}^{n_R} \times \mathbb{R}^{n_R}$. In this case there is no coupling between $(\underset{\sim}{v}_R,\underset{\sim}{i}_R)$ and $(\underset{\sim}{v}_C,\underset{\sim}{v}_L,\underset{\sim}{i}_C,\underset{\sim}{i}_L)$. Therefore $\underset{\sim}{f}$ of (16) is independent of $(\underset{\sim}{v}_C,\underset{\sim}{v}_L,\underset{\sim}{i}_C,\underset{\sim}{i}_L)$. Let $U_R \triangleq U \cap \left(\mathbb{R}^{n_R} \times \mathbb{R}^{n_R}\right)$ and define $\underset{\sim}{f}_R : U_R \to \mathbb{R}^{n_R}$ simply by

$$\underset{\sim}{f}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \triangleq \underset{\sim}{f}(\underset{\sim}{v},\underset{\sim}{i}). \tag{21}$$

Next let $\underset{\sim}{\pi}'_R : \mathbb{R}^b \times \mathbb{R}^b \to \mathbb{R}^{n_R} \times \mathbb{R}^{n_R}$ be the projection map defined by

$$\underset{\sim}{\pi}'_R(\underset{\sim}{v},\underset{\sim}{i}) = (\underset{\sim}{v}_R,\underset{\sim}{i}_R) \tag{22}$$

and let

$$\underset{\sim}{\pi}_R \triangleq \underset{\sim}{\pi}'_R \circ \underset{\sim}{\imath} \tag{23}$$

where $\underset{\sim}{\imath}$ is defined by (8). Decompose $\underset{\sim}{v}$ and $\underset{\sim}{i}$ as follows:

$$\left.\begin{aligned} \underset{\sim}{v} &= (\underset{\sim}{v}_{\mathcal{L}} \vdots \underset{\sim}{v}_{\mathcal{J}}) = (\underset{\sim}{v}_{R_{\mathcal{L}}}, \underset{\sim}{v}_{C_{\mathcal{L}}}, \underset{\sim}{v}_{L_{\mathcal{L}}} \vdots \underset{\sim}{v}_{R_{\mathcal{J}}}, \underset{\sim}{v}_{C_{\mathcal{J}}}, \underset{\sim}{v}_{L_{\mathcal{J}}}) \\ \underset{\sim}{i} &= (\underset{\sim}{i}_{\mathcal{L}} \vdots \underset{\sim}{i}_{\mathcal{J}}) = (\underset{\sim}{i}_{R_{\mathcal{L}}}, \underset{\sim}{i}_{C_{\mathcal{L}}}, \underset{\sim}{i}_{L_{\mathcal{L}}} \vdots \underset{\sim}{i}_{R_{\mathcal{J}}}, \underset{\sim}{i}_{C_{\mathcal{J}}}, \underset{\sim}{i}_{L_{\mathcal{J}}}) \end{aligned}\right\} \tag{24}$$

where R, C and L denote resistors, capacitors and inductors, respectively. Decompose $\underset{\sim}{B}_{\mathcal{J}}$ of (15) accordingly;

$$\underset{\sim}{B}_{\mathcal{J}} = \begin{bmatrix} \underset{\sim}{B}_{RR} & \underset{\sim}{B}_{RC} & \underset{\sim}{B}_{RL} \\ \underset{\sim}{B}_{CR} & \underset{\sim}{B}_{CC} & \underset{\sim}{B}_{CL} \\ \underset{\sim}{B}_{LR} & \underset{\sim}{B}_{LC} & \underset{\sim}{B}_{LL} \end{bmatrix} \tag{25}$$

<u>Corollary 1</u> Let $\Lambda$ be given by (20). Then $\Lambda \bar{\pitchfork} K$ if and only if for each $(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \underset{\sim}{\pi}_R(\Sigma)$

$$\text{rank } \underset{\sim}{\mathcal{J}}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = n_R \tag{26}$$

where

$$\begin{aligned} \underset{\sim}{\mathcal{J}}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \triangleq \Big[ & D_{\underset{\sim}{v}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R - (D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R) \underset{\sim}{B}_{RR} \vdots -(D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R)[\underset{\sim}{B}_{RC}\ \underset{\sim}{B}_{RL}] \vdots \\ & D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R + (D_{\underset{\sim}{i}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R) \underset{\sim}{B}^T_{RR} \vdots (D_{\underset{\sim}{i}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R)[\underset{\sim}{B}^T_{CR}\ \underset{\sim}{B}^T_{LR}] \Big]_{(\underset{\sim}{v}_R, \underset{\sim}{i}_R)} \end{aligned} \tag{27}$$

<u>Proof</u> Observe the following:

$$D_{\underset{\sim}{v}_{\mathcal{J}}} \underset{\sim}{f} = [D_{\underset{\sim}{v}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R\ \underset{\sim}{0}\ \underset{\sim}{0}],\quad D_{\underset{\sim}{v}_{\mathcal{L}}} \underset{\sim}{f} = [D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R\ \underset{\sim}{0}\ \underset{\sim}{0}]$$

$$D_{\underset{\sim}{i}_{\mathcal{J}}} \underset{\sim}{f} = [D_{\underset{\sim}{i}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R\ \underset{\sim}{0}\ \underset{\sim}{0}],\quad D_{\underset{\sim}{i}_{\mathcal{L}}} \underset{\sim}{f} = [D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R\ \underset{\sim}{0}\ \underset{\sim}{0}].$$

Substituting these into (19), we obtain (27). Since $(\underset{\sim}{v}, \underset{\sim}{i}) \in \Sigma$, the vector $(\underset{\sim}{v}_R, \underset{\sim}{i}_R)$ must belong to $\underset{\sim}{\pi}_R(\Sigma)$. ⌑

<u>Remark</u> Note that $\mathcal{J}$ is arbitrary. However, assumption (e) implies that there exists a proper tree.[3] Consequently, if we choose a proper tree, then submatrices $\underset{\sim}{B}_{RL}$, $\underset{\sim}{B}_{CR}$, $\underset{\sim}{B}_{CC}$, $\underset{\sim}{B}_{CL}$ and $\underset{\sim}{B}_{LL}$ are $0 \times 0$ matrices and the matrix of (27) tkaes on a particularly simple form.

Next suppose that $\Lambda_R$ admits a <u>generalized port coordinate</u> [3], i.e., $\Lambda_R$ is represented by

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[3] A tree is called a proper tree if it contains all the capacitors and its associated cotree contains all the inductors.

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} v_R \\ i_R \end{bmatrix}, \quad \xi = F(\eta) \tag{28}$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are $n_R \times n_R$ matrices and $F : \mathbb{R}^{n_R} \times \mathbb{R}^{n_R}$ is a $C^2$ function. Recall the partition $v_R = (v_{R_\mathcal{L}} \vdots v_{R_\mathcal{J}})$, $i_R = (i_{R_\mathcal{L}} \vdots i_{R_\mathcal{J}})$ and partition $\alpha$, $\beta$, $\gamma$ and $\delta$ accordingly;

$$\alpha = [\alpha_1 \vdots \alpha_2], \ \beta = [\beta_1 \vdots \beta_2], \ \gamma = [\gamma_1 \vdots \gamma_2], \ \delta = [\delta_1 \vdots \delta_2]. \tag{29}$$

Also recall that $\Lambda_R$ is said to be <u>globally voltage controlled</u> [3] if $\xi = i_R$, $\eta = v_R$ and <u>globally current controlled</u> if $\xi = v_R$, $\eta = i_R$.

<u>Corollary 2</u>. Let $\Lambda_R$ admit a generalized port coordinate representation. Then

$$\mathcal{F}_R(v_R, i_R) \triangleq \Big[ \big(\alpha_2 - (DF)\gamma_2\big) - \big(\alpha_1 - (DF)\gamma_1\big) B_{RR} \vdots -\big(\alpha_1 - (DF)\gamma_1\big)[B_{RC} \ B_{RL}] \vdots \\ \big(\beta_1 - (DF)\delta_1\big) + \big(\beta_2 - (DF)\delta_2\big) B_{RR}^T \vdots \big(\beta_2 - (DF)\delta_2\big)[B_{CR}^T \ B_{LR}^T] \Big]_{(v_R, i_R)}. \tag{30}$$

In particular, if $\Lambda_R$ is globally voltage controlled, then

$$\mathcal{F}_R(v_R, i_R) = \left[ (DF) \begin{bmatrix} B_{RR} \\ 1 \end{bmatrix} \vdots (DF) \begin{bmatrix} B_{RC} & B_{RL} \\ 0 & 0 \end{bmatrix} \vdots \begin{bmatrix} 1 \\ B_{RR}^T \end{bmatrix} \vdots \begin{bmatrix} 0 & 0 \\ B_{CR}^T & B_{LR}^T \end{bmatrix} \right]_{v_R} \tag{31}$$

and if $\Lambda_R$ is globally current controlled, then

$$\mathcal{F}_R(v_R, i_R) = \left[ \begin{bmatrix} -B_{RR} \\ 1 \end{bmatrix} \vdots \begin{bmatrix} -B_{RC} & -B_{RL} \\ 0 & 0 \end{bmatrix} \vdots -(DF) \begin{bmatrix} 1 \\ B_{RR}^T \end{bmatrix} \vdots -(DF) \begin{bmatrix} 0 & 0 \\ B_{CR}^T & B_{LR}^T \end{bmatrix} \right]_{i_R}. \tag{32}$$

Suppose that $\Lambda_R$ is <u>globally parametrizable</u> [3], i.e., $\Lambda_R$ is globally diffeomorphic to $\mathbb{R}^{n_R}$ and write

$$\big(v_R(\rho), i_R(\rho)\big) \triangleq \psi^{-1}(\rho), \ \rho \in \mathbb{R}^{n_R} \tag{33}$$

where $\psi : \Lambda_R \to \mathbb{R}^{n_R}$ is a global coordinate.

<u>Proposition 2</u> Let $\Lambda_R$ be globally parametrized by $\rho$ as in (33). Then $\Lambda \pitchfork K$ if and only if for each $\rho \in \mathbb{R}^{n_R}$ with $\big(v_R(\rho), i_R(\rho)\big) \in \pi_R(\Sigma)$,

$$\operatorname{rank} \mathcal{J}^*(\rho) = n_R \tag{34}$$

where

$$\mathcal{J}^*(\rho) = \left[\begin{array}{cc|cccc} Dv_{R_\mathcal{L}} + B_{RR}\, Dv_{R_\mathcal{J}} & & -B_{RC} & -B_{RL} & 0 & 0 \\ Di_{R_\mathcal{J}} - B_{RR}^T\, Di_{R_\mathcal{L}} & & 0 & 0 & B_{CR}^T & B_{LR}^T \end{array}\right]_{\rho} \quad . \tag{35}$$

Remark Formula (35) holds even when $\Lambda_R$ is locally parametrized by $\rho$ at each point. In fact (35) holds if and only if rank $J = 2n_R$ where $J$ is the matrix defined by Desoer and Wu [6].

Suppose now that $\Lambda \not\pitchfork K$. Then it would be helpful if one can perturb $\mathcal{N}$ in an appropriate way such that the resulting network satisfies transversality. In the following we give two perturbation results. The first method involves element perturbation and consists of perturbing the existing resistor constitutive relations $\Lambda$. The second method involves network perturbations and consists of augmenting $\mathcal{N}$ by adding arbitrarily small linear inductors and arbitrarily large linear capacitors by pliers-type entry, and by adding arbitrarily large linear inductors and arbitrarily small linear capacitors by soldering-iron entry. Therefore, in the limit we recover the original network. Notice that this procedure consists of adding parasitic capacitors and inductors at appropriate locations.

In order to give a transversalization result via element perturbation, let us first define a $C^2$ perturbation of $\Lambda$. Let $M$ be a $C^2$ submanifold of $\mathbb{R}^n$ and let $C^2(M,\mathbb{R}^n)$ be the set of all $C^2$ maps from $M$ into $\mathbb{R}^n$. Let $F \in C^2(M,\mathbb{R}^n)$ and consider

$$\mathcal{U}^2(F;\varepsilon(\cdot)) \triangleq \left\{ G : M \to \mathbb{R}^n \;\middle|\; \begin{array}{l} G \in C^2(M,\mathbb{R}^n) \\ \|F(x)-G(x)\| + \|(dF)_x - (dG)_x\| \\ + \|(d^2F)_x - (d^2G)_x\| < \varepsilon(x) \\ \text{for all } x \in M \end{array} \right\}$$

where $\varepsilon(x)$ is an arbitrary continuous function from $M$ into the set of positive numbers and $d^2F$ and $d^2G$ are the second derivatives. These sets generate the strong $C^2$ topology for $C^2(M,\mathbb{R}^n)$ [11]. The set $\mathrm{Emb}^2(M,\mathbb{R}^n)$ of all $C^2$ embeddings of $M$ into $\mathbb{R}^n$ is open with respect to this topology [11]. Let $\mathcal{U}^2(\iota_M)$ be a neighborhood of the inclusion map such that all elements of $\mathcal{U}^2(\underset{\sim}{\imath}_M)$ are embeddings. Then a $C^2$ perturbation $\hat{M}$ of M is defined by $\hat{M} \triangleq \underset{\sim}{G}(M)$, where $\underset{\sim}{G} \in \mathcal{U}^2(\underset{\sim}{\imath}_M)$. The following is our first transversalization result via element perturbation. Although the proof is similar to that of Theorem 3 of [3], there is a technical difference because of the $C^2$ perturbations instead of $C^1$ perturbations. Proof is given in the Appendix.

Proposition 3 Given a nonlinear network $\mathcal{N}$ let $\Lambda \cap K \neq \phi$ and $\Lambda \not\pitchfork K$. Then there is a perturbation $\hat{\Lambda}$ of $\Lambda$ arbitrarily close to $\Lambda$ in the strong $C^2$ topology such that $\hat{\Lambda} \cap K \neq \phi$ and $\hat{\Lambda} \pitchfork K$.

The next result gives a transversalization procedure via network perturbation. The proof is similar to that of Theorem 4 in [3].

Proposition 4 Given a nonlinear network $\mathcal{N}$, let $\Lambda \cap K \neq \phi$ and $\Lambda \not\pitchfork K$. Let $\mathcal{T}$ be an arbitrary tree for $\mathcal{G}$ and let $\mathcal{L}$ be its associated cotree. Insert a small linear capacitor in parallel with each branch of $\mathcal{T}$ and insert a small linear inductor in series with each branch of $\mathcal{L}$. Then the perturbed network $\hat{\mathcal{N}}$ has the following properties: (i) $\hat{\Lambda} \cap \hat{K} \neq \phi$, (ii) $\hat{\Lambda} \pitchfork \hat{K}$, where $\hat{K}$ is the Kirchhoff space of $\hat{\mathcal{N}}$.

If $\Lambda$ has simpler forms, then the number of reactive elements added can be reduced.

Proposition 5 Given a nonlinear network $\mathcal{N}$ let $\Lambda$ be represented by (20). Suppose $\Lambda \cap K \neq \phi$ and $\Lambda \not\pitchfork K$. Let $\mathcal{T}$ be an aribtrary tree for $\mathcal{G}$ and let $\mathcal{L}$ be its associated cotree. Partition $\mathcal{T}$ and $\mathcal{L}$ as $\mathcal{T} = \mathcal{T}_R \cup \mathcal{T}_C \cup \mathcal{T}_L$ and $\mathcal{L} = \mathcal{L}_R \cup \mathcal{L}_C \cup \mathcal{L}_L$ respectively, where R, C and L denote resistors, capacitors and inductors, respectively. Insert a small linear capacitor in parallel with each branch of $\mathcal{T}_R$ and insert a small linear inductor in series with each branch of $\mathcal{L}_R$. Then the perturbed network $\hat{\mathcal{N}}$ satisfies the following properties: (i) $\hat{\Lambda} \cap \hat{K} \neq \phi$, (ii) $\hat{\Lambda} \pitchfork \hat{K}$.

Proof. (i) Let $\mathcal{T}_1$ denote the branches representing the capacitors added in parallel with $\mathcal{T}_R$ and let $\mathcal{L}_1$ denote the branches representing the inductors added in series with $\mathcal{L}_R$. Then $\hat{\mathcal{T}} \triangleq \mathcal{T}_C \cup \mathcal{T}_L \cup \mathcal{T}_1 \cup \mathcal{L}_R$ is a tree for $\hat{\mathcal{N}}$ and $\hat{\mathcal{L}} \triangleq \mathcal{L}_C \cup \mathcal{L}_L \cup \mathcal{L}_1 \cup \mathcal{T}_R$ is its associated cotree. Let

$$\left.\begin{aligned} \hat{\underset{\sim}{v}} &= (\underset{\sim}{v}_{R_{\mathcal{T}}}, \underset{\sim}{v}_{C_{\mathcal{L}}}, \underset{\sim}{v}_{L_{\mathcal{L}}}, \underset{\sim}{v}_{\mathcal{L}_1} \,\vdots\, \underset{\sim}{v}_{R_{\mathcal{L}}}, \underset{\sim}{v}_{C_{\mathcal{T}}}, \underset{\sim}{v}_{\mathcal{T}_1}, \underset{\sim}{v}_{L_{\mathcal{T}}}) \\ \hat{\underset{\sim}{i}} &= (\underset{\sim}{i}_{R_{\mathcal{T}}}, \underset{\sim}{i}_{C_{\mathcal{L}}}, \underset{\sim}{i}_{L_{\mathcal{L}}}, \underset{\sim}{i}_{\mathcal{L}_1} \,\vdots\, \underset{\sim}{i}_{R_{\mathcal{L}}}, \underset{\sim}{i}_{C_{\mathcal{T}}}, \underset{\sim}{i}_{\mathcal{T}_1}, \underset{\sim}{i}_{L_{\mathcal{T}}}) \end{aligned}\right\} \quad (36)$$

be the variables of $\hat{\mathcal{N}}$. Let

$$(\underset{\sim}{v}_0, \underset{\sim}{i}_0) \in \Lambda \cap K \neq \phi, \tag{37}$$

$$\left.\begin{aligned} \underset{\sim}{v}_0 &= \left(\underset{\sim}{v}_{R_{\mathcal{L}_0}}, \underset{\sim}{v}_{C_{\mathcal{L}_0}}, \underset{\sim}{v}_{L_{\mathcal{L}_0}} \,\vdots\, \underset{\sim}{v}_{R_{\mathcal{J}_0}}, \underset{\sim}{v}_{C_{\mathcal{J}_0}}, \underset{\sim}{v}_{L_{\mathcal{J}_0}}\right) \\ \underset{\sim}{i}_0 &= \left(\underset{\sim}{i}_{R_{\mathcal{L}_0}}, \underset{\sim}{i}_{C_{\mathcal{L}_0}}, \underset{\sim}{i}_{L_{\mathcal{L}_0}} \,\vdots\, \underset{\sim}{i}_{R_{\mathcal{J}_0}}, \underset{\sim}{i}_{C_{\mathcal{J}_0}}, \underset{\sim}{i}_{L_{\mathcal{J}_0}}\right) \end{aligned}\right\} . \tag{38}$$

We first claim that with

$$\left.\begin{aligned} \hat{\underset{\sim}{v}}_0 &\triangleq \left(\underset{\sim}{v}_{R_{\mathcal{J}_0}}, \underset{\sim}{v}_{C_{\mathcal{L}_0}}, \underset{\sim}{v}_{L_{\mathcal{L}_0}}, \underset{\sim}{0} \,\vdots\, \underset{\sim}{v}_{R_{\mathcal{L}_0}}, \underset{\sim}{v}_{C_{\mathcal{J}_0}}, \underset{\sim}{v}_{R_{\mathcal{J}_0}}, \underset{\sim}{v}_{L_{\mathcal{J}_0}}\right) \\ \hat{\underset{\sim}{i}}_0 &\triangleq \left(\underset{\sim}{i}_{R_{\mathcal{J}_0}}, \underset{\sim}{i}_{C_{\mathcal{L}_0}}, \underset{\sim}{i}_{L_{\mathcal{L}_0}}, \underset{\sim}{i}_{R_{\mathcal{L}_0}} \,\vdots\, \underset{\sim}{i}_{R_{\mathcal{L}_0}}, \underset{\sim}{i}_{C_{\mathcal{J}_0}}, \underset{\sim}{0}, \underset{\sim}{i}_{L_{\mathcal{J}_0}}\right) \end{aligned}\right\} \tag{39}$$

we have

$$(\hat{\underset{\sim}{v}}_0, \hat{\underset{\sim}{i}}_0) \in \hat{K}. \tag{40}$$

Since $(\hat{\underset{\sim}{v}}_0, \hat{\underset{\sim}{i}}_0)$ corresponds to open circuiting $\mathcal{J}_1$ and short circuiting $\mathcal{L}_1$ and since such a situation is contained in $\hat{K}$, we have (40). Next, since no resistors are added, we have

$$\hat{\Lambda} = \{(\hat{\underset{\sim}{v}}, \hat{\underset{\sim}{i}}) \mid (\underset{\sim}{v}, \underset{\sim}{i}) \in \Lambda\}. \tag{41}$$

This implies that

$$(\hat{\underset{\sim}{v}}_0, \hat{\underset{\sim}{i}}_0) \in \hat{\Lambda}. \tag{42}$$

which together with (38) implies (i).

