INTERCONNECTIONS OF MARKOV CHAINS AND QUASI-REVERSIBLE QUEUING NETWORKS

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ABSTRACT

A quasi-reversible queue can be associated with certain types of transition of a Markov chain. It is shown that if Markov chains are coupled in a certain way, then to the resulting chain can be associated a queuing network which is itself quasi-reversible and the stationary distribution of the chain takes the product form. The product form for mixed networks is derived from the result for open networks.

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1. Introduction

Kelly [1,2] has recently shown that for a large class of queuing networks in equilibrium, the state of the queue at any single node is independent of the state at other nodes, and hence the distribution of the state has the product form. Furthermore, the number of customers of a given type who leave the network form a given node form independent Poisson processes whose history is independent of the present state of the network. Motivated by this result, Kelly proposed to call a single queue quasi-reversible if, in equilibrium, the customers departing from it have the above-mentioned properties. The particular networks which Kelly studied are obtained by interconnecting quasi- reversible queues.

In this paper we show that a network obtained by interconnecting queues, each of which is quasi-reversible when considered in isolation, is itself quasi-reversible; moreover, in equilibrium, the distribution of the state has the product form. We show this in the context of a fairly general model for Markov chains to which queues can be naturally associated. Such a model, together with a characterization of quasi-reversibility, was introduced earlier [3,4] and is summarized in Section 2. In Section 3 we propose a way of connecting two such chains such that the interconnection is quasi-reversible when its components are. In Section 4 we apply the result to queuing networks. In Section 5 we show how the product form obtained for the open networks of Section 4 also gives the same result for mixed networks.

2. A Markov Chain Model for Networks

The model presented below was introduced in [3,4] where a detailed analysis of its properties may be found. Let X, I be countable sets. X is the state space, I indexes possible state transitions. For each i in I is given a subset E_i of X, a mapping (or transition) $T_i : E_i \to X$, and a Poisson process $N^i = (N^i_t)$, t _ 0, with intensity $\lambda^i \geq 0$. The processes N^i and the initial state X_0 are all independent.

Assume that $\sum_i \lambda^i 1(x \in E_i)$ is bounded. ($1(\cdot)$ is the indicator of $(\cdot)$.) Then a Markov chain (X_t), $t \geq 0$, can be defined in the following manner.

Let $\xi_t(t) = 1(X_t = x)$, $\xi_t(A) = 1(X_t \in A)$, $x \in X$, $A \subset X$.

$$d\xi_t(x) = \sum_i [\xi_{t-}(T_i^{-1}x) - \xi_{t-}(x)]\xi_{t-}(E_i)dN^i_t, \quad t \geq 0, \tag{2.1}$$

$$\xi_0(x) = 1(X_0 = x). \tag{2.2}$$

Thus (X_t) has right-continuous piecewise constant sample paths. It starts at X_0 and if $X_{t-} = x \in E_i$ and $dN^i_t = N^i_t - N^i_{t-} = 1$, then $X_t = T_i(x)$.

We denote this Markov chain by (X_t) or $M = \{X, I, E_i, N^i, \lambda^i\}$. Later it will be convenient to vary the intensities $\lambda = \{\lambda^i\}$ and then we will discriminate among different chains by writing $M(\lambda)$.

Lemma 2.1 [3]. For $J \subset I$ define the counting process $Y = (Y_t)$, $t \geq 0$ by $Y_0 = 0$,

$$dY_t = \sum_{j \in J} \xi_{t-}(E_j)dN^j_t.$$

Suppose (X_t) is in equilibrium, and let $P(A) = \text{Prob}(X_t \in A)$ be the stationary distribution. Then X_t and $F^Y_t = \sigma(Y_s, s \leq t)$ are independent if and only if

$$\sum_{j\in J} \lambda^j P(E_j)P(x) = \sum_{j\in J} \lambda^j P(T_j^{-1}x), \ x \in X \tag{2.3}$$

Moreover, when this holds, Y is Poisson with intensity $\sum_{j\in J} \lambda^j P(E_j)$.

