STABILITY THEOREMS FOR STOCHASTIC INTEGRAL EQUATIONS DRIVEN BY RANDOM MEASURES AND SEMIMARTINGALES

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Introduction

In this work we want to study stochastic integral equations of the form

$$(1) \qquad \xi_t = V_t + \int_{]0,t]} a_s^1 \xi dS_s + \int_{]0,t]\times \underline{E}} a^2(s,\xi,x)q(ds,dx)$$

where V is a given process, S a semimartingale, q a quite general white random measure (i.e., such that the measure valued process $q(]0,t],\cdot)$ is a martingale), $a^1\xi$ and $a^2(\cdot,\xi,\cdot)$ are functionals of the process ξ, which may at time s, depend on the whole past of ξ. Equations of this type naturally include those which have been considered by various authors: K. Ito, A.V. Skorokhod, Ph. Protter, C. Doleans-Dade, L. Galtchouk and others.

Our main idea is to show that, by the way, this general integral equation can be considered as an equation of the type

$$(2) \qquad \xi_t = V_t + \int_{]0,t]} \underset{\sim}{a}_s \xi dZ_s$$

where Z is a Banach valued process, ξ a Hilbert valued one, which can be dealt with in a surprisingly simple way, to get existence, uniqueness non explosion theorems and stability theorems as well, in presence of Lipschitz-type hypothesis.

The tools for doing that are mostly those which have been developed in [10] and [11] by J. Pellaumail and the author. The main difference here is

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that in (2) Z cannot be assumed to be Hilbert valued as it is the case in [10] and [11] because we have to consider measure valued processes to include the case of a random-measure driving term in (1). Moreover the process $\underset{\sim}{a}\xi$ is an operator valued process, but its values may be unbounded operators. But these new difficulties are easily circumvented by looking at things properly.

The paper is organized in the following way.

As we consider random measures which are not necessarily integer valued, we devote the first section to a short review of definitions and mainly to introduce the notion of "control-couple" of a random measure (see proposition 1) which plays a dominant role.

To help the reader in understanding our methods and motivations we consider in section 2 an equation of the particular type:

$$(3) \qquad \xi_t(\omega) = V_t(\omega) + \int_{]0,t]\times \underline{E}} \underset{\sim}{a}_s(\omega,\xi,x)q(\omega,ds,dx)$$

with simplified hypotheses on $\underset{\sim}{a}$, q being a general "white optional random measure."

The results of this section 2 are thus only introductory results to section 3. Nevertheless these are new in many respects (particularly theorem 3), and the method of proofs exemplifies the simplicity of the ideas and the power of the method.

In section 3 we give a general formulation, introducing the notion of Λ-spaces. This notion makes possible a one treatment of apparently different situations and in some sense provides us with a tool for fabricating a variety of existence, uniqueness non explosion and stability theorems in particular situations, by introducing the convenient Λ-space.

I should thank very much Professors D. Angelakos and E. Wong for the possibility they gave me to work out this paper during a stay at the Electronics Research Laboratory in Berkeley and the many fruitful discussions I had during this time with faculty and students.

Notations. Through all the paper (Ω,F,P) is a fixed probability space and $(F_t)_{t\geq 0}$ is an increasing family of sub σ-algebras of F. All the notation and notions used are now classical, and the reader who has doubts is referred to the first chapter of [6] or [11].

If $\mathbb{B}$ is a Banach space, $\|\cdot\|_{\mathbb{B}}$ denotes the norm in $\mathbb{B}$, $\langle\cdot,\cdot\rangle$ expresses the duality between $\mathbb{B}$ and $\mathbb{B}'$, and if $\mathbb{H}$ is a Hilbert space, $\langle x,y\rangle_{\mathbb{H}}$ stands for the scalar product of x and y.

A regular process is an adapted process, the paths of which are right continuous and have left limits.

1. Random measures

1.1 Random measures and measure valued processes. Preliminaries

We refer to J. Jacod (see [6]) for details on random measures. We give the main definitions a slightly different form here, but it will be readily checked that they are equivalent to J. Jacod's one in all the cases.

$\underline{E}$ being an open subspace of some space $\mathbb{R}^d$ (or more generally a Lusin space!) a random measure μ is a family $\{\mu(\omega;ds,du)\colon \omega\in\Omega\}$ of measures on the measure-space $(\mathbb{R}^+\times\underline{E}, \mathcal{B}(\mathbb{R}^+)\otimes\mathcal{B}(\underline{E}))$, where $\mathcal{B}(T)$ denotes the Borel σ-algebra of the topological space T. It is said to be positive, if $\mu(\omega;ds,du)$ is a positive measure for every ω.

One can equivalently say, that, for every ω, $\mu(\omega;ds,\cdot)$ is a measure on $(\mathbb{R}^+,\mathcal{B}(\mathbb{R}^+))$ taking its values in some locally convex space of measures.

In all the examples usually considered $\mu(\omega,ds,\cdot)$ restricted to $([0,t], \mathcal{B}[0,t])$ takes actually its values, for all t, in a space $\mathcal{M}^{\mu}$ of measures which is naturally endowed with a structure of Banach space for a suitable norm (this norm may be much smaller than the variation norm). Quite typically the space of measures considered is the space of measures m on $\mathbb{R}^d - \{0\}$ such that $\int \frac{|x|^r}{|x|^r+1} |m|(dx) < \infty$ † for a given r. On this space we consider precisely the norm $m \rightsquigarrow \int \frac{|x|^r}{|x|^r+1} |m|(dx)$.

More generally, let p be a strictly positive bounded function on $\underline{E}$, we denote by M^p the space of measures m on $(\underline{E}, \mathcal{B}(\underline{E}))$ such that $\int p(x)|m|(dx) < \infty$ with the norm $\|m\|_p := \int p(x)|m|(dx)$. This Banach space is clearly the dual of the space C^p of continuous functions Φ on $\underline{E}$ such that $\sup_{x\in\underline{E}} (|\Phi(x)|/p(x)) < \infty$ with the norm $\|\Phi\|_p := \sup_{x\in\underline{E}} (|\Phi(x)|/p(x))$.

We consider then random measures μ such that for every ω the random measure $\mu(\omega, ds; du)$ can be expressed as the difference of two positive valued random measures such that $\int_{[0,t]\times\underline{E}} p(x)|\mu|(\omega, ds, dx) < \infty$. Therefore the M^p valued random measure $\mu(\omega, ds, \cdot)$ restricted to $[0,t]$ has clearly bounded variation (for the norm in M^p).

We write $F^\mu(t)$ for the M^p-valued function on Ω defined by $F^\mu(t,\omega) := \mu(\omega,]0,t], \cdot)$ and call F^μ the <u>primitive process</u> of μ.

The measure μ will be called <u>adapted</u> (resp. optional, resp. predictable) if the process F^μ is the difference of two M_+^p-valued processes F_+^μ and F_-^μ such that for any $\Phi \in C^p$ the real valued processes $\int_{\underline{E}} \Phi(x) F_+^\mu(\cdot, t, dx)$ and $\int_{\underline{E}} \Phi(x) F_-^\mu(\cdot, t, dx)$ are adapted (resp. optional, resp. predictable).

A random measure μ will be called <u>white</u> if F^μ is a weak martingale; i.e. for every $\Phi \in C^p$, $\left(\int_{\underline{E}} \Phi(x) F^\mu(\cdot, t, dx)\right)_{t\in\mathbb{R}^+}$ is a real martingale.

The "predictable dual projection" ν of μ is the unique (up to P-equivalence) predictable ν such that $\mu - \nu$ is white. For the existence of such a ν either see [6], or use [8] 24.

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† $|m|$ denotes the variation of m.

It is to be remarked that, because of the separability of C^p, the real process $(\|F^\mu(\cdot,t,\cdot)\|_p)_{t\in\mathbb{R}^+}$ is optional (resp. predictable) if μ is optional (resp. predictable).