(ii) To prove $\hat{\Lambda} \pitchfork \hat{K}$ we compute matrix $\hat{\mathcal{F}}(\underset{\sim}{v}, \underset{\sim}{i})$ of (19) for $\hat{\mathcal{N}}$. Observe that the fundamental loop matrix $\hat{\underset{\sim}{B}}$ for $\hat{\mathcal{N}}$ associated with the tree $\hat{\mathcal{J}}$ assumes the following form:

$$\begin{array}{cccc:cccc} \underset{\sim}{v}_{R_\mathcal{J}} & \underset{\sim}{v}_{C_\mathcal{L}} & \underset{\sim}{v}_{L_\mathcal{L}} & \underset{\sim}{v}_{\mathcal{L}_1} & \underset{\sim}{v}_{R_\mathcal{L}} & \underset{\sim}{v}_{C_\mathcal{J}} & \underset{\sim}{v}_{\mathcal{J}_1} & \underset{\sim}{v}_{L_\mathcal{J}} \end{array}$$

$$\left[\begin{array}{c:cccc} & \underset{\sim}{0} & \underset{\sim}{0} & -\underset{\sim}{1} & \underset{\sim}{0} \\ & \underset{\sim}{0} & \underset{\sim}{B}_{CC} & \underset{\sim}{B}_{CR} & \underset{\sim}{B}_{CL} \\ \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{B}_{LC} & \underset{\sim}{B}_{LR} & \underset{\sim}{B}_{LL} \\ & \underset{\sim}{1} & \underset{\sim}{B}_{RC} & \underset{\sim}{B}_{RR} & \underset{\sim}{B}_{RL} \end{array}\right] \tag{43}$$

where the submatrices in (43) correspond to those of $\underset{\sim}{B}$ for $\mathcal{N}$. (See (25)). The sign of the identity matrices in (43) are chosen merely for convenience and involves no loss of generality. Next notice that

$$\hat{\underset{\sim}{f}}_R = \underset{\sim}{f}_R,\ \hat{\underset{\sim}{B}}_{RR} = \underset{\sim}{0},\ \hat{\underset{\sim}{B}}_{RC} = [\underset{\sim}{0}\ \ -\underset{\sim}{1}],\ \hat{\underset{\sim}{B}}_{LR}^T = [\underset{\sim}{0}\ \ \underset{\sim}{1}]$$

$$\hat{B}_{RL} = \underset{\sim}{0},\ \ \hat{\underset{\sim}{B}}_{CR}^T = \underset{\sim}{0},\ (\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) = (\underset{\sim}{v}_R,\underset{\sim}{i}_R),$$

$$D_{\hat{\underset{\sim}{v}}_{R_{\mathcal{T}}}} \hat{\underset{\sim}{f}}_R = \underset{\sim}{D}_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R,\ D_{\hat{\underset{\sim}{v}}_{R_{\mathcal{L}}}} \hat{\underset{\sim}{f}}_R = \underset{\sim}{D}_{\underset{\sim}{v}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R,$$

$$D_{\hat{\underset{\sim}{i}}_{R_{\mathcal{T}}}} \hat{\underset{\sim}{f}}_R = \underset{\sim}{D}_{\hat{\underset{\sim}{i}}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R,\ \underset{\sim}{D}_{\hat{\underset{\sim}{i}}_{R_{\mathcal{L}}}} \hat{\underset{\sim}{f}}_R = \underset{\sim}{D}_{\underset{\sim}{i}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R.$$

Substituting these and (43) into (27) we have

$$\hat{\underset{\sim}{\mathcal{T}}}_R(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) = \left[ \underset{\sim}{D}_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R \vdots \underset{\sim}{D}_{\underset{\sim}{v}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R \vdots \underset{\sim}{D}_{\underset{\sim}{i}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R \vdots \underset{\sim}{D}_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R \right]_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)}$$

$$= (\underset{\sim}{D}\underset{\sim}{f}_R)_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)} \quad . \tag{44}$$

This is exactly the same as the matrix of (17) where $\underset{\sim}{f} = \underset{\sim}{f}_R$. Consequently it has rank $n_R$. By <u>Corollary 1</u> we have $\hat{\Lambda} \pitchfork \hat{K}$. ¤

<u>Example 1</u> Consider the circuit of Fig. 1(a) where the resistor constitutive relations are given in Fig. 1(b) with $i_{R_k} = f_{R_k}(v_{R_k})$, $k = 1,2$. Choose $\mathcal{T} = \{R_1,R_2\}$ to be our tree. Then $\underset{\sim}{B}_{RR} = \underset{\sim}{B}_{RL} = \underset{\sim}{B}_{LL} = \phi$,[4] $\underset{\sim}{B}_{LR} = [1\ \ 1]$,

$$\underset{\sim}{D}_{\underset{\sim}{v}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{bmatrix} -Df_{R_1} & 0 \\ 0 & -Df_{R_2} \end{bmatrix},\ \underset{\sim}{D}_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \phi$$

$$\underset{\sim}{D}_{\underset{\sim}{i}_{R_{\mathcal{T}}}} \underset{\sim}{f}_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \underset{\sim}{D}_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R = \phi \quad .$$

[4] We denote a $0 \times 0$ matrix by $\phi$.

Therefore (27) is given by

$$\mathcal{J}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = \begin{bmatrix} -Df_{R_1} & 0 & \vdots & 1 \\ & & \vdots & \\ 0 & -Df_{R_2} & \vdots & 1 \end{bmatrix}_{\underset{\sim}{v}_R} .$$

Now, for the value $i_R^*$ in Fig. 1(b) we have $i_R^* = f_{R_1}(v_{R_{10}}) = f_2(v_{R_{20}})$ and $(Df_{R_1})_{v_{R_{10}}} = (Df_{R_2})_{v_{R_{20}}} = 0$. It is clear that the point $(\underset{\sim}{v}_{R_0},\underset{\sim}{i}_{R_0}) \triangleq (v_{R_{10}},v_{R_{20}}, i_R^*,i_R^*)$ belongs to $\underset{\sim}{\pi}_R(\Sigma)$. Therefore rank $\mathcal{J}_R(\underset{\sim}{v}_{R_0},\underset{\sim}{i}_{R_0}) = 1 < 2$ and hence $\Lambda \not\pitchfork K$. Now insert $\hat{C}_1$ and $\hat{C}_2$ as in Fig. 1(a), then (44) tells us that

$$\mathcal{J}_R(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) = (\underset{\sim}{D}\underset{\sim}{f}_R)_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)} = \begin{bmatrix} -Df_{R_1} & 0 & \vdots & 1 & 0 \\ & & \vdots & & \\ 0 & -Df_{R_2} & \vdots & 0 & 1 \end{bmatrix}_{\underset{\sim}{v}_R}$$

which has rank $2 = n_R$. Therefore $\hat{\Lambda} \pitchfork \hat{K}$. ¤

The transversalization procedure is further simplified if $\Lambda$ is described by (20) and $\Lambda_R$ is locally voltage controlled [3], i.e., there is a $C^2$ function $\underset{\sim}{f}_R : \mathbb{R}^{n_R} \times \mathbb{R}^{n_R} \to \mathbb{R}^{n_R}$ such that

$$\Lambda_R = \underset{\sim}{f}_R^{-1}(\underset{\sim}{0}) \tag{45}$$

and

$$\text{rank } (D_{\underset{\sim}{i}_R}\underset{\sim}{f}_R)_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)} = n_R \text{ for all } (\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R . \tag{46}$$

<u>Proposition 6</u> Let $\Lambda$ be described by (20) and let $\Lambda_R$ be locally voltage controlled. Suppose that $\Lambda \cap K \neq \phi$ and $\Lambda \not\pitchfork K$. Let $\mathcal{T}$ be an arbitrary tree for $\mathcal{G}$ and let $\mathcal{L}$ be its associated cotree. As in <u>Proposition 5</u>, let $\mathcal{T} = \mathcal{T}_R \cup \mathcal{T}_C \cup \mathcal{T}_L$, $\mathcal{L} = \mathcal{L}_R \cup \mathcal{L}_C \cup \mathcal{L}_L$. Insert a small linear capacitor in parallel with each

branch of $\mathcal{T}_R$. Then the perturbed network $\hat{\mathcal{N}}$ has properties (i) and (ii) of Proposition 5.

Proof (i) can be proved in a manner similar to that of Proposition 5.
(ii) It is clear that $\hat{\mathcal{T}} \triangleq \mathcal{T}_C \cup \mathcal{T}_1 \cup \mathcal{T}_L$ is a tree for $\hat{\mathcal{N}}$ and $\hat{\mathcal{L}} \triangleq \mathcal{L}_R \cup \mathcal{T}_R \cup \mathcal{L}_C \cup \mathcal{L}_L$ is its associated cotree where $\mathcal{T}_1$ represents the branches of the capacitors added. To compute $\hat{\mathcal{F}}_R(\hat{v}_R,\hat{i}_R)$ observe that the fundamental loop matrix $\hat{B}$ for $\hat{\mathcal{N}}$ is given by

$$
\begin{array}{cccc:ccc}
v_{R_{\mathcal{L}}} & v_{R_{\mathcal{T}}} & v_{C_{\mathcal{L}}} & v_{L_{\mathcal{L}}} & v_{C_{\mathcal{T}}} & v_{\mathcal{T}_1} & v_{L_{\mathcal{T}}}
\end{array}
$$

$$
\left[\begin{array}{c:ccc}
 & 0 & -1 & 0 \\
1 & B_{RC} & B_{RR} & B_{RL} \\
 & B_{CC} & B_{CR} & B_{CL} \\
 & B_{LC} & B_{LR} & B_{LL}
\end{array}\right]
$$

where the submatrices are those of $B$ for $\mathcal{N}$. Therefore $\hat{B}_{RR} = \hat{B}_{CR}^T = \hat{B}_{LR}^T = \phi$,

$$
\hat{B}_{RC} = \begin{bmatrix} 0 & -1 \\ B_{RC} & B_{RR} \end{bmatrix}, \quad \hat{B}_{RL} = \begin{bmatrix} 0 \\ B_{RL} \end{bmatrix}
$$

$$
D_{\hat{v}_{R_{\mathcal{T}}}} f_R = \phi, \quad D_{\hat{v}_{R_{\mathcal{L}}}} f_R = D_{v_R} f_R,
$$

$$
D_{\hat{i}_{R_{\mathcal{T}}}} f_R = \phi, \quad D_{\hat{i}_{R_{\mathcal{L}}}} f_R = D_{i_R} f_R, \quad (\hat{v}_R,\hat{i}_R) = (v_R,i_R).
$$

Substituting these into (27) we have

$$
\hat{\mathcal{F}}_R(\hat{v}_R,\hat{i}_R) = \left[-(D_{v_R} f_R)[\hat{B}_{RC} \quad \hat{B}_{RL}] \vdots D_{i_R} f_R\right]_{(v_R,i_R)} . \tag{47}
$$

Since $\Lambda_R$ is locally voltage controlled, (46) implies that the matrix of (47) has rank $n_R$. ⌑

A dual argument shows the following:

Proposition 7 Let $\Lambda$ be described by (20) and let $\Lambda_R$ be locally current controlled, i.e., (45) holds and

$$\text{rank }(D_{\underset{\sim}{v}_R} \underset{\sim}{f}_R)_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)} = n_R \text{ for all } (\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R. \tag{48}$$

Suppose that $\Lambda \cap K \neq \phi$ and $\Lambda \not\pitchfork K$. Insert a small linear inductor in series with each branch of $\mathcal{L}_R$, where $\mathcal{L}_R$ is as in Proposition 6. Then the perturbed network $\hat{\mathcal{N}}$ has properties (i) and (ii) of Proposition 5.

## IV. Well-Posedness

Recall that transversality of $\Lambda$ and K is a static condition in the sense that it has nothing to do with the dynamics of $\mathcal{N}$. In order to motivate the discussion of this section we first consider the following example.

Example 2 Consider the circuit of Fig. 2(a) where $\Lambda_R$ is given by Fig. 2(b) with $v_R = g(i_R)$. If we choose $\mathcal{T} = \{C\}$ to be our tree, then $B_{RC} = 1$, $D_{v_{R_{\ell}}} f_R = 1$, $D_{i_{R_{\ell}}} f_R = -Dg$ and rank $\mathcal{F}_R(v_R,i_R) = \text{rank } [-1 \vdots -Dg]_{i_R} = 1$. It follows from Corollary 1 that $\Lambda \pitchfork K$ and $\Sigma$ is a perfectly well-defined 1-dimensional submanifold. The dynamics, however, has points where it is not well defined. To show this observe that $\underset{\sim}{i}_R$ serves as a global coordinate for $\Sigma$, i.e., $(v_R,v_C,i_R,i_C) = \left(g(i_R),\ -g(i_R),\ i_R,\ i_R\right)$. In terms of this coordinate, the dynamics is given by

$$C(Dg)_{i_R} \frac{di_R}{dt} = -i_R\ . \tag{49}$$

Since $(Dg)_{i_{R_0}} = 0$, differential equation (49) is undefined at $i_R = i_{R_0}$. Therefore (49) cannot define a unique vector field at $i_{R_0}$. This observation naturally leads to the following definition.

Definition 1 Given a nonlinear network $\mathcal{N}$ assume that $\Sigma$ is an $(n_C+n_L)$-dimensional $C^2$ submanifold. Then $\mathcal{N}$ is said to be well-posed if at each point $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$, (10) determines a unique $C^1$ vector field $\underset{\sim}{X}_{(\underset{\sim}{v},\underset{\sim}{i})} \in T_{(\underset{\sim}{v},\underset{\sim}{i})}\Sigma$. If $\mathcal{N}$ violates this property, it is said to be ill-posed. Any point $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$ violating the above condition is said to be an impasse point.

Remarks 1. Observe that "well-posedness" is a coordinate-free definition corresponding to the concept of "local solvability" in Chua and Wang [7]. Smale [4] uses "regularity" instead of well-posedness. Since well-posedness is a coordinate-free property, one needs to check it in terms of only one coordinate. On the other hand, if ill-posedness occurs with respect to one choice of coordinate, then no matter how cleverly one chooses another coordinate, one cannot eliminate ill-posedness. For example, in Example 2, there is no choice of coordinate which avoids $(Dg)i_{R_0} = 0$.

2. Let us explain why $\Lambda$ must be $C^2$ in order to define a $C^1$ vector field. This stems from the fact that, in general, a $C^r$ vector field can be defined only on a $C^s$ manifold with $s > r$. Consider, for example, the circuit of Fig. 2(a), where g is a global $C^1$ diffeomorphism (not that of Fig. 2(b)). Hence $i_R = h(v_R)$ where $h = g^{-1}$ and h is also a global $C^1$ diffeomorphism. The sets $\Lambda$ and $\Sigma$ are $C^1$ submanifolds. Capacitor voltage $v_C$ serves as a global coordinate for $\Sigma$ and the dynamics is given by

$$\frac{dv_C}{dt} = \frac{h(-v_C)}{C} .$$

The right hand side is $C^1$. Now it is clear that $i_R$ is another global coordinate for $\Sigma$ and the dynamics is given by

$$\frac{di_R}{dt} = \frac{i_R}{(Dg)_{i_R}} .$$

Since g is $C^1$, the right hand side is only $C^0$. This gives rise to a problem because the differentiability depends on the choices of coordinates. If we assume, however, g is $C^2$, then the right hands side of each equation is <u>at least $C^1$</u>. Therefore $C^1$-ness does not depend on the choices of coordinates. More generally, let $\underset{\sim}{X}$ be a vector field on $\Sigma$ and let $(U\cap\Sigma,\underset{\sim}{\psi})$ be a local chart at $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$. Then a natural coordinate representation is

$$(\underset{\sim}{X}(\underset{\sim}{\psi}))_{\underset{\sim}{\psi}(\underset{\sim}{v},\underset{\sim}{i})} = (d\underset{\sim}{\psi})_{(\underset{\sim}{v},\underset{\sim}{i})}\underset{\sim}{X}(v,i) .$$

If $(V\cap\Sigma,\underset{\sim}{\phi})$ is another chart, then for $(\underset{\sim}{v},\underset{\sim}{i}) \in U \cap V \cap \Sigma$ one has

$$\begin{aligned}(\underset{\sim}{X}(\underset{\sim}{\phi}))_{\underset{\sim}{\phi}(\underset{\sim}{v},\underset{\sim}{i})} &= \underset{\sim}{d}(\underset{\sim}{\phi}\circ\underset{\sim}{\psi}^{-1}\circ\underset{\sim}{\psi})_{(\underset{\sim}{v},\underset{\sim}{i})}\underset{\sim}{X}(\underset{\sim}{v},\underset{\sim}{i}) \\ &= (d\underset{\sim}{\phi}\circ\underset{\sim}{\psi}^{-1})_{\underset{\sim}{\psi}(\underset{\sim}{v},\underset{\sim}{i})}(d\underset{\sim}{\psi})_{(\underset{\sim}{v},\underset{\sim}{i})}\underset{\sim}{X}(\underset{\sim}{v},\underset{\sim}{i}) \\ &= (d\underset{\sim}{\phi}\circ\underset{\sim}{\psi}^{-1})_{\underset{\sim}{\psi}(\underset{\sim}{v},\underset{\sim}{i})}(\underset{\sim}{X}(\underset{\sim}{\psi}))_{\underset{\sim}{\psi}(\underset{\sim}{v},\underset{\sim}{i})}\end{aligned}$$

Therefore, if we want $\underset{\sim}{X}(\underset{\sim}{\psi})$ to be $C^r$ independent of the choice of coordinates, we must require the change of coordinates $d\underset{\sim}{\psi} \circ \underset{\sim}{\psi}^{-1}$ to be $C^r$. This requires $\underset{\sim}{\phi} \circ \underset{\sim}{\psi}^{-1}$ to be at least $C^{r+1}$. But this is exactly the condition required for $\Sigma$ to be at least $C^{r+1}$.