We call Y an <u>output</u> of the chain M, and if J contains only one element we call Y an <u>elementary output</u>.

<u>Remark</u>. When (2.3) holds we will say that M is quasi-reversible (QR) (with respect to the output Y). For future reference observe that P satisfies $P(A) = E\xi_t(A)$, and since $E(dN_t^i) = \lambda^i dt$, (2.1) gives

$$\sum_i \lambda^i [P(T_i^{-1}x \cap E_i) - P(x \cap E_i)] = 0, \ x \in X,$$

when we remember that $\xi_t(A)\xi_t(B) = \xi_t(A \cap B)$. Further since $T_i^{-1}x \subset E_i$ by definition of T_i, we get the "balance equations" that characterize P,

$$\sum_i \lambda^i [P(T_i^{-1}x) - P(x \cap E_i)] = 0. \tag{2.4}$$

It may be worth signalling at this point that we will associate a queue to a chain M by identifying certain outputs of M with processes which count arriving and departing customers.

3. <u>Interconnection of two Markov Chains</u>

Let $M^u(\lambda^u) = \{X^u, I^u, E_i^u, T_i^u, N^{ui}, \lambda^{ui}\}$, u = 1,2 be two independent Markov chains i.e., X_0^1, X_0^2, N^{1i}, N^{2i} are all independent. Denote the elementary outputs of M^u by S^{ui}, where $S_0^{ui} = 0$ and

$$dS_t^{ui} = 1(X_{t-}^u \in E_i^u)dN_t^{ui}.$$

Assume given for each u a set of outputs Y_t^{uk} , $k \in K^u$, of the form

$$Y_t^{uk} = \sum_{i \in J_k^u} S_t^{ui},$$

such that

$$J_k^u \cap J_\ell^u = \phi \qquad \text{if} \quad k \neq \ell. \tag{3.1}$$

We now construct a chain $M = \{X, I, E_i, T_i, N^i, \lambda^i\}$ by coupling M^1 and M^2 in such a way that changes in the outputs of one of the M^u randomly trigger transitions in the other. This is made precise as follows.

Let 0 be a point not in $I^1 \cup I^2$. Let

$$I = I^1 \times I^2 \cup I^1 x\{0\} \cup \{0\} x I^2,$$

and for each (i,j) in I define

$$E_{ij} = E_i^1 \times E_j^2, \text{ with } E_0^u = X^u,$$

$$T_{ij}(x^1, x^2) = (T_i^1(x^1), T_j^2(x^2)), \text{ with } T_0^u \text{ as identity map,}$$

$N^{ij} = (N_t^{ij})$, $t \geq 0$, an independent Poisson process with rate

$$\lambda^{ij} = \lambda^{1i} c_{ij}^1 + \lambda^{2j} c_{ji}^2, \ \lambda^{u0} = 0, \tag{3.2}$$

where the c_{ij}^1, c_{ji}^2 are prespecified nonnegative numbers satisfying the following conditions

$$\sum_{I^2 \cup \{0\}} c_{ij}^1 1(x^2 \in E_j^2) = \sum_{j \in I^1 \cup \{0\}} c_{ij}^2 1(x^1 \in E_j^1) \equiv 1, \tag{3.3}$$

$$c^u_{ij} = c^u_{hj} \quad \text{if} \quad i,h \text{ are in } J^u_k \text{ for some } k \text{ and } c^u_{ij} = 0 \text{ if } i \notin \bigcup_k J^u_k \tag{3.4}$$

if Y^{uk} is not an elementary output and $c^u_{ij} > 0$ for $i \in J^u_k$

$$\text{then } c^v_{ji} = 0, \quad \text{for } v \neq u. \tag{3.5}$$

For the chain M define for each (i,j) in I the elementary output S^{ij} by $S^{ij}_0 = 0$ and

$$dS^{ij}_t = 1(X_{t-} \in E_{ij})dN^{ij}_t.$$

For $k \in K^1$, $\ell \in K^2$ let

$$Y^{k0}_t = \sum_{i \in J^1_k} S^{i0}_t, \quad Y^{0\ell}_t = \sum_{j \in J^2_\ell} S^{0j}_t. \tag{3.6}$$

