A GLOBALLY CONVERGENT, IMPLEMENTABLE MULTIPLIER METHOD WITH AUTOMATIC PENALTY LIMITATION

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ABSTRACT

This paper deals with penalty function and multiplier methods for the solution of constrained nonconvex nonlinear programming problems. Starting from an idea introduced several years ago by Polak, we develop a class of implementable methods which, under suitable assumptions, produce a sequence of points converging to a strong local minimum for the problem, regardless of the location of the initial guess. In addition, for sequential minimization type multiplier methods, we make use of a rate of convergence result due to Bertsekas and Polyak, to develop a test for limiting the growth of the penalty parameter and thereby prevent ill-conditioning in the resulting sequence of unconstrained optimization problems.

Key Words: nonlinear programming, multiplier methods, penalty methods global convergence, penalty limitation

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1. Introduction

Since their introduction in 1969, independently, by Hestenes [H1] and Powell [P1], multiplier methods have become a very popular tool for constrained optimization. At present, we find a sizeable literature dealing with the two main forms of these methods: those of the sequential unconstrained minimization type, which was originally proposed by Hestenes [H1] and Powell [P1], and those of the continuous multiplier update type first proposed by Fletcher [F1]. An excellent review of the literature on sequential minimization type methods can be found in the survey papers by Rockafellar [R1], Fletcher [F2], Bertsekas [B1] and Powell [P2] as well as in the book by Pierre and Lowe [P3]. A number of major results on continuous multiplier update type methods can be found in the work of Fletcher and his collaborators [F3,F4] and of Mukai and Polak [M1] and Glad and Polak [G1]. For the sequential minimization type methods, we find results on local convergence, rate of convergence, with both increasing and finite penalty, and the effects of approximate unconstrained minimization [B2,B3,B5,P5,P6], but no theoretical results on automatic penalty limitation. For continuous multiplier update methods we find results on global convergence, rate of convergence and automatic penalty limitation [M1,G1].

At least in the case of Hestenes, sequential minimization type multiplier methods have evolved from much earlier attempts [H2,H3] to obtain stronger second order conditions of optimality for problems of the form

$$\min\{f(x)\,|\,h(x)=0\}, \qquad (*)$$

with $f:\mathbb{R}^n \to \mathbb{R}^1$ and $h:\mathbb{R}^n \to \mathbb{R}^\ell$ twice continuously differentiable, by replacing (*) with the equivalent problem

$$\min\{f(x)+\frac{c}{2}\|h(x)\|^2 \mid h(x)=0\} \qquad (**)$$

Specifically, it was shown [H1] that if $\hat{x}$ is a local minimizer for (*) satisfying second order sufficiency conditions of optimality, with multiplier $\hat{\psi}$, then there exists a $\hat{c} \geq 0$ such that for the augmented Lagrangian

$$\ell_c(x,\psi) = f(x) + \langle \psi, h(x) \rangle + \frac{c}{2}\|h(x)\|^2,$$

the Hessian $\frac{\partial^2 \ell_c(\hat{x},\hat{\psi})}{\partial x^2}$ is positive semi-definite for all $c \geq \hat{c}$, so that $\hat{x}$ is a local minimizer for $\ell_c(\cdot,\hat{\psi})$ also. Consequently, this suggested that a solution to (*) could be obtained by sequential minimization of $\ell_c(\cdot,\psi_i)$, for $i = 0,1,2,\ldots$, for c sufficiently large, with ψ_i updated so as to force convergence of ψ_i to $\hat{\psi}$. Both Hestenes and Powell proposed the update formula $\psi_{i+1} = \psi_i + ch(x_i)$, $i = 0,1,2,\ldots$, where x_i is a minimizer of $\ell_c(\cdot,\psi_i)$. Powell also suggested a scheme for determining a satisfactory, finite c by means of a test on the rate of decrease of $\|h(x_i)\|$, but has offered no analytic evidence as to when and what sense his scheme is guaranteed to work, though empirical evidence supports his claim that the scheme is well conceived. Bersekas in [B1] claims, without proof, that Powell's scheme can be proved to work, which we assume to mean locally, in a neighborhood of a strong local minimizer. Thus, in the theory of sequential minimization type multiplier methods, the construction of a globally convergent, limited penalty method has remained until now one of the major open challenges.

In this paper, we present a class of sequential minimization methods, for both equality and inequality constrained problems which are characterized by the following features (i) the multipliers are updated as in Hestenes [H1]; (ii) the penalty is increased according to a new recursive relation, on the basis of a new test on the norm of the difference of successive multipliers; (iii) the maximum step length of the unconstrained minimization algorithm to be used is geared to decrease when the multipliers are updated, and (iv) for best results it is necessary to use an unconstrained minimization algorithm which converges only to points satisfying both first and second order necessary conditions of optimality. These features ensure (under certain conditions) that (i) the triplets (x_i,λ_i,ψ_i) (consisting of approximate solution x_i and approximate multipliers λ_i,ψ_i) constructed by our methods converge to a Kuhn-Tucker triplet and (ii) that the penalty growth is automatically arrested.

Although the proofs whch are required to show that our methods perform as claimed are quite complex, the ideas underlying our work are quite simple. First, we recall that it was shown by Bertsekas [B3] that if $\hat{x}$ is a local minimizer for (*) satisfying second order sufficiency conditions, with corresponding multiplier $\hat{\psi}$, and if S is any compact set in $\mathbb{R}^{\ell}$ containing $\hat{\psi}$, then there exists a neighborhood B of $\hat{x}$ (which we show in Appendix 1 to be independent of c) and a $\hat{c} \geq 0$ such that for any $c \geq \hat{c}$ and $\psi \in S$, there exists a unique $x \in B$ which is a local minimizer of $\ell_c(\cdot,\psi)$. Furthermore, for some $M \in (0,\infty)$

$$\left.\begin{aligned} \|\tilde{\psi}-\hat{\psi}\| &\leq \frac{M}{c}\|\psi-\hat{\psi}\| \\ \|x-\hat{x}\| &\leq \frac{M}{c}\|\psi-\tilde{\psi}\| \end{aligned}\right\} \forall c \geq \hat{c}, \quad \psi \in S \qquad (***)$$

where $\tilde{\psi} = \psi + ch(x)$. This shows that the method of multipliers converges locally, and that $\psi_i \to \hat{\psi}$ linearly, with the rate constant proportional to $1/c$. We shall show that when conditions for (***) to be valid hold, $\|\psi_{i+1}-\psi_i\| \to 0$ linearly, also, with the rate constant also proportional to $1/c$ and hence that a test of the type $\|\psi_{i+1}-\psi_i\| \leq M\gamma^i$, with $M > 0$, $\gamma \in (0,1)$, arbitrary, will eventually be satisfied for c large enough. We shall use such a test to detect when the penalty c is large enough.

Next, to ensure that the conditions for (***) to hold are eventually satisfied, we have to devise a scheme for forcing the pair (x_i,ψ_i) (triplet (x_i,λ_i,ψ_i) for equality and inequality constraints) to converge to $(\hat{x},\hat{\psi})$ ($(\hat{x},\hat{\lambda},\hat{\psi})$, respectively), a Kuhn-Tucker pair (triplet) with $\hat{x}$ a strong local minimizer of (*). This is achieved by observing that as c is increased, the level sets of $\ell_c(\cdot,\psi)$ develop "dimples" around strong local minimizers of (*). Hence if the step size of an unconstrained method is kept sufficiently small, the requirement of cost decrease in $\ell_c(\cdot,\psi)$ keeps the sequence $\{x_i\}$ within a single "dimple" and hence the sequence must converge (c.f. theorem (1.3.66) in [P4]). When combined, as in our paper, the penalty limitation test and step size limitation rule result in a globally convergent multiplier method with limited penalty growth. Since by design, the <u>entire</u> sequence constructed by our methods converges to a solution, rather than only subsequences of those sequences, as is common to claim in algorithm convergence theorems, we coin a phrase by saying that our algorithms are totally convergent.

To conclude, we hope that our work on automatic penalty limitation and total convergence will prove to be both of theoretical and of practical interest.

2. A Scheme for Forcing Total Convergence

In this section we shall deal with penalty and augmented Lagrangian methods in which the penalty is driven to infinity. We shall show that under suitable assumptions one can force the sequences constructed by such a method to converge to a local minimizer of the problem

$$\min\{f(x) \mid g(x) \leq 0, h(x) = 0\}, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^1$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^\ell$.

We shall make use of the following assumptions and definitions, as the need arises.

<u>Assumption 1</u>: The functions $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are twice locally Lipschitz continuously differentiable. ¤

Let $\underline{m} \triangleq \{1,2,\ldots,m\}$, $\underline{\ell} \triangleq \{1,2,\ldots,\ell\}$ and for any $x \in \mathbb{R}^n$, let

$$J(x) \triangleq \{j \in \underline{m} \mid g^j(x) \geq 0\}. \tag{2}$$

<u>Definition 1</u>: We shall say that $\hat{x}$ is a Kuhn-Tucker point for (1) if

$$\hat{x} \in \mathcal{F} = \{x \mid g(x) \leq 0, h(x) = 0\} \tag{3}$$

and there exist multipliers $\hat{\lambda}^j \geq 0$, $j \in \underline{m}$, with $\hat{\lambda}^j = 0$ for all $j \notin J(\hat{x})$, and multipliers $\hat{\psi}^k$, $k \in \underline{\ell}$, such that

$$\nabla f(\hat{x}) + \sum_{j \in J(\hat{x})} \hat{\lambda}^j \nabla g^j(x) + \sum_{k \in \underline{\ell}} \hat{\psi}^j \nabla h^j(\hat{x}) = 0. \tag{4}$$

We shall denote by Δ the set of all Kuhn-Tucker points for (1).