Let us now look at well-posedness from a more general view point. Recall (6), (9) and (10). Observe that $\pi^*G_{(v,i)}(\cdot,\cdot)$ is simply a symmetric bilinear form on $T_{(v,i)}\Sigma \times T_{(v,i)}\Sigma$, i.e., $\pi^*G_{(v,i)}(\xi_1,\cdot)$ and $\pi^*G_{(v,i)}(\cdot,\xi_2)$ are linear and $\pi^*G_{(v,i)}(\xi_1,\xi_2) = \pi^*G_{(v,i)}(\xi_2,\xi_1)$ for all $\xi_1,\xi_2 \in T_{(v,i)}\Sigma$. In Example 2, in terms of the coordinate $i_R$, we have $\pi^*G_{(v,i)} = C(Dg)_{i_R} di_R \otimes di_R$.

<u>Proposition 8</u> Suppose that $\Sigma$ is an $(n_C+n_L)$-dimensional $C^2$ submanifold. Then $\mathcal{N}$ is well-posed if and only if at each point $(v,i) \in \Sigma$, $\pi^*G_{(v,i)}$ is nonsingular, i.e.,

$$\pi^*G_{(v,i)}(\xi_1,\xi_2) = 0 \text{ for all } \xi_1 \in T_{(v,i)}\Sigma \text{ implies } \xi_2 = 0. \tag{50}$$

<u>Proof</u> We look at $\pi^*G_{(v,i)}(\cdot,\cdot)$ in a slightly different manner. Consider the map $J_{(v,i)}$ defined by

$$J_{(v,i)} : \xi_1 \mapsto \pi^*G_{(v,i)}(\xi_1,\cdot). \tag{51}$$

To each $\xi_1$, the map $J_{(v,i)}$ assigns the linear functional[5] $\pi^*G_{(v,i)}(\xi_1,\cdot)$ on $T_{(v,i)}\Sigma$. A linear functional on $T_{(v,i)}\Sigma$ belongs to its dual $T^*_{(v,i)}\Sigma$. This means that $J_{(v,i)} = \pi^*G_{(v,i)}(\cdot,\cdot)$ maps $T_{(v,i)}\Sigma$ into $T^*_{(v,i)}\Sigma$. It is clear that (50) implies that $J_{(v,i)}$ is an isomorphism and therefore it is invertible. It follows from (10) that the vector field $X_{(v,i)}$ is uniquely determined by

$$X_{(v,i)} = J^{-1}_{(v,i)}\omega_{(v,i)} . \tag{52}$$

In order to show that $X$ is $C^1$, recall the definition of $\pi^*G$;

$$\pi^*G_{(v,i)}(\xi_1,\xi_2) = G_{\pi(v,i)}((d\pi)_{(v,i)}\xi_1,(d\pi)_{(v,i)}\xi_2) \tag{53}$$

where $\xi_1$, $\xi_2 \in T_{(v,i)}\Sigma$. Since $\Sigma$ is $C^2$, the map $(d\pi)$ is $C^1$. Therefore $J^{-1}$ is $C^1$. Similarly, $\omega$ is $C^1$. This implies that $X$ determined by (52) is $C^1$. Conversely if $J_{(v,i)}$ is not an isomorphism, (10) cannot determine a unique vector field. ¤

<u>Corollary 3</u> Let $\tilde{G}_{(v,i)}$ be the matrix representation of $\pi^*G_{(v,i)}$ with respect to a particular choice of coordinate for $\Sigma$. Then $\mathcal{N}$ is well posed if and only if $\tilde{G}_{(v,i)}$ is nonsingular for all $(v,i) \in \Sigma$.

<u>Proof</u> The map $J_{(v,i)}$ defined by (51) is an isomorphism if and only if its matrix representation with respect to a particular choice of coordinate is nonsingular. ¤

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[5] A linear functional is a real-valued linear function.

<u>Proposition 9</u> Let $\Sigma$ be an $(n_C+n_L)$-dimensional $C^2$ submanifold. Then $\mathcal{N}$ is well-posed if and only if for each $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$, the following map is an isomorphism:

$$(d\underset{\sim}{\pi})_{(\underset{\sim}{v},\underset{\sim}{i})} : T_{(\underset{\sim}{v},\underset{\sim}{i})}\Sigma \to T_{\underset{\sim}{\pi}(\underset{\sim}{v},\underset{\sim}{i})}\mathbb{R}^n, \tag{54}$$

i.e., $\underset{\sim}{\pi}$ is a local diffeomorphism, where $\underset{\sim}{\pi}$ is defined by (9).

<u>Proof</u> Recall the definition (53) of $\underset{\sim}{\pi}^*\underset{\sim}{G}$. Since $\underset{\sim}{G}$ defined by (6) is nonsingular in the sense of <u>Proposition 8</u>, we see that $\underset{\sim}{\pi}^*\underset{\sim}{G}$ is nonsingular if and only if $(d\underset{\sim}{\pi})_{(\underset{\sim}{v},\underset{\sim}{i})}$ is an isomorphism. ¤

In Example 2, in terms of the coordinate $i_R$, we have $\pi=g$. Therefore $\underset{\sim}{\pi}^*\underset{\sim}{G}_{(\underset{\sim}{v},\underset{\sim}{i})} = C\,(Dg)_{i_R}\, di_R \otimes di_R$ becomes singular when $(Dg)_{i_{R_0}} = 0$.

The following is essentially a restatement of <u>Proposition 9</u>.

<u>Corollary 4</u> Let $\Sigma$ be an $(n_C+n_L)$-dimensional $C^2$ submanifold. Then $\mathcal{N}$ is well-posed if and only if at each point $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$, $(\underset{\sim}{v}_C,\underset{\sim}{i}_L)$ serves as a local coordinate for $\Sigma$.

<u>Remark</u> Because of its coordinate-free property, <u>Corollary 4</u> has an interesting circuit theoretic consequence. It says that if (10) fails to determine a unique $C^1$ vector field with respect to capacitor voltages and inductor currents, then there is no choice of variables in the network in terms of which (10) defines a unique $C^1$ vector field. Conversely, if (10) specifies a unique $C^1$ vector field with respect to one coordinate system, then it specifies a unique $C^1$ vector field with respect to capacitor voltages and inductor currents also. This shows that capacitor voltages and inductor currents are a good choice of a coordinate system to describe the dynamics. Of course, this is not true if capacitors and inductors are not described by (2) and (3), respectively. Consider, for example, the circuit of Fig. 2(a) where the capacitor is described by $v_C = f(q_C)$, the resistor is described by $i_R = g(v_R)$ and f is <u>not</u> injective. Then the dynamics is perfectly well defined in terms of $q_C$; $\dot{q}_C = q(-f(q_C))$. It is clear, however, that the dynamics cannot be described in terms of $v_C$.

<u>Theorem 1</u> Let $\Sigma$ be an $(n_C+n_L)$-dimensional $C^2$ submanifold and pick a proper tree $\mathcal{T}$. Then $\mathcal{N}$ is well-posed if and only if for each $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$,

$$\det \underset{\approx}{\mathcal{H}}(\underset{\sim}{v},\underset{\sim}{i}) \neq 0 \tag{55}$$

where

$$\mathcal{H}(\underset{\sim}{v},\underset{\sim}{i}) \triangleq \Big[ D_{\underset{\sim}{v}_{R_{\mathcal{J}}}} \underset{\sim}{f} - (D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}) \underset{\sim}{B}_{RR} - (D_{\underset{\sim}{v}_{L_{\mathcal{L}}}} \underset{\sim}{f}) \underset{\sim}{B}_{LR} \; \vdots$$

$$D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f} + (D_{\underset{\sim}{i}_{R_{\mathcal{J}}}} \underset{\sim}{f}) \underset{\sim}{B}_{RR}^T + (D_{\underset{\sim}{i}_{C_{\mathcal{J}}}} \underset{\sim}{f}) \underset{\sim}{B}_{RC}^T \Big]_{(\underset{\sim}{v},\underset{\sim}{i})} \quad . \tag{56}$$

<u>Proof</u> Let $(\psi, \Sigma \cap U)$ be a local chart for $\Sigma$ at $(\underset{\sim}{v},\underset{\sim}{i})$. Then $(\underset{\sim}{d\pi})_{(\underset{\sim}{v},\underset{\sim}{i})}$ is an isomorphism if and only if $(\underset{\sim}{D\pi} \circ \underset{\sim}{\psi}^{-1})_{\underset{\sim}{\psi}(\underset{\sim}{v},\underset{\sim}{i})}$ is a nonsingular matrix. Since $\underset{\sim}{\pi} \circ \underset{\sim}{\psi}^{-1} = \underset{\sim}{\pi}_{CL} \circ \underset{\sim}{\iota} \circ \underset{\sim}{\psi}^{-1}$, we have

$$(\underset{\sim}{D\pi} \circ \psi^{-1})_{\underset{\sim}{\psi}(\underset{\sim}{v},\underset{\sim}{i})} = (\underset{\sim}{D\pi}_{CL})_{(\underset{\sim}{v},\underset{\sim}{i})} \circ (\underset{\sim}{d\iota})_{(\underset{\sim}{v},\underset{\sim}{i})} \circ (\underset{\sim}{D\psi}^{-1})_{\underset{\sim}{\psi}(\underset{\sim}{v},\underset{\sim}{i})} . \tag{57}$$

Since $(\underset{\sim}{d\iota})_{(\underset{\sim}{v},\underset{\sim}{i})}$ is a linear inclusion map, the matrix of (57) is nonsingular if and only if

$$\mathrm{Ker}(\underset{\sim}{D\pi}_{CL})_{(\underset{\sim}{v},\underset{\sim}{i})} \cap \mathrm{Im}(\underset{\sim}{D\psi}^{-1})_{\underset{\sim}{\psi}(\underset{\sim}{v},\underset{\sim}{i})} = \{\underset{\sim}{0}\} \; . \tag{58}$$

Let $\underset{\sim}{g} : U \to \mathbb{R}^{b+n_R}$ be defined by

$$\underset{\sim}{g}(\underset{\sim}{v},\underset{\sim}{i}) = \begin{bmatrix} \underset{\sim}{B}\,\underset{\sim}{v} \\ \underset{\sim}{Q}\,\underset{\sim}{i} \\ \underset{\sim}{f}(\underset{\sim}{v},\underset{\sim}{i}) \end{bmatrix} \tag{59}$$

where $\underset{\sim}{B}$ and $\underset{\sim}{Q}$ are fundamental loop and cut set matrices, respectively and $\underset{\sim}{f}$ is as in (16). Since $\Sigma \cap U = \underset{\sim}{g}^{-1}(\underset{\sim}{0})$, we have [3]

$$T_{(\underset{\sim}{v},\underset{\sim}{i})}\Sigma = \mathrm{Im}(\underset{\sim}{D\psi}^{-1})_{\underset{\sim}{\psi}(\underset{\sim}{v},\underset{\sim}{i})} = \mathrm{Ker}(\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})} . \tag{60}$$

It follows from (58) and (60) that the matrix of (57) is nonsingular if and only if

$$\mathrm{Ker}(\underset{\sim}{D\pi}_{CL})_{(\underset{\sim}{v},\underset{\sim}{i})} \cap \mathrm{Ker}(\underset{\sim}{Dg})_{(\underset{\sim}{v},\underset{\sim}{i})} = \{\underset{\sim}{0}\} \tag{61}$$

which is equivalent to

$$\mathrm{rank} \begin{bmatrix} \underset{\sim}{Dg} \\ \underset{\sim}{D\pi}_{CL} \end{bmatrix}_{(\underset{\sim}{v},\underset{\sim}{i})} = 2b \; . \tag{62}$$

Computing the matrix of (62) one can easily show that it has rank 2b if and only if the following matrix has rank $b+n_R$:

$$\begin{array}{c} \begin{array}{cccccc} \underset{\sim}{v}_{R_\mathcal{L}} & \underset{\sim}{v}_{L_\mathcal{L}} & \underset{\sim}{v}_{R_\mathcal{J}} & \underset{\sim}{i}_{R_\mathcal{L}} & \underset{\sim}{i}_{R_\mathcal{J}} & \underset{\sim}{i}_{C_\mathcal{J}} \end{array} \\ \left[\begin{array}{ccc|ccc} \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{B}_{RR} & & \underset{\sim}{O} & \\ \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{LR} & & & \\ \hline & & & -\underset{\sim}{B}_{RR}^T & \underset{\sim}{1} & \underset{\sim}{0} \\ & \underset{\sim}{O} & & -\underset{\sim}{B}_{RC}^T & \underset{\sim}{0} & \underset{\sim}{1} \\ \hline D_{\underset{\sim}{v}_{R_\mathcal{L}}}\underset{\sim}{f} & D_{\underset{\sim}{v}_{L_\mathcal{L}}}\underset{\sim}{f} & D_{\underset{\sim}{v}_{R_\mathcal{J}}}\underset{\sim}{f} & D_{\underset{\sim}{i}_{R_\mathcal{L}}}\underset{\sim}{f} & D_{\underset{\sim}{i}_{R_\mathcal{J}}}\underset{\sim}{f} & D_{\underset{\sim}{i}_{C_\mathcal{J}}}\underset{\sim}{f} \end{array}\right] \end{array} \tag{63}$$

By further elementary operations, one can show that this matrix has rank $b+n_R$ if and only if (55) holds. ¤

Corollary 5  Suppose that $\Lambda$ is described by (20) and that $\Sigma$ is an $(n_C+n_L)$-dimensional $C^2$ submanifold. Then $\mathcal{N}$ is well-posed if and only if for each $(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \underset{\sim}{\pi}_R(\Sigma)$

$$\det \underset{\sim}{\mathcal{H}}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \neq 0 \tag{64}$$

where

$$\underset{\sim}{\mathcal{H}}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \triangleq \left[D_{\underset{\sim}{v}_{R_\mathcal{J}}}\underset{\sim}{f}_R - (D_{\underset{\sim}{v}_{R_\mathcal{L}}}\underset{\sim}{f}_R)\underset{\sim}{B}_{RR} \,\vdots\, D_{\underset{\sim}{i}_{R_\mathcal{L}}}\underset{\sim}{f}_R + (D_{\underset{\sim}{i}_{R_\mathcal{J}}}\underset{\sim}{f}_R)\underset{\sim}{B}_{RR}^T\right]_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)} \tag{65}$$

and $\underset{\sim}{f}_R$ is as in (21).

Proof  If $\Lambda$ is described by (20), then $\underset{\sim}{f}$ is independent of $(\underset{\sim}{v}_C,\underset{\sim}{v}_L,\underset{\sim}{i}_C,\underset{\sim}{i}_L)$. This implies the result. ¤

Example 3  Consider the circuit of Example 2. Since $\mathcal{H}_R(v_R,i_R) = -(Dg)_{i_R}$, it fails to have rank 1 at $i_R = i_{R_0}$ and therefore this circuit is ill-posed.

Example 4  Consider the circuit of Example 1, where $\Lambda_R$ is given in Fig. 3(a) with $i_{R_k} = f_{R_k}(v_{R_k})$, k = 1,2,. Since

$$\underset{\sim}{\mathcal{F}}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) = \begin{bmatrix} -Df_{R_1} & 0 & \vdots & 1 \\ 0 & -Df_{R_2} & \vdots & 1 \end{bmatrix}_{\underset{\sim}{v}_R}$$

and since $Df_{R_1}$ and $Df_{R_2}$ never vanish simultaneously, rank $\mathcal{F}_R(v_R, i_R) = 2$. Consequently $\Lambda \pitchfork K$ and $\Sigma$ is a 1-dimensional submanifold. Since

$$\mathcal{H}_R(v_R, i_R) = \begin{bmatrix} -Df_{R_1} & 0 \\ 0 & -Df_{R_2} \end{bmatrix}_{v_R}$$

there are points where $\det \mathcal{H}_R(v_R, i_R) = 0$. Therefore the circuit is ill-posed. If we use Corollary 4, we can see this more clearly. Consider the projection $\mathcal{R}$ of $\Sigma$ onto the $(v_L, i_L)$-space given in Fig. 3(b). If we further project $\mathcal{R}$ onto the $i_L$-axis, we see that $i_L$ cannot be a local coordinate where the curve intersects itself. Therefore $\mathcal{N}$ is ill-posed.