Assume that there exists probability measures P^u on X^u such that

$$P^u \text{ is a stationary distribution for } M^u(\rho^u), \ u = 1,2, \tag{3.7}$$

$$\rho^{ui} = \lambda^{ui} + \mu^{ui} = \lambda^{ui} + \sum_{j \in I^v} \lambda^{vj} P^v(E^v_j)c^v_{ji}, \ i \in I^u, \ u = 1, 2. \tag{3.8}$$

The proof of the next result is given in the Appendix.

<u>Theorem 3.1</u>. Suppose that $M^u(\rho^u)$ is QR with respect to Y^{uk}, $k \in K^u$. Then

$$P(x^1,x^2) = P^1(x^1)P^2(x^2), \quad (x^1,x^2) \in X \tag{3.9}$$

is a stationary distribution for M. Furthermore M in QR with respect to Y^{k0}, $Y^{0\ell}$, $k \in K^1$, $\ell \in K^2$.

Remark. For $i \in J_k^u$, c_{ij}^u is the probability that a change in Y^{uk} induces a transition in S^{vj}. The constraints (3.1), (3.3) and (3.4) mean that all changes in the Y^{uk} are accounted for. In queuing applications this corresponds to the conservation of flow of customers between nodes. Condition (3.5) is essential although technical and is needed because a chain can be QR with respect to an output, without being QR with respect to the elementary outpus whose sum it is. It is easy to give examples for which (3.9) does not hold when (3.5) is violated.

It is not necessary that the chains $M^u(\lambda^u)$ have a stationary distribution. The existence of a solution to (3.7), (3.8) can often be readily established. We can see that for $\lambda = \Sigma\lambda^{1i} + \Sigma\lambda^{2j}$, $|\rho^u| = \Sigma\rho^{ui} \leq \lambda$, so that if λ is finite then ρ^1, ρ^2 belong to bounded, hence weakly compact spheres in ℓ_1. Suppose now that for each ρ^u with $|\rho^u| \leq \lambda$ the chain $M^u(\rho^u)$ has a unique stationary distribution $P^u(\rho^u)$ which varies continuously with ρ^u. Then the existence of a solution follows from the usual fixed point argument. Note further that since the transition rates of $M^u(\rho^u)$ vary continuously with ρ^u, therefore the continuity of $P^u(\rho^u)$ holds if it is unique.

4. Network of QR queues

Consider independent Markov chains $M^u(\lambda^u) = \{X^u, I^u, E_i^u, T_i^u, N^{ui}, \lambda^{ui}\}$, $u = 1,\ldots,n$. As before let S^{ui}, $i \in I^u$ denote the elementary outputs of M^u. Let K be an index set. For each k in K we distinguish certain outputs A^{uk} and D^{uk} of M^u such that

$$A_t^{uk} = S_t^{ui_u(k)}, \quad D_t^{uk} = \sum_{i \in J_k^u} S_t^{ui}.$$

Thus A_t^{uk} is elementary. To anticipate the application A^{uk} and D^{uk} are called respectively the arrival and departure process of type k at u. We assume that the condition corresponding to (3.1) is satisfied i.e., for each u, the sets $\{i_u(k)\}$ and $\{J_\ell^u\}$ are all disjoint.