<u>Assumption 2</u>: (i) For any $x \in \mathbb{R}^n$

$$\sum_{j \in J(x)} \lambda^j \nabla g^j(x) + \sum_{j \in \underline{\ell}} \psi^j \nabla h^j(x) = 0, \tag{5}$$

with $\lambda^j \geq 0$ for all $j \in J(x)$, implies that $\lambda^j = 0$ for all $j \in J(x)$ and $\psi^j = 0$ for all $j \in \underline{\ell}$. (ii) For any Kuhn-Tucker point $\hat{x}$, the vectors $\nabla h^j(\hat{x})$, $j \in \underline{\ell}$, together with the vectors $\nabla g^j(\hat{x})$, $j \in J(\hat{x})$ are linearly independent. ¤

The first part of Assumption 2 is rather strong; unfortunately it is required by any penalty function method which computes points of zero gradient rather than global minimizers, by sequential unconstrained optimization. Since in the absence of convexity one cannot be sure of computing a global minimizer in the sequential minimization process, one is inevitably forced to invoke Assumption 2 (c.f. [P4,M1,G1]).

Definition 2: We shall say that $\hat{x} \in \mathcal{F}$ is a strict local minimizer for (1) if there exists a $\hat{\rho} > 0$ such that $f(\hat{x}) < f(x)$ for all $x \neq \hat{x}$ such that $x \in \mathcal{F} \cap B(\hat{x},\hat{\rho})$, where $B(\hat{x},\hat{\rho}) \triangleq \{x \mid \|x-x\| \leq \hat{\rho}\}$, and we shall call such a $\hat{\rho}$ a radius of attraction for $\hat{x}$. ¤

Definition 3: We shall say that $\hat{x} \in \mathcal{F}$ is a strong local minimizer for (1) if for some $\hat{\lambda}^j \geq 0$, $j \in \underline{m}$, and ψ^k, $k \in \underline{\ell}$, such that $\hat{\lambda}^j = 0$ for $j \in J(\hat{x})$,

(i) Equation (4) holds,

(ii) $\hat{\lambda}^j > 0$ for all $j \in J(\hat{x})$ (strict complementary slackness condition),

(iii) For

$$L(x,\lambda,\psi) \triangleq f(x) + \sum_{j\in\underline{m}} \lambda^j g^j(x) + \sum_{k\in\ell} \psi^k h^k(x) \tag{6}$$

there exists $m > 0$ such that

$$\langle y, \frac{\partial^2 L(\hat{x},\hat{\lambda},\hat{\psi})}{\partial x^2} y\rangle \geq m\|y\|^2 \tag{7}$$

for all

$$y \in H_{\hat{x}} \triangleq \{y \mid \frac{\partial h(x)}{\partial x} y = 0;\ \langle \nabla g^j(x), y\rangle = 0,\ j \in J(\hat{x})\}. \tag{8}$$

¤

We note that strong local minimizers are strict local minimizers which satisfy a second order sufficient condition of optimality, i.e. they are a subset of the strict local minimizers. Our final assumptions is

Assumption 3: Every Kuhn-Tucker point $\hat{x}$ which satisfies the second order necessary condition of optimality (7) and (8), with m = 0, is a strong local minimizer for (1). ¤

Next, we introduce the usual augmented Lagrangian (see [R2,B5,G1]) $F: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \times \mathbb{R}^1 \to \mathbb{R}^1$ defined by

$$F(x,\lambda,\psi,c) \triangleq f(x) + \frac{1}{2c}\{\|(cg(x)+\lambda)_+\|^2 - \|\lambda\|^2\}$$
$$+ \langle \psi, h(x) \rangle + \frac{c}{2}\|h(x)\|^2 , \qquad (9)$$

where for any $y \in \mathbb{R}^m$, y_+ is a vector whose jth component is $\max\{0,y^j\}$, $j \in \underline{m}$.

We note at this point that $\nabla_x F(x,\lambda,\psi,c)$ is given by

$$\nabla_x F(x,\lambda,\psi,c) = \nabla f(x) + \frac{\partial g(x)}{\partial x}^T (\lambda+cg(x))_+ + \frac{\partial h(x)}{\partial x}^T (\psi+ch(x)) \qquad (9a)$$

and that the Hessian matrix $\frac{\partial^2 F(x,\lambda,\psi,c)}{\partial x^2}$ is well defined provided $g^j(x) \neq \lambda^j/c$ for all $j \in \underline{m}$, in which case it is given by

$$\frac{\partial^2 F(x,\lambda,\psi,c)}{\partial x^2} = \frac{\partial^2 f(x)}{\partial x^2} + \sum_{j\in\underline{m}} \frac{\partial^2 g^j(x)}{\partial x^2} (\lambda^j+cg^j(x))_+$$
$$+ \sum_{j\in\underline{\ell}} \frac{\partial^2 h(x)}{\partial x^2} (\psi^j+ch^j(x))$$
$$+ c\{\sum_{j\in I(x,\lambda,c)} \frac{\partial g^j(x)}{\partial x}^T \frac{\partial g^j(x)}{\partial x} + \frac{\partial h(x)}{\partial x}^T \frac{\partial h(x)}{\partial x}\}, \qquad (9b)$$

where

$$I(x,\lambda,c) \triangleq \{j \in \underline{m} \,|\, g^j(x) > -\lambda^j/c\}. \qquad (9c)$$

Definition 4: Let $\hat{x} \in \mathcal{F}$ be any strong local minimizer for (1), with $\hat{\rho} > 0$ a radius of attraction. Then, for any $\lambda \in \mathbb{R}^m$, $\psi \in \mathbb{R}^\ell$, $c \geq 0$, $\varepsilon \geq 0$, we define the level sets $C_{\hat{x}}(\lambda,\psi,c,\varepsilon)$ by

$$C_{\hat{x}}(\lambda,\psi,c,\varepsilon) \triangleq \{x \mid F(x,\lambda,\psi,c) \leq f(\hat{x}) + \varepsilon\}, \tag{10}$$

and their intersection with the ball $B(\hat{x},\hat{\rho})$ by

$$N_{\hat{x},\hat{\rho}}(\lambda,\psi,c,\varepsilon) \triangleq B(\hat{x},\hat{\rho}) \cap C_{\hat{x}}(\lambda,\psi,c,\varepsilon). \tag{11}$$

¤

We are finally ready for our first result, which shows that the set $C_{\hat{x}}(\lambda,\psi,c,\varepsilon)$ is not empty.

<u>Lemma 1</u>: Let x be a strict local minimizer for (1) and let $S \subset \mathbb{R}^{m+\ell}$ be a compact set, then

$$\hat{x} \in C_{\hat{x}}(\lambda,\psi,c,\varepsilon) \quad \forall(\lambda,\psi) \in S,\ \forall c \geq 0,\ \forall \varepsilon \geq 0. \tag{12}$$

<u>Proof</u>: Since $\hat{x} \in \mathcal{F}$,

$$F(\hat{x},\lambda,\psi,c) = f(\hat{x}) + \frac{1}{2c}\{\|(cg(\hat{x})+\lambda)_+\|^2 - \|\lambda\|^2\} \tag{13}$$

and, since $cg(\hat{x}) \leqq 0$, $cg(\hat{x}) + \lambda \leqq \lambda$ and therefore $(cg(x)+\lambda)_+ \leqq \lambda_+$ which implies that $\|(cg(\hat{x})+\lambda)_+\|^2 \leq \|\lambda\|^2$ and hence $F(\hat{x},\lambda,\psi,c) \leq f(\hat{x})$ for all $c \geq 0$, which completes the proof. ¤

The next lemma shows that when c is large enough and ε is small enough, the "dimple" $N_{\hat{x},\hat{\rho}}(\lambda,\psi,c,\varepsilon)$ becomes contained in any given ball $B(\hat{x},\delta)$ about $\hat{x}$.