We will make use of the following lemma:

<u>Lemma 1</u>. Let $\mathbb{B}$ be a separable Banach space, $\mathbb{B}'$ its dual, U a $\mathbb{B}'$ valued function on $\mathbb{R}^+\times\Omega$ such that for every $y\in\mathbb{B}$, the real process $\langle y,U\rangle$ is a process with finite variation. Let us assume that Q is an increasing adapted process such that for every predictable (resp. optional) subset A of $\mathbb{R}^+\times\Omega$ and every $y\in\mathbb{B}$ with $\|y\|\le 1$, the following inequality holds:

$$(1.1.1)\qquad E\left(\int 1_A(s,\omega)\langle y,dU_s\rangle\right)\le E\left(\int 1_A(s,\omega)dQ(s)\right).$$

Then there exists a $\mathbb{B}'$ valued process u such that for every $y\in\mathbb{B}$ $\langle y,u\rangle$ is predictable (resp. optional) and for every A predictable (resp. optional)

$$E\left(\int 1_A(s,\omega)\langle y,dU_s\rangle\right)=E\left(\int 1_A(s,\omega)\langle y,u(s,\omega)\rangle dQ(s)\right)$$

Moreover, $\|u\|_{B'}\le 1$.

<u>Proof</u>. The inequality (1.1.1) expresses that $E\left(\int 1_A(s,\omega)dU_s\right)$ as a function of A is a B' valued measure, the variation of which is smaller than the positive measure $A\rightsquigarrow E\left(\int 1_A(s,\omega)dQ(s)\right)$. One has then only to apply a weak Radon-Nikodym theorem of the type of theorem 4 in [9], to get the function u.

For our convenience we will agree to call <u>weakly predictable</u> (resp. weakly optional) a $\mathbb{B}'$-valued process U such that $\langle y,U\rangle$ is a real predictable (resp. optional) process for every $y\in\mathbb{B}$.

1.2 Isometric L^2-stochastic integral with respect to white random measures

Let q be an optional random measure with associated primitive process F^q, with values in $\mathcal{M}^p$, where p is a weight-function as in 1.1. We assume that for every t,

$$E\{|\int_{\underline{E}} p(x)q(\cdot,]0,t],dx)|^2\} < \infty$$

According to the definitions F^q is the difference of two $\mathcal{M}^p$-valued processes F^{q_+} and F^{q_-} and for every (t,ω)

$$\|F^q(t,\omega)\|_p = \|F^{q_+}(t,\omega)\|_p + \|F^{q_-}(t,\omega)\|_p \quad \text{(increasing processes)}$$

We see immediately that $\sum_{\tau\leq t} 1_{\{\tau=s\}} q(\omega,ds,dx)\otimes q(\omega,ds,dy)$ defines for every t a random measure with values in the dual of the space of continuous functions on $\underline{E}\times\underline{E}$ weighted by $p\otimes p$, and we call β the dual predictable projection of this measure. We denote by $b(t,\omega)$ the variation on the interval $[0,t]$ of the measure $\beta(\omega,ds,\cdot)$ for the norm $\|\ \|_{p\otimes p}$.

Applying lemma 1, we define an $\mathcal{M}^{p\otimes p}(\underline{E}\times\underline{E})$-valued process $\mathring{q}$ with the following properties: for every $\Phi \in C^{p\otimes p}(\underline{E}\times\underline{E})$ the real process $\int_{\underline{E}\times\underline{E}} \Phi(x,y)\mathring{q}(s,\omega,dx\otimes dy)$ is predictable and for every real bounded predictable process Y and any t

$$(1.2.1) \quad \int_{]0,t]\times\underline{E}\times\underline{E}} Y(s,\omega)\Phi(x,y)\beta(\omega,s,dx\otimes dy) = \int_{]0,t]} db(s)[\int_{\underline{E}\times\underline{E}} Y(s,\omega)\Phi(x,y)\mathring{q}(s,\omega)\ dx\otimes dy]$$

We write this as an equality between measure valued processes

$$(1.2.2) \quad \int_{]0,t]\times\underline{E}\times\underline{E}} Y(s,\omega)\beta(\omega,s,dx\otimes dy) = \int_{]0,t]} db(s)[\int_{\underline{E}\times\underline{E}} Y(s,\omega)\mathring{q}(s,\omega,dx\otimes dy)]$$

Let $\mathbb{H}$ be a separable Hilbert space, and let us consider the Banach space $\mathbb{L}_1$ of $\mathbb{H}$-valued Borel functions f on $\underline{E}$ such that $\sup_{x\in\underline{E}} (\|f(x)\|/p(x)) < \infty$. $\mathbb{L}_1$ is thus a subspace of $L(M^p,\mathbb{H})$, the space of all bounded linear operators from M^p into $\mathbb{H}$.

We call $\mathbb{L}_1$-simple predictable process an $\mathbb{L}$-valued process Y, which can be written:

$$Y = \sum_i 1_{]s_i,t_i]\times F_i} \cdot a_i$$

where $a_i \in \mathbb{L}_1 \subset L(M^p,\mathbb{H})$, $F_i \in F_i$. The rectangles $]s_i,t_i] \times F_i$ may moreover be assumed disjoint.

We define the $\mathbb{H}$-valued process N by:

$$\text{(1.2.3)} \quad N_t(\omega) := \sum_i 1_{F_i} a_i (F^q_{t_i\wedge t}(\omega) - F^q_{s_i\wedge t}(\omega)) = \sum_i \int_{]s_i\wedge t, t_i\wedge t]\times\underline{E}} a_i(x)q(\omega,ds,dx)$$

If we <u>assume</u> <u>q</u> <u>white</u> then it is easily derived from (1.2.3) that N is an $\mathbb{H}$-valued square integrable martingale. The martingale property of N then implies

$$E\|N_t\|^2 = \sum_i E\left\{1_{F_i} <\int_{]s_i\wedge t,t_i\wedge t]} a_i\circ q(\cdot,ds,\cdot), \int_{]s_i\wedge t,t_i\wedge t]} a_i\circ q(\cdot,ds',\cdot)>_{\mathbb{H}}\right\}$$

But the martingale property for F^q also shows, that the real measure on the Borel subsets of $]s_i,\infty] \times]s_i,\infty]$ generated by the set function

$$]s,t] \times]u,v] \rightsquigarrow E\left\{1_{F_i} <a_i\circ q(\cdot,]s,t],\cdot), a_i\circ q(\cdot,]u,v],\cdot)>_{\mathbb{H}}\right\}$$

gives measure zero to the rectangles $]s,t] \times]u,v]$ as soon as $]s,t] \cap]u,v] = \emptyset$. This measure is therefore concentrated on the diagonal and

$$E\|N_t\|^2 = \sum_i E\left\{1_{F_i} \sum_{s_i<s\leq t_i} <a_i\circ q(\cdot,\{s\},\cdot), a_i\circ q(\cdot,\{s\},\cdot)>_{\mathbb{H}}\right\}$$

Using the fact that the a_i are functions on $\underline{E}$, we can write

$$(1.2.4)\qquad E\|N_t\|^2 = E\left\{\sum_{s\leq t} \langle Y(s,\omega,x),Y(s,\omega,y)\rangle_{\mathbb{H}}\, q(\omega,\{s\},dx)\otimes q(\omega,\{s\},dy)\right\}$$

If we then use the definitions of $\mathring{q}$ and b as given at the beginning of this section 1.1, we may write:

$$(1.2.5)\qquad E\|N_t\|^2 = E\left\{\int_{]0,t]}\left(\int_{\underline{E}\times\underline{E}} \langle Y(s,\omega,x),Y(s,\omega,y)\rangle \mathring{q}(s,\omega,dx\otimes dy)\right)db(s)\right\}$$

Denoting by $\mathcal{S}(\mathbb{L})$ the spaces of $\mathbb{L}$-valued simple predictable processes, we write $\lambda(Y)$ for the process defined by:

$$(1.2.6)\qquad \lambda_s(Y) := \int_{\underline{E}\times\underline{E}} \langle Y(s,\cdot,x),Y(s,\cdot,y)\rangle_{\mathbb{H}}\, \mathring{q}(s,\cdot,dx\otimes dy),\ \forall \in \mathcal{S}(\mathbb{L}_1)\,.$$

From the definition of $\mathring{q}$ it is clear that for every Hilbert valued function Φ, $\int_{\underline{E}\times\underline{E}} \langle\Phi(x),\Phi(y)\rangle_{\mathbb{H}}\, \mathring{q}(s,\cdot,dx\otimes dy)$ is a positive finite or infinite number. We may therefore define $\lambda_s(Y)$ for any function Y on $(\mathbb{R}^+\times\Omega\times\underline{E})$ which is measurable for $\mathcal{P}\otimes\mathcal{B}(\underline{E})$, and consider the space Λ of $\mathbb{H}$-valued $\mathcal{P}\otimes\mathcal{B}(\underline{E})$ measurable functions Y such that

$$\|Y\|_\Lambda := [E(\int_{[0,\ ]}\lambda_s(Y)\,db(s))]^{1/2} < \infty$$

It is not difficult to see that $\|Y\|$ is an Hilbertian seminorm on Λ and that $\mathcal{S}(\mathbb{L}_1)$ is dense in Λ. The equality (1.2.5) shows that the mapping $Y \rightsquigarrow N$ extends into an isometry from Λ into the space of right continuous square integrable martingales for its usual norm. All the properties of a usual stochastic integral are derived for this mapping $Y \rightsquigarrow N$ through the same standard considerations.