Assume given nonnegative numbers (routing probabilities) $r_{k\ell}^{uv}$, r_k^{u0}, $1 \leq u, v \leq n$, k,ℓ in K such that $r_{k\ell}^{uu} = 0$, and for all u,k

$$r_k^{u0} + \sum_{v=1}^{n} \sum_{\ell \in K} r_k^{uv} 1(x^v \in E^v_{i_v(\ell)}) = 1, \quad x^v \in X^v \tag{4.1}$$

We now define the interconnected chain $M = (X, I, E_i, T_i, N^i, \lambda^i)$ as follows. Let

$$X = X^1 \times \ldots \times X^n,$$

$$I = \bigcup_{u \neq v} I^u \times (I^v \cup \{0\}),$$

and for each $u \neq v$, $i \in I^u$, $j \in I^v$ define

$$E_{ij}^{uv} = \{x=(x^1,\ldots,x^n) \mid x^u \in E_i^u, x^v \in E_j^v\}, \quad E_i^{u0} = \{x \mid x^u \in E_i^u\},$$

$$T_{ij}^{uv}(x^1,\ldots,x^n) = (x^1,\ldots,T_i^u(x^u)\ldots T_j^v(x^v)\ldots,x^n),$$

$$T_i^{u0}(x^1,\ldots,x^n) = (x^1,\ldots,T_i^u(x^u),\ldots,x^n),$$

and let N_{ij}^{uv}, N_i^{u0} be independent Poisson processes with respective rates

$$\lambda_{ij}^{uv} = \lambda^{ui} c_{ij}^{uv} + \lambda^{vj} c_{ji}^{vu}, \quad \lambda_i^{u0} = \lambda^{ui} c_i^{u0} \tag{4.2}$$

where

$$c_{ij}^{uv} = r_{k\ell}^{uv}, \text{ if for some } k,\ell \quad i \in J_k^u \text{ and } j = i_v(\ell), \tag{4.3}$$

$$= 0, \text{ otherwise},$$

$$c_i^{u0} = r_k^{u0}, \text{ if for some } k \quad i \in J_k^u \tag{4.4}$$

(Thus changes in the departure processes trigger changes in the arrival processes.) Let S_{ij}^{uv}, S_i^{u0} denote the elementary outpus of M and define the outputs D_k^{u0}, for $1 \leq u \leq n$, $k \in K$, by

$$D_{kt}^{u0} = \sum_{i \in J_k^u} S_{it}^{u0} = \sum_{J_k^u} S_{it}^{u0} \tag{4.5}$$

Suppose now that there exist probability measures P^u on X^u such that

$$P^u \text{ is a stationary distribution for } M^u(\rho^u),\ u = 1,\ldots,n, \tag{4.6}$$

$$\rho^{ui} = \lambda^{ui} + \sum_{v=1}^{n} \sum_{j \in I^v} \lambda^{vj} P^v(E_j^v) c_{ji}^{vu},\ i \in I^u,\ u = 1,\ldots,n. \tag{4.7}$$

The result below is proved in the Appendix.

Theorem 4.1. Suppose that each $M^u(\rho^u)$ is QR with respect to its departure processes D^{uk}, $k \in K$. Then

$$P(x^1,\ldots,x^n) = P^1(x^1),\ldots,P^n(x^n) \tag{4.8}$$

is a stationary distribution for M. Furthermore M is QR with respect to the departure processes D_k^{u0}, $1 \leq u \leq n$, $k \in K$.

Remark. M can be interpreted as the chain describing the network built from the component queuing systems $M^1,\dots,M^n$. Each M^u serves customers of class $k \in K$, and A_t^{uk} is the number of customers who arrive at M^u, or leave from it, during $[0,t]$. Depending on the specification of M^u customers may or may not change class when in service at u. Class k customers can arrive at M^u whenever its state $x^u \in E^u_{i_u(k)}$ either form outside the network in a Poisson stream with rate $\lambda^{ui_u(k)}$ or from inside the network. The movement of the latter is determined as follows. If a customer who has just completed service at u is of class k (indicated by a change in D_t^{uk}), he either leaves the network with probability r^u_{k0} or he changes into a class ℓ customer and moves to M^v with probability $r^{uv}_{k\ell}$ provided M^v is in state $x^v \in E^v_{i_v(\ell)}$. The number of customers of class k who leave the network from u during $[0,t]$ is D^{u0}_{kt}, and the theorem asserts that M is QR with respect to these departure processes.