<u>Lemma 2</u>: Suppose that Assumption 1 holds. Let $\hat{x}$ be a strict local minimizer for (1), with $\hat{\rho} > 0$ a radius of attraction. Then for any $\delta > 0$ and any $S \subset \mathbb{R}^{m+\ell}$, a compact set, there exist $\hat{c} \geq 0$ and $\hat{\varepsilon} > 0$ such that

$$N_{\hat{x},\hat{\rho}}(\lambda,\psi,c,\varepsilon) \subset B(\hat{x},\delta) \quad \forall c \geq \hat{c},\ \forall(\lambda,\psi) \in S,\quad \forall \varepsilon \in (0,\hat{\varepsilon}]. \tag{14}$$

<u>Proof</u>: For the sake of contradiction, suppose that there exist $\delta^* > 0$, $S^* \subset \mathbb{R}^{m+\ell}$ compact and sequences $\{c_i\}_{i=0}^{\infty}$, $\{\varepsilon_i\}_{i=0}^{\infty}$, $\{(\lambda_i,\psi_i)\}_{i=0}^{\infty} \subset S^*$ and $\{x_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$ such that $c_i \nearrow \infty$ as $i \to \infty$, $\varepsilon_i \searrow 0$ as $i \to \infty$,

$x_i \in N_{\hat{x},\hat{\rho}}(\lambda_i,\psi_i,c_i,\varepsilon_i)$ and $x_i \notin B(\hat{x},\hat{\delta})$ for $i = 0,1,2,\ldots$, i.e., for $i = 0,1,2,\ldots$

$$f(x_i) + \frac{1}{2c_i}\{\|(c_i g(x_i)+\lambda_i)_+\|^2 - \|\lambda_i\|^2\}$$
$$+ \langle \psi_i,h(x_i)\rangle + \frac{c_i}{2}\|h(x_i)\|^2 \leq f(\hat{x}) + \varepsilon_i \tag{15a}$$

and

$$x_i \in \mathcal{A} \triangleq \{x \mid \delta^* \leq \|x-\hat{x}\| \leq \hat{\rho}\}. \tag{15b}$$

Now, since $\varepsilon_i \searrow 0$, $f(\hat{x}) + \varepsilon_i$ is bounded for all i. Since S^* and $B(\hat{x},\hat{\rho})$ are compact and $c_i \nearrow \infty$, as $i \to \infty$, $\|\lambda_i\|^2/2c_i$, $\langle \psi_i,h(x_i)\rangle$ and $f(x_i)$ are all bounded for $i = 0,1,2,\ldots$, and hence, from (15a), $\frac{1}{2c_i}\|(c_i g(x_i)+\lambda_i)\|^2 = \frac{c_i}{2}\|((g(x_i) + \frac{1}{c_i}\lambda_i))_+\|^2$ and $\frac{c_i}{2}\|h(x_i)\|^2$ must be bounded for $i = 0,1,2,\ldots$ Consequently

$$g(x_i)_+ \to 0 \text{ as } i \to \infty\ , \tag{16a}$$

$$h(x_i) \to 0 \text{ as } i \to \infty\ . \tag{16b}$$

Since $\{x_i\}_{i=0}^{\infty}$ is bounded, there exists an infinite subset $K \subset \{0,1,2,\ldots\}$ such that $x_i \overset{K}{\to} \tilde{x}$, as $i \to \infty$, for some $\tilde{x} \in \mathcal{A}$. Next, since $\|(c_i g(x_i)+\lambda_i)_+\|^2 \geq 0$ and $\|h(x_i)\|^2 \geq 0$, it follows from (15a) that for $i = 0,1,2,\ldots$

$$f(x_i) \leq f(\hat{x}) + \varepsilon_i + \frac{1}{2c_i}\|\lambda_i\|^2 - \langle \psi_i,h(x_i)\rangle\ . \tag{17}$$

Hence, since $c_i \to \infty$ as $i \to \infty$, S^* is compact and $h(x_i) \to 0$ as $i \to \infty$, it follows, by continuity, from (17), that (16a) and (16b) that

$$f(\tilde{x}) \leq f(\hat{x})\ , \tag{18a}$$

$$g(\hat{x}) \leq 0\ , \tag{18b}$$

$$h(\tilde{x}) = 0\ . \tag{18c}$$

But by definition of $\hat{\rho}$ there is no $\tilde{x} \in \mathcal{A}$, such that (18a-18c) holds and thus we have a contradiction. The proposition must therefore be true. $\square$

<u>Lemma 3</u>: Suppose that Assumptions 1 and 2 are satisfied, that $S \subset \mathbb{R}^{m+\ell}$ is a compact set, and that the sequences $\{c_i\}_{i=0}^{\infty}$, $\{\gamma_i\}_{i=0}^{\infty}$, $\{x_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$, $\{(\lambda_i,\psi_i)\}_{i=0}^{\infty} \subset S$ are such that (i) $c_i \geq 0$, $\gamma_i \geq 0$ for all i, $c_i \to \infty$ and $\gamma_i \to 0$ as $i \to \infty$, and (ii)

$$\|\nabla_x F(x_i,\lambda_i,\psi_i,c_i)\| \leq \gamma_i \quad \text{for } i = 0,1,2,\ldots \tag{19}$$

If $x_i \to \hat{x}$ as $i \to \infty$, then

(i) $\hat{x}$ is a Kuhn-Tucker point;

(ii) the sequences $\{c_i g(x_i)_+\}_{i=0}^{\infty}$ and $\{c_i h(x_i)\}_{i=0}^{\infty}$ are all bounded;

(iii) the following subsequences converge:

$$(\psi_i+c_i h(x_i)) \to \hat{\psi}, \text{ as } i \to \infty, \tag{20a}$$

$$(c_i g(x_i)+\lambda_i)_+ \to \hat{\lambda}, \text{ as } i \to \infty, \tag{20b}$$

$$F(x_i,\lambda_i,\psi_i,c_i) \to f(\hat{x}), \text{ as } i \to \infty, \tag{20c}$$

where $(\hat{\lambda},\hat{\psi})$ is a pair of Kuhn-Tucker multipliers for $\hat{x}$.[(1)]

<u>Proof</u>: Since λ_i is bounded and $c_i \to \infty$, it follows that there exists an $i_o \geq 0$ such that

$$(g(x_i)+\lambda_i/c_i)_+^j = (g^j(x_i)+\lambda_i^j/c_i)_+ = 0 \quad \text{for all } i \geq i_o \tag{21}$$

for all $j \in J^c(x) \triangleq \{j \in \underline{m} \mid j \notin J(\hat{x})\}$ (i.e., for j such that $g^j(\hat{x}) < 0$). Hence, by (9a) and (19)

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(1) If $\gamma_i = 0$ for all i, it can be proved that, together with (20c), we have $F(x_i,\lambda_i,\psi_i,c_i) \leq f(\hat{x})$ for i large enough.

$$\lim_{\substack{i\to\infty \\ i\in K}} \{\nabla f(x_i) + \sum_{j\in J(\hat{x})} (c_i g^j(x_i)+\lambda_i^j)_+ \nabla g^j(x_i)$$

$$+ \sum_{j\in \underline{\ell}} (c_i h^j(x_i)+\psi_i^j)\nabla h^j(x_i)\} = 0. \tag{22}$$

It now follows from Assumptions 1 and 2(i) that $(c_i g^j(x_i)+\lambda_i^j)_+$, $j \in \underline{m}$ and $(c_i h^j(x_i)+\psi_i^j)$, $j \in \underline{\ell}$, must all be bounded and hence that there exists an infinite subsequence indexed by $K \subset \{0,1,2,\ldots\}$ such that

$$(c_i g^j(x_i)+\lambda_i^j)_+ \overset{K}{\to} \hat{\lambda} \text{ as } i \to \infty,\ j \in \underline{m}\ , \tag{23a}$$

$$(c_i h^j(x_i)+\psi_i^j) \overset{K}{\to} \hat{\psi} \text{ as } i \to \infty,\ j \in \underline{\ell}, \tag{23b}$$

with $(\hat{\lambda},\hat{\psi})$ such that $\hat{\lambda} \geqq 0$, and

$$\nabla f(\hat{x}) + \frac{\partial g(\hat{x})}{\partial x}^T \hat{\lambda} + \frac{\partial h(\hat{x})}{\partial x}^T \hat{\psi} = 0. \tag{24}$$

Next, it follows from (23a,b) and the compactness of S that $c_i g^j(x_i)_+$ $j \in \underline{m}$ and $c_i h^j(x_i)$, $j \in \underline{\ell}$ are all bounded for $i \in K$ and hence, since $c_i \nearrow \infty$ as $i \to \infty$, that $g(\hat{x})_+ = 0$ and $h(\hat{x}) = 0$. Therefore, $\langle \hat{\lambda},g(\hat{x})\rangle = 0$, which proves that $\hat{x}$ is a Kuhn Tucker point with multipliers $(\hat{\lambda},\hat{\psi})$. It now follows from Assumption 2(ii) that the multipliers $(\hat{\lambda},\hat{\psi})$ are unique and hence that we may choose $K = \{0,1,2,\ldots\}$. Finally, since $(c_i g(x_i)+\lambda_i)_+$, λ_i, ψ_i and $c_i h(x_i)$ are all bounded and $h(x_i) \to 0$ and $c_i \to \infty$, as $i \to \infty$, (20c) follows directly from the definition of F in (9), which completes our proof. ¤

Normally, we only require from a sequential minimization algorithm, based on penalty functions, that all the accumulation points of the sequences that it constructs be "acceptable" solutions. The next proposition lays out conditions for such a method to be totally convergent in the sense that the entire sequences that it constructs converge to an "acceptable" solution, in contrast to merely constructing acceptable accumulation points.

Proposition 1: Suppose that Assumptions 1 and 2 are satisfied.