Corollary 6 Let $\Lambda$ be described by (20) and let $\Lambda_R$ admit a generalized port coordinate representation. Suppose that $\Sigma$ is an $(n_C+n_L)$-dimensional $C^2$ submanifold. Then

$$\mathcal{H}_R(v_R, i_R) = \Big[\left(\alpha_2 - (DF)\,\gamma_2\right) - \left(\alpha_1 - (DF)\gamma_1\right)B_{RR} \vdots \left(\beta_1 - (DF)\delta_1\right) + \left(\beta_2 - (DF)\delta_2\right)B_{RR}^T\Big]_{(v_R, i_R)} \tag{66}$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are as in (29). In particular, if $\Lambda_R$ is globally voltage controlled, then

$$\mathcal{H}_R(v_R, i_R) = \left[ (DF)\begin{bmatrix} B_{RR} \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 1 \\ B_{RR}^T \end{bmatrix} \right]_{v_R} \tag{67}$$

and if $\Lambda_R$ is globally current controlled, then

$$\mathcal{H}_R(v_R, i_R) = \left[ \begin{bmatrix} -B_{RR} \\ 1 \end{bmatrix} \vdots -(DF) \begin{bmatrix} 1 \\ B_{RR}^T \end{bmatrix} \right]_{i_R} . \tag{68}$$

Recall (33). If $\Lambda_R$ is globally parametrizable, then the following holds:

<u>Proposition 10</u> Let $\Lambda$ be described by (20) and let $\Lambda_R$ be globally parametrizable. Suppose that $\Sigma$ is an $(n_C+n_L)$-dimensional $C^2$ submanifold. Then $\mathcal{N}$ is well-posed if and only if for each $\underset{\sim}{\rho} \in \mathbb{R}^{n_R}$ with $\left(\underset{\sim}{v}_R(\underset{\sim}{\rho}), \underset{\sim}{i}_R(\underset{\sim}{\rho})\right) \in \underset{\sim}{\pi}_R(\Sigma)$,

$$\text{rank } \underset{\sim}{\mathcal{H}}^*(\underset{\sim}{\rho}) = n_R \tag{69}$$

where

$$\underset{\sim}{\mathcal{H}}^*(\underset{\sim}{\rho}) \triangleq \begin{bmatrix} D\underset{\sim}{v}_{R_{\mathcal{L}}} + \underset{\sim}{B}_{RR}\left(D\underset{\sim}{v}_{R_{\mathcal{J}}}\right) \\ D\underset{\sim}{i}_{R_{\mathcal{J}}} - \underset{\sim}{B}^T_{RR}\left(D\underset{\sim}{i}_{R_{\mathcal{L}}}\right) \end{bmatrix}_{\underset{\sim}{\rho}} . \tag{70}$$

<u>Proof</u> Substitute (33) into KVL and KCL;

$$\left[\begin{array}{cc:cc} \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{B}_{RR} & \underset{\sim}{B}_{RC} \\ \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{B}_{LR} & \underset{\sim}{B}_{LC} \end{array}\right] \begin{bmatrix} \underset{\sim}{v}_{R_{\mathcal{L}}}(\underset{\sim}{\rho}) \\ \underset{\sim}{v}_{L_{\mathcal{L}}} \\ \underset{\sim}{v}_{R_{\mathcal{J}}}(\underset{\sim}{\rho}) \\ \underset{\sim}{v}_{C_{\mathcal{J}}} \end{bmatrix} = \underset{\sim}{0} \tag{71}$$

$$\left[\begin{array}{cc:cc} -\underset{\sim}{B}^T_{RR} & -\underset{\sim}{B}^T_{LR} & \underset{\sim}{1} & \underset{\sim}{0} \\ -\underset{\sim}{B}^T_{RC} & -\underset{\sim}{B}^T_{LC} & \underset{\sim}{0} & \underset{\sim}{1} \end{array}\right] \begin{bmatrix} \underset{\sim}{i}_{R_{\mathcal{L}}}(\underset{\sim}{\rho}) \\ \underset{\sim}{i}_{L_{\mathcal{L}}} \\ \underset{\sim}{i}_{R_{\mathcal{J}}}(\underset{\sim}{\rho}) \\ \underset{\sim}{i}_{C_{\mathcal{J}}} \end{bmatrix} = \underset{\sim}{0} . \tag{72}$$

Let us write (71) and (72) as

$$\underset{\sim}{h}(\underset{\sim}{\rho}, \underset{\sim}{v}_{L_{\mathcal{L}}}, \underset{\sim}{v}_{C_{\mathcal{J}}}, \underset{\sim}{i}_{L_{\mathcal{L}}}, \underset{\sim}{i}_{C_{\mathcal{J}}}) = \underset{\sim}{0} . \tag{73}$$

Then $\Sigma = \underset{\sim}{h}^{-1}(\underset{\sim}{0})$. By a similar argument to that of the proof of <u>Theorem 1</u> we see that (58) holds if and only if for each $(\underset{\sim}{\rho}, \underset{\sim}{v}_{L_{\mathcal{L}}}, \underset{\sim}{v}_{C_{\mathcal{J}}}, \underset{\sim}{i}_{L_{\mathcal{L}}}, \underset{\sim}{i}_{C_{\mathcal{J}}})$,

$$\text{rank} \begin{bmatrix} D\underset{\sim}{h} \\ D\underset{\sim}{\pi}_{CL} \end{bmatrix}_{(\underset{\sim}{\rho}, \underset{\sim}{v}_L, \underset{\sim}{v}_C, \underset{\sim}{i}_L, \underset{\sim}{i}_C)} = b + n_C + n_L . \tag{74}$$

More explicitly this matrix turns out to be the following:

$$\begin{array}{c} \\ \\ \end{array}\begin{array}{ccccc} \rho & v_{L_\mathcal{L}} & v_{C_\mathcal{J}} & i_{L_\mathcal{L}} & i_{C_\mathcal{J}} \end{array}$$

$$\begin{bmatrix} Dv_{R_\mathcal{L}}+B_{RR}(Dv_{R_\mathcal{J}}) & \cdot & B_{RC} & \cdot & \cdot \\ B_{LR}(Dv_{R_\mathcal{J}}) & 1 & B_{LC} & \cdot & \cdot \\ B_{RR}^T(Di_{R_\mathcal{L}})+Di_{R_\mathcal{J}} & \cdot & \cdot & -B_{LR}^T & \cdot \\ -B_{RC}^T(Di_{R_\mathcal{L}}) & \cdot & \cdot & -B_{LC}^T & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix} \qquad (75)$$

where $\cdot$ denotes a zero matrix of appropriate dimension. By elementary operations one can show that (74) holds if and only if (69) holds. ¤

Remark The above result holds even when $\Lambda_R$ is locally parametrized by $\rho$. In fact (69) holds if and only if $K$ is nonsingular, where $K$ is the matrix defined in [6].

Now observe that $\mathcal{H}(v,i)$ is a submatrix of $\mathcal{F}(v,i)$. This implies the following:

Proposition 11 If (55) holds, then $\Lambda \pitchfork K$ and $\mathcal{N}$ is well-posed.

Remark Similar results hold for Corollaries 5 and 6 and Proposition 10. While Theorem 1 assumes that $\Sigma$ is an $(n_C+n_L)$-dimensional $C^2$ submanifold, Proposition 11 does not assume it.

In many practical networks, $\pi$ is a global diffeomorphism, i.e., all variables in the network can be globally expressed as a function of $(v_C, i_L)$ and hence $\Sigma$ is globally diffeomorphic to $\mathbb{R}^{n_C+n_L}$. Of course $\mathcal{N}$ is well-posed. In Example 4, $\mathcal{N}$ is not well-posed and $\Sigma$ is not diffeomorphic to $\mathbb{R}^1$. A question arises; are there networks such that $\Sigma$ is a submanifold not diffeomorphic to $\mathbb{R}^{n_C+n_L}$, yet they are well-posed? The answer is affirmative as the following example shows.

Example 5 Consider the map $F: \mathbb{R}^3 \to \mathbb{R}^6$ defined by

$F(x,y,z) \triangleq (e^y\cos x, e^y\sin x, z, y, \cos x, \sin x)$.

For $x, x' \in \mathbb{R}$, define the equivalence relation $x \sim x'$ by $x - x' = 2k\pi$ where k is an integer. Clearly, then, the quotient space of $\mathbb{R}$ with respect to this equivalence relation can be regarded as the unit circle $S^1$ in $\mathbb{R}^2$; $\mathbb{R}/\sim = S^1$. Let [x] denote the equivalence class. Then $\underset{\sim}{F}$ naturally induces the map $\tilde{\underset{\sim}{F}} : S^1 \times \mathbb{R}^2 \to \mathbb{R}^6$ by

$$\tilde{\underset{\sim}{F}}\Big([x],y,z\Big) \triangleq \underset{\sim}{F}(x,y,z). \tag{76}$$

Since

$$(d\tilde{\underset{\sim}{F}})\Big([x],y,z\Big) = \begin{bmatrix} e^y(-\sin x) & e^y\cos x & 0 \\ e^y(\cos x) & e^y \sin x & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ -\sin x & \cos x & 0 \end{bmatrix}_{([x],y,z)}$$

and since

$$\det = \begin{vmatrix} e^y(-\sin x) & e^y\cos x & 0 \\ e^y(\cos x) & e^y \sin x & 0 \\ 0 & 0 & 1 \end{vmatrix} = e^{2y} \neq 0$$

we have rank $(d\tilde{\underset{\sim}{F}})_{([x],y,z)} = 3$ for all $\Big([x],y,z\Big) \in S^1 \times \mathbb{R}^2$. Therefore $\tilde{\underset{\sim}{F}}$ is an immersion [11]. Clearly $\tilde{\underset{\sim}{F}}$ is injective. Since $\|\tilde{\underset{\sim}{F}}\Big([x],y,z\Big)\|^2 = e^{2y} + 2^2 + y^2 + 1$, we have $\|\tilde{\underset{\sim}{F}}\Big([x],y,z\Big)\| \to \infty$ as $\|(y,z)\| \to \infty$. Hence $\tilde{\underset{\sim}{F}}$ is proper [11]. Consequently it is an embedding [11]. Define

$$\left.\begin{aligned} &\Lambda_R = \tilde{\underset{\sim}{F}}(S^1 \times \mathbb{R}^2), \\ &v_{R_1} = e^y \cos x,\ v_{R_2} = e^y \sin x,\ v_{R_3} = z \\ &i_{R_1} = y,\ i_{R_2} = \cos x,\ i_{R_3} = \sin x \end{aligned}\right\} . \tag{77}$$

This is a parametric representation of $\Lambda_R$. Consider the circuit of Fig. 4 where $\Lambda_R$ is described by (77). It follows from the above argument that $\Lambda_R$ is a 3-dimensional submanifold diffeomorphic to $S^1 \times \mathbb{R}^2$. It is clear that $\Sigma$ is diffeomorphic to $\Lambda_R$ and therefore diffeomorphic to $S^1 \times \mathbb{R}^2$. Note that $(x,y,z)$ is always a local coordinate for $\Sigma$. (not a global coordinate). Choose $\mathcal{T} = \{C_1, C_2, C_3\}$ to be our tree. Then, with $\underset{\sim}{\rho} = (x,y,z)$, the matrix of (70) is given by

$$\underset{\sim}{\mathcal{H}}^*(\underset{\sim}{\rho}) = \begin{bmatrix} -e^y \sin x & e^y \cos x & 0 \\ e^y \cos x & e^y \sin x & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\underset{\sim}{\rho}} .$$

Since $\det \underset{\sim}{\mathcal{H}}^*(\underset{\sim}{\rho}) = -e^{2y} \neq 0$, $\mathcal{N}$ is well-posed. Consequently, the dynamics of $\mathcal{N}$ is perfectly well defined everywhere on $\Sigma$, yet there is no global coordinate in terms of which dynamics admits a global state equation because $\Sigma \approx S^1 \times \mathbb{R}^2 \neq \mathbb{R}^3$.

We next give a perturbation result on well-posedness. Recall that $\Lambda_R$ is said to be locally hybrid [3] if there is a $C^2$ function $\underset{\sim}{f}_R : \mathbb{R}^{n_R} \times \mathbb{R}^{n_R} \to \mathbb{R}^{n_R}$ such that

$$\Lambda_R = \underset{\sim}{f}_R^{-1}(\underset{\sim}{0}) \tag{78}$$

and

$$\det\left((D\underset{\sim}{f}_R)\underset{\sim}{A}\right)_{(\underset{\sim}{v}_R, \underset{\sim}{i}_R)} \neq 0 \text{ for all } (\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R \tag{79}$$

for some fixed $2n_R \times n_R$ matrix $\underset{\sim}{A}$, where each column of $\underset{\sim}{A}$ has either of the following forms:

$$(0,\ldots,0,1,0,\ldots,0,0,\ldots\ldots\ldots,0)$$

$$(\underbrace{0,\ldots\ldots\ldots,0}_{n_R},\underbrace{0,\ldots,0,1,0,\ldots,0}_{n_R}) .$$

Let

$$\left((D\underset{\sim}{f}_R)\underset{\sim}{A}\right)_{(\underset{\sim}{v}_R, \underset{\sim}{i}_R)} = [\underset{\sim}{F}_1, \ldots, \underset{\sim}{F}_{n_R}] \tag{80}$$

and suppose that $\underset{\sim}{F}_k$ corresponds to $i_{R_k}$ (resp., $v_{R_k}$). Then that particular resistor is said to be locally voltage controlled (resp., locally current controlled).

Remark Observe that in (45) and (46), local controlledness is defined for $\Lambda_R$, whereas in the above definition, local controlledness is defined for each resistor provided that $\Lambda_R$ is locally hybrid.

Theorem 2 Given a nonlinear network $\mathcal{N}$ assume the following:

(i) $\Lambda$ is described by (20) and $\Lambda_R$ is locally hybrid.

(ii) $\Lambda \cap K \neq \phi$.

Then, by adding small linear capacitors and small linear inductors appropriately we can obtain a new network $\hat{\mathcal{N}}$ such that (1) $\hat{\Lambda} \cap \hat{K} \neq \phi$, (2) $\hat{\Lambda} \pitchfork \hat{K}$, (3) $\hat{\mathcal{N}}$ is well-posed.

Proof Pick a proper tree $\mathcal{T}$ containing a maximum number of locally voltage controlled resistors and a minimum number of locally current controlled resistors. Let $\mathcal{L}$ denote its associated cotree. Partition $(\underset{\sim}{v},\underset{\sim}{i})$ in the following manner:

| elements | voltages | currents |
|---|---|---|
| locally voltage controlled resistors in $\mathcal{L}$ | $\underset{\sim}{v}_{V_{\mathcal{L}}}$ | $\underset{\sim}{i}_{V_{\mathcal{L}}}$ |
| locally current controlled resistors in $\mathcal{L}$ | $\underset{\sim}{v}_{I_{\mathcal{L}}}$ | $\underset{\sim}{i}_{I_{\mathcal{L}}}$ |
| inductors in $\mathcal{L}$ | $\underset{\sim}{v}_L$ | $\underset{\sim}{i}_L$ |
| locally voltage controlled resistors in $\mathcal{T}$ | $\underset{\sim}{v}_{V_{\mathcal{T}}}$ | $\underset{\sim}{i}_{V_{\mathcal{T}}}$ |
| locally current controlled resistors in $\mathcal{T}$ | $\underset{\sim}{v}_{I_{\mathcal{T}}}$ | $\underset{\sim}{i}_{I_{\mathcal{T}}}$ |
| capacitors in $\mathcal{T}$ | $\underset{\sim}{v}_C$ | $\underset{\sim}{i}_C$ |

The fundamental loop matrix has the following form:

$$\begin{array}{cccccc} \underset{\sim}{v}_{V_{\mathcal{L}}} & \underset{\sim}{v}_{I_{\mathcal{L}}} & \underset{\sim}{v}_L & \underset{\sim}{v}_{V_{\mathcal{T}}} & \underset{\sim}{v}_{I_{\mathcal{T}}} & \underset{\sim}{v}_C \end{array}$$
$$\left[\begin{array}{c:ccc} & \underset{\sim}{B}_{VV} & \underset{\sim}{0} & \underset{\sim}{B}_{VC} \\ \underset{\sim}{1} & \underset{\sim}{B}_{IV} & \underset{\sim}{B}_{II} & \underset{\sim}{B}_{IC} \\ & \underset{\sim}{B}_{LV} & \underset{\sim}{B}_{LI} & \underset{\sim}{B}_{LC} \end{array}\right] . \qquad (81)$$

The submatrix $\underset{\sim}{B}_{VI} = \underset{\sim}{0}$ because of the choice of the tree. Now insert a small linear capacitor in parallel with each locally voltage controlled resistor in $\mathcal{T}$ and insert a small linear inductor in series with each locally current controlled resistor in $\mathcal{L}$. Let $\hat{\mathcal{T}} \triangleq \mathcal{T}_I \cup \mathcal{L}_I \cup \mathcal{T}_C \cup \mathcal{T}_1$, where $\mathcal{T}_1$ is the branches of the capacitors added, I denotes locally current controlled resistors and C denotes capacitors. It is clear that $\hat{\mathcal{T}}$ is a proper tree for the new network. Statement (1) can be proved in a similar manner to the one in Proposition 5. To prove (3) observe that the fundamental loop matrix for $\hat{\mathcal{N}}$ with respect to $\hat{\mathcal{T}}$ has the following form:

$$
\begin{array}{cccc|cccc}
\underset{\sim}{v}_{V_{\mathcal{L}}} & \underset{\sim}{v}_{V_{\mathcal{T}}} & \underset{\sim}{v}_{L} & \underset{\sim}{v}_{\mathcal{L}_1} & \underset{\sim}{v}_{I_{\mathcal{T}}} & \underset{\sim}{v}_{I_{\mathcal{L}}} & \underset{\sim}{v}_{C} & \underset{\sim}{v}_{\mathcal{T}_1} \\
\end{array}
$$

$$
\left[\begin{array}{c|cccc}
 & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}_{VC} & \underset{\sim}{B}_{VV} \\
 & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & -\underset{\sim}{1} \\
\underset{\sim}{1} & \underset{\sim}{B}_{LI} & \underset{\sim}{0} & \underset{\sim}{B}_{LC} & \underset{\sim}{B}_{LV} \\
 & \underset{\sim}{B}_{II} & \underset{\sim}{1} & \underset{\sim}{B}_{IC} & \underset{\sim}{B}_{IV}
\end{array}\right] \tag{82}
$$

where $\mathcal{L}_1$ represents the inductors added. Since no resistors are added, $\hat{\Lambda}_R$ is the same as $\Lambda_R$. We compute the matrix $\hat{\underset{\sim}{\mathcal{H}}}_R(\hat{v}_R,\hat{i}_R)$ of (65) for $\hat{\mathcal{N}}$. It follows from (82) that $\hat{\underset{\sim}{B}}_{RR} = \underset{\sim}{0}$. Since $\hat{\underset{\sim}{v}}_{R_{\mathcal{L}}} = (\underset{\sim}{v}_{V_{\mathcal{L}}},\underset{\sim}{v}_{V_{\mathcal{T}}})$, $\hat{\underset{\sim}{v}}_{R_{\mathcal{T}}} = (\underset{\sim}{v}_{I_{\mathcal{T}}},\underset{\sim}{v}_{I_{\mathcal{L}}})$, $\hat{\underset{\sim}{i}}_R = (\underset{\sim}{i}_{V_{\mathcal{L}}},\underset{\sim}{i}_{V_{\mathcal{T}}})$ and $\hat{\underset{\sim}{i}}_{R_{\mathcal{T}}} = (\underset{\sim}{i}_{I_{\mathcal{T}}},\underset{\sim}{i}_{I_{\mathcal{L}}})$, we have

$$\hat{\underset{\sim}{\mathcal{H}}}_R(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) = [\underset{\sim}{D}_{(\underset{\sim}{v}_{I_{\mathcal{T}}},\underset{\sim}{v}_{I_{\mathcal{L}}})}\underset{\sim}{f}_R \vdots \underset{\sim}{D}_{(\underset{\sim}{i}_{V_{\mathcal{L}}},\underset{\sim}{i}_{V_{\mathcal{T}}})}\underset{\sim}{f}_R](\underset{\sim}{v}_R,\underset{\sim}{i}_R) \tag{83}$$

Observe that the matrix (83) depends only on $(\underset{\sim}{v}_R,\underset{\sim}{i}_R)$ and if $(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) \in \hat{\pi}_R(\hat{\Sigma})$, then $(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R) \in \Lambda_R$. Since (83) is obtained simply by exchanging columns of the matrix of (80), it follows from (79) that $\hat{\underset{\sim}{\mathcal{H}}}_R(\hat{\underset{\sim}{v}}_R,\hat{\underset{\sim}{i}}_R)$ is nonsingular. By Theorem 1, $\hat{\mathcal{N}}$ is well-posed. This proves (3). By Proposition 11, $\hat{\Lambda} \,\bar{\pitchfork}\, \hat{K}$. This proves (2). ◻

Example 6 Consider the circuit of Example 1. Since the resistors are voltage controlled, insertion of $\hat{C}_1$ and $\hat{C}_2$ yields well-posedness of the circuit. Therefore the perturbation in Example 1 was already good enough to ensure not only transversality but also well-posedness.

Example 7 Consider the circuit of Example 2. Inserting a small linear inductor in series, one can make the circuit well-posed.