In many cases the rates ρ^{ui} given in (4.7) can be obtained without solving for the P^v simultaneously. Observe first that if i does not correspond to an arrival i.e., $i \neq i_u(k)$ for all k, then $c^{vu}_{ji} = 0$ by (4.3), and so (4.7) simplifies to

$$\rho^{ui} = \lambda^{ui}, \quad i \notin \{i_u(k) | k \in K\},\ u = 1,\dots,n. \tag{4.9}$$

But if $i = i_u(k)$ then, by (4.3),

$$\rho^{ui_u(k)} = \lambda^{ui_u(k)} + \sum_{v=1}^{n} \sum_{\ell \in K} r^{vu}_{\ell k} \sum_{j \in J^v_\ell} \lambda^{vj} P^v(E^v_j). \tag{4.10}$$

Now the last sum is just the average rate of departures of class ℓ from node v. Suppose that at each node and for each class the average rate of arrivals equals the average rate of departures. (This always holds for queuing networks but need not hold for the more general chains considered here.) Then the last sum in (4.10) simply equals $\rho^{vi_v(\ell)}$, the average arrival rate of class ℓ at node v and so (4.10) simplifies to

$$\rho^{ui_u(k)} = \lambda^{ui_u(k)} + \sum_{v=1}^{n} \sum_{\ell \in K} r_{\ell k}^{vu} \rho^{vi_v(\ell)}, \quad k \in K,\ u = 1,\ldots,n. \tag{4.11}$$

Thus, under the above-mentioned assumption (4.7) is replaced by the simpler (4.9), (4.11), which do not involve the P^v.

5. Mixed Networks

Consider now an interconnection of chains as in the preceding section and suppose in addition that (4.7) is replaced by (4.9), (4.11). Recall that these equations are used to solve for the $\{\rho^u\}$, the $\{\lambda^u\}$ being given exogenuously and corresponding, in the network interpretation, to prespecified external arrival rates and service rates. If (4.11) gives a unique solution to the $\{\rho^u\}$, the network is said to be open, otherwise it is called mixed. In particular, if $\lambda^{ui_u(k)} = 0$ and $r_k^{u0} = 0$, for all u,k the network is closed, and in this case (4.11) is homogeneous and the $\{\rho^u\}$ are at best determined uniquely only up to a multiplicative constant.

Suppose now that the network is mixed. It is then possible, neglecting some trivial cases (see [5]), to partition the set of node-class pairs $\{1,\ldots,n\} \times K$ into two disjoint sets 0 and C such that $r_{k\ell}^{uv} = 0$, $r_{\ell k}^{vu} = 0$ if $(u,k) \in 0$, $(v,\ell) \in C$, and $\lambda^{ui_u(k)} = 0$, $r_k^{u0} = 0$, $(u,k) \in C$. Then (4.11) splits as follows.

$$\rho^{ui_u(k)} = \lambda^{ui_u(k)} + \sum_{(v,\ell)\in 0} r_{\ell k}^{vu} \rho^{vi_v(\ell)}, \quad (u,k) \in 0 \tag{5.1}$$

$$\rho^{ui_u(k)} = \sum_{(v,\ell)\in C} r_{\ell k}^{vu} \rho^{ui_v(\ell)}, \qquad (u,k) \in C \tag{5.2}$$

Assume that (5.1) gives a unique solution denoted $\{\bar{\rho}^{ui_u(k)}\}$ (which depends upon the $\{\lambda^u\}$), and that (5.2) gives a positive solution denoted $\bar{\rho}^{ui_u(k)}$ which is unique up to a multiplicative constant. Suppose that for $u = 1,\ldots,n$, $\bar{P}^u$ is a stationary distribution for the component chain $M^u(\bar{\rho}^u)$, and suppose it is QR with respect to its departure processes D^{uk}, $k \in K$. Then, by Theorem 4.1, $\bar{P}(x^1,\ldots,x^n) = \bar{P}^1(x^1,\ldots,\bar{P}^n(x^n)$ is a stationary distribution for the interconnected chain, or the mixed network, M.