(i) Let $\{\gamma_i\}_{i=0}^{\infty}$, $\{\rho_i\}_{i=0}^{\infty}$ be such that $\gamma_i \geq 0$, $\rho_i > 0$ for all i and $\gamma_i \to 0$, $\rho_i \to 0$ as $i \to \infty$. (ii) For i = 1,2,..., let c_i be constructed by the formula $c_{i+1} = a_i c_i + b_i$, where $c_o \geq 0$, and a_i, b_i are bounded reals such that $c_i \nearrow \infty$ as $i \to \infty$ (e.g., $a_i \geq 1$, $b_i \geq 0$ and $a_i + b_i \geq 1+\delta$ with $\delta > 0$). (iii) Let $S \subset \mathbb{R}^{m+\ell}$ be a compact set and let $\{(\lambda_i,\psi)\}_{i=0}^{\infty} \subset S$.
(iv) Let $\{x_i\}_{i=0}^{\infty}$ be an infinite sequence such that

$$\|\nabla_x F(x_i,\lambda_i,\psi_i,c_i)\| \leq \gamma_i, \ \forall i; \tag{25}$$

(b) for i = 0,1,2,..., x_{i+1} is constructed by a step size limited descent method, i.e.,

$$x_{i+1} = x_i + \sum_{j=0}^{k_i} (z_{j+1}-z_j) \tag{26}$$

where $z_0 = x_i$, and

$$\|z_{j+1}-z_j\| \leq \rho_i, \quad \forall j, \tag{27a}$$

$$F(z_{j+1},\lambda_i,\psi_i,c_i) < F(z_j,\lambda_i,\psi_i,c_i), \quad \forall j. \tag{27b}$$

Under these conditions, if $\{x_i\}_{i=0}^{\infty}$ has an accumulation point $\hat{x}$ which is a strong local minimizer for (1), then $x_i \to \hat{x}$ as $i \to \infty$.

Proof: Suppose $x_i \xrightarrow{K} \hat{x}$, with $\hat{x}$ a strong local minimizer for (1). Then there exists a radius of attraction $\hat{\rho} > 0$ for $\hat{x}$ such that $\hat{x}$ is the only Kuhn-Tucker point in $B(\hat{x},\hat{\rho})$. Let K' be the infinite subset of {0,1,2,...} defined by

$$K' \triangleq \{i \mid x_i \in B(x,\rho)\} \tag{28}$$

Then $x_i \xrightarrow{K'} \hat{x}$, since otherwise $x_i \xrightarrow{K''} \tilde{x}$ with $\tilde{x} \neq \hat{x}$, $K'' \subset K$ and $\tilde{x} \in B(\hat{x},\hat{\rho})$ a Kuhn-Tucker point by Lemma 3 (i), which is clearly impossible by definition of $\hat{\rho}$. Next, by Lemma 3 (ii), the subsequences $\{c_i h(x_i)\}_{i\in K'}$, and $\{c_i g(x_i)_+\}_{i\in K'}$ are bounded and hence, since $c_i \nearrow \infty$ as $i \to \infty$

$$h(x_i) \overset{K'}{\to} 0, \text{ as } i \to \infty \tag{29}$$

and

$$g(x_i)_+ \overset{K'}{\to} 0, \text{ as } i \to \infty. \tag{30}$$

Now,

$$F(x_i,\lambda_{i+1},\psi_{i+1},c_{i+1}) - F(x_i,\lambda_i,\psi_i,c_i)$$

$$= \frac{1}{2} \{c_{i+1}\|(g(x_i) + \frac{1}{c_{i+1}} \lambda_{i+1})_+\|^2 - c_i\|(g(x_i) + \frac{1}{c_i} \lambda_i\|_+^2$$

$$- \frac{1}{c_{i+1}} \|\lambda_{i+1}\|^2 + \frac{1}{c_i} \|\lambda_i\|^2 + [(a_i-1)c_i+b_i] \|h(x_i)\|^2\}$$

$$+ \langle \psi_{i+1}-\psi_i, h(x_i)\rangle. \tag{31}$$

Consequently, since a_i, b_i, λ_i and ψ_i are all bounded, it follows that $\varepsilon_i \searrow 0$ as $i \to \infty$, where for $i = 0,1,2,\ldots,$

$$\varepsilon_i \triangleq \max_{\substack{j\in K' \\ j\geq i}} \{\max\{0,F(x_i,\lambda_{i+1},\psi_{i+1},c_{i+1}) - F(x_i,\lambda_i,\psi_i,c_i)\}\} \tag{32a}$$

so that

$$F(x_i,\lambda_{i+1},\psi_{i+1},c_{i+1}) - F(x_i,\lambda_i,\psi_i,c_i) \leq \varepsilon_i \text{ for all } i \in K' \tag{32b}$$

Next, for $i = 1,2,3,\ldots,$ let

$$C_i \triangleq C_{\hat{x}}(\lambda_i,\psi_i,c_i,\varepsilon_{i-1}+\eta_{i-1}), \tag{33a}$$

$$N_i \triangleq N_{\hat{x},\hat{\rho}}(\lambda_i,\psi_i,e_i,\varepsilon_{i-1}+\eta_{i-1}), \tag{33b}$$

where for $i = 0,1,2,\ldots$

$$\eta_i \triangleq \max_{\substack{j\in K' \\ j\geq i}} \{\max\{0,F(x_i,\lambda_i,\psi_i,c_i)-f(\hat{x})\}\}, \tag{34a}$$

and

$$F(x_i,\lambda_i,\psi_i,c_i) \leq f(x) + \eta_i \quad \text{for all } i \in K'. \tag{34b}$$

Since it follows from Proposition 2 (iii) that $\eta_i \searrow 0$ as $i \to \infty$, we conclude that $\varepsilon_i + \eta_i \searrow 0$ as $i \to \infty$, $c_i \nearrow \infty$ as $i \to \infty$, and $(\lambda_i,\psi_i) \in S$, a compact set, for $i = 0,1,2,\ldots$. It now follows from Lemma 2 that there exists an i_1 such that

$$N_i \subseteq B(\hat{x},\hat{\rho}/2) \quad \forall i \geq i_1. \tag{35}$$

Next, since $\rho_i \to 0$ as $i \to \infty$, there exists an $i_2 \geq i_1$ such that $\rho_1 < \hat{\rho}/2$ for all $i \geq i_2$. Let $i_3 \in K'$ be such that $i_3 \geq i_2$. We now prove by induction that $x_i \in N_i$ for all $i \geq i_3$ and that $i \in K'$ for all $i \geq i_3$. First, by definition of K', $x_{i_3} \in B(\hat{x},\hat{\rho})$ and by (34b)

$$F(x_{i_3},\lambda_{i_3},\psi_{i_3},c_{i_3}) \leq f(\hat{x}) + \eta_{i_3} < f(\hat{x}) + \varepsilon_{i_3-1} + \eta_{i_3}$$

$$\leq f(\hat{x}) + \varepsilon_{i_3-1} + \eta_{i_3-1}, \tag{36}$$

since $\eta_{i_3-1} \geq \eta_{i_3}$ by construction. Hence $x_{i_3} \in N_{i_3}$. Now, suppose that for $i \geq i_3$, $i \in K'$ and $x_i \in N_i$, we shall show that $i+1 \in K'$ and $x_{i+1} \in N_{i+1}$ to complete the proof. Indeed, with $\{z_j\}$ defined as in (26), we have because of (27b) that for $j = 0,1,2,\ldots,k_i$,

$$z_j \in A_i \triangleq \{x \mid F(x,\lambda_{i+1},\psi_{i+1},c_{i+1}) \leq F(x_i,\lambda_{i+1},\psi_{i+1},c_{i+1})\}. \tag{37}$$

Now, since $x_i \in N_i \subset B(\hat{x},\hat{\rho}/2)$ and since $i \in K'$, it follows from (32b) and (34b) that

$$F(x_i,\lambda_{i+1},\psi_{i+1},c_{i+1}) \leq F(x_i,\lambda_i,\psi_i,c_i) + \varepsilon_i \leq f(x) + \varepsilon_i + \eta_i \tag{38}$$

and hence we see that $z_j \in A_i \subset C_{i+1}$. Now, since $i \geq i_1$, we must have

$$A_i \cap B(\hat{x},\hat{\rho}) \subset C_{i+1} \cap B(\hat{x},\hat{\rho}) = N_{i+1} \subset B(\hat{x},\rho/2). \tag{39}$$

Consequently,

$$\min\{\|z'-z''\| \mid z' \in A_i \cap B(\hat{x},\hat{\rho}),\ z'' \in B(\hat{x},\hat{\rho})^c\} \geq \hat{\rho}/2 \tag{40}$$

and therefore, since $\rho_i < \hat{\rho}/2$, if $z_j \in A_i \cap B(\hat{x},\hat{\rho})$, then by (40) and (27a) $z_{j+1} \in A_i \cap B(\hat{x},\hat{\rho})$ for $j = 0,1,2,\ldots,k_i$. Therefore, $x_{i+1} = z_{k_i+1} \in A_i \cap B(\hat{x},\hat{\rho}) \subset N_{i+1}$ and $i+1 \in K'$. Consequently, $i \in K'$ for $i \geq i_3$ and therefore $x_i \to \hat{x}$, which completes the proof. ¤

Now suppose that for $i = 0,1,2,\ldots$ we apply an unconstrained minimization algorithm to $F(x,\lambda_i,\psi_i,c_i)$, that we limit the step size as in Proposition 1 and that we stop when (25) is satisfied. We shall model this sequence of operations by a map $A(\cdot,\cdot,\cdot,\cdot,\cdot)$ from $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \times \mathbb{R}^1 \times \mathbb{R}^1$ into $\mathbb{R}^n$, $x_i = A(x_{i-1},\lambda_i,\psi_i,\gamma_i,\rho_i)$ satisfying (25)-(27b). Now consider the following algorithm.