In the same way we have expressed $E(\|N_t\|^2)$. We may write, for any predictable rectangle $]s,t]\times E$:

$$E(1_F \cdot \|N_t - N_s\|^2) = E\left\{1_F \cdot \int_{]s,t]} db(s)\left[\int_{\underline{E}\times\underline{E}} \langle Y(s,\cdot,x), Y(s,\cdot,y)\rangle_{\mathbb{H}}\, \mathring{q}(s,\cdot,dx\otimes dy)\right]\right\}$$

which gives immediately:

$$(1.2.7) \qquad \langle N\rangle_t = \int_{]0,t]} \lambda_s(Y)\, db(s)$$

for the Meyer predictable process $\langle N\rangle$ of $\|N\|^2$ (see [11]).

A process Y will be said to be <u>locally in</u> Λ if there exists an increasing sequence (τ_n) of stopping times, such that $\lim_n \tau_n = +\infty$ and the stopped processes $\bar{Y}^{\tau_n}(\omega,t,x) := Y(\omega, t\wedge\tau_n, x)$ belongs to Λ. For such a Y the stochastic process $(\int Y dF^q)$ is, from what precedes, uniquely defined up to P-equivalence and is locally a square integrable martingale.

1.3 <u>Examples</u>

1. Let $m(\omega,ds,dx)$ be a Poisson stationary random measure with Levy measure α and order r (see [13]). The measures $m(\omega,\cdot,\cdot)$ which are denumerable sums of distinct unit discrete masses and the measure α are elements of $\mathcal{M}^p$ where $p(x) = \frac{|x|^r}{|x|^{r+1}}$. For every Borel set A in $\underline{E}$ and $t\in\mathbb{R}^+$ the random variable $\int_{]0,t]\times A} m(\omega,ds,dx)$ is by assumption a Poisson random variable with average $t\cdot\alpha(A)$. We consider the random measure

$$q(\omega,ds,dx) = m(\omega,ds,dx) - ds\otimes\alpha(dx)$$

In this special case it is easily calculated that $\mathring{q}(\omega,s,dx\otimes dy)$ is a measure, independent of ω and s, defined by: $\int \Phi(x,y)\mathring{q}(\omega,s,dx\otimes dy) := \int \Phi(x,x)\alpha(dx)$ and the function $\lambda_t(Y)$ is nothing but

$$\lambda_t(Y) = \int \|Y(\omega,s,y)\|^2 \alpha(dy)$$

The Λ-stochastic integral is therefore nothing else but the one defined by A.V. Skorokhod ([13]).

2. Let μ be an integer valued random measure as described in [6]. Take its predictable dual projection μ^p and consider the white random measure $q = \mu - \mu^p$.

Using the fact that in this particular situation, we have, for all s such that $\mu^p(\omega,\{s\},\underline{E}) = 0$:

$$\int_{\underline{E}\times\underline{E}} \langle Y(s,\omega,x),Y(s,\omega,y)\rangle_{\mathbb{H}}\, q(\omega,\{s\},dx)\otimes q(\omega,\{s\},dy)$$
$$= \int_{\underline{E}} \|Y(s,\omega,x)\|^2_{\mathbb{H}}\mu(\omega,\{s\},dx)$$

A simple calculation shows that the increasing process $\int_{]0,t]} \lambda_s(Y)db_s$ is exactly the process $C^{\infty}(Y,q)$ in [6] Ch. III. The stochastic integral for processes in $G^2(\mu)$ as defined in [6] is therefore the Λ-stochastic integral and the class of processes which are locally in Λ is the class of integrable processes in [6].

1.4 The control couple of a random measure

If the white random measure q is such that, for every predictable stopping time τ the variation $|q|(\omega,\{\tau(\omega)\},E)$ is zero a.s,[†] the process $N := (\int YdF^q)$ has no predictable jump, and we know, as an immediate consequence of the Doob-inequality and the continuity of $\langle N\rangle$, that

$$(1.4.1) \quad E(\sup_{s<\tau} \|N_s\|^2) \leq 4E\langle N\rangle_\tau = 4E\langle N\rangle_{\tau-}$$

From formula (1.2.7) we may write

$$(1.4.2) \quad E(\sup_{s<\tau} \|N_s\|^2) \leq 4E\left(\int_{[0,\tau[} \lambda_s(Y)\,db(s)\right)$$

[†]This is the case in the above example 1.

for all $Y \in \mathcal{E}(\mathbb{L})$ and the same inequality holds by continuity for all Y locally in Λ.

If, on the contrary, the process F^q has jumps on a denumerable family $\{\tau_n\}$ of predictable stopping times, the process $N = (\int YdF^q)$ has predictable jumps only at times τ_n, and, if we introduce as in [10] the pure jump part N^j of N

$$N^j := \sum_n \Delta M_{\tau_n} 1_{[\tau_n,\infty[}$$

We have the following expression for the quadratic variation of N^j

$$\int <Y(\omega,t_n(\omega),x),Y(\omega,t_n(\omega),y>_{\mathbb{H}} q(\omega,\{t_n\}dx) \otimes q(\omega,\{t_n\},dy)$$

$$(1.4.2) \qquad [N^j]_t := \sum_n \|Y(\omega,\tau_n(\omega),x)q(\omega,\{\tau_n\},dx)\|^2_{\mathbb{H}}$$

Then introducing the measure valued process

$$q^j(t,\omega,dx\otimes dy) = \sum_n 1_{\{\tau_n\}}(t)\frac{q(\omega,\{\tau_n\},dx \otimes q(\omega,\{\tau_n\},dy)}{\|q(\omega,\{\tau_n\},\cdot) \otimes q(\omega,\{\tau_n\},dy\|_{\mathcal{M}^{p\otimes p}}}$$

and the increasing process

$$a(t) := \sum_{\tau_n \leq t} \|q(\omega,\{\tau_n\},\cdot) \otimes q(\omega,\{\tau_n\},\cdot)\|_{\mathcal{M}^{p\otimes p}}$$

we see that

$$(1.4.3) \qquad [N^j]_t := \int_{[0,t]} da(s) \int_{\underline{E}\times\underline{E}} <Y(\cdot,s,x),Y(\cdot,s,y)>_{\mathbb{H}} q^j(s,\cdot,dx\otimes dy)$$

Now, it follows from Lemma 1 with $\mathbb{B} = C^{p\otimes p}(\underline{E}\times\underline{E})$

$$U(t,\omega) = 4\int_{]0,t]} \mathring{q}(t,\omega,\cdot)db(s) + 4\int_{]0,t]} q^j(t,\omega,\cdot)da(s)$$

and

$$Q(s,\omega) = 4\int_{]0,t]} \|\mathring{q}(t,\omega,\cdot)\|_{M^{p\otimes p}} db(s) + 4\int_{]0,t]} \|q^j(t,\omega,\cdot)\|_{M^{p\otimes p}} da(s)$$

that there exists a weakly optional process γ, which values in $M^{p\otimes p}$ and a positive increasing regular process A such that for any optional subset G of $\mathbb{R}^+ \times \Omega$:

$$(1.4.4) \qquad 4E\int 1_G(s,\cdot)d(\langle N\rangle_i + [N^j]_s)$$
$$= E\int dA_s \int_{\underline{E}\times\underline{E}} \langle Y(s,\cdot,x),Y(s,\cdot,y)\rangle \gamma(s,\cdot,dx\otimes dy)$$

Moreover $\|\gamma(s,\omega,\cdot)\|_{M^{p\otimes p}} \leq 1$ for all s and ω.