Remark The number of reactive elements added in Theorem 2 is no greater than the number of reactive elements added in Proposition 5. Notice, however, that in Theorem 2, $\Lambda_R$ is required to be locally hybrid, whereas in Proposition 5, the only restriction on $\Lambda_R$ is that it should be an $n_R$-dimensional $C^2$ submanifold. The local hybridness assumption cannot be relaxed as the following example shows.

Example 8 Consider the circuit of Fig. 1(a) where the resistor constitutive relation is given by the unit circle $S^1$. (Fig. 5) It is easy to check $\Lambda \bar{\pitchfork} K$. In fact $\Sigma$ is diffeomorphic to $S^1$. Since $f_R(v_R, i_R) = v_R^2 + i_R^2 - 1$ and since $\mathcal{H}_R(v_R, i_R) = (D_{i_R} f_R)_{(v_R, i_R)}$ we see that $\det \mathcal{H}_R(v_R, i_R) = 0$ at points A and B. Therefore A and B are impasse points and hence the circuit is ill-posed. Observe that $\Lambda_R$ is not locally hybrid since there is no function $f_R$ satisfying (78) and (79) for a fixed matrix $\underset{\sim}{A}$. We claim that there is no way of making the circuit well-posed by adding reactive elements. To show this let $\hat{\mathcal{N}}$ be a circuit obtained by adding arbitrary number of reactive elements to the original circuit $\mathcal{N}$. Then, by (65), either $\hat{\mathcal{H}}_R(\hat{v}_R, \hat{i}_R) = (D_{i_R} f_R)_{(v_R, i_R)}$ or $\hat{\mathcal{H}}_R(\hat{v}_R, \hat{i}_R) = (D_{v_R} f_R)_{(v_R, i_R)}$ depending on how the reactive elements are added. In any case there are points such that $\det \hat{\mathcal{H}}(\hat{v}_R, \hat{i}_R) = 0$. Since well-posedness is coordinate free, $\hat{\mathcal{N}}$ is not well-posed with respect to any coordinate.

Remarks 1. Note that the perturbation in Theorem 2 is a network perturbation. It is not known if one can give element perturbations as in Proposition 3 in such a manner that $\hat{\mathcal{N}}$ is well-posed.

2. The above argument, of course, depends on the assumption that the incremental capacitance matrix and the incremental inductance matrix are nonsingular. If at least one of them is singular somewhere, then the above procedure does not work. To show this recall (6) and (54). If at least one of the incremental capacitance matrix or incremental inductance matrix is singular at a point $\underset{\sim}{\pi}(\underset{\sim}{v}, \underset{\sim}{i})$, then $\underset{\sim}{G}_{\underset{\sim}{\pi}(\underset{\sim}{v}, \underset{\sim}{i})}$ is singular. By (54), $\underset{\sim}{\pi}^* \underset{\sim}{G}_{(\underset{\sim}{v}, \underset{\sim}{i})}$ is singular even if $(d\underset{\sim}{\pi})_{(\underset{\sim}{v}, \underset{\sim}{i})}$ is an isomorphism. This singularity cannot be removed by adding reactive elements because the incremental capacitance or/and incremental inductance matrix of $\hat{\mathcal{N}}$ contains the original incremental matrix as a submatrix and therefore it is still singular. Consequently $\hat{\mathcal{N}}$ is ill-posed.

We next discuss relationship between well-posedness of $\mathcal{N}$ and transversality of $(n_C+n_L)$-port N derived from $\mathcal{N}$, under certain excitations. Replace capacitors and inductors of $\mathcal{N}$ by ports. The resulting network is called $(n_C+n_L)$-port N derived from $\mathcal{N}$. For the purpose of convenience we keep the same notation for N as $\mathcal{N}$. For simplicity, we assume that $\Lambda$ is represented by (20). Drive the capacitor ports by independent voltage sources $\underset{\sim}{v}_a^*$ and drive the inductor ports by independent current sources $\underset{\sim}{i}_b^*$. Let

$$\Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*) \triangleq \left\{(\underset{\sim}{v},\underset{\sim}{i}) \in \mathbb{R}^b \times \mathbb{R}^b \,\middle|\, (\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R, \underset{\sim}{v}_C = \underset{\sim}{v}_a^*, \underset{\sim}{i}_L = \underset{\sim}{i}_b^*\right\}. \tag{84}$$

This set represents the internal resistor constitutive relations of N under the excitation $(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*)$. Since $\Lambda$ is of dimension $2b-n_R$ and since $\underset{\sim}{v}_C = \underset{\sim}{v}_a^*$ and $\underset{\sim}{i}_L = \underset{\sim}{i}_b^*$ add $n_C+n_L$ more constraints, $\Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*)$ is a b-dimensional submanifold of $\mathbb{R}^b \times \mathbb{R}^b$. Recall $\underset{\sim}{\pi}$ of (9).

<u>Theorem 3</u> Given a nonlinear network $\mathcal{N}$, assume that $\Lambda$ is represented by (20) and let $\Sigma$ be an $(n_C+n_L)$-dimensional $C^2$ submanifold. Then $\mathcal{N}$ is well-posed if and only if for the $(n_C+n_L)$-port N derived from $\mathcal{N}$ the following holds:

$$\Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*) \pitchfork K \text{ for all } (\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*) \in \underset{\sim}{\pi}(\Sigma). \tag{85}$$

<u>Proof</u> <u>Sufficiency</u> Let $(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*) \in \underset{\sim}{\pi}(\Sigma)$ and let

$$\underset{\sim}{G}(\underset{\sim}{v},\underset{\sim}{i}) \triangleq \begin{bmatrix} \underset{\sim}{f}(\underset{\sim}{v},\underset{\sim}{i}) \\ \underset{\sim}{\pi}_{CL}(\underset{\sim}{v},\underset{\sim}{i}) - \begin{bmatrix} \underset{\sim}{v}_a^* \\ \underset{\sim}{i}_b^* \end{bmatrix} \end{bmatrix} \tag{86}$$

where $\underset{\sim}{f}$ is defined in (16). It is clear by the definition of $\Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*)$ that for each $(\underset{\sim}{v}_0,\underset{\sim}{i}_0) \in \Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*) \subset \Lambda$,

$$\left.\begin{aligned} &\Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*) \cap U = \underset{\sim}{G}^{-1}(\underset{\sim}{0}) \\ &\text{rank}(\underset{\sim}{D}\underset{\sim}{G})_{(\underset{\sim}{v},\underset{\sim}{i})} = b \text{ for all } (\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*) \cap U \end{aligned}\right\} \tag{87}$$

where U is as in (16). Using (85), (86) and an argument similar to that of the proof of <u>Proposition 1</u>, one sees that for each $(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*) \in \underset{\sim}{\pi}(\Sigma)$ and for $(\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*) \cap K$,

$$\text{rank}\begin{bmatrix} \underset{\sim}{B} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{Q} \\ \underset{\sim}{D}_{\underset{\sim}{v}}\underset{\sim}{f} & \underset{\sim}{D}_{\underset{\sim}{i}}\underset{\sim}{f} \\ \underset{\sim}{D}_{\underset{\sim}{v}}\underset{\sim}{\pi}_{CL} & \underset{\sim}{D}_{\underset{\sim}{i}}\underset{\sim}{\pi}_{CL} \end{bmatrix}_{(\underset{\sim}{v},\underset{\sim}{i})} = 2b. \tag{88}$$

But since

$$\Lambda \cap K = \bigcup_{(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*)\in\underset{\sim}{\pi}(\Sigma)} \Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*)\cap K \tag{89}$$

it follows from (88) that for each $(\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda \cap K$, (62) holds. This implies well-posedness.

Necessity If (62) holds for each $(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma$, it follows from (89) that (88) holds for each $(\underset{\sim}{v},\underset{\sim}{i}) \in \Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*)$. Since $(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*) \in \underset{\sim}{\pi}(\Sigma)$, (85) holds. ◻

Remark If $\Lambda$ is not represented by (20), the set $\Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*)$ may not be a submanifold. If we assume, however, that $(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*)$ is a regular value [11] of the map $\underset{\sim}{\pi}_{CL}|\Lambda : \Lambda \to \mathbb{R}^{n_C} \times \mathbb{R}^{n_L}$, then $\Lambda(\underset{\sim}{v}_a^*,\underset{\sim}{i}_b^*)$ is a b-dimensional submanifold.

## V. Eventual Strict Passivity

Eventual strict passivity is an important qualitative property of electrical networks, because it guarantees that all trajectories eventually approach a fixed compact subset of the configuration space [8-10]. Roughly speaking, the results of this section say the following: suppose that the resistors are eventually strictly passive and that every capacitor is in parallel with a large linear resistor and every inductor is in series with a small linear resistor. Then all trajectories approach a fixed compact subset of the configuration space. Since the above assumption is satisfied by most practical networks, this result guarantees that the voltage and current waveforms are bounded in most networks of practical interest.

In order to simplify our notation, we assume that $\Lambda$ is described by (20). The general case is not difficult to derive, however. Let $W_R' : \mathbb{R}^{n_R} \times \mathbb{R}^{n_R} \to \mathbb{R}$ be defined by

$$W_R'(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \triangleq \sum_{k=1}^{n_R} v_{R_k} i_{R_k}. \tag{90}$$

Recall the map $\underset{\sim}{\pi}_R'$ defined by (22). Let $W_R$ and $W$ be defined by Fig. 6. Clearly $W$ is the power dissipated by resistors. Let $E_{CL} : \mathbb{R}^{n_C} \times \mathbb{R}^{n_L} \to \mathbb{R}$ be defined by

$$E_{CL}(\underset{\sim}{v}_C,\underset{\sim}{i}_L) \triangleq \int_{\Gamma_C}^{\underset{\sim}{v}_C} \sum_{m,n=1}^{n_C} C_{mn}(\underset{\sim}{v}_C')v_{C_m}'\, dv_{C_n}' + \int_{\Gamma_L}^{\underset{\sim}{i}_L} \sum_{m,n=1}^{n_L} L_{mn}(\underset{\sim}{i}_L')i_{L_m}'\, di_{L_n}' \tag{91}$$

where $\Gamma_C$ (resp., $\Gamma_L$) denotes a smooth curve connecting $\underset{\sim}{v}_C$ (resp., $\underset{\sim}{i}_L$) with the origin of $\mathbb{R}^{n_C}$ (resp., $\mathbb{R}^{n_L}$). Since the incremental capacitance matrix and the incremental inductance matrix are always symmetric, $E_{CL}(\underset{\sim}{v}_C,\underset{\sim}{i}_L)$ does not depend on particular curves $\Gamma_C$ and $\Gamma_L$ but depend only on the end points $\underset{\sim}{v}_C$ and $\underset{\sim}{i}_L$. Therefore (91) is well defined. Set

$$E \triangleq E_{CL} \circ \underset{\sim}{\pi} \tag{92}$$

where $\underset{\sim}{\pi}$ is defined by (9). The function E is the <u>energy</u> stored in the memory elements. It follows from Tellegen's theorem that

$$\frac{dE(\underset{\sim}{v}(t),\underset{\sim}{i}(t))}{dt} = - W(\underset{\sim}{v}(t),\underset{\sim}{i}(t)). \tag{93}$$

Recall that a network $\mathcal{N}$ is said to be <u>eventually strictly passive</u> [8-10] if there is a compact subset $\Omega \subset \Sigma$ such that

$$W(\underset{\sim}{v},\underset{\sim}{i}) > 0 \text{ for all } (\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma - \Omega. \tag{94}$$

The following two propositions show the importance of eventual strict passivity.

<u>Proposition 12</u> [8-10] Let E be proper, i.e., for every $\alpha \geq 0$, the set $\{(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma | E(\underset{\sim}{v},\underset{\sim}{i}) \leq \alpha\}$ is a compact set, and let the network $\mathcal{N}$ be eventually strictly passive. Then the set defined by

$$\mathcal{E} \triangleq \{(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma | E(\underset{\sim}{v},\underset{\sim}{i}) \leq \alpha_1\} \tag{95}$$

$$\alpha_1 = \max_{(v,i)\in\Omega} E(\underset{\sim}{v},\underset{\sim}{i}) \tag{96}$$

is compact, and for any initial state $(\underset{\sim}{v}(0),\underset{\sim}{i}(0))$, either one of the following happens:

(i) there is a $t_1 > 0$ such that $(\underset{\sim}{v}(t),\underset{\sim}{i}(t)) \in \mathcal{E}$ for all $t \geq t_1$,

(ii) $(\underset{\sim}{v}(t),\underset{\sim}{i}(t)) \notin \mathcal{E}$ for all $t \geq 0$ and $\lim_{t\to\infty}(\underset{\sim}{v}(t),\underset{\sim}{i}(t)) \in \mathcal{E}$.

The set $\mathcal{E}$ contains most of the important information concerning the dynamics. In particular, the following holds.

<u>Proposition 13</u> Under the same setting as <u>Proposition 12</u>, we have

(i) all periodic orbits and equilibria are in $\mathcal{E}$,

(ii) in particular, equilibria lie in the set

$$\{(\underset{\sim}{v},\underset{\sim}{i}) \in \Sigma | W(\underset{\sim}{v},\underset{\sim}{i}) = 0\}. \tag{97}$$

<u>Proof</u> (i) It follows from (94) and the eventual strict passivity hypothesis that for any $(\underset{\sim}{v}(t),\underset{\sim}{i}(t)) \in \Sigma - \Omega$, the instantaneous energy $E(\underset{\sim}{v}(t),\underset{\sim}{i}(t))$ is strictly decreasing. This implies that for $(\underset{\sim}{v}(0),\underset{\sim}{i}(0)) \in \Sigma - \Omega$, the trajectory

$(\underset{\sim}{v}(t),\underset{\sim}{i}(t))$ cannot come back to $(\underset{\sim}{v}(0),\underset{\sim}{i}(0))$. Similarly, $(\underset{\sim}{v}(t),\underset{\sim}{i}(t))$ cannot remain at $(\underset{\sim}{v}(0),\underset{\sim}{i}(0))$, (ii) since $E(\underset{\sim}{v}(t),\underset{\sim}{i}(t))$ is either strictly increasing or strictly decreasing outside the set defined by (97), the equilibria must be located as specified in (97). ◻

The set $\mathcal{E}$ in (95) is sometimes called a <u>set of attraction</u> since it attracts all trajectories.

Eventual strict passivity is a property of W on $\Sigma$, while $W_R$ is defined on $\Lambda_R$. These two functions may behave very differently depending on the properties of $\underset{\sim}{\iota}$ and $\underset{\sim}{\pi}_R'$. (see Fig. 6) The properties of $W_R$ are easier to check than those of W because $W_R$ depends only on $\Lambda_R$. We need the following:

<u>Definition 2</u> The resistor constitutive relations represented by $\Lambda_R$ are said to be <u>eventually strictly passive</u> if there is a compact subset $\Omega_R$ of $\Lambda_R$ such that

$$W_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) > 0 \text{ for all } (\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R - \Omega_R. \tag{98}$$

As we will show shortly, (98) does not necessarily imply (94). Smale [4] poses a problem related to the present one. In terms of our terminology, the problem in [4] can be rephrased as follows: Suppose there is a number $\beta > 0$ such that

$$W_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \geq \beta \sum_{k=1}^{n_R} (v_{R_k}^{\;2}+i_{R_k}^{\;2}) \tag{99}$$

for all $(\underset{\sim}{v}_R,\underset{\sim}{i}_R)$ with $\|(\underset{\sim}{v}_R,\underset{\sim}{i}_R)\|$ sufficiently large. If condition (99) holds, does the network have a compact set of attraction? The answer is <u>no</u> as the following example shows.

<u>Example 9</u> Consider the circuit of Fig. 7(a) where all elements are linear and element values are all equal to one. Observe that

$$W_R(v_R,i_R) = v_R i_R = v_R^2 = i_R^2 \geq \beta(v_R^2+i_R^2) \tag{100}$$

for $0 < \beta < \frac{1}{2}$, $(v_R,i_R) \neq (0,0)$. Therefore (99) is satisfied. Notice that (98) is also satisfied. We claim that this circuit does not have a compact set of attraction. To this end let us write the dynamics in terms of $(v_{C_1},v_{C_2},i_{L_1},i_{L_2})$;

$$\begin{aligned} C_1 \frac{dv_{C_1}}{dt} &= i_{L_1}, \quad C_2 \frac{dv_{C_2}}{dt} = i_{L_2} \\ L_1 \frac{di_{L_1}}{dt} &= -v_{C_1} + R(-i_{L_1}+i_{L_2}), \quad L_2 \frac{di_{L_2}}{dt} = -v_{C_2} - R(-i_{L_1}+i_{L_2}). \end{aligned} \tag{101}$$

Since all element values are equal to one, for any $a \in \mathbb{R}$, the following is a solution to (101):

$$v_{C_1}(t) = a \sin t,\ v_{C_2}(t) = a \sin t,\ i_{L_1}(t) = a \cos t,\ i_{L_2}(t) = a \cos t.$$

Since $a \in \mathbb{R}$ is arbitrary, the solution can have an arbitrarily large magnitude. Consequently the circuit does not have a compact set of attraction. In fact, in terms of the above coordinate we have

$$W(\underset{\sim}{v}_C, \underset{\sim}{i}_L) = R(i_{L_2} - i_{L_1})^2 \qquad (102)$$

and hence it does not satisfy the inequality in (94). Fig. 7(b) shows a rough description of the trajectories. Notice that any trajectory starting outside the linear subspace $W = 0$, approaches the origin.

Since (98) is satisfied by most resistors of practical interest, it is natural for us to seek conditions under which (98) implies (94). The following is a generalization of a recent result by Chua and Green [8] for a general manifold. We assume that $\Lambda_R$ is closed for technical reasons. This is not a restrictive condition , however.

Lemma 1 Let $\Lambda_R$ be closed and eventually strictly passive. Let $\Sigma$ be an $(n_C+n_L)$-dimensional $C^2$ submanifold. Then $\mathcal{N}$ is eventually strictly passive if the following fundamental topological hypothesis is satisfied:

There are no loops and no cut sets consisting only of capacitors and inductors, or equivalently

(i) there is a tree $\mathcal{T}(R)$ consisting only of resistors.

(ii) there is a tree $\mathcal{T}(CL)$ containing all capacitors and inductors.