<u>Algorithm 1</u>:

<u>Data</u>: $x_0' \in \mathbb{R}^n$, $c_0 \geq 0$, $a \geq 1$, $b \geq 0$, $a+b > 1$, $S \subset \mathbb{R}^{m+\ell}$ compact, $(\lambda_0,\psi_0) \in S$, $\{\gamma_i\}_{i=0}^{\infty}$, $\{\rho_i\}_{i=0}^{\infty}$, with $\gamma_i \searrow 0$, $\rho_i \searrow 0$ as $i \to \infty$.

<u>Step 0</u>: Set $i = 0$ and compute $x_0 = A(x_0',\lambda_0,\psi_0,c_0,\gamma_0,\rho_0)$.

<u>Step 1</u>: Compute $x_{i+1} = A(x_i,\lambda_i,\psi_i,c_i,\gamma_i,\rho_i)$.

<u>Step 2</u>: Compute $c_{i+1} = ac_i+b$ and $(\lambda_{i+1},\psi_{i+1}) \in S$.

<u>Step 3</u>: Set $i = i+1$ and go to step 1. ¤

<u>Theorem 1</u>: Suppose that Assumptions 1 and 2 are satisfied, and that the computation in step 2 is well defined for all i. Consider the sequence $\{x_i\}_{i=0}^{\infty}$ constructed by Algorithm 1. For $i = 1,2,3,\ldots$, let

$$\mathcal{L}_i \triangleq \{x \mid F(x,\lambda_i,\psi_i,c_i) \leq F(x_{i-1},\lambda_i,\psi_i,c_i)\} \tag{41}$$

If every Kuhn-Tucker point in $\overline{\lim}\, \mathcal{L}_i$ is a strong local minimizer and $\{x_i\}_{i=0}^{\infty}$ has accumulation points, then $x_i \to \hat{x}$ as $i \to \infty$, with $\hat{x}$ a strong local minimizer.

<u>Proof</u>: First, by assumption, $\{x_i\}_{i=0}^{\infty}$ has accumulation points, all of which, by Lemma 3, are Kuhn-Tucker points. Then, since any limit point $\hat{x}$ of $\{x_i\}_{i=0}^{\infty}$ satisfies $\hat{x} \in \overline{\lim}\, \mathcal{L}_i$ any such $\hat{x}$ is a strong local minimizer and hence the Theorem follows from Proposition 1. ¤

Corollary 1: Suppose that the hypotheses is Theorem 1 hold, that $S = S_\lambda \times S_\psi$ is such that all the Kuhn-Tucker multipliers $(\hat{\lambda},\hat{\psi})$ corresponding to Kuhn-Tucker points $\hat{x} \in \overline{\lim}\, \mathcal{L}_i$ are in the interior of S, and that for i = 1,2,3,..., (λ_i,ψ_i) are constructed in Step 2 of Algorithm 1 according to the rule

$$\lambda_{i+1} = \begin{cases} (c_i g(x_i)+\lambda_i)_+ \triangleq \lambda_i' & \text{if } \lambda_i' \in S_\lambda' \\ \lambda_i & \text{otherwise} \end{cases}, \tag{41a}$$

$$\psi_{i+1} = \begin{cases} (c_i h(x_i)+\psi_i) = \psi_i' & \text{if } \psi_i' \in S_\psi \\ \psi_i & \text{otherwise} \end{cases}. \tag{41b}$$

Under these conditions, $x_i \to \hat{x}$ as $i \to \infty$, with $\hat{x}$ a strong local minimizer, $(\lambda_i,\psi_i) \to (\hat{\lambda},\hat{\psi})$ as $i \to \infty$, with $(\hat{\lambda},\hat{\psi})$ corresponding multipliers, and there exists a constant $M \in (0,\infty)$, such that

$$\|x_i-\hat{x}\| \le \frac{M}{c_i} \{\|\lambda_i-\hat{\lambda}\|^2 + \|\psi_i-\hat{\psi}\|^2 + \gamma_i^2\}^{1/2} , \tag{42a}$$

$$\|\lambda_{i+1}-\hat{\lambda}\| + \|\psi_{i+1}-\hat{\psi}\| \le \frac{M}{c_i} \{\|\lambda_i-\hat{\lambda}\|^2 + \|\psi_i+\hat{\psi}\|^2 + \gamma_i^2\}^{1/2}. \tag{42b}$$

Proof: This corollary follows directly from Theorem 1 and Theorem A1 in Appendix 1. ⌑

3. A Scheme for Automatic Penalty Limitation

We are now going to augment Algorithm 1 by a test which will result in an automatic limitation of the penalty growth. Before we do so, we note that in Algorithm 1, we postulated the use of an unconstrained optimization algorithm which stops when the gradient is small enough. This required us to assume that all the Kuhn-Tucker points in a certain set were strong local minimizers. In [M2], by Mukai and Polak, we find an extension of Newton's method which converges only to stationary points that satisfy second order necessary conditions of optimality and hence are much more likely to be strong local minimizers. The Mukai-Polak algorithm in [M2] requires that the function to be minimized have continuous second order derivatives. As we saw in (9b), the Hessian of F is not defined everywhere and hence is discontinuous. Nevertheless, the Mukai-Polak algorithm [M2] can be extended to this case by suitably smearing the Hessian, as in methods of feasible directions [P4]. We present this new algorithm in Appendix 2, where we see that it replaces the Hessian of F by the matrix, with $\mu > 0$,

$$H(x,\lambda,\psi,c,\mu) = \frac{\partial^2 f(x)}{\partial x^2} + \sum_{j\in \underline{m}} \frac{\partial^2 g^j(x)}{\partial x^2} (\lambda^j + cg^j(x))_+ + \sum_{j\in\underline{\ell}} \frac{\partial^2 h(x)}{\partial x^2} (\psi^j + ch^j(x)) + c\, \frac{\partial h(x)}{\partial x}^T \frac{\partial h(x)}{\partial x} + c \sum_{j\in \tilde{I}(x,\lambda,c,\mu)} \frac{\partial g^j(x)}{\partial x}^T \frac{\partial g^j(x)}{\partial x} , \tag{43a}$$

where

$$\tilde{I}(x,\lambda,c,\mu) \triangleq \{j \in \underline{m} \,|\, g^j(x) \geq (-\lambda^j/c) - \mu\} . \tag{43b}$$

In particular, for given $\lambda_i \in \mathbb{R}^m$, $\psi_i \in \mathbb{R}^\ell$, $c_i \geq 0$, $\gamma_i \geq 0$, $\mu_i > 0$, $\rho_i > 0$, the algorithm in Appendix 2 will yield, under suitable assumptions, after a finite number of iterations, a point x_i such that

$$\|\nabla_x F(x_i,\lambda_i,\psi_i,c_i)\| \leq \gamma_i , \tag{44a}$$

$$\langle y, H(x_i,\lambda_i,\psi_i,c_i,\mu_i)y\rangle \geq -\mu_i \|y\|^2, \ \forall y \in \mathbb{R}^n . \tag{44b}$$

In addition, if initialized at $x_i \triangleq z_0$,

$$x_{i+1} = x_i + \sum_{j=0}^{k_i} z_{j+1} - z_j \tag{44c}$$

and

$$\|z_{j+1} - z_j\| \leq \rho_i \quad \forall j , \tag{44d}$$

$$F(z_{j+1},\lambda_i,\psi_i,c_i) < F(z_j,\lambda_i,\psi_i,c_i) \quad \forall j \; . \tag{44e}$$

We shall denote the result of the computation by the algorithm in Appendix 2 by the map $A: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \times \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^n$, so that

$$x_i = A(x_{i-1},\lambda_i,\psi_i,c_i,\gamma_i,\mu_i,\rho_i) \; . \tag{45}$$

We shall state our multiplier method with automatic penalty limitation in terms of this new unconstrained minimization algorithm. However, any convergent unconstrained minimization algorithm can be used, and our convergence Theorem 2, to be stated later, will remain valid, provided its assumptions are strengthened as indicated in Corollary 2.