<u>Proposition 1</u>. If q is a white optional random measure, with values in M^p, and γ and A are the above defined processes, for every $\mathbb{H}$-valued process, locally in Λ, and every stopping time τ, we have

$$E(\sup_{t<\tau} \|\int_{]0,t]\times\underline{E}} Y(s,\cdot,x)q(\cdot,ds,dx)\|^2)$$
$$\leq E\left(\int_{]0,\tau[} dA_s \int_{\underline{E}\times\underline{E}} \langle Y(s,\cdot,x),Y(s,\cdot,y)\rangle_{\mathbb{H}} \gamma(s,\cdot,dx\otimes dy)\right)$$

<u>Proof</u>. We have only to use the formula (1.4.4) and the "stopped Doob's Inequality" proved in [10] (see also [11]) which says:

$$\sup_{t<\tau} \|N_t\|^2 \leq 4E[\langle N\rangle_{\tau-} + [N^j]_{\tau-}] \ .$$

<u>Definition</u>. A couple (γ,A) having the properties of the proposition 1 with respect to the random measure q will be called a "<u>control couple</u>" for q.

2. A particular example of a stochastic equation driven by a white random measure: Introductory results

2.1 $\mathcal{M}^P$ is again the dual of the space of continuous functions with bounded weight $p(x)$ on $\underline{E}$. We consider an optional white random measure $q(\omega, ds, dx)$, such that, for every t, $q(\omega,]0,t], \cdot) \in {}^{P}$.

$\mathbb{H}$ being a separable Hilbert space we consider the class of $\mathbb{H}$-valued processes, which are adapted, and the paths of which are right continuous and have left limits in every point (we call them regular $\mathbb{H}$-valued processes).

We consider a "functional $\underset{\sim}{a}$" which, by definition, to each regular $\mathbb{H}$-valued predictable process ξ, and to each $x \in \underline{E}$ associates an $\mathbb{H}$-valued predictable process $(a_s(\omega,\xi,x))_{s\in\mathbb{R}^+}$ with the following properties:

(i) For every s, ω, ξ and $h \in \mathbb{H}$ with $\|h\| \leq 1$, the real function $x \rightsquigarrow \langle h, a_s(\omega,\xi,x)\rangle$ is Borel and such that $\sup_{x\in\underline{E}} |\langle h, a_s(\omega,\xi,x)\rangle|/p(x) < \infty$.

(ii) For every stopping time τ, $a_\tau(\omega,\xi,x) = a_\tau(\omega,\xi',x)$, as soon as $\xi_s(\omega) = \xi'_s(\omega)$ for all $s < \tau$.

(iii) The following "Lipschitz-condition" is fulfilled for every $t \in \mathbb{R}^+$:

$$\text{(L)} \qquad \sup_{x\in\underline{E}} \left(\|a_t(\cdot,\xi,x) - a_t(\cdot,\xi',x)\|_{\mathbb{H}}/p(x)\right) \leq L_t \sup_{s<t} \|\xi_s - \xi'_s\|_{\mathbb{H}} .$$

where $(L_t)_{t\in\mathbb{R}^+}$ is an increasing positive adapted process.

V_t being an $\mathbb{H}$-valued regular process we consider the following stochastic integral equation:

$$\text{(2.1)} \qquad \xi_t = V_t + \int_{]0,t]}\int_{\underline{E}} a_s(\omega,\xi,x)\, q(\omega,ds,dx) .$$

A process ξ defined on a stochastic interval $[0,\tau]$ (resp. $[0,\tau[$) will be called a strong solution of (2.1) if processes on both sides of (2.1) are equal up to P-equivalence on $[0,\tau]$ (resp. $[0,\tau[$).

2.2 Remark. Although, in many respects, this equation is more general than the ones considered in [12], [6], [4], because of the generality on q, the boundedness assumption (i), which will be removed later, is a restriction compared to the hypothesis made in the just mentioned papers, when considering their particular settings. We add it here in order to demonstrate in a simple way, the method which will be applied in a more general setting in Section 4.

2.3 Existence, uniqueness, non explosion and stability statements

Theorem 1. (1) Under the assumptions made in 2.1 above, there exists a unique stopping time τ and a unique (up to P-equivalence) process ξ on $[0,\tau[$ such that

(i) τ is predictable and on the set $\{\tau < \infty\}$ we have

$$\limsup_{t\to\tau} \|\xi_t\|_{\mathbb{H}} = +\infty$$

(i.e. τ is, when finite, an explosion time).

(ii) ξ is a strong solution of (2.1) on $[0,\tau[$.

(2) If, to the previous hypothesis, we add the following one: there exists a constant such that

$$\sup_{x\in \underline{E}} [\|a_t(\omega,\xi,x)\|_{\mathbb{H}}/p(x)] \leq d(1+\sup_{s\leq t} \|\xi_s\|_{\mathbb{H}})$$

then $P\{\tau = +\infty\} = 1$ (no explosion).

Theorem 2 (Stability theorem). Let us consider the equation (2.1) and the equation:

$$(2.2) \qquad \xi'_t = V_t + \int_{[0,t]}\int_{\underline{E}} a_s(\omega,\xi',x)q'(\omega,ds,dx)$$

where q' is another white random measure in the same space $\mathbb{B}'$, the functional a being submitted to hypotheses (i), (ii), (iii).

We consider (γ,A) the control couple of q, as defined in 1.3 and (γ^1,A^1) the control couple of $q-q'$.

Then, if ξ is a solution of (2.1) on $[0,\tau[$, equation (2.2) admits a solution ξ' on $[0,\tau[$ and moreover, for every $\varepsilon > 0$, there exists a stopping time, τ_ε, such that $P\{\tau_\varepsilon < \tau\} \leq \varepsilon$ and

$$E\{\sup_{t\leq u_\varepsilon} \|\xi_t-\xi'_t\|^2_{\mathbb{H}}\} \leq R_\varepsilon(\rho_\varepsilon)$$

where ρ_ε is any number such that $P\{A^1_{\tau-} \geq \rho_\varepsilon\} \leq \frac{\varepsilon}{2}$ for some function R_ε such that $\lim_{\rho\to 0} R_\varepsilon(\rho) = 0$. (This function R_ε depends only on ε, A and L.)

2.4 Proof of Theorem 1.1. This proof follows the same line as the proof of the existence and uniqueness theorem in [11]. We will therefore omit a few details.

Everything is based on the following "local existence and uniqueness lemma":

Lemma 2. Let us assume that equation (2.1) has a solution ξ^o on the stochastic interval $[0,\tau]$ (τ may be identically zero); then there exists a stopping time σ such that $P\{\sigma > \tau\} > 0$ and ξ^o can be extended into a solution ξ of (2.1) on $[0,\sigma]$.

Proof. Call $\hat{\xi}^o$ the process equal to ξ^o on $[0,\tau]$ and to ξ_τ on $[\tau,\infty[$, and $\eta(s) = \sup_{x\in E} (\|a_s(\omega,\hat{\xi}^o,x)\|_{\mathbb{H}}/p(x))$. We can clearly choose a positive number ℓ, such that the stopping time σ:

$$\sigma := \begin{cases} \inf\{t:\ t\geq\tau,\ (1\vee L_t)(A_t-A_\tau) > \frac{1}{2}\} & \text{on } \{\sup_{s\leq\tau} (\|\xi^o_s\|_{\mathbb{H}}+\eta(s)) \leq \ell\} \\ \tau \quad \text{on the set } \{\sup_{s\leq\tau} \|\xi_s\|_{\mathbb{H}} + \eta(s) > \ell\} \end{cases}$$

has the property $P\{\sigma > \tau\} > 0$.