Proof Recall the map $\underset{\sim}{\pi}_R$ defined by (23). Suppose that $\Lambda_R$ is eventually strictly passive and let $\Omega_R$ be as in (98). If $\underset{\sim}{\pi}_R$ is proper, then the preimage $\underset{\sim}{\pi}_R^{-1}(\Omega_R)$ is compact because the preimage of a compact set under a proper map is compact. It is, then, clear that the inequality in (94) holds with respect to $\underset{\sim}{\pi}_R^{-1}(\Omega_R)$. So we show that the fundamental topological hypothesis implies that $\underset{\sim}{\pi}_R$ is proper. To this end let

$$\underset{\sim}{\imath}_K : \Sigma \to K \qquad (103)$$

be the inclusion map and consider the map $\underset{\sim}{\chi}$ defined by Fig. 8. Since $\Lambda_R$ is assumed to be closed, $\Lambda$ is also closed. Therefore $\Sigma = \Lambda \cap K$ is a closed submanifold of K. Consequently, for any compact subset A of K, the preimage $\underset{\sim}{\imath}_K^{-1}(A)$ is compact. This shows that $\underset{\sim}{\imath}_K$ is proper. Therefore we need only show that $\underset{\sim}{\chi}$ is proper. Since $\underset{\sim}{\chi}$ is obviously continuous, we need only show that the preimage of a bounded subset of $\mathbb{R}^{n_R} \times \mathbb{R}^{n_R}$ is bounded. Suppose that the fundamental topological hypothesis holds and let $\underset{\sim}{v}_R$ (resp., $\underset{\sim}{i}_R$) be the tree branch voltages (resp., link currents) for $\mathcal{J}(R)$ (resp., links associated with $\mathcal{J}(CL)$). It follows from (15) that for $(\underset{\sim}{v},\underset{\sim}{i}) \in K$,

$$\underset{\sim}{v} = \tilde{\underset{\sim}{Q}}^T \underset{\sim}{v}_R \quad , \quad \underset{\sim}{i} = \bar{\underset{\sim}{B}}^T \underset{\sim}{i}_R \tag{104}$$

where $\tilde{\underset{\sim}{Q}}$ and $\bar{\underset{\sim}{B}}$ are the fundamental cut set matrix and the fundamental loop matrix associated with $\mathcal{J}(C)$ and $\mathcal{J}(CL)$, respectively. Equation (104) implies

$$\|(\underset{\sim}{v},\underset{\sim}{i})\| \to \infty, \; (\underset{\sim}{v},\underset{\sim}{i}) \in K \Rightarrow \|(\underset{\sim}{v}_{R_{\mathcal{J}}}, \underset{\sim}{i}_{R_{\mathcal{L}}})\| \to \infty. \tag{105}$$

Since $(\underset{\sim}{v}_{R_{\mathcal{J}}}, \underset{\sim}{i}_{R_{\mathcal{L}}})$ is a subvector of $(\underset{\sim}{v}_R, \underset{\sim}{i}_R)$, we have

$$\|(\underset{\sim}{v},\underset{\sim}{i})\| \to \infty \; , \; (\underset{\sim}{v},\underset{\sim}{i}) \in K \Rightarrow \|(\underset{\sim}{v}_R, \underset{\sim}{i}_R)\| \to \infty. \tag{106}$$

This shows that the preimage of a bounded subset under $\underset{\sim}{\chi}$ is bounded. Since the properties of $\underset{\sim}{\chi}$ do not depend on a particular choice of a tree, $\underset{\sim}{\chi}$ is proper.¤

Remark Observe that in the above proof we took full advantage of the coordinate-free property, since in (104)-(106) we are using two different trees simultaneously.

Now, experiences tell us that most networks of practical interest have a compact set of attraction. We next justify this observation formally by carrying out a slight network perturbation. The perturbation we make is simply a formalization of the following hypothesis: "Every capacitor is in parallel with a large linear resistor and every inductor is in series with a small linear resistor." Before stating the results, we need the following:

Definition 3 A nonlinear network $\mathcal{N}$ is said to be strongly well-posed if

(i) there is a $C^2$ function $\underset{\sim}{f}_R : \mathbb{R}^{n_R} \times \mathbb{R}^{n_R} \to \mathbb{R}^{n_R}$ such that $\Lambda_R = \underset{\sim}{f}_R^{-1}(\underset{\sim}{0})$ and $\text{rank}(D\underset{\sim}{f}_R)_{(\underset{\sim}{v}_R,\underset{\sim}{i}_R)} = n_R$ for all $(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R$.

(ii) $\det \underset{\sim}{\mathcal{H}}_R(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \neq 0$ for all $(\underset{\sim}{v}_R,\underset{\sim}{i}_R) \in \Lambda_R$. (107)

Remark Condition (ii) is stronger than (64) because the determinant should be nonzero on $\Lambda_R$ and since $\pi_R(\Sigma) \subset \Lambda_R$. This condition is satisfied by many networks, however. For example, the circuit of Fig. 1 with the capacitors added, satisfies this condition because

$$\mathcal{H}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = D_{\underset{\sim}{i}_R} \underset{\sim}{f}_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

The perturbed network $\hat{\mathcal{N}}$ of Theorem 2 is strongly well-posed because the matrix of (83) is nonsingular for all $(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R$.

Theorem 4 Given a nonlinear network $\mathcal{N}$ assume the following:

(i) $\mathcal{N}$ is strongly well-posed.

(ii) $\Lambda_R$ is closed and eventually strictly passive.

Insert a large linear resistor $g_k$, $k = 1, \ldots, n_C$, in parallel with every capacitor and insert a small linear resistor $r_k$, $k = 1, \ldots, n_L$, in series with every inductor. Then

(1) $\hat{\Lambda} \cap \hat{K} \neq \phi$ and $\hat{\Sigma} = \hat{\Lambda} \cap \hat{K}$ is an $(n_C + n_L)$ - dimensional $C^2$ submanifold.

(2) $\hat{\mathcal{N}}$ is well-posed.

(3) $\hat{\mathcal{N}}$ is eventually strictly passive. Consequently $\hat{\mathcal{N}}$ has a compact set of attraction.

Proof (1) In terms of a proper tree $\mathcal{T}$, the original network $\mathcal{N}$ is described by

$$\underset{\sim}{v}_{R_{\mathcal{L}}} + \underset{\sim}{B}_{RR} \underset{\sim}{v}_{R_{\mathcal{T}}} + \underset{\sim}{B}_{RC} \underset{\sim}{v}_C = \underset{\sim}{0} \tag{108}$$

$$\underset{\sim}{v}_L + \underset{\sim}{B}_{LR} \underset{\sim}{v}_{R_{\mathcal{T}}} + \underset{\sim}{B}_{LC} \underset{\sim}{v}_C = \underset{\sim}{0} \tag{109}$$

$$\underset{\sim}{i}_{R_{\mathcal{T}}} - \underset{\sim}{B}_{RR}^T \underset{\sim}{i}_{R_{\mathcal{L}}} - \underset{\sim}{B}_{LR}^T \underset{\sim}{i}_L = \underset{\sim}{0} \tag{110}$$

$$\underset{\sim}{i}_C - \underset{\sim}{B}_{RC}^T \underset{\sim}{i}_{R_{\mathcal{L}}} - \underset{\sim}{B}_{LC}^T \underset{\sim}{i}_L = \underset{\sim}{0} \tag{111}$$

$$(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R. \tag{112}$$

Let $\hat{\mathcal{T}} \triangleq \mathcal{T} \cup \mathcal{T}_r$ where $\mathcal{T}_r$ represents branches of $r_k$'s. It is clear that $\hat{\mathcal{T}}$ is a proper tree for $\hat{\mathcal{N}}$. Decompose $\hat{\underset{\sim}{v}}$ and $\hat{\underset{\sim}{i}}$ as

$$\hat{\underset{\sim}{v}} = (\underset{\sim}{v}_{R_{\mathcal{L}}}, \underset{\sim}{v}_g, \underset{\sim}{v}_L \vdots \underset{\sim}{v}_{R_{\mathcal{T}}}, \underset{\sim}{v}_r, \underset{\sim}{v}_C)$$

$$\hat{\underset{\sim}{i}} = (\underset{\sim}{i}_{R_{\mathcal{L}}}, \underset{\sim}{i}_g, \underset{\sim}{i}_L \vdots \underset{\sim}{i}_{R_{\mathcal{J}}}, \underset{\sim}{i}_r, \underset{\sim}{i}_C)$$

where g and r denote the variables associated with $g_k$'s and $r_k$'s. Then $\hat{\mathcal{N}}$ is described by

$$\underset{\sim}{v}_{R_{\mathcal{L}}} + \underset{\sim}{B}_{RR}\underset{\sim}{v}_{R_{\mathcal{J}}} + \underset{\sim}{B}_{RC}\underset{\sim}{v}_C = \underset{\sim}{0} \tag{113}$$

$$\underset{\sim}{v}_L + \underset{\sim}{B}_{LR}\underset{\sim}{v}_{R_{\mathcal{J}}} + \underset{\sim}{B}_{LC}\underset{\sim}{v}_C + \underset{\sim}{v}_r = \underset{\sim}{0} \tag{114}$$

$$\underset{\sim}{v}_g + \underset{\sim}{v}_C = \underset{\sim}{0} \tag{115}$$

$$\underset{\sim}{i}_{R_{\mathcal{J}}} - \underset{\sim}{B}_{RR}^T\underset{\sim}{i}_{R_{\mathcal{L}}} - \underset{\sim}{B}_{LR}^T\underset{\sim}{i}_L = \underset{\sim}{0} \tag{116}$$

$$\underset{\sim}{i}_C - \underset{\sim}{B}_{RC}^T\underset{\sim}{i}_{R_{\mathcal{L}}} - \underset{\sim}{B}_{LR}^T\underset{\sim}{i}_L - \underset{\sim}{i}_g = \underset{\sim}{0} \tag{117}$$

$$\underset{\sim}{i}_r - \underset{\sim}{i}_L = \underset{\sim}{0} \tag{118}$$

$$(\underset{\sim}{v}_R, \underset{\sim}{i}_R) \in \Lambda_R \tag{119}$$

$$\underset{\sim}{i}_g = \underset{\sim}{g}^{-1}\underset{\sim}{v}_g \tag{120}$$

$$\underset{\sim}{v}_r = \underset{\sim}{r}\underset{\sim}{i}_r \tag{121}$$

where

$$\underset{\sim}{g} \triangleq \text{diag}(g_1, \ldots, g_{n_C}),\ \underset{\sim}{r} = \text{diag}(r_1, \ldots, r_{n_L}) \tag{122}$$

Eliminating $\underset{\sim}{v}_g$, $\underset{\sim}{v}_r$, $\underset{\sim}{i}_g$ and $\underset{\sim}{i}_r$, we see that $\hat{\mathcal{N}}$ is described by (113), (116), (119) and

$$\underset{\sim}{v}_L + \underset{\sim}{B}_{LR}\underset{\sim}{v}_{R_{\mathcal{J}}} + \underset{\sim}{B}_{LC}\underset{\sim}{v}_C + \underset{\sim}{r}\underset{\sim}{i}_L = \underset{\sim}{0} \tag{123}$$

$$\underset{\sim}{i}_C - \underset{\sim}{B}_{RC}^T\underset{\sim}{i}_{R_{\mathcal{L}}} - \underset{\sim}{B}_{LR}^T\underset{\sim}{i}_L + \underset{\sim}{g}^{-1}\underset{\sim}{v}_C = \underset{\sim}{0} \tag{124}$$

$$\underset{\sim}{v}_g = -\underset{\sim}{v}_C \tag{125}$$

$$\underset{\sim}{i}_g = -\underset{\sim}{g}^{-1}\underset{\sim}{v}_C \tag{126}$$

$$\underset{\sim}{v}_r = \underset{\sim}{r}\underset{\sim}{i}_L \tag{127}$$

$$\underset{\sim}{i}_r = \underset{\sim}{i}_L. \tag{128}$$

Let us rewrite (108)-(112) and (123)-(128) more concisely. Let $\underset{\sim}{B}$ and $\underset{\sim}{Q}$ be the fundamental loop matrix and the fundamental cut set matrix for $\hat{\mathcal{N}}$, respectively.

Then, of course, (108)-(112) are written as

$$\begin{bmatrix} \tilde{Q}^T & \tilde{0} \\ \tilde{0} & \tilde{B}^T \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{i} \end{bmatrix} = \tilde{0} \tag{129}$$

$$(\tilde{v}_R, \tilde{i}_R) \in \Lambda_R. \tag{130}$$

Comparing (108)-(112) with (113), (116), (119), (123)-(130), we see that the differences between $\mathcal{N}$ and $\hat{\mathcal{N}}$ are in the last two terms of (123), (124) and (125)-(128). Therefore $\hat{\mathcal{N}}$ is described by

$$\begin{bmatrix} \tilde{Q}^T & \tilde{0} \\ \tilde{0} & \tilde{B}^T \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{i} \end{bmatrix} + \begin{bmatrix} \tilde{0} & \tilde{F} \\ \tilde{G} & \tilde{0} \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{i} \end{bmatrix} = \tilde{0} \tag{131}$$

$$(\tilde{v}_R, \tilde{i}_R) \in \Lambda_R \tag{132}$$

$$(\tilde{v}_g, \tilde{v}_r, \tilde{i}_g, \tilde{i}_r) = \tilde{H} \begin{bmatrix} \tilde{v}_C \\ \tilde{i}_L \end{bmatrix} \tag{133}$$

where

$$\tilde{F} = \begin{array}{c} \begin{array}{cccc} \tilde{i}_{R_\mathcal{I}} & \tilde{i}_{L_\mathcal{J}} & \tilde{i}_{R_\mathcal{J}} & \tilde{i}_C \end{array} \\ \left[\begin{array}{cc:cc} \tilde{0} & \tilde{0} & \tilde{0} & \tilde{0} \\ \tilde{0} & \tilde{r} & \tilde{0} & \tilde{0} \end{array}\right] \end{array}, \quad \tilde{G} = \begin{array}{c} \begin{array}{cccc} \tilde{v}_{R_\mathcal{I}} & \tilde{v}_L & \tilde{v}_{R_\mathcal{J}} & \tilde{v}_C \end{array} \\ \left[\begin{array}{cc:cc} \tilde{0} & \tilde{0} & \tilde{0} & \tilde{0} \\ \tilde{0} & \tilde{0} & \tilde{0} & \tilde{g}^{-1} \end{array}\right] \end{array} \tag{134}$$

$$H = \begin{bmatrix} -\tilde{1} & \tilde{0} \\ \tilde{0} & \tilde{r} \\ -\tilde{g}^{-1} & \tilde{0} \\ \tilde{0} & \tilde{1} \end{bmatrix} . \tag{135}$$

Now let $(\tilde{v}_0, \tilde{i}_0) \in \Lambda \cap K \neq \phi$ and let U be a bounded neighborhood of $(\tilde{v}_0, \tilde{i}_0)$ in $\mathbb{R}^b \times \mathbb{R}^b$. Since the set $(\Lambda \cap K) \cap U$ is a bounded submanifold, small perturbations of $K \cap U$ do not destroy transversality of $\Lambda \cap U$ and $K \cap U$ and hence they do not destroy nonemptiness of $(\Lambda \cap U) \cap (K \cap U)$. Therefore if $g_k$ is large enough and if $r_k$ is small enough, then $\|\tilde{F}\|$ and $\|\tilde{G}\|$ in (131) are small enough to guarantee nonemptiness of the intersection of (131) and (132). Since (133) does not destroy this nonemptiness, we have $\hat{\Lambda} \cap \hat{K} \neq \phi$. We show $\hat{\Lambda} \overline{\pitchfork} \hat{K}$ later.

(2) Observe that

$$\hat{f}_R(\hat{v}_R, \hat{i}_R) = \begin{bmatrix} f_R(v_R, i_R) \\ i_g - g^{-1} v_g \\ v_r - r i_r \end{bmatrix} \tag{136}$$

and that the fundamental loop matrix $\hat{B}$ for $\hat{\mathcal{N}}$ is given by

$$\begin{array}{ccc:ccc} v_{R_\mathcal{L}} & v_L & v_g & v_{R_\mathcal{J}} & v_C & v_r \end{array}$$

$$\left[\begin{array}{ccc:ccc} & & & B_{RR} & B_{RC} & 0 \\ & 1 & & B_{LR} & B_{LC} & 1 \\ & & & 0 & 1 & 0 \end{array}\right] . \tag{137}$$

It follows from this that

$$\hat{B}_{RR} = \begin{array}{c} \begin{array}{cc} v_{R_\mathcal{J}} & v_r \end{array} \\ \begin{bmatrix} B_{RR} & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} v_{R_\mathcal{L}} \\ v_g \end{array}, \quad B_{RC} = \begin{array}{c} v_C \\ \begin{bmatrix} B_{RC} \\ 1 \end{bmatrix} \end{array} \begin{array}{c} v_{R_\mathcal{L}} \\ v_g \end{array}, \tag{138}$$

$$\hat{B}_{LR} = \begin{array}{c} \begin{array}{cc} v_{R_\mathcal{J}} & v_r \end{array} \\ \begin{bmatrix} B_{LR} & 1 \end{bmatrix} \end{array} \; v_L .$$

Let $\hat{v}_{R_\mathcal{L}} = \left(v_R, v_g\right)$, $\hat{v}_{R_\mathcal{J}} = \left(v_{R_\mathcal{J}}, v_r\right)$, $\hat{i}_{R_\mathcal{L}} = \left(i_{R_\mathcal{L}}, i_g\right)$, and $\hat{i}_{R_\mathcal{J}} = \left(i_{R_\mathcal{J}}, i_r\right)$. Then

$$D_{\hat{v}_{R_\mathcal{J}}} \hat{f}_R - \left(D_{\hat{v}_{R_\mathcal{L}}} \hat{f}_R\right) \hat{B}_{RR}$$

$$= \begin{bmatrix} D_{v_{R_\mathcal{J}}} f_R & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} D_{v_{R_\mathcal{L}}} f_R & 0 \\ 0 & -g^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_{RR} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} D_{\underset{\sim}{v}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R - \left(D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R\right) \underset{\sim}{B}_{RR} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} \end{bmatrix}, \tag{139}$$

$$D_{\hat{\underset{\sim}{i}}_{R_{\mathcal{L}}}} \hat{\underset{\sim}{f}}_R + \left(D_{\hat{\underset{\sim}{i}}_{R_{\mathcal{J}}}} \hat{\underset{\sim}{f}}_R\right) \underset{\sim}{B}_{RR}^T$$

$$= \begin{bmatrix} D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} + \begin{bmatrix} D_{\underset{\sim}{i}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & -\underset{\sim}{r} \end{bmatrix} \begin{bmatrix} \underset{\sim}{B}_{RR}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix}$$

$$= \begin{bmatrix} D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R + \left(D_{\underset{\sim}{i}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R\right) \underset{\sim}{B}_{RR}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \tag{140}$$

It follows from (139), (140) and (65) that

$$\hat{\underset{\sim}{\mathcal{H}}}_R(\hat{\underset{\sim}{v}}_R, \hat{\underset{\sim}{i}}_R) = \left[\begin{array}{cc|cc} D_{\underset{\sim}{v}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R - \left(D_{\underset{\sim}{v}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R\right) \underset{\sim}{B}_{RR} & \underset{\sim}{0} & D_{\underset{\sim}{i}_{R_{\mathcal{L}}}} \underset{\sim}{f}_R + \left(D_{\underset{\sim}{i}_{R_{\mathcal{J}}}} \underset{\sim}{f}_R\right) \underset{\sim}{B}_{RR}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} \\ \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} \end{array}\right] (\hat{\underset{\sim}{v}}_R, \hat{\underset{\sim}{i}}_R). \tag{141}$$

It is clear that $\hat{\underset{\sim}{\mathcal{H}}}_R$ depends only on $(\underset{\sim}{v}_R, \underset{\sim}{i}_R)$ and that

$$|\det \hat{\underset{\sim}{\mathcal{H}}}_R(\hat{\underset{\sim}{v}}_R, \hat{\underset{\sim}{i}}_R)| = |\det \hat{\underset{\sim}{\mathcal{H}}}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R)|. \tag{142}$$

Now if $(\hat{v}_R,\hat{i}_R) \in \hat{\Lambda}_R$, then $(v_R,i_R) \in \Lambda_R$, because

$$\hat{\Lambda}_R = \{(\hat{v}_R,\hat{i}_R) \mid (v_R,i_R) \in \Lambda_R, i_g = g^{-1} v_g, v_r = r i_r\}. \tag{143}$$

By the strong well-posedness assumption, we have

$$|\det \mathcal{H}_R(v_R,i_R)| > 0 \text{ for all } (v_R,i_R) \in \Lambda_R. \tag{144}$$

This and (142) imply

$$|\det \hat{\mathcal{H}}_R(\hat{v}_R,\hat{i}_R)| > 0 \text{ for all } (\hat{v}_R,\hat{i}_R) \in \hat{\Pi}_R(\hat{\Sigma}). \tag{145}$$

It follows from Corollary 5 that $\hat{\mathcal{N}}$ is well-posed. By Proposition 11 we have $\hat{\Lambda} \pitchfork \hat{K}$. Therefore $\Sigma$ is an $(n_C+n_L)$-dimensional $C^2$ submanifold.