We shall use the notation

$$a(x,\lambda,c) \triangleq (g(x) + \frac{1}{c}\lambda)_+ - \lambda \tag{46a}$$

which leads, via (9b) to

$$\nabla_x F(x,\lambda,\psi,c) = \nabla f(x) + \frac{\partial g(x)}{\partial x}^T (\lambda + c a(x,\lambda,c))$$
$$+ \frac{\partial h(x)}{\partial x}^T (\psi + c h(x)) \; . \tag{46b}$$

<u>Algorithm 2</u>

<u>Data</u>: $S = S_\lambda \times S_\psi \subset \mathbb{R}^{m+\ell}$ a compact set containing all possible Kuhn-Tucker multiplier pairs (λ,ψ) in its interior; $x_{-1} \in \mathbb{R}^n$, $(\lambda_0,\psi_0) \in S$, $c_0 \geq 0$.

<u>Parameters</u>: $a \geq 1$, $b \geq 0$ such that $a+b > 1$; $\theta \in (0,1)$; sequences $\{\gamma_i\}_{i=0}^\infty$, $\{\rho_i\}_{i=0}^\infty$, $\{\mu_i\}_{i=0}^\infty$ such that $\gamma_i > 0$, $\rho_i > 0$, $\mu_i > 0$ for all i, $\gamma_i \to 0$, $\rho_i \to 0$, $\mu_i \to 0$ as $i \to \infty$, and $\sum_{i=0}^\infty \gamma_i < \infty$.

<u>Step 0</u>: Set $i = 0$, $j = 0$, $\xi_0 = 1$.

Step 1: Compute $x_i = A(x_{i-1},\lambda_i,\psi_i,c_j,\gamma_i,\mu_i,\rho_i)$. If $\nabla_x F(x_i,\lambda_i,\psi_i,c_j) = 0$, $a(x_i,\lambda_i,c_j) = 0$, $h(x_i) = 0$, and $H(x_i,\lambda_i,\psi_i,c_j,0)$ is positive semidefinite, stop. Else, go to step 2.

Step 2: Set $\tilde{\psi} = \psi_i + c_j h(x_i)$, $\tilde{\lambda} = (\lambda_i + c_j g(x_i))_+ \; (=\lambda_i + c_j a(x_i,\lambda_i,c_j))$.

Step 3: a) If $\tilde{\lambda} \in S_\lambda$ set $\lambda_{i+1} = \lambda$. Else, set $\lambda_{i+1} = \lambda_i$.

b) If $\tilde{\psi} \in S_\psi$, set $\psi_{i+1} = \tilde{\psi}$. Else, set $\psi_{i+1} = \psi_i$.

Step 4: If

$$c_j^2(\|a(x_i,\lambda_i,c_j)\|^2 + \|h(x_i)\|^2) \leq \xi_i, \tag{47a}$$

go to step 5. Else, set $c_{j+1} = ac_j + b$ and go to step 5.

Step 5: Set

$$\xi_{i+1} = \theta\xi_i + \frac{1+\theta}{\theta}\gamma_{i+1}, \tag{47b}$$

set $j(i+1) = j$, $i = i+1$ and go to step 1. ¤

The properties of Algorithm 2 are summarized in the following

Theorem 2: Suppose that Assumptions 1-3 are satisfied, that the construction of x_i in step 1 of algorithm 2 is well defined for all i and consider a sequence $\{x_i\}$ constructed by Algorithm 2.

(i) If $\{x_i\}$ is finite, then the last element, say x_k, is a strong local minimizer.

(ii) If $\{x_i\}$ is infinite and $\{j(i)\}$ is bounded and $\{x_i\}$ has at least one accumulation point $\hat{x}$, then $x_i \to \hat{x}$, as $i \to \infty$, and $\hat{x}$ is a strong local minimizer for (1).

(iii) If $\{x_i\}$ is infinite and $\{j(i)\}$ is unbounded, then $\{x_i\}$ is unbounded. ¤

To prove this theorem we shall need the following results.

Lemma 4: Suppose (x,λ,ψ,c), with $c > 0$ are such that

$$\nabla F(x,\lambda,\psi,c) = 0\ , \tag{48a}$$

$$a(x,\lambda,c) = 0 \ , \tag{48b}$$

$$h(x) = 0 \ . \tag{48c}$$

Then x is a Kuhn-Tucker point and (λ,ψ) is a corresponding multiplier pair. Furthermore, if $H(x,\lambda,\psi,c,0)$ is positive semidefinite, then second order necessary conditions for (1) are satisfied at (x,λ,ψ).

Proof: Since

$$0 = a^j(x,\lambda,c) = \max\{g^j(x), -\frac{\lambda^j}{c}\} \tag{49}$$

we must have $g(x) \leqq 0$, $\lambda \geqq 0$, and $\langle \lambda, g(x)\rangle = 0$. Furthermore, from (9a) and (48a), since $(\lambda + cg(x))_+ = \lambda + ca(x,\lambda,c))$ we obtain that

$$\nabla f(x) + \frac{\partial g(x)}{\partial x}^T \lambda + \frac{\partial h(x)}{\partial x}^T \psi = 0. \tag{50}$$

Hence (x,λ,ψ) is a Kuhn-Tucker triplet. Now,

$$\tilde{I}(x,\lambda,c,0) = \{j \mid g^j(x) \geq -\frac{\lambda^j}{c}\} = \{j \mid g^j(x) = 0\} \triangleq I(x) \tag{51}$$

and hence

$$\begin{aligned} H(x,\lambda,\psi,c,0) &= \frac{\partial^2 f(x)}{\partial x^2} + \sum_{j\in\underline{m}} \frac{\partial^2 g^j(x)}{\partial x^2} \lambda^j \\ &\quad + \sum_{j\in\underline{\ell}} \frac{\partial^2 h^j(x)}{\partial x^2} \psi^j + c\left[\sum_{j\in I(x)} \frac{\partial g^j(x)}{\partial x}^T \frac{\partial g^j(x)}{\partial x} + \frac{\partial h(x)}{\partial x}^T \frac{\partial h(x)}{\partial x}\right] \\ &= \frac{\partial^2 L(x,\lambda,\psi)}{\partial x^2} + c\left[\sum_{j\in I(x)} \frac{\partial g^j(x)}{\partial x}^T \frac{\partial^j g(x)}{\partial x} + \frac{\partial h(x)}{\partial x}^T \frac{\partial h(x)}{\partial x}\right] \end{aligned} \tag{52}$$

Hence, for any y such that $\frac{\partial h(x)}{\partial x} y = 0$ and $\langle \nabla g^j(x), y\rangle = 0$, for all $j \in I(x)$,

$$0 \leq \langle y, H(x,\lambda,\psi,c,0)y\rangle = \langle y, \frac{\partial^2 L(x,\lambda,\psi)}{\partial x^2} y\rangle \tag{53}$$

which shows that the second order necessary condition of optimality is satisfied at (x,λ,ψ). ¤

The following result shows that under certain conditions a sequence of approximate strong local minimizer for the unconstrained problems $\min_x F(x,\lambda_i,\psi_i,c_j)$ converge to a strong local minimizer for (1).

<u>Lemma 5</u>: Suppose that Assumptions 1 and 2 are satisfied. Consider an infinite sequence $\{x_i\}_{i=0}^{\infty}$ constructed by Algorithm 2, with $A(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ satisfying (44a-44c) and suppose that $x_i \overset{K}{\to} \hat{x}$ as $i \to \infty$ (i.e. $\hat{x}$ is an accumulation point of $\{x_i\}$). If (i) $\hat{x}$ is a Kuhn-Tucker point, (ii) $((\lambda_i+c_{j(i)}g(x_i))_+, \psi_i+c_{j(i)}h(x_i)) \overset{K}{\to} (\hat{\lambda},\hat{\psi})$ as $i \to \infty$, a multiplier pair for $\hat{x}$, and (iii) either $j(i) \to \infty$ or $(\lambda_i,\psi_i) \overset{K}{\to} (\hat{\lambda},\hat{\psi})$ as $i \to \infty$, then $\hat{x}$ satisfies second order necessary conditions of optimality for (1).