Then we introduce the following complete metric space $\mathbb{M}$ of $\mathbb{H}$-valued regular processes on $[0,\sigma[$:

$$\mathbb{M} := \{\xi\colon \xi\cdot 1_{[0,\tau]} = \xi\cdot 1^{o}_{[0,\tau]},\ E\{\sup_{0\leq t<\sigma} \|\xi_t\|^2_{\mathbb{H}}\cdot 1_{\{\sigma>\tau\}}\} < \infty\}$$

with the metric

$$\zeta(\xi,\xi') = [E\{\sup_{\tau\leq t<\sigma} \|\xi_t - \xi'_s\|^2_{\mathbb{H}}\}]^{1/2}$$

According to the choice of σ this space is not empty. With each $\xi \in \mathbb{M}$ we may then associate the process $\Phi\xi$ defined by

$$\Phi_t\xi := V_t + \int_{]0,t]} ds \int_{\underline{E}} a_s(\omega,\xi,x)q(\omega,ds,dx)$$

The properties of the control couple (γ,A) for q give:

$$\begin{aligned}
&E(\sup_{0\leq t<\sigma} \|\Phi_t\xi\|^2\cdot 1_{\{\sigma>\tau\}}) \\
&\quad\leq 2E(\sup_{0\leq t\leq\tau} \|\xi_t\|^2_{\mathbb{H}}\cdot 1_{\{\tau<\sigma\}}) \\
&\qquad + 2E(\int_{[\tau,\sigma[} dA_t(\int_{\underline{E}\times\underline{E}} \langle a_t(\omega,\xi,x),a_t(\omega,\xi,y)\rangle_{\mathbb{H}}\,\gamma(\omega,t,dx\otimes dy))) \\
&\quad\leq 2\ell^2 + 2E(\int_{[\tau,\sigma[} (L_t(\sup_{s<t} \|\xi_s - \hat{\xi}^{o}_s\|^2_{\mathbb{H}}) + \ell^2))dA_t \\
&\quad\leq 2\ell^2 + E(\sup_{\tau\leq s<\sigma} \|\xi_s\|^2) + \ell^2 < \infty
\end{aligned}$$

Therefore Φ maps $\mathbb{M}$ into $\mathbb{M}$ and the same usage of (γ,A) and of the definition of σ leads to:

$$
\begin{aligned}
E\Big(\sup_{0\le t<\sigma}\|\Phi_t\xi-\Phi_t\xi'\|^2_{\mathbb{H}}\Big) &\le E\int_{[\tau,\sigma[}dA_t\Big(\int_{\underline{E}\times\underline{E}}\langle a_t(\omega,\xi,x)-a_t(\omega,\xi',x),\\
&\qquad a_t(\omega,\xi,y)-a_t(\omega,\xi',y)\rangle_{\mathbb{H}}\gamma(\omega,t,dx\otimes dy)\Big)\\
&\le E\Big(\int_{[\tau,\sigma[}L_t\sup_{s<t}\|\xi_s-\xi'_s\|^2_{\mathbb{H}}\,dA_t\Big)\\
&\le \tfrac{1}{2}E\big(\sup_{s<\sigma}\|\xi_s-\xi'_s\|^2_{\mathbb{H}}\big) \quad .
\end{aligned}
$$

This inequality shows that the mapping Φ is a contraction in $\mathbb{M}$ and there is therefore a unique fixed point ξ for the mapping Φ: $\xi = \phi(\xi)$. But noticing, from property (ii) in 2.1 that the process $a_s(\omega,\xi,x)$ is then defined on $[0,\sigma]$, and setting

$$\xi_\tau := V_\tau + \int_{]0,\tau]}\int_{\underline{E}} a_s(\omega,\xi,x)q(\omega,ds,dx)$$

we get the unique extension ξ of ξ^o which is a solution of (2.1) on $[0,\sigma]$.

We can now conclude the proof of theorem 1. We skip details which may be found in [11] Ch. III.§6 or in [10]. The class of couples (ξ,σ) where ξ is a solution of (2.1) on $[0,\sigma]$ is non empty according to the above lemma (apply it with $\tau = 0$, $\xi_0 = V_0$). We define τ as the essential supremum of the family of stopping times σ, and choose among the above σ's an increasing denumerable family (σ_n) such that $[0,\tau[\ = \bigcup_n [0,\sigma_n]$. Because of the uniqueness of the solution on every $[0,\sigma_n]$ (take two solutions ξ, ξ' on $[0,\sigma_n]$, consider the stopping time $\tau = \inf\{t: \|\xi_t-\xi'_t\| \ge 0\}\cap\sigma_n$: it follows immediately from the above lemma that $\tau = \sigma_n$) the process ξ is uniquely defined on $[0,\tau[$ by saying that its restriction to $[0,\sigma_n]$ is the unique solution on $[0,\sigma_n]$. Defining the stopping times σ'_n by

$$\sigma'_n := \inf\{t: \|\xi_t\|_{\mathbb{H}} \ge n\}\cap\sigma_n$$

we see that $[0,\tau[\ = \bigcup_n [0,\sigma'_n]$.

If for some n the inequality $P\{\sigma_n' = \tau < \infty\} > 0$ were to hold, we could find, using the lemma, a stopping time σ' such that $P\{\sigma' > \tau\} > 0$ and a solution on $[0,\sigma']$. This would contradict the definition of τ. Therefore $\sigma_n' < \tau$ on $\{\tau < \infty\}$ with probability one which implies: either $\tau = +\infty$ or $\lim_n \|\xi_{\sigma_n'}\| = +\infty$. This proves too that τ is predictable.

2.5 Proof of theorem 1.2°

The proof, as in [11], rests on a lemma which we have proved in [11], and which we state again:

Lemma 3. Let A be an adapted increasing positive process defined on the stochastic interval $[0,\tau[$, and bounded by ℓ. Then, for every adapted increasing positive process Φ such that

$$E(\Phi_{\sigma-}) \leq K + \rho E\left(\int_{[0,\sigma[} \Phi_{s-} dA_s\right) \quad (\rho \text{ and } K \text{ constants})$$

for all stopping time $\sigma \leq \tau$, the following inequality holds

$$E(\Phi_{\tau-}) \leq 2K \sum_{j=0}^{[2\rho\ell]} (2\rho\ell)^j$$

where $[x]$ denotes the integer part of x.

The proof of the non explosion part of the theorem (compare with [11], Ch. III §2) consists in setting $\Phi_t = \sup_{s<t} \|\xi_s\|$, where ξ is the maximal solution of (2.1) defined on $[0,\tau[$ as above, and to use again the control couple (γ,A) to write for every $\sigma \leq \tau$:

$$(2.5.1) \qquad E(\Phi_{\sigma-}) \leq 2E(\sup_{s<\sigma} \|V_s\|^2_{\mathbb{H}}) + 2E\left(\int_{]0,\sigma[} d(1 + \sup_{s<t} \|\xi_s\|_{\mathbb{H}})\, dA_t\right)$$

If $\sigma_n := \inf\{t: (A_t + \sup_{s\leq t} \|V_s\|^2_{\mathbb{H}}) > n\} \cap \tau$, we have for every stopping time $\sigma \leq \sigma_n$:

(2.5.2) $\quad E(\Phi_{\sigma^-}) \leq 2n(1+d) + 2d\int_{]0,\sigma[} (\Phi_{t^-})dA_t$.

This last inequality together with the lemma 2 implies $E(\Phi_{\sigma_n^-}) < \infty$. But, since $\lim_{n\to\infty} P\{\sigma_n = \tau\} = 1$, τ cannot be an explosion time and $P\{\tau = \infty\} = 1$.

3. Λ-spaces of processes and associated stochastic integrals

This concept of Λ-space is suggested by different approaches to stochastic integration, in order to propose a unique model in situations apparently as different as the isometric Hilbert valued integral (see [8]) and integration with respect to random measures as described above.

3.1 Λ-spaces

Let $\mathbb{B}$ be a Banach space, $\mathbb{H}$ a Hilbert space, $\mathbb{L}$ a closed subspace of the Banach space of bounded linear operators from $\mathbb{B}$ into $\mathbb{H}$ (with the uniform norm) A and $\tilde{A}$ two positive increasing adapted processes. We consider a vector space of processes, the values of which are (possibly unbounded) operators from $\mathbb{B}$ into $\mathbb{H}$. This vector space will be called a <u>Λ-space associated with $\mathbb{L}$, A, $\tilde{A}$, and the functional λ</u>, if there exists an increasing sequence (τ_n) of stopping times such that $\lim_n \tau_n = +\infty$ a.s. and

(i) for each $\Phi \in \Lambda$, $(\lambda_t(\Phi))_{t\in\mathbb{R}}$ is a positive adapted process such that for each n, $E(\tilde{A}_{\tau_n^-} \cdot \int_{]0,\tau_n[} \lambda_t(\Phi)dA_t) < \infty$.

(ii) The mappings $\Phi \rightsquigarrow E(\tilde{A}_{\tau_n^-} \cdot \int_{]0,\tau_n[} \lambda_t(\Phi)dA_t)$, $n \geq 0$ are seminorms on Λ giving to Λ a structure of complete vector space.

(iii) The set of simple predictable $\mathbb{H}$-valued processes is a dense subspace of Λ.

We denote by $\Lambda(\mathbb{L},A,\tilde{A},\lambda)$ such a Λ-space associated with $\mathbb{L}$, Λ and λ.