(3) The resistor constitutive relations $\hat{\Lambda}_R$ for $\hat{\mathcal{N}}$ is described by (143) where $\hat{v}_R = (v_R,v_g,v_r)$, $\hat{i}_R = (i_R,i_g,i_r)$. Therefore the function $\hat{W}_R$ corresponding to $W_R$ is given by

$$\hat{W}_R(\hat{v}_R,\hat{i}_R) = W_R(v_R,i_R) + v_g^T g^{-1} v_g + i_r^T r i_r. \tag{146}$$

It follows from condition (ii) that there is a compact set $\Omega_R \subset \Lambda_R$ such that (96) holds. For any $\alpha > 0$, let

$$\Omega_{gr} \triangleq \left\{ (v_g,v_r,i_g,i_r) \;\middle|\; \begin{array}{l} i_g = g^{-1} v_g, v_r = r i_r \\ \|(v_g,v_r,i_g,i_r)\| \le \alpha \end{array} \right\}. \tag{147}$$

Then the set $\hat{\Omega}_R \triangleq \Omega_R \times \Omega_{gr}$ has the property that

$$\hat{W}_R > 0 \text{ on } \hat{\Lambda}_R - \hat{\Omega}_R \tag{148}$$

because $g$ and $r$ are diagonal matrices with positive elements. Finally, to show that the fundamental topological hypothesis is satisfied, let $\mathcal{T}_R, \mathcal{T}_L$, $\mathcal{T}_r$ and $\mathcal{T}_g$ represent the branches of the resistors in $\mathcal{T}$, inductors in $\mathcal{L}$, the added resistors $r_k$'s and the added resistors $g_k$'s. Then $\hat{\mathcal{T}}(R) \triangleq \mathcal{T}_R \cup \mathcal{T}_r \cup \mathcal{L}_g$ is a tree for $\hat{\mathcal{N}}$ which consists only of resistors. Also $\hat{\mathcal{T}}(CL) = \mathcal{T} \cup \mathcal{T}_L$ is a tree for $\hat{\mathcal{N}}$ which contains all capacitors and inductors.

It follows from Lemma 1 that $\hat{\mathcal{N}}$ is eventually strictly passive. ◻

Example 10 Consider the network of Example 9. Since the circuit is linear, all the conditions of Theorem 4 are satisfied. The perturbed network is shown in Fig. 9. It follows from Theorem 4 that this network has a compact set of attraction. In fact the linear subspace W = 0 in Fig. 7(b) degenerates into the origin and any closed ball centered at the origin serves as a compact set of attraction.

Remark As we have seen, $\hat{\mathcal{H}}_R(\hat{v}_R,\hat{i}_R) = \mathcal{H}_R(v_R,i_R)$ for $(\hat{v}_R,\hat{i}_R) \in \hat{\pi}_R(\hat{\Sigma})$. But $(\hat{v}_R,\hat{i}_R) \in \hat{\pi}_R(\hat{\Sigma})$ does not necessarily imply $(v_R,i_R) \in \pi_R(\Sigma)$ even though $(v_R,i_R) \in \Lambda_R$. Recalling (64) and (107), one sees why we needed the strong well-posedness hypothesis.

We next replace strong well-posedness by another condition.

Theorem 5 Replace the "strong well-posedness" hypothesis in Theorem 4 by the following hypothesis:

(i)' $\pi$ is global diffeomorphism.

Then, under the same perturbation as in Theorem 4, the same conclusion holds.

Proof The preceding proof for Theorem 4 remains applicable except for the fact that $\Sigma$ is an $(n_C+n_L)$-dimensional $C^2$ submanifold and that $\hat{\mathcal{N}}$ is well-posed. In order to prove this, recall (108)-(112). Assumption (i)' implies that $(v,i)$ is expressible as a $C^2$ function of $(v_C,i_L)$;

$$(v,i) = \pi^{-1}(v_C,i_L).$$

Recall (113)-(121) and set $v_L' \triangleq v_L + v_r, i_C' \triangleq i_C - i_g$. Then (113), (114), (116), (117) and (119) are exactly the same as (108)-(112). Therefore

$$(v_R,v_C,v_L',i_R,i_C',i_L) = \pi^{-1}(v_C,i_L).$$

It follows from (115), (118), (120) and (121) that $(v_r,v_g,i_r,i_g)$ is also a $C^2$ function of $(v_C,i_L)$. Therefore all variables of $\hat{\mathcal{N}}$ are expressible as a $C^2$ function of $(v_C,i_L)$;

$$(\hat{v},\hat{i}) = \hat{\pi}^{-1}(v_C,i_L).$$

It follows from the way $\hat{\pi}^{-1}$ was determined that $\hat{\pi}^{-1}$ is a global diffeomorphism and hence so is $\hat{\pi}$. Therefore $\hat{\Sigma}$ is an $(n_C+n_L)$-dimensional $C^2$ submanifold.

Since $\hat{\pi}$ is a global diffeomrophism, it is a local diffeomorphism. It follows from Proposition 9 that $\hat{\mathcal{N}}$ is well-posed. ¤

Example 11 Consider the network of Fig. 10(a) where the resistor is described by Fig. 10(b). The resistor is eventually strictly passive. It is easy to show that $\pi$ is a global diffeomorphism. Therefore we can make the same perturbation as in Example 10 so that the network will have a compact set of attraction.

We will next relax the "strong well-posedness" hypothesis and the global diffeomorphism assumption, while imposing a stronger condition on $\Lambda_R$ to derive a different perturbation result. Recall that $\Lambda_R$ is said to be globally hybrid [3] if

$$\Lambda_R = \{(v_R, i_R) \mid y = h(x)\} \tag{149}$$

where $y = (y_1, \ldots, y_{n_R})$, $x = (x_1, \ldots, x_{n_R})$ and if $y_k$ is the current (resp., voltage) of the k-th resistor then $x_k$ is the voltage (resp., current) of the k-th resistor. If $y_k$ is the current (resp., voltage), then that particular resistor is called voltage controlled (resp., current controlled). The following result says that most practical networks can be perturbed in such a manner that the resulting network is well-posed and has a compact set of attraction.

Theorem 6 Given a nonlinear network $\mathcal{N}$ assume the following:

(i) $\Lambda_R$ is closed and is globally hybrid.

(ii) $\Lambda_R$ is eventually strictly passive.

(iii) $\Lambda \cap K \neq \phi$.

Perturb $\mathcal{N}$ in the following manner:

(a) Let $\mathcal{T}$ be a proper tree containing a maximum number of voltage controlled resistors and a minimum number of current controlled resistors and let $\mathcal{L}$ be its associated cotree. Insert a small linear capacitor in parallel with each voltage controlled resistor in $\mathcal{T}$ and insert a small linear inductor in series with each current controlled resistor in $\mathcal{L}$. Call the resulting network $\hat{\mathcal{N}}$

(b) Insert a large linear resistor $g_k$ in parallel with each capacitor of $\hat{\mathcal{N}}$ and insert a small linear resistor $r_k$ in series with each inductor of $\hat{\mathcal{N}}$. Call the resulting network $\bar{\mathcal{N}}$.

Then the following hold:

(1) $\bar{\Lambda} \cap \bar{K} \neq \phi$ and $\bar{\Sigma} = \bar{\Lambda} \cap \bar{K}$ is an $(n_C+n_L+k)$-dimensional $C^2$ submanifold where

k is the number of reactive elements added, $\bar{\Lambda}$ and $\bar{K}$ are the resistor constitutive relations and the Kirchhoff space of $\bar{\mathcal{N}}$, respectively.

(2) $\bar{\mathcal{N}}$ is well-posed.

(3) $\bar{\mathcal{N}}$ is eventually strictly passive. Consequently $\bar{\mathcal{N}}$ has a compact set of attraction.

Proof (1) It is clear that one can prove $\bar{\Lambda} \cap \bar{K} \neq \phi$ in a similar manner to the proof of Theorem 4. We prove $\bar{\Lambda} \pitchfork \bar{K}$ later.

(2), (3) We first claim that $\hat{\mathcal{N}}$ is strongly well-posed. To this end partition $(\underset{\sim}{v},\underset{\sim}{i})$ of $\mathcal{N}$ as in the proof of Theorem 4. Since $\Lambda_R$ is assumed to be globally hybrid, it can be represented as follows:

$$\underset{\sim}{i}_{V_{\mathcal{J}}} - \underset{\sim}{f}_{V_{\mathcal{J}}}\left(\underset{\sim}{v}_{V_{\mathcal{J}}}, \underset{\sim}{v}_{V_{\mathcal{L}}}, \underset{\sim}{i}_{I_{\mathcal{J}}}, \underset{\sim}{i}_{I_{\mathcal{L}}}\right) = \underset{\sim}{0} \tag{150}$$

$$\underset{\sim}{i}_{V_{\mathcal{L}}} - \underset{\sim}{f}_{V_{\mathcal{L}}}\left(\underset{\sim}{v}_{V_{\mathcal{J}}}, \underset{\sim}{v}_{V_{\mathcal{L}}}, \underset{\sim}{i}_{I_{\mathcal{J}}}, \underset{\sim}{i}_{I_{\mathcal{L}}}\right) = \underset{\sim}{0} \tag{151}$$

$$\underset{\sim}{v}_{I_{\mathcal{J}}} - \underset{\sim}{f}_{I_{\mathcal{J}}}\left(\underset{\sim}{v}_{V_{\mathcal{J}}}, \underset{\sim}{v}_{V_{\mathcal{L}}}, \underset{\sim}{i}_{I_{\mathcal{J}}}, \underset{\sim}{i}_{I_{\mathcal{L}}}\right) = \underset{\sim}{0} \tag{152}$$

$$\underset{\sim}{v}_{I_{\mathcal{L}}} - \underset{\sim}{f}_{I_{\mathcal{L}}}\left(\underset{\sim}{v}_{V_{\mathcal{J}}}, \underset{\sim}{v}_{V_{\mathcal{L}}}, \underset{\sim}{i}_{I_{\mathcal{J}}}, \underset{\sim}{i}_{I_{\mathcal{L}}}\right) = \underset{\sim}{0} \tag{153}$$

where V and I denote voltage controlled and current controlled resistors, respectively. We write these equations as

$$\underset{\sim}{f}_R(\underset{\sim}{v}_R, \underset{\sim}{i}_R) = \underset{\sim}{0}. \tag{154}$$

It follows from (83) that for $\hat{\mathcal{N}}$ we have

$$\underset{\sim}{\hat{\mathcal{H}}}_R(\hat{\underset{\sim}{v}}_R, \hat{\underset{\sim}{i}}_R) = \left[\underset{\sim}{D}_{(\underset{\sim}{v}_{I_{\mathcal{J}}}, \underset{\sim}{v}_{I_{\mathcal{L}}})}\underset{\sim}{f}_R \vdots \underset{\sim}{D}_{(\underset{\sim}{i}_{V_{\mathcal{L}}}, \underset{\sim}{i}_{V_{\mathcal{J}}})}\underset{\sim}{f}_R\right]_{(\hat{\underset{\sim}{v}}_R, \hat{\underset{\sim}{i}}_R)}$$

$$= \begin{array}{c} \begin{matrix} \quad \underset{\sim}{v}_{I_{\mathcal{J}}} & \underset{\sim}{v}_{I_{\mathcal{L}}} & \underset{\sim}{i}_{V_{\mathcal{L}}} & \underset{\sim}{i}_{V_{\mathcal{J}}} \end{matrix} \\ \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \end{array}. \tag{155}$$

Therefore $\mathcal{H}_R(\hat{v}_R, \hat{i}_R)$ is a constant nonsingular matrix. Therefore $\hat{\mathcal{N}}$ is strongly well-posed. Clearly $\hat{\Lambda}_R = \Lambda_R$ because no resistors are added in (a). This implies that $\hat{\Lambda}_R$ is eventually strictly passive. Since $\hat{\mathcal{N}}$ satisfies the hypotheses of Theorem 5, by taking procedure (b), we obtain $\bar{\mathcal{N}}$ which is well-posed, $\bar{\Lambda} \bar{\pitchfork} \bar{K}$, and eventually passive. ¤

Example 12 Consider the network of Fig. 11(a), where $R_1$ and $R_2$ are as in Fig. 1(b). Other elements are linear. By a similar reasoning to that of Example 1, one can show that $\Lambda \not\pitchfork K$. Pick the proper tree $\mathcal{T} = \{C_1, C_2, C_3, R_2\}$. Then applying procedure (a) of Theorem 6, we obtain $\hat{\mathcal{N}}$ which is strongly well-posed (Fig. 11(b)). The network $\hat{\mathcal{N}}$ of Fig. 11(b) does not satisfy the fundamental topological hypothesis, however, because there is a capacitor-only cut set. Insert large linear resistors $g_1$, $g_2$ and $g_4$ according to procedure (b) and obtain $\bar{\mathcal{N}}$ (Fig. 11(c)). Theorem 6 says that $\bar{\mathcal{N}}$ has a compact set of attraction.

Remark The elements added in Theorem 6 can be thought of as parasitic elements of $\mathcal{N}$. Therefore Theorem 6 formally justifies the fact that in most networks of practical interest, voltage and current waveforms eventually approach a fixed compact set.

APPENDIX

Proof of Proposition 3 The proof is similar to that of Theorem 3 of [3] which is the same as the proof of (ii-a) of Theorem 2 of [3]. Proof of (ii-a) of Theorem 2 uses Lemmas 1, 2 and 4 of [3]. It is easy to show that Lemma 1 is true for $C^2$ submanifolds. Lemma 2 has nothing to do with differentiability. Therefore we need to only show that Lemma 4 is true in the $C^2$ category. We state this in the following:

Lemma A Let $\underset{\sim}{A}$ be an n×n matrix such that $\|\underset{\sim}{A}-\underset{\sim}{1}\|$ is sufficiently small. Then there are neighborhoods $U_1$ and $U_2$ of the origin of $\mathbb{R}^n$ with $\bar{U}_1 \subset U_2$ and there is a $C^2$ function $\underset{\sim}{G} : \mathbb{R}^n \to \mathbb{R}^n$ such that

(i) $\underset{\sim}{G} = \underset{\sim}{A}$ on $U_1$

(ii) $\underset{\sim}{G} = i_d$ off $U_2$, where $i_d$ is the identity map of $\mathbb{R}^n$

(iii) $\underset{\sim}{G}$ is arbitrarily close to $i_d$ in the strong $C^2$ topology.

Proof Let $\mathcal{U}^2(i_d;\varepsilon(\cdot))$ be a sufficiently small neighborhood of $i_d$ in $C^2(\mathbb{R}^n, \mathbb{R}^n)$ with respect to the strong $C^2$ topology. Since $\varepsilon(\underset{\sim}{x}) > 0$ for all $\underset{\sim}{x} \in \mathbb{R}^n$, there are numbers $\varepsilon > 0$ and $\delta > 0$ such that $\varepsilon(\underset{\sim}{x}) \geq \varepsilon$ for all $\underset{\sim}{x}$ with $\|\underset{\sim}{x}\| < \delta$. Let $\delta_o$ satisfy $0 < \delta_o < \delta$. Then there is a $C^2$ function (bump function [11]) $\mu : \mathbb{R}^n \to \mathbb{R}$ such that

$$\text{(i)} \quad \mu(\underset{\sim}{x}) = \begin{cases} 1 \text{ if } \|\underset{\sim}{x}\| < \delta_o \\ 0 \text{ if } \|\underset{\sim}{x}\| \geq \delta \end{cases} \tag{A.1}$$

(ii) there is a $k > 0$ such that

$$\|(\underset{\sim}{D}\mu)_{\underset{\sim}{x}}\| < k, \quad \|(\underset{\sim}{D}^2\mu)_{\underset{\sim}{x}}\| < k \tag{A.2}$$

for all $\underset{\sim}{x} \in \mathbb{R}^n$.

Now, choose $\underset{\sim}{A}$ close enough to $i_d$ so that

$$\|\underset{\sim}{A}-\underset{\sim}{1}\| < \frac{\varepsilon}{(1+\delta)(1+2k)} \tag{A.3}$$

and define

$$\underset{\sim}{G}(\underset{\sim}{x}) \triangleq \mu(\underset{\sim}{x})\underset{\sim}{A}\underset{\sim}{x} + (1-\mu(\underset{\sim}{x}))\underset{\sim}{x}. \tag{A.4}$$

We will show that $\underset{\sim}{G} \in \mathcal{U}^2(i_d;\varepsilon(\cdot))$. Since $\mu(\underset{\sim}{x}) \equiv 0$ for $\|\underset{\sim}{x}\| \geq \delta$, we need to check the $C^2$ size of $\underset{\sim}{G} - i_d$ only for $\|\underset{\sim}{x}\| < \delta$. Since $\underset{\sim}{G}(\underset{\sim}{x}) - \underset{\sim}{x} = \mu(\underset{\sim}{x})(\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{x})$, we have, using (A.1)-(A.3), that

$$\|\underset{\sim}{G}(\underset{\sim}{x})-\underset{\sim}{x}\| + \|(\underset{\sim}{D}\underset{\sim}{G})_{\underset{\sim}{x}}-\underset{\sim}{1}\| + \|(\underset{\sim}{D}^2\underset{\sim}{G})_{\underset{\sim}{x}}\| \le \mu(\underset{\sim}{x})\|\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{x}\| + \|(\underset{\sim}{D}\mu)_{\underset{\sim}{x}}\| \; \|\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{x}\|$$

$$+ \mu(\underset{\sim}{x}) \; \|\underset{\sim}{A}-\underset{\sim}{1}\| + \|(\underset{\sim}{D}^2\mu)_{\underset{\sim}{x}}\| \; \|\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{x}\| + 2\|(\underset{\sim}{D}\mu)_{\underset{\sim}{x}}\| \; \|\underset{\sim}{A}-\underset{\sim}{1}\|$$

$$\le \|A-1\| \; (\|\underset{\sim}{x}\|+k\|\underset{\sim}{x}\|+1+k\|\underset{\sim}{x}\|+2k) \le \|\underset{\sim}{A}-\underset{\sim}{1}\| \; (1+\delta) \; (1+2k) < \varepsilon.$$

Take $U_1 \triangleq \{\underset{\sim}{x} \in \mathbb{R}^n \,|\, \|\underset{\sim}{x}\| < \delta\}$ and $U_2 \triangleq \{\underset{\sim}{x} \in \mathbb{R}^n \,|\, \|\underset{\sim}{x}\| < \delta_o\}$. Then all the properties are satisfied. ⌑

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Figure Captions

Fig. 1. A nonlinear circuit with $\Lambda \not\pitchfork K$.