<u>Proof</u>: To establish a contradiction, suppose that there exists a $\hat{y} \in \mathbb{R}^n$ such that $\frac{\partial h(\hat{x})}{\partial x}\hat{y} = 0$, $\langle \nabla g^j(\hat{x}),\hat{y}\rangle = 0$ for all $j \in I(\hat{x})$, and $\langle \hat{y}, \frac{\partial^2 L(\hat{x},\hat{\lambda},\hat{\psi})}{\partial x^2}\hat{y}\rangle = -\delta < 0$. Since by Assumption 2, the vectors $\nabla h^j(\hat{x})$, $j \in \ell$ together with the vectors $\nabla g^j(\hat{x})$, $j \in I(\hat{x})$ are linearly independent, it follows from the Implicit Function Theorem that there exists a sequence of vectors $\{y_i\}_{i\in K}$ in $\mathbb{R}^n$ such that $y_i \overset{K}{\to} \hat{y}$ as $i \to \infty$, and

$$\frac{\partial h(x_i)}{\partial x} y_i = 0, \ \langle \nabla g^j(x_i),y_i\rangle = 0 \quad \text{for all } j \in I(\hat{x}), \quad \text{for all } i \in K . \tag{54a}$$

Hence, from continuity of the scalar product and of $\frac{\partial^2 L(\cdot,\cdot,\cdot)}{\partial x^2}$, it follows that there exists an $i_o \in K$ such that

$$\langle y_i, \frac{\partial^2 L(x_i,(\lambda_i+c_{j(i)}g(x_i))_+,\psi_i+c_{j(i)}h(x_i))}{\partial x^2} y_i\rangle < -\frac{\delta}{4} \quad \text{for all } i \in K,\ i \geq i_o \tag{54b}$$

(since $(\lambda_i+c_{j(i)}g(x_i))_+ \overset{K}{\to} \hat{\lambda}$ and $(\psi_i+c_{j(i)}h(x_i)) \overset{K}{\to} \hat{\psi}$ as $i \to \infty$ by assumption). Now by definition,

$$H(x_i,\lambda_i,\psi_i,c_{j(i)},\mu_i) = \frac{\partial^2 L(x_i(\lambda_i+c_{j(i)}g(x_i))_+,\psi_i+c_{j(i)}h(x_i))}{\partial x^2}$$

$$+ c_{j(i)}\left[\sum_{j\in I(x_i,\lambda_i,c_{j(i)},\mu_i)} \frac{\partial g^j(x_i)^T}{\partial x}\frac{\partial g^j(x_i)}{\partial x} + \frac{\partial h(x_i)^T}{\partial x}\frac{\partial h(x_i)}{\partial x}\right] \tag{55}$$

So, suppose that $j(i) \to \infty$, then $c_{j(i)} \to \infty$ and hence $\lambda_i^j/c_{j(i)} \to 0$ as $i \to \infty$, because λ_i is bounded. Since the $g^j(\cdot)$ are continuous and $\mu_i \to 0$ as $i \to \infty$, $\lambda_i \geqq 0$ for all i and

$$I(x_i,\lambda_i,c_{j(i)},\mu_i) = \{j \mid g^j(x_i) \geq -\lambda_i^j/c_{j(i)} - \mu_i\}, \tag{56a}$$

we see that in this case there exists an $i_1 \geq i_o$ such that

$$I(x_i,\lambda_i,c_{j(i)},\mu_i) \subset I(\hat{x}) \text{ for all } i \in K,\ i \geq i_1 \tag{56b}$$

and hence, from (54a,b) for some $i_2 \geq i_1$,

$$\langle y_i, H(x_i,\lambda_i,\psi_i,c_{j(i)},\mu_i)y_i\rangle < -\delta/4 < -\mu_i, \text{ for all } i \in K,\ i \geq i_2, \tag{57}$$

which contradicts the definition of A. Similarly if $(\lambda_i,\psi_i) \overset{K}{\to} (\hat{\lambda},\hat{\psi})$, as $i \to \infty$, $\lambda_i^j \overset{K}{\to} 0$ as $i \to \infty$ for all $j \notin I(\hat{x})$ and hence (56b) must again hold for a suitable $i_1 \geq i_0$. But then (57) must hold, which again leads to a contradiction. This completes our proof. ¤

The next two results are analogs of Lemma 2 and Proposition 1 for the case case where c is kept constant rather than driven to infinity.

<u>Lemma 6</u>: Suppose that Assumption 1 is satisfied and that $\hat{x}$ is a strict local minimizer for (1), with $\hat{\rho} > 0$ a radius of attraction. Then for any $\delta > 0$, $S \subset \mathbb{R}^{m+\ell}$ compact and $c > 0$ there exist $\hat{\varepsilon} > 0$ and $\hat{\gamma} > 0$ such that

$$N_{\hat{x},\hat{\rho}}(\lambda,\psi,c,\varepsilon) \cap \{x \mid \|h(x)\| \leq \hat{\gamma},\ \|a(x,\lambda,c)\| \leq \hat{\gamma}\} \subset B(\hat{x},\delta)$$

$$\text{for all } \varepsilon \in [0,\hat{\varepsilon}],\ (\lambda,\psi) \in S\ . \tag{58}$$

¤

We omit a proof of this lemma since its proof is almost identical to that of Lemma 2.

The following proposition can be proved using Lemma 6 in more or less the same way as Proposition 1 was proved using Lemma 2.

Proposition 2: Suppose that Assumptions 1 and 2 are satisfied. Let $c > 0$ and let $\{\rho_i\}_{i=0}^{\infty}$ be such that $\rho_i > 0$ for all i and $\rho_i \to 0$ as $i \to \infty$. Let $\{x_i\}_{i=0}^{\infty}$ be such that $h(x_i) \to 0$ as $i \to \infty$ and $x_i \overset{K}{\to} \hat{x}$, for some $K \subset \{0,1,2,\dots\}$, with $\hat{x}$ a strong local minimizer for (1), with corresponding Kuhn-Tucker multipliers $(\hat{\lambda},\hat{\psi})$. Let $\{(\lambda_i,\psi_i)\}_{i=0}^{\infty} \subset \mathbb{R}^{m+\ell}$ be such that $(\lambda_i,\psi_i) \to (\hat{\lambda},\hat{\psi})$ as $i \to \infty$, and $a(x_i,\lambda_i,c) \to 0$ as $i \to \infty$. Furthermore, suppose that for $i = 1,2,3,\dots$, x_i is constructed by a step size limited descent method, i.e.,

$$x_i = x_{i-1} + \sum_{j=0}^{k_i} (z_{j+1}-z_j) \tag{59a}$$

where $z_0 = x_{i-1}$ and

$$\|z_{j+1}-z_j\| \leq \rho_i, \ \forall j \tag{59b}$$

Under these conditions, $x_i \to \hat{x}$ as $i \to \infty$. ¤

We are finally ready to give a proof of Theorem 2.

Proof of Theorem 2:

(i) Suppose $\{x_i\}$ is finite, with last element x_k. Then it satisfies the test in step 1 of Algorithm 2 and hence $\nabla_x F(x_k,\lambda_k,\psi_k,c_{j(k)}) = 0$, $a(x_k,\lambda_k,c_{j(k)}) = 0$, $h(x_k) = 0$, and $H(x_k,\lambda_k,\psi_k,c_{j(k)},0)$ is positive semidefinite. It now follows from Lemma 4 that (x_k,λ_k,ψ_k) is a Kuhn-Tucker triplet satisfying the second order necessary conditions of optimality for (1). Hence, by Assumption 3, $\hat{x}$ is a strong local minimizer.

(ii) Next we suppose that $\{x_i\}$ is infinite, that $j(i)$ is bounded and that $x_i \overset{K}{\to} \hat{x}$ as $i \to \infty$. Then there exists an i_o such that $j(i) = j_o \triangleq j(i_o)$ for all $i \geq i_o$, and by (47a)

$$c_{j_o}^2 (\|a(x_i,\lambda_i,c_{j_o})\|^2 + \|h(x_i)\|^2) \leq \xi_i, \text{ for all } i \geq i_o \tag{60}$$

Since by construction in steps 2 and 3 we always have

$$\|\lambda_{i+1}-\lambda_i\|^2 \leq c_{j(i)}^2 \|a(x_i,\lambda_i,c_{j(i)})\| \tag{61a}$$

and

$$\|\psi_{i+1}-\psi_i\| \leq c_{j(i)}^2 \|h(x_i)\|^2 \tag{61b}$$

it follows from (60) that

$$\|\lambda_{i+1}-\lambda_i\|^2 + \|\psi_{i+1}-\psi_i\|^2 \leq \xi_i \text{ for all } i \geq i_o \tag{62a}$$

so that for all $k \geq 1$,

$$\|\lambda_{i+k}-\lambda_i\| + \|\psi_{i+k}-\psi_i\|^2 \leq \sum_{j=i}^{i+k-1} \xi_j \tag{62b}$$

Next, we show that by construction in (47b), ξ_i is bounded for all i. Indeed, since $\gamma_i \to 0$ as $i \to \infty$, there exists an i_1 such that

$$\frac{1+\theta}{\theta} \gamma_{i+1} < 1-\theta, \text{ for all } i \geq i_1 \tag{63a}$$

Hence, from (47b)

$$\xi_{i+1} \leq \theta\xi_i + (1-\theta) \text{ for all } i \geq i_1 \tag{63b}$$

Now, if $\xi_i \leq 1$, then, from (63b), $\xi_{i+1} \leq 1$ and if $\xi_i > 1$, then from (63b), $\xi_{i+1} \leq \theta\xi_i + (1-\theta)\xi_i = \xi_i$. Consequently, $\xi_i \leq \min\{1,\xi_o\}$ for all i. Returning to (62b), we see that

$$\sum_{j=i}^{i+k-1} \xi_j = \theta \sum_{j=i}^{i+k-1} \xi_{j-1} + \frac{1+\theta}{\theta} \sum_{j=i}^{i+k-1} \gamma_j = \theta \sum_{j=i-1}^{i+k-2} \xi_i + \frac{1+\theta}{\theta} \sum_{j=i}^{i+k-1} \gamma_j \tag{64a}$$

Hence

$$(1-\theta) \sum_{j=i}^{i+k-2} \xi_j = -\theta\xi_{i-1} - \xi_{i-k-1} + \frac{1+\theta}{\theta} \sum_{j=i}^{i+k-1} \gamma_j \, . \tag{64b}$$