A process X is said to be <u>locally in Λ</u>, if there exists an increasing sequence (σ_n) of stopping times, such that $\lim_n \sigma_n = +\infty$ and the process $1_{]0,\sigma_n]}X$ is in Λ for every n.

3.2 <u>Λ-stochastic integral</u>

Let Z be a $\mathbb{B}$-valued regular process. We say that the <u>Λ-space $\Lambda(\mathbb{L},A,\tilde{A},\lambda)$ is associated with $\mathbb{Z}$</u> if, for every simple predictable $\mathbb{L}$-valued process Y and every stopping time σ the following inequality holds:

$$E(\|\int YdZ^{\sigma}\|^2_{\mathbb{H}}) \leq E(\tilde{A}_{\sigma-}\cdot\int_{]0,\sigma[}\lambda_s(Y)dA_s)$$

where, by definition, for $Y := \sum_i a_i\cdot 1_{]s_i,t_i]\times F_i}$ the $\mathbb{H}$-valued random variable $\int YdZ^{\sigma}$ is given by

$$\int YdZ^{\sigma} := \sum_i 1_{F_i}\cdot a_i(Z^{\sigma}_{t_i}-Z^{\sigma}_{s_i})$$

where Z^{σ} is the process <u>stopped strictly before σ</u>. (See [11], Ch. I.)

$$Z^{\sigma}_t(\omega) := \begin{cases} Z_t(\omega) & \text{if } t < \sigma(\omega) \\ Z_{\sigma-}(\omega) & \text{if } t \geq \sigma(\omega) \end{cases}$$

The definition of the Λ-stochastic integral for every process Y which belongs locally to Λ is immediate from the inequality (3.2.1) (see [11] Ch. I §2).

We say that the Λ-space $\Lambda(\mathbb{L},A,\tilde{A},\lambda)$ is $*$-associated with $\mathbb{Z}$, or that $\mathbb{Z}$ is controlled in $\Lambda(\mathbb{L},A,\tilde{A},\lambda)$ if for every stopping time σ and every single $\mathbb{L}$-valued predictable process Y (therefore, for every $Y \in \Lambda(\mathbb{L},A,\tilde{A},\lambda)$) the following inequality holds:

$$E(\sup_{s<\tau} \|\int_{]0,s]} YdZ\|^2_{\mathbb{H}}) \leq E(\tilde{A}_{\tau-}\cdot\int_{[0,\tau[} \lambda_s(Y)dA_s) .$$

4. Examples of Λ-spaces

4.1 Example 1. Let A be an increasing adapted, right continuous process. The space Λ of all predictable processes Φ with values in $\mathcal{L}(\mathbb{H}:\mathbb{G})$ such that $A_t\cdot\int_{]0,t]} \|\Phi(s)\|^2 dA_s$ is a finite random variable for all t is clearly a Λ-space $\Lambda(\mathcal{L}(\mathbb{H};\mathbb{G}),A,A,\lambda)$ with $\lambda_t(\Phi) = \|\Phi_t\|^2$.

Let us assume that Z is a regular $\mathbb{H}$-valued semimartingale. Let A be a $*$-dominating process for Z: this means (cf. [11]) that, for every stopping time σ and every simple $\mathcal{L}(\mathbb{H};\mathbb{G})$-valued predictable process Y, we have

$$E\{\sup_{0\leq t<\sigma} \|\int_{]0,t]} YdZ\|^2_{\mathbb{G}}\} \leq E\{A_{\sigma-}\cdot\int_{]0,\sigma]} \|Y_s\|^2 dA_s\}$$

Therefore Z is controlled in $\Lambda(\mathcal{L}(\mathbb{H},\mathbb{G}),A,A,\lambda)$.

It is clear, from the definition, that every locally bounded predictable process Y belongs to Λ and its stochastic integral, as defined in section 3.1 above, is nothing but the usual one.

4.2 Example 2. The isometric integral with respect to martingales.

Let M be an $\mathbb{H}$-valued right continuous square integrable martingale, $\mathbb{H}$ being an Hilbert space. It was proved (see [12]) that the set function

$]s,t] \times F \rightsquigarrow E\{1_F \cdot (M_t - M_s)^{\otimes 2}\}$ can be extended into a measure μ_M on P, with values in $\mathbb{H} \hat{\otimes}_1 \mathbb{H}$ and is absolutely continuous with respect to the real measure α_M: $]s,t] \times F \rightsquigarrow E(1_F \cdot (\|M_t - M_s\|^2))$ which actually turns out to be the variation of μ_M. It was shown in [12] that there exists a predictable $\mathbb{H} \hat{\otimes}_1 \mathbb{H}$ valued process Q_M, defined up to an equivalence as the density of μ_M with respect to α_M, taking its values in the set of positive symmetric elements of $\mathbb{H} \hat{\otimes}_1 \mathbb{H}$ and such that trace $Q_M(\omega,t) = \|Q_M(\omega,t)\|_{\mathbb{H}\hat{\otimes}_1 \mathbb{H}} = 1$, α_M a.e.

Denoting by $\tilde{Q}_M(\omega,t)$ the nuclear operator from $\mathbb{H}$ into $\mathbb{H}$ defined by $\langle h, \tilde{Q}_M(\omega,t) g\rangle_{\mathbb{H}} = \langle h \otimes g, Q_M(\omega,t)\rangle$, what we proved in [12] can be rephrased by saying that the following space is a Λ-space: a process X belongs to Λ iff for every (ω,t) the domain of the operator $X(\omega,t)$ contains $\tilde{Q}_M^{1/2}(\omega,t)(\mathbb{H})$,

$$\int_{\mathbb{R}^+ \times \Omega} \operatorname{trace}(X \circ \tilde{Q}_M \circ X^*)\, d\alpha_M = \int_{\mathbb{R}^+ \times \Omega} \|X \circ \tilde{Q}_M\|_{H.S}^{1/2} d\alpha_M < \infty$$

(where $\|\cdot\|_{H.S}$ denotes the Hilbert-Schmidt norm for operators) and X lies in the closure of the simple predictable $\mathcal{L}(\mathbb{H};\mathbb{G})$-valued processes for the Hilbert norm $[E(\int_{]0,\infty[} \lambda_s(X)\, d\langle M_s\rangle)]^{1/2}$, where

$$\lambda_s(X) = \operatorname{trace}(X_s \circ \tilde{Q}_M(s) \circ X_s^*) .$$

We have moreover in this case the isometry formula:

$$E\left\|\int_{]0,\infty[} X dM\right\|^2 = E\left(\int_{]0,\infty[} \lambda_s(X)\, d\langle M\rangle_s\right) .$$

The space $\Lambda(\mathcal{L}(\mathbb{H};\mathbb{G}), 1, \langle M\rangle, \lambda)$ is therefore the Λ-space associated with M.

If we use the fact that $N := (\int Y dZ)$ is a square integrable martingale for every simple Y, and

$$\langle N\rangle_t = \int_{]0,t]} \mathrm{trace}(X\circ\tilde{Q}_M\circ X^*)\,d\langle M\rangle \qquad ([11]\ §14)$$

$$[N^j]_t = \sum_{\tau_n \leq t} \|Y_{\tau_n}\circ\Delta Z_{\tau_n}\|^2$$

where the τ_n's are predictable stopping times, it is easily seen that one can write

$$\langle M\rangle_t + [N^j]_t = \int_{]0,t]} \mathrm{trace}(X\circ\tilde{\tilde{Q}}_M\circ X^*)\,d(\langle M\rangle + [M^j])$$

for some optional nuclear operator valued process $\tilde{\tilde{Q}}_M$. Using the basic inequality in [10] (or [11] §10), and defining $\mu_t(Y) = \mathrm{trace}(X\circ\tilde{\tilde{Q}}_M\circ X^*)$, we may write

$$E\|\sup_{s<\tau} \int_{]0,s]} XdM\|^2_{\mathbb{G}} \leq 4E\int_{]0,\tau[} (\mu_s X)\,d[\langle M\rangle_s + [M^j]_s]$$

M is therefore controlled in $\Lambda(\mathcal{L}(\mathbb{H};\mathbb{G});4,\langle M\rangle+[M^j],\mu)$.