(a) The circuit diagram.

(b) Resistor constitutive relations.

Fig. 2. A nonlinear circuit which is not well posed.

(a) The circuit diagram.

(b) Resistor constitutive relations.

Fig. 3. A nonlinear circuit which is not well posed.

(a) Resistor constitutive relations.

(b) Projection of $\Sigma$ onto the $(v_L, i_L)$-space.

Fig. 4. A nonlinear circuit which is well posed, yet $\Sigma$ is not diffeomorphic to $\mathbb{R}^n$.

Fig. 5. Resistor constitutive relation for the circuit of Example 8.

Fig. 6. Diagram defining the two functions $W$ and $W_R$.

Fig. 7. A network which is not eventually strictly passive.

(a) The circuit diagram.

(b) Trajectories on the linear subspace $W = 0$.

Fig. 8. Diagram defining the two functions $\chi$ and $\iota_K$.

Fig. 9. Perturbation of the network of Fig. 7(a).

Fig. 10. A nonlinear network which becomes eventually strictly passive after perturbation.

(a) The circuit diagram.

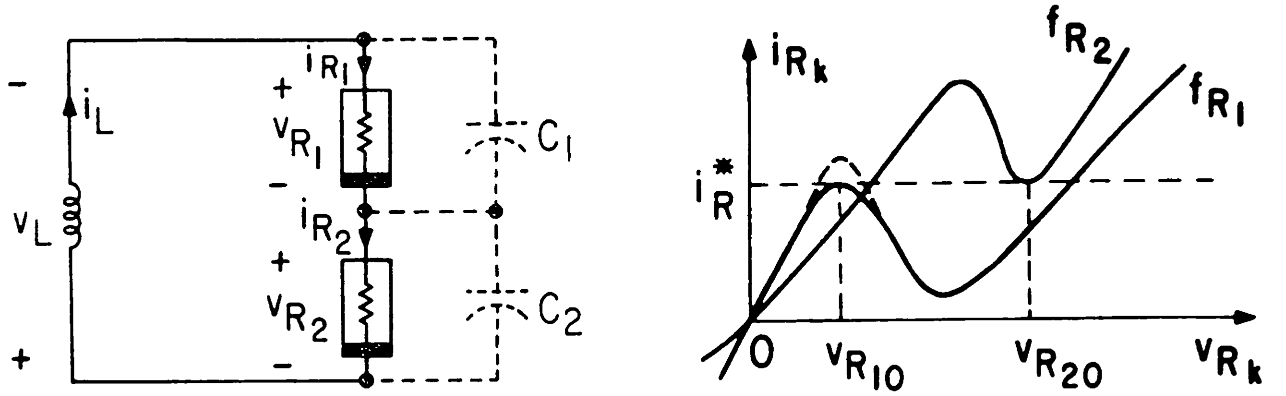
(b) Resistor constitutive relation.

Fig. 11. A nonlinear network which becomes well-posed and eventually strictly passive after perturbations.

(a) Original network $\mathcal{N}$.

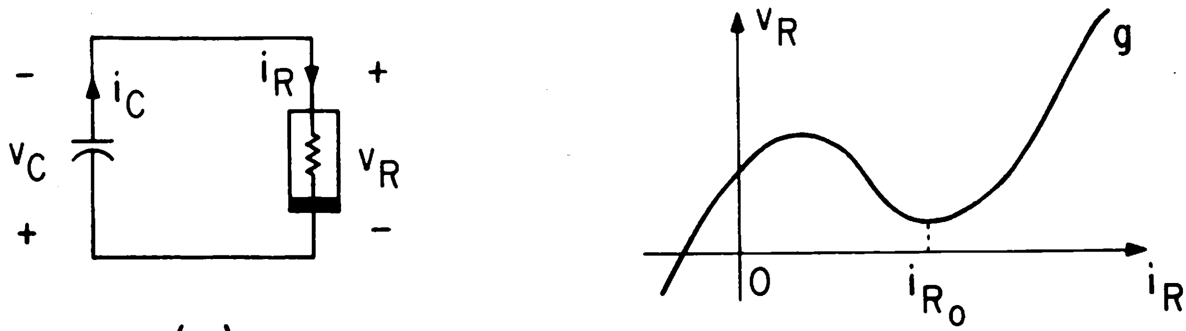
(b) Perturbed network $\hat{\mathcal{N}}$.

(c) Perturbed network $\bar{\mathcal{N}}$.



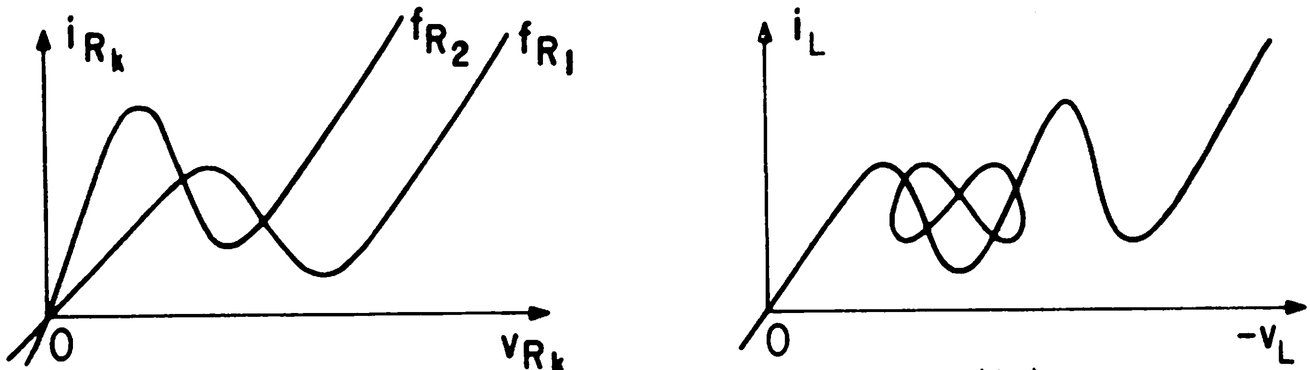

(a) (b)

Fig. 1




(a) (b)

Fig. 2




(a) (b)

Fig. 3

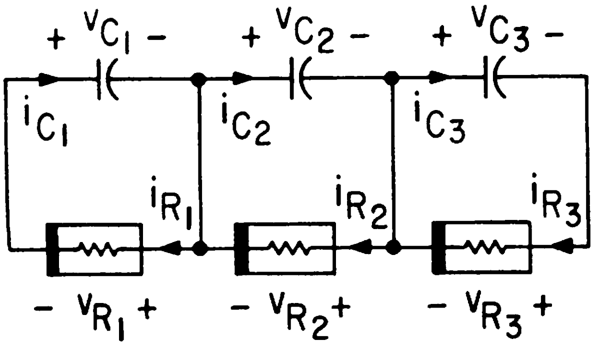



Fig. 4

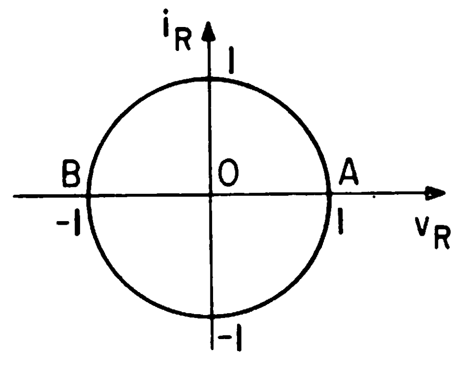



Fig. 5

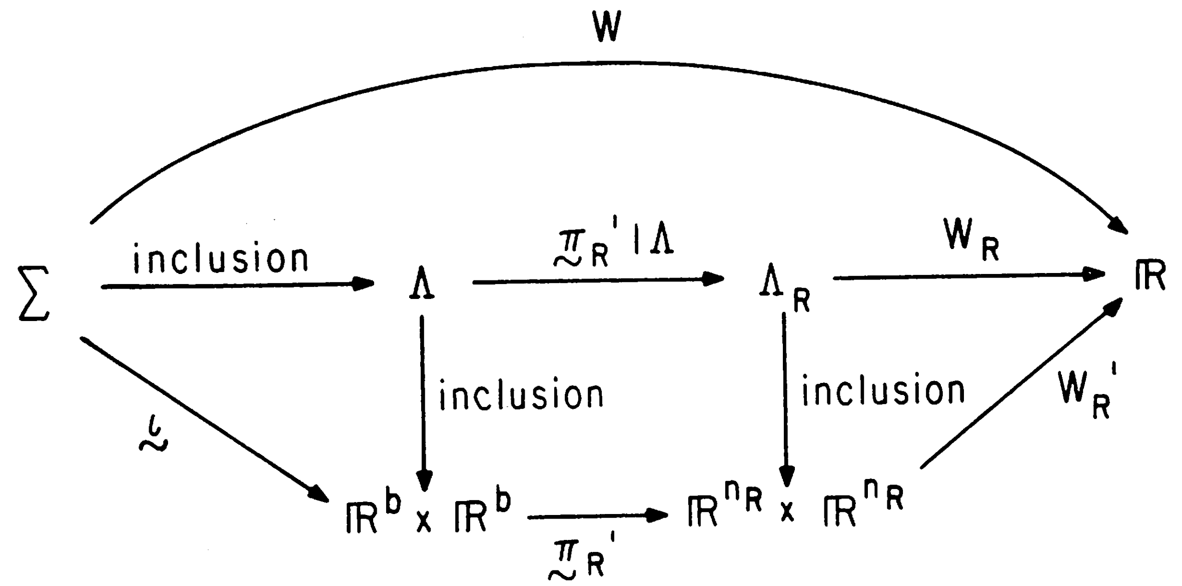



Fig. 6

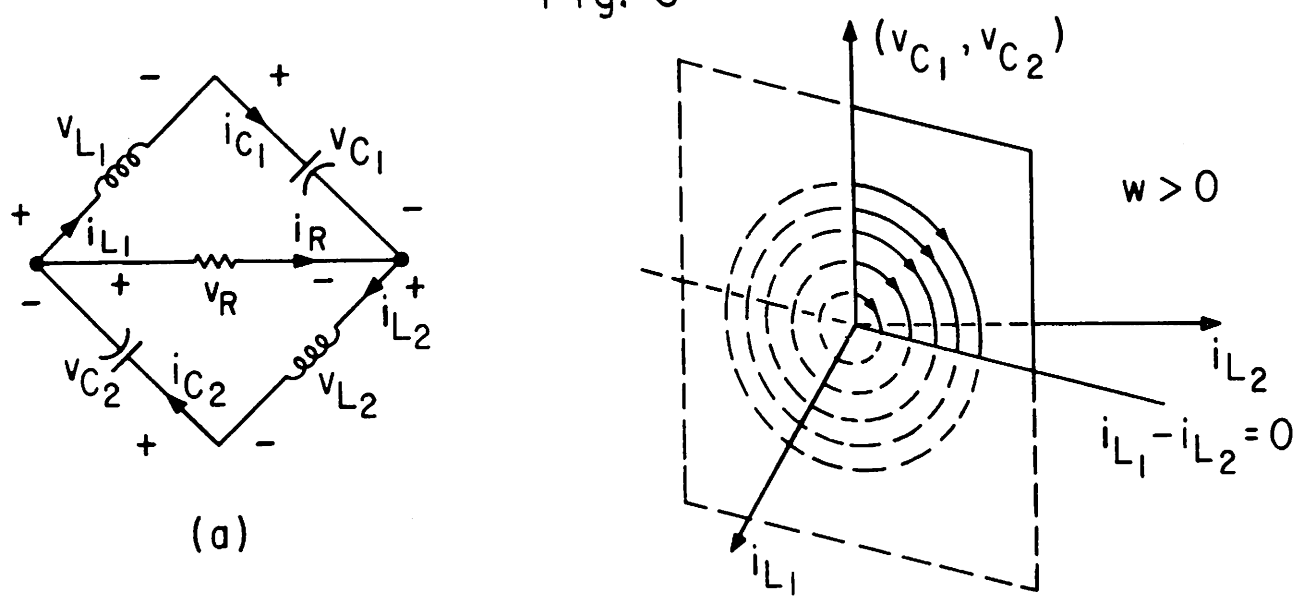



Fig. 7

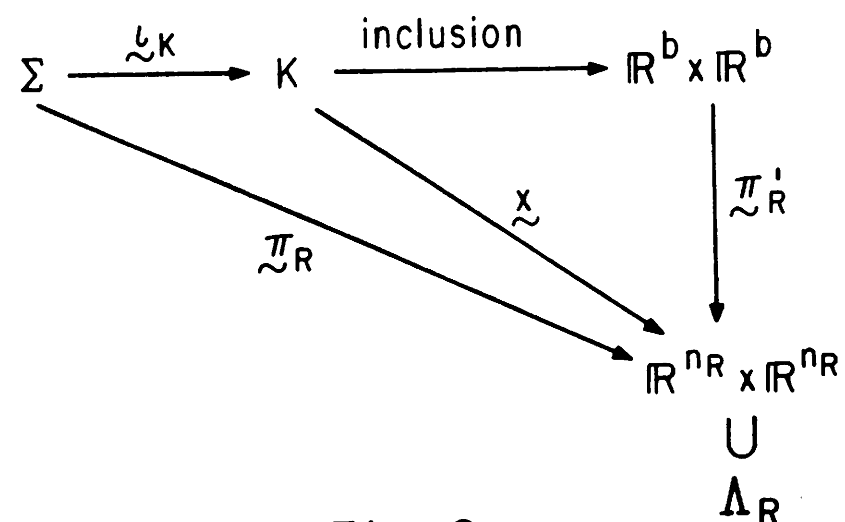



Fig. 8

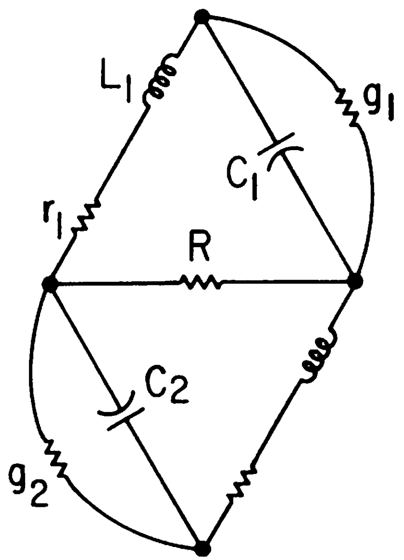
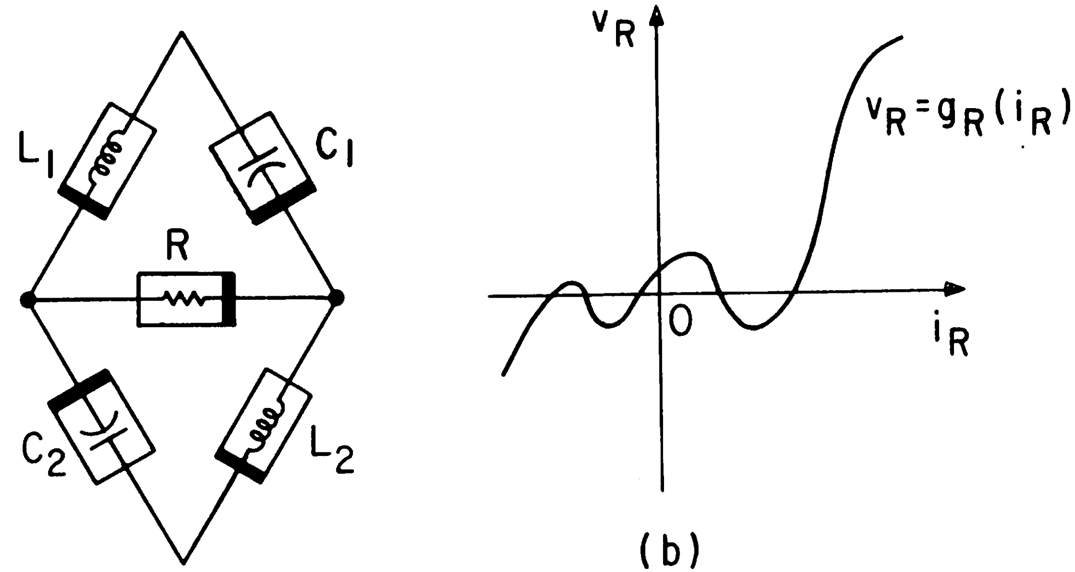


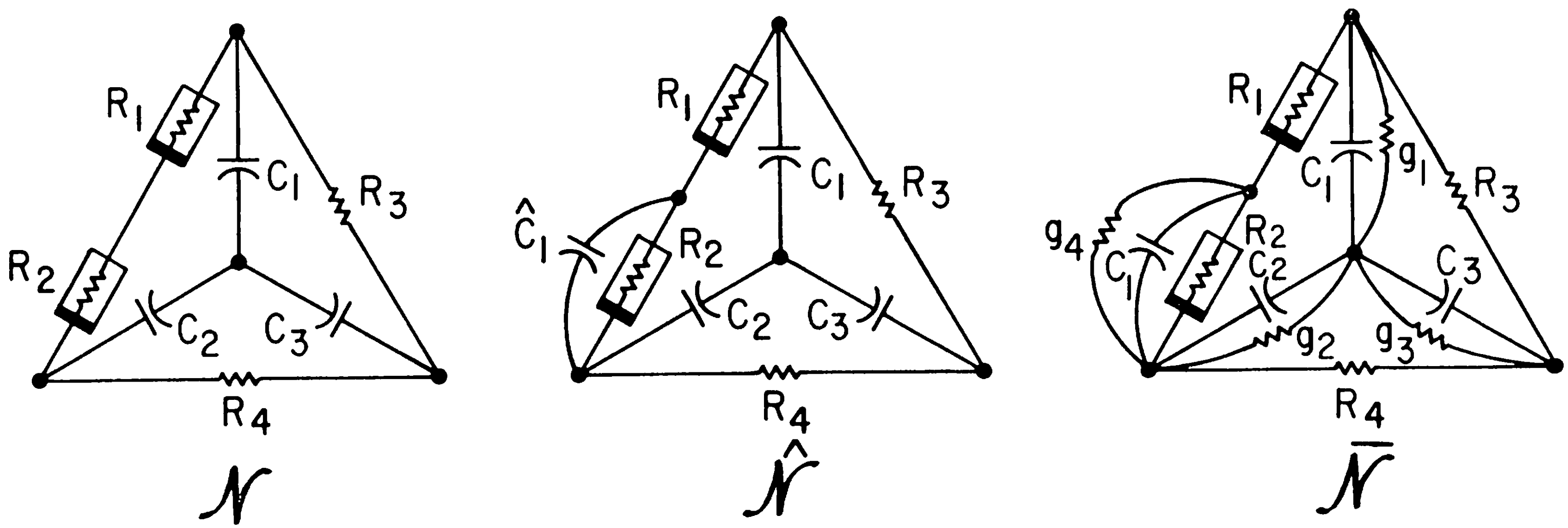

Fig. 9




(a)

(b)

Fig. 10




(a)

(b)

(c)

Fig. 11