However, the mixed network has many stationary distributions and $\bar{P}$ is usually the least interesting one of them. To see this interpret the sets C,0 as follows: if a customer of class k enters node u then he is "trapped" in the network if $(u,k) \in C$, whereas if $(u,k) \in 0$ then he will eventually leave the network. Thus the stationary distribution of the mixed network depends in particular on the number and type of trapped customers.

Let $\tilde{X}$ be a subset of X and suppose that there is a <u>unique</u> stationary probability measure on $\tilde{X}$ i.e., if $\text{Prob}\{X_0 = x\} = \tilde{P}(x)$, $x \in \tilde{X}$, then

$$\text{Prob}\{X_t = x\} = \tilde{P}(x),\ x \in \tilde{X},\ t \geq 0.$$

We will show that $\tilde{P}$ is just the restriction of $\bar{P}$ to $\tilde{X}$. To see this we first "open" the mixed network. Fix $0 < \varepsilon < 1$, and consider the component chains $M^u(\tilde{\lambda}^u)$ where $\tilde{\lambda}^{ui} = \lambda^{ui}$ if $i \notin \{i_u(k) | k \in K\}$, and

$$\tilde{\lambda}^{ui_u(k)} = \lambda^{ui_u(k)}, \qquad (u,k) \in 0 \tag{5.3}$$

$$\tilde{\lambda}^{ui_u(k)} = \varepsilon\bar{\rho}^{ui_u(k)}, \qquad (u,k) \in C \tag{5.4}$$

Now interconnect these chains using the routing probabilities $\tilde{r}^{vu}_{\ell k}$ where

$$\tilde{r}^{vu}_{\ell k} = r^{vu}_{\ell k}, \qquad (v,\ell) \in 0,\ (u,k) \in 0 \tag{5.5}$$

$$\tilde{r}^{vu}_{\ell k} = (1-\varepsilon)r^{vu}_{\ell k}, \quad (v,\ell) \in C,\ (u,k) \in C \tag{5.6}$$

We now apply Theorem 4.1 to obtain a stationary distribution for the interconnected chain $M(\varepsilon)$. We first solve for the rates $\{\tilde{\rho}^u\}$,

$$\tilde{\rho}^{ui_u(k)} = \tilde{\lambda}^{ui_u(k)} + \sum_{(v,\ell)\in 0} \tilde{r}^{vu}_{\ell k}\, \tilde{\rho}^{vi_v(\ell)}, \qquad (u,k) \in 0$$

$$\tilde{\rho}^{ui_u(k)} = \tilde{\lambda}^{ui_u(k)} = \sum_{(v,\ell)\in C} \tilde{r}^{vu}_{\ell k}\, \tilde{\rho}^{vi_v(\ell)}, \qquad (u,k) \in C.$$

Substituting from (5.3)(5.6) into (5.1), (5.2) shows that $\tilde{\rho}^{ui} \equiv \rho^{-ui}$. Hence $\bar{P}(x^1,\dots,x^n) = \bar{P}^1(x^1,\dots,\bar{P}^n(x^n)$ is again a stationary distribution for the chain $M(\varepsilon)$ i.e. $\bar{P}$ satisfies the balance equations (2.4) for the chain $M(\varepsilon)$ which we may write as

$$\sum_i \tilde{\lambda}^i_\varepsilon[\bar{P}(T_i^{-1}x) - \bar{P}(x \cap E_i)] = 0, \qquad x \in X. \tag{5.7}$$

Observe that in (5.7) the term in [ ] does not depend on ε.