4.3 <u>Example 3</u>. The §1 provides us immediately with a third example of two Λ-spaces: in this example $\mathbb{L}_1$ is the set of continuous mappings from $\mathbb{B}'$ into $\mathbb{H}$, which are defined by Borelian mappings Φ from $\underline{E}$ into $\mathbb{H}$ such that $\|\Phi(x)\|/p(x)$ is bounded. If we define

$$\lambda_s(Y) = \int_{\underline{E}} \langle Y(\omega,s,x),Y(\omega,s,y)\rangle\, \mathring{q}\,(\omega,s,dx\otimes dy)$$

$$\mu_s(Y) = \int_{\underline{E}} \langle Y(\omega,s,x),Y(\omega,s,y)\rangle \gamma(\omega,s,dx\otimes dy)$$

and b, A and F^q as in §1.2, then we see that $\Lambda(\mathbb{L},1,b,\lambda)$ is a Λ-space associated with F^q, while F^q is controlled in $\Lambda(\mathbb{L},1,A,\mu)$ (with $A = b$ and $\lambda = \mu$ where F^q has no predictable jump).

4.4 Remark. With these three examples as building blocks, the reader can produce a variety of Λ-spaces associated with processes and stochastic integrals.

5. Theorems for general stochastic integral equations

5.1 The general equation under consideration

We consider a Banach valued process Z (with values in $\mathbb{B}$). We assume Z controlled in the Λ-space $\Lambda(\mathbb{L},\tilde{A},A,\lambda)$ where $\mathbb{L}$ is a closed subspace of $\mathcal{L}(\mathbb{B};\mathbb{H})$ ($\mathbb{H}$ separable Hilbert space), and V is a regular $\mathbb{H}$-valued process.

The equation under study is the following:

$$(5.1.1) \qquad \xi_t = V_t + \int_{]0,t]} \underset{\sim}{a}_s \xi \, dZ_s$$

where the functional $\underset{\sim}{a}$ has the following properties:

(i) For every regular $\mathbb{H}$-valued process ξ, $\underset{\sim}{a}\xi$ is a process locally in $\Lambda(\mathbb{L},\tilde{A},A,\lambda)$.

(ii) For every stopping time τ the random variable $a_\tau \xi$ depends only on the values of ξ on $[0,\tau[$.

(iii) For every $\beta > 0$, there exists an increasing adapted positive process L^β such that for every couple (ξ,ξ') of $\mathbb{H}$-valued regular processes for which $\sup\limits_s \|\xi_s\|_{\mathbb{H}} \leq \beta$, and $\sup\limits_s \|\xi'_s\| \leq \beta$, and for every $t \in \mathbb{R}^+$ the following Lipschitz condition holds:

$$(L_1) \qquad \lambda_t(\underset{\sim}{a}\xi-\underset{\sim}{a}\xi') \leq L_t^\beta \sup_{s<t} \|\xi_s-\xi'_s\|^2$$

5.2 Typical example

A typical example of equation (5.1.1) is the following:

$$\xi_t = V_t + \int_{]0,t]} \underset{\sim}{a}^1_s \xi dS_s + \int_{]0,t]} \underset{\sim}{a}^2_s \xi dM_s + \int_{]0,t]}\int_{\underset{\sim}{E}} \underset{\sim}{a}^3(\cdot,s,\xi,x)q(\cdot,ds,dx)$$

where S is a $\mathbb{G}$-valued semi-martingale, M a $\mathbb{K}$-valued square integrable martingale ($\mathbb{G}$ and $\mathbb{K}$ Hilbert) and q is a white random measure of some order α. To S, M and F^q we associate the Λ-spaces defined in examples of §3, and assume for the functionals $\underset{\sim}{a}^1$, $\underset{\sim}{a}^2$ and $\underset{\sim}{a}^3$ properties (i) to (iii) above.

By considering the process Z, with components S, M and F^q, taking its values in $\mathbb{G}\times\mathbb{K}\times\mathcal{M}^r$ where $\mathcal{M}^r$ is the Banach space of measures weighted by $\dfrac{|x|^r}{|x|^r+1}$, we see immediately that the situation reduces to the one described in 5.1. It is to be noted here that, in this situation, $\underset{\sim}{a}^3(\omega,s,\xi,\cdot)$ is no longer necessarily continuous in x as in §1 above.

The reader will check for himself that the Lipschitz condition on $\underset{\sim}{a}^3$ expresses in our general context the one considered by A.V. Skorokhod [cf. [13]) and others ([4], [6]).

5.3 Existence, uniqueness, non explosion theorems

Theorem 3. Under the assumptions made in 5.1, there exists a unique stopping time τ and a unique (up to P-equivalence) process ξ on $[0,\tau[$ such that

(1) τ is predictable and on the set $\{\tau<\infty\}$ we have $\limsup_{t\uparrow\tau} \|\xi_t\|_{\mathbb{H}} = +\infty$ (i.e. when finite $\tau(\omega)$ is an explosion time).

(2) ξ is a strong solution of (5.1.1) on $[0,\tau[$.

Proof. As in the proof of Theorem 1 the core of the proof consists in proving a "local existence and uniqueness lemma" which reads exactly as lemma 2. Because the Lipschitz coefficient process L^β in (L_1) depends on the bound β for the processes ξ and ξ', a slight modification in the proof has to be made.

We define $\hat{\xi}^o$ as in the proof of lemma 2 and $\eta(s) = \lambda_s(a\hat{\xi}^o)$. We choose a $\beta > 0$ such that

$$P\{\sup_{s\leq\tau}(\|\xi^o_s\|_{\mathbb{H}} + \eta(s)) \leq \beta\} > 0$$

and then define

$$\sigma := \inf\{t:\ t \geq \tau, (1 \vee L^{2\beta}_t)\tilde{A}_t(A_t - A_\tau) > \tfrac{1}{2}\} \text{ on the set } \{\sup_{s\leq\tau} \|\xi^o_s\|_{\mathbb{H}} + \eta(s)) \leq \beta\}$$

and

$$\sigma := \tau \text{ on } \{\sup_{s\leq\tau} (\|\xi^o_s\|_{\mathbb{H}} + \eta(s)) > \beta\}$$

For every regular $\mathbb{H}$-valued process ξ we set $\xi^{2\beta} = (1 \wedge \frac{2\beta}{\|\xi\|_{\mathbb{H}}})\cdot\xi$ and consider the functional

$$\underset{\sim}{a}^{2\beta}(\xi) := \underset{\sim}{a}(\xi^{2\beta}) .$$

The same reasoning as in the proof of lemma 1 shows that $\xi \rightsquigarrow \Phi^{2\beta}_t\xi := V_t + \int_{]0,t]} \underset{\sim}{a}^{2\beta}_s(\xi)dZ_s$ maps $\mathbb{M}$ into $\mathbb{M}$ and

$$E(\sup_{0\leq t<\sigma} \|\Phi^{2\beta}_t\xi - \Phi^{2\beta}_t\xi'\|_{\mathbb{H}}) \leq E\{\tilde{A}_{\sigma-}\cdot\int_{[\tau,\sigma[} L^{2\beta}_t \sup_{s<t} \|\xi_s - \xi'_s\|^2_{\mathbb{H}}\, dA_s\}$$

in view of the properties of $\Lambda(\mathbb{L},\tilde{A},A,\lambda)$ and of property (L_1).

Therefore

$$\delta(\Phi^{2\beta}_t\xi, \Phi^{2\beta}_t\xi') \leq \tfrac{1}{2}\delta(\xi,\xi')$$

and $\Phi^{2\beta}$ is a contraction.

If we define then

$$\sigma' =: \inf\{t: t \geq \tau, \|\xi_t\|_{\mathbb{H}} > 2\beta\} \wedge \sigma$$

and notice that $P\{\sigma' > \tau\} > 0$ and $\Phi^{2\beta}(\xi)$, $V_t + \int_{]0,t]} \underset{\sim}{a}_s \xi \, dZ_s$ are two P-equivalent processes on $[\tau, \sigma'[$, we see the lemma is proved. The end of the proof of the theorem goes very much like the proof of theorem 1, and details are left to the reader.

<u>Theorem 4</u>. If, to the assumptions given in §5.1, we add the following one:

$$\lambda_t(\underset{\sim}{a}\xi) \leq d(1 + \sup_{s<t} \|\xi_s\|^2_{\mathbb{H}})$$

for some constant d, then $P\{\tau = \infty\} = 1$ (no explosion).

<u>Proof</u>. Doing the same as in the proof of theorem 1.2° if we set $\Phi_t = \sup_{s \leq t} \|\xi_s\|^2_{\mathbb{H}}$, we may write for every stopping time $\sigma \leq \tau$

$$E(\Phi_{\sigma-}) \leq 2E(\sup_{s<\sigma} \|V_s\|^2_{\mathbb{H}}) + 2E(\sup_{s<\sigma} \|\int \underset{\sim}{a}_s \xi \, dZ_s\|^2_{\mathbb{H}})$$

$$\leq 2E(\sup_{s<\sigma} \|V_s\|^2_{\mathbb{H}}) + 2E(\tilde{A}_{\sigma-} \cdot \int_{]0,\sigma[} \lambda_s \xi \, dA_s) .$$

Defining then for every n

$$\sigma_n := \inf\{t: A_t \vee \tilde{A}_t + \sup_{s \leq t} \|V_s\|^2_{\mathbb{H}} > n\} \wedge \tau$$

we have for every stopping time $\sigma \leq \sigma_n$:

$$E(\Phi_{\sigma-}) \leq 2n(1+d) + 2d \int_{]0,\sigma[} \Phi_t \, dA_t$$

We use the lemma 3 to see that $E(\Phi_{\sigma-}) < \infty$ and therefore, since $\lim_{n\to\infty} P\{\sigma_n = \tau\} = 1$, τ is a.s. not an explosion time.

5.4 Stability theorems

We consider two equations:

$$(5.4.1) \qquad \xi_t = V_t + \int_{]0,t]^{\sim}} \underset{\sim}{a}\xi_s \, dZ_s$$

$$(5.4.2) \qquad \xi'_t = V'_t + \int_{]0,t]^{\sim}} \underset{\sim}{a}'\xi'_s \, dZ'_s$$

of the type considered in 5.1.

We assume more precisely that Z and Z' are $\mathbb{B}$-valued regular processes controlled respectively in $\Lambda(\mathbb{L},\tilde{A},A,\lambda)$ and $\Lambda(\mathbb{L},\tilde{A}',A',\lambda')$ where $\mathbb{L}$ is a closed subspace of $\mathcal{L}(\mathbb{B};\mathbb{H})$ and V and V' are regular $\mathbb{H}$-valued processes. The functionals $\underset{\sim}{a}$ and $\underset{\sim}{a}'$ verify the conditions (i) and (ii) with respect to the Λ-spaces considered for Z and Z' and the condition (iii) with the same Lipschitz coefficient-processes L (independent of $\beta > 0$).

ξ being the solution of (5.4.1) on $[0,\sigma[$ we define and assume:

$$(5.4.3) \qquad d_1 := E\{\sup_{s<\sigma} \lambda_s(a\xi - a'\xi)\} < \infty$$

$$(5.4.4) \qquad d' := E \sup_{s<\sigma} \|V_s - V'_s\|^2_{\mathbb{H}} < \infty$$

The proximity of Z and Z' will be expressed through the consideration of $Z-Z'$ and making the following assumptions:

(iv) $Z - Z'$ is controlled in $\Lambda(\mathbb{L};\tilde{Q},Q,\mu)$

(v) for every regular processes ξ and ξ'

$$\mu_t(a'\xi - a'\xi') \le L_t \sup_{s<t} \|\xi_s - \xi'_s\|^2_{\mathbb{H}}$$

(vi) For the solution ξ of (5.4.1) on $[0,\sigma[$

$$d_2 := E\{\sup_{s<\sigma} \mu_s(a\xi - a'\xi)\} < \infty$$

(vii) $c := E\{\sup_{s<\sigma} \mu_s(\underset{\sim}{a}\xi)\} < \infty$

<u>Theorem 5</u>. Under the above assumptions and the hypothesis that the positive random variables $A_{\sigma-}$, $\tilde{A}_{\sigma-}$, $A'_{\sigma-}$, $\tilde{A}'_{\sigma-}$, $Q_{\sigma-}$, $\tilde{Q}_{\sigma-}$ are finite, the equation (5.4.2) has a (unique) strong solution ξ' on $[0,\sigma[$. Let $\varepsilon > 0$ be given and q be a positive number such that $P\{L_{\sigma-}\tilde{Q}_{\sigma-} \vee \tilde{Q}_{\sigma-}Q_{\sigma-} \vee Q_{\sigma-} \geq q\} < \frac{\varepsilon}{2}$.

Then there exists a function $R_\varepsilon(d_1,d_2,q)$, determined by the functional $\underset{\sim}{a}$ and the processes A, $\tilde{A}$ and L only, such that $\lim_{d_1,d_2,q\to 0} R_\varepsilon(d_1,d_2,q) = 0$, and a stopping time $\sigma_\varepsilon \leq \sigma$ such that:

(a) $P\{\sigma_\varepsilon < \sigma\} \leq \varepsilon$

(b) $E(\sup_{t<\sigma_\varepsilon} \|\xi_t - \xi'_t\|^2) \leq R_\varepsilon(d_1,d_2,q)$

Let $\ell > 0$ such that

$$P\{\tilde{A}_{\sigma-}A_\sigma \vee L_{\sigma-}A'_{\sigma-} \vee A'_{\sigma-} \geq \ell\} \leq \frac{\varepsilon}{2}$$

A function R_ε is given by

$$R_\varepsilon(d_1,d_2,q) := 2K \sum_{j=0}^{[4\rho\ell]} [4\rho\ell]^j$$

where

$$K := 6(d' + d_1\ell + d_2 q + cq) \ , \quad \rho := \ell + q$$

<u>Proof</u>. In the same way as in [11] §7, and assuming that (5.4.2) has a solution ξ' on $[0,\sigma[$, we define the stopping time σ_ε by setting

$$\sigma_\varepsilon := \sigma \wedge \inf\{t: \tilde{A}_t A_t \vee L_t\tilde{A}'_t \vee A'_t \geq \ell\} \wedge \inf\{t: \tilde{Q}_t Q_t \vee L_t\tilde{Q}_t \vee Q_t > q\}$$

From the definition of q and ℓ it is clear that

$$P\{\sigma_\varepsilon < \sigma\} \leq \varepsilon .$$

We write then $\Phi_s := \sup_{t<s} \|\xi_t - \xi'_t\|^2_{\mathbb{H}}$ and use J. Pellaumail's decomposition:

$$\begin{aligned}\xi_t - \xi'_t = V_t - V'_t &+ \int_{]0,t]} (\underset{\sim}{a}_s\xi - a'_s\xi)dZ_s + \int_{]0,t]} (\underset{\sim}{a}'_s\xi - a'_s\xi')dZ_s \\ &+ \int_{]0,t]} (\underset{\sim}{a}'_s\xi' - \underset{\sim}{a}'_s\xi)d(Z_s - Z'_s) + \int_{]0,t]} (\underset{\sim}{a}'_s\xi - \underset{\sim}{a}_s\xi)d(Z_s - Z'_s) \\ &+ \int_{]0,t]} \underset{\sim}{a}_s\xi \, d(Z_s - Z'_s) .\end{aligned}$$

Using the control Λ-spaces we may write for every $\tau < \sigma_\varepsilon$

$$\begin{aligned}E(\Phi_{\tau-}) \leq 6d' &+ 6E[\tilde{A}_{\tau-}\int \lambda_s(a\xi - a'\xi)dA_t + L_{\tau-}\tilde{A}'_{\tau-}\int_{[0,\tau[} \Phi_{t-}dA'_t] \\ &+ 6E[L_{\tau-}\tilde{Q}_{\tau-}\int_{[0,\tau[} \Phi_{t-}dQ_t + \tilde{Q}_{\tau-}\int_{[0,\tau[} \mu_s(a'\xi - a\xi)dQ_s] \\ &+ 6E[\tilde{Q}_{\tau-}\cdot\int_{[0,\tau[} \mu_s(a\xi)dQ_s]\end{aligned}$$

From there on we derive

$$E(\Phi_{\tau-}) \leq 6(d' + d_1\ell + d_2 q + cq) + (\ell+q)\int_{[0,\tau[} \Phi_{t-}d(A'_t + Q_t)$$

Now we apply lemma 3 to get the inequality (b) of the theorem and the expression for R_ε.

We are only left to prove that ξ' actually exists on $[0,\sigma[$. For this we consider any stopping time σ' such that the two equations have solution on $[o,\sigma'[$. The above reasoning shows that (5.4.2) has no explosion as long as the solution of (5.4.1) has none. This completes the proof.

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