Theorem 5.1. Suppose $\bar{P}(\tilde{X}) > 0$. Then $\tilde{P}$ is just the restriction of $\bar{P}$ to $\tilde{X}$ i.e.,

$$\tilde{P}(x) = [\sum_{y\in\tilde{X}} \bar{P}(y)]^{-1}\bar{P}(x),\ x \in \tilde{X}.$$

Proof. From (5.3), (5.4) it is easy to see that the rates $\{\lambda^i\}$ in (5.7) are continuous in ε and, so $\bar{P}$ satisfies the balance equations for the mixed network which is obtained by setting $\varepsilon = 0$,

$$\sum_i \tilde{\lambda}_0^i[\bar{P}(T_i^{-1}x) - \bar{P}(x \cap E_i)] = 0, \ x \in X \tag{5.8}$$

By assumption $\tilde{P}$ is the unique solution to the balance equations

$$\sum_i \tilde{\lambda}_0^i[\tilde{P}(T_i^{-1}x) - \tilde{P}(x \cap E_i)] = 0, \qquad x \in \tilde{X}$$

$$\sum_{x \in \tilde{X}} \tilde{P}(x) = 1$$

From (5.8) it follows that the restriction of $\bar{P}$ to $\tilde{X}$ is an invariant measure for the mixed network and so the result follows from the uniqueness of $\tilde{P}$. □

Remark. From Theorem 4.1 it follows also that the mixed network is QR with respect to the departure processes D_k^{u0}, $(u,k) \in 0$.

Appendix

Proof of Theorem 3.1. Using (2.3) and (3.8) the hypothesis may be expressed as

$$\sum_{J_k^u} (\lambda^{ui}+\mu^{ui})P^u((T_i^u)^{-1}x^u) = \sum_{J_k^u} (\lambda^{ui}+\mu^{ui})P^u(E_i^u)P^u(x^u), x^u \in X^u,\ k \in K^u. \tag{A1}$$

Consider k in K^u. If Y_k^u is elementary, then J_k^u is a singleton, and so (A1) implies

$$\sum_{J_k^u} c_{ij}^u \lambda^{ui} P^u((T_i^u)^{-1}x^u) = \sum_{J_k^u} c_{ij}^u \lambda^{ui} P^u(E_i^u)P^u(x^u), \text{ all } j, \tag{A2}$$

where, if Y_k^u is not elementary, then $c_{ij}^u \mu^{ui} = 0$ by (3.5), and once again (A2) holds.

From (2.4) P^u satisfies

$$\sum_{I^u} (\lambda^{ui}+\mu^{ui})[P^u((T_i^u)^{-1}x^u)-P^u(x^u \cap E_i^u)] = 0. \tag{A3}$$

Also from (2.4), P is an equilibrium distribution for M if

$$\sum_I \lambda^{ij}[P(T_{ij}^{-1}x)-P(x \cap E_{ij})] = 0.$$

In particular, the P given by (3.9) is an equilibrium distribution if

$$\sum_{i\in I^1} \sum_{j\in I^2} (\lambda^{1i}c_{ij}^1+\lambda^{2j}c_{ji}^2)[P^1((T_i^1)^{-1}x^1)P^2((T_j^2)^{-1}x^2)-P^1(x^1 \cap E_i^1)P^2(x^2 \cap E_j^2)]$$

$$+ \sum_{u\neq v} \sum_{I^u} \lambda^{ui} c^u_{io} [P^u((T^u)^{-1}x^u) - P^u(x^u \cap E^u_i)]P^v(x^v) = 0. \qquad \text{(A4)}$$

Now, by (3.3),

$$c^u_{i0} = 1 - \sum_{I^v} c^u_{ij} 1(x^v \in E^v_j),$$

and so

$$c^u_{i0} P^v(x^v) = P^v(x^v) - \sum_{I^v} c^u_{ij} P^v(x^v \cap E^v_j).$$

Substituting this into (A4) gives the equivalent condition

$$\sum_{u\neq v} \sum_{I^u} \lambda^{ui} [P^u((T^u_i)^{-1}x^u) - P^u(x^u \cap E^u_i)]P^v(x^v)$$

$$+ \sum_{u\neq v} \sum_{i\in I^u} \sum_{j\in I^v} \lambda^{ui} c^u_{ij} P^u((T^u_i)^{-1}x^u)[P^v((T^v_j)^{-1}x^v) - P^v(x^v \cap E^v_j)] = 0 \qquad \text{(A5)}$$

Because of (3.1) and (3.4), (A2) allows us to replace in the second sum above the expression $P^u((T^u_i)^{-1}x^u)$ by $P^u(E^u_i)P^u(x^u)$. After we do this and rearrange terms, we can rewrite (A5) as

$$\sum_{u\neq v} \sum_{I^u} [\lambda^{ui} + \sum_{I^v} \lambda^{vj} c^v_{ji} P^v(E^v_j)][P^u((T^u_i)^{-1}x^u) - P^u(x^u \cap E^u_i)]P^v(x^v) = 0,$$

and since this condition is implied by (A3) it follows that $P^1(x^1)P^2(x^2)$ is an equilibrium distribution as asserted.

Next, using (2.3) and (3.6), M is QR with respect to Y^{k0} if

$$\sum_{i\in J_k^1} \lambda^{i0} P(E_{i0}) P(x) = \sum_{i\in J_k^1} \lambda^{i0} P(T_{i0}^{-1} x).$$

By (3.9) and (3.2) this can be rewritten as

$$\sum_{i\in J_k^1} \lambda^{1i} c_{i0}^1 P^1(E_i^1) P^1(x^1) P^2(x^2) = \sum_{i\in J_k^1} \lambda^{1i} c_{i0}^1 P^1((T_i^1)^{-1} x^1) P^2(x^2)$$

which is immediate from (A2). In a similar way it can be shown that M is QR with respect to $Y^{0\ell}$. The theorem is proved.

<u>Proof of Theorem 4.1</u>. The proof is obtained by induction on n. Consider the chain $\bar{M}$ obtained by interconnecting the chains $M^u(\bar{\lambda}^u)$, $u = 1,\dots,n-1$ with routing probabilities $\bar{r}_{k\ell}^{uv}$, $\bar{r}_k^{u0}$ where

$$\bar{r}_{k\ell}^{uv} = r_{k\ell}^{uv}, \quad \bar{r}_k^{u0} = r_k^{u0} + \sum_{\ell} r_k^{un}, \quad 1 \le u, v \le n-1$$

and

$$\bar{\lambda}^{ui} = \lambda^{ui} + \sum_{j\in I^n} \lambda^{nj} P^n(E_j^n) c_{ji}^{nu}, \quad 1 \le u \le n-1, \ i \in I^u.$$

Using the restrictions implied by (4.3) and the assumption that the sets $\{i_u(k)\}$ and $\{J_\ell^u\}$ are disjoint it is easy to check that the $\bar{\lambda}^u$ satisfy

$$\rho^{ui} = \bar{\lambda}^{ui} + \sum_{v=1}^{n-1} \sum_{j\in I^v} \bar{\lambda}^{vj} P^v(E_j^v) c_{ji}^{vu}, \quad i \in I^u, \ u = 1,\dots,n-1$$

By the induction hypothesis

$$\bar{P}(x^1,\dots,x^{n-1}) = P^1(x^1),\dots,P^{n-1}(x^{n-1})$$

is a stationary distribution for the chain $\bar{M}$ and $\bar{M}$ is QR with respect to its departure processes $\bar{D}_k^{u0}$, $1 \leq u \leq n-1$, $k \in K$ which we defined in the natural way.

Now the chain M is obtained in a straightforward way by interconnecting the two chains $\bar{M}$ and $M^n(\lambda^n)$. The chains $\bar{M}$ and $M^n(\rho^n)$ are QR with respect to the relevant outputs and so it follows by Theorem 3.1 that

$$P(x^1,\ldots,x^n) = \bar{P}(x^1,\ldots,x^{n-1})P^n(x^n)$$

is a stationary distribution for M and M in QR with respect to the D_k^{u0}.

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