Since ξ_i is bounded for all i and $\sum_{j=0}^{\infty} \gamma_i < \infty$, it follows from (64b) that $\sum_{j=i}^{\infty} \xi_i < \infty$ for all i and hence that

$$\sum_{j=i}^{\infty} \xi_j \to 0 \text{ as } i \to \infty \, . \tag{65a}$$

We therefore conclude from (62b) that for all $k \geq 1$,

$$\|\lambda_{i+k}-\lambda_i\|^2 + \|\psi_{i+k}-\psi_i\|^2 \to 0 \text{ as } i \to \infty, \tag{65b}$$

and hence that $\{(\lambda_i,\psi_i)\}$ is Cauchy, so that $\lambda_i \to \hat{\lambda}$, $\psi_i \to \hat{\psi}$ as $i \to \infty$, for some $(\hat{\lambda},\hat{\psi}) \in \mathbb{R}^{m+\ell}$. Now, from (65a), $\xi_i \to 0$ as $i \to \infty$ and hence from (60), we conclude that

$$a(x_i,\lambda_i,c_{j_o}) \to 0 \text{ as } i \to \infty, \tag{66a}$$

$$h(x_i) \to 0 \text{ as } i \to \infty \, . \tag{66b}$$

Hence, since $x_i \overset{K}{\to} \hat{x}$, by continuity,

$$a(\hat{x},\hat{\lambda},c_{j_o}) = 0, \; h(\hat{x}) = 0 \text{ and } \nabla_x F(\hat{x},\hat{\lambda},\hat{\psi},c_{j_o}) = 0.$$

Therefore, by Lemma 4, $(\hat{x},\hat{\lambda},\hat{\psi})$ is a Kuhn-Tucker triplet. Since it also follows that $(\lambda_i+c_{j_o}g(x_i))_+ \to \hat{\lambda}$ and $(\psi_i+c_{j_o}h(x_i)) \to \hat{\psi}$ as $i \to \infty$, we conclude from Lemma 5 that the Kuhn-Tucker triplet $(\hat{x},\hat{\lambda},\hat{\psi})$ satisfies second order necessary conditions of optimality for (1). But by Assumption 3, $\hat{x}$ is then a strong local minimizer for (1). Finally, it follows from Proposition 2 that $x_i \to \hat{x}$ as $i \to \infty$.

(iii) We now suppose that $j(i) \to \infty$ as $i \to \infty$ and we will show that $\{x_i\}$ has no accumulation points. To obtain a contradiction, suppose that $x_i \overset{K}{\to} \hat{x}$ as $i \to \infty$. Then, by Lemma 3, $\hat{x}$ is a Kuhn-Tucker point, and

and $(\lambda_i+c_i g(x_i))_+ \to \hat{\lambda}$ and $\psi_i + c_i h(x_i) \to \hat{\psi}$ as $i \to \infty$, with $(\hat{\lambda},\hat{\psi})$ a corresponding Kuhn-Tucker multiplier pair. It now follows from Lemma 5 that $\hat{x}$ satisfies second order necessary conditions of optimality for (1). Hence, from Assumption 3, $\hat{x}$ is a strong local minimizer for (1) and therefore, by Proposition 1, $x_i \to \hat{x}$ as $i \to \infty$. Now, since by assumption on S, $(\hat{\lambda},\hat{\psi})$ must be in the interior of $S_\lambda \times S_\psi$, there exists an i_o such that $(\lambda_i+c_{j(i)}g(x_i))_+ \in S_\lambda$ and $(\psi_i+c_{j(i)}h(x_i)) \in S_\psi$ for all $i \geq i_o$ and $\psi_{i+1} = \psi_i + c_{j(i)}h(x_i)$ for all $i \geq i_o$. It now follows from Theorem A1 that there exists an $i_1 \geq i_o$ and an $M \in (0,\infty)$ such that

$$\|\lambda_{i+1}-\hat{\lambda}\|^2 + \|\psi_{i+1}-\hat{\psi}\|^2 \leq \frac{M}{c_{j(i)}^2} (\|\lambda_i-\hat{\lambda}\|^2 + \|\psi_i-\hat{\psi}\|^2) + M\gamma_i^2 \quad \text{for all } i \geq i_1 \tag{67a}$$

Let $i_2 \geq i_1$ be such that $M/c_{j(i)} < \theta$ and $M\gamma_i < 1$ for all $i \geq i_2$, then, from (67a)

$$\|\lambda_{i+1}-\hat{\lambda}\|^2 + \|\psi_{i+1}-\hat{\psi}\|^2 \leq \eta(\|\lambda_i-\hat{\lambda}\|^2 + \|\psi_i-\hat{\psi}\|^2) + \gamma_i \quad \text{for all } i \geq i_2, \tag{67b}$$

for some $\eta < \theta$. Consequently, for $k = 1,2,3,\ldots,$

$$\|\lambda_{i_2+k}-\hat{\lambda}\|^2 + \|\psi_{i_2+k}-\hat{\psi}\|^2 \leq \eta^k(\|\lambda_{i_2}-\hat{\lambda}\|^2 + \|\psi_{i_2}-\hat{\psi}\|^2) + \sum_{j=i_2}^{i_2+k-1} \eta^{i_2+k-1-j} \gamma_j \tag{68a}$$

and

$$\|\lambda_{i_2+k+1}-\hat{\lambda}\|^2 + \|\psi_{i_2+k+1}-\hat{\psi}\|^2 \leq \eta^{k+1}(\|\lambda_{i_2}-\hat{\lambda}\|^2 + \|\psi_{i_2}-\hat{\psi}\|^2) + \sum_{j=i_2}^{i_2+k} \eta^{i_2+k-j} \gamma_j \,. \tag{68b}$$

It therefore follows that

$$\|\lambda_{i_2+k+1}-\lambda_{i_2+k}\|^2 + \|\psi_{i_2+k+1}-\psi_{i_2+k}\|^2 \leq \eta^k(1+\eta)(\|\lambda_{i_2}-\hat{\lambda}\|^2 + \|\psi_{i_2}-\hat{\psi}\|^2)$$

$$+ \sum_{j=i_2}^{i_2+k-1} \eta^{i_2+k-1-j}(1+\eta)\gamma_j + \gamma_{i_2+k} \tag{68c}$$

Since $i_2+k > i_2$ for $k = 1,2,\ldots$, we conclude from (68c) and the construction of $(\lambda_{i+1},\psi_{i+1})$ in step 3, that

$$c^2_{j(i_2+k)}(\|a(x_{i_2+k},\lambda_{i_2+k},c_{j(i_2+k)})\|^2 + \|h(x_{i_2+k})\|^2)$$

$$\leq \eta^k(1+\eta)(\|\lambda_{i_2}-\hat{\lambda}\|^2 + \|\psi_{i_2}-\hat{\psi}\|^2$$

$$+ \sum_{j=i_2}^{i_2+k-1} \eta^{i_2+k-1-j}(1-\eta)\gamma_i + \gamma_{i_2+k} \tag{68d}$$

Now, from (47b), and since from Step 0 of Algorithm 2, $\xi_0 = 1$,

$$\xi_{i_2+k} = \theta^{i_2+k} + \frac{(1+\theta)}{\theta}\sum_{j=0}^{i_2+k-1} \theta^{i_2+k-1=j}\gamma_{j+1}$$

$$= \theta^{i_2+k} + (1+\theta)\sum_{j=1}^{i_2+k-1} \theta^{i_2+k-1-j}\gamma_j + \frac{1+\theta}{\theta}\gamma_{i_2+k}\,. \tag{69}$$

Since $\theta < 1$, $\theta < (1+\theta)/\theta$ and since $\eta < \theta$, it follows from (68d) and (69) that

$$c^2_{j(i_2+k)}\|a(x_{i_2},\lambda_{i_2+k},c_{j(i_2+k)})\|^2 + \|h(x_{i_2+k})\|^2 \leq \xi_{i_2+k}$$

$$\text{for } k = 1,2,\ldots \tag{70}$$

which shows that the test in step 4 of Algorithm 2 must have been satisfied for all $i > i_2$ and therefore $j(i)$ could not have been increased an infinite number of times, as hypothesized. Hence, if $j(i) \to \infty$ as $i \to \infty$, then $\{x_i\}$ cannot have any accumulation points. This completes our proof. ¤

The following result should be obvious.

Corollary 2: Suppose that all the assumptions of Theorem 2 are satisfied and, in addition, that every Kuhn-Tucker point in $\overline{\lim}\ \mathcal{L}_i$ is a strong local minimizer. Then the conclusions of Theorem 2 remain valid even when the condition (44b) is removed from the qualification of A (i.e. even when only a first order stopping rule is used). ¤

Conclusion

Our initial intention was to construct a scheme for limiting the penalty growth for multiplier methods. However, the mechanisms that presented themselves worked only if the sequence $\{x_i\}$, constructed by the multiplier method without penalty growth limitation, converged, and moreover, the limit point had to be a strong local minimum. Now, in general, a sequence constructed by a standard penalty function method may have several accumulation points and hence it was necessary to devise a modification to standard practice in order to force the entire sequence to converge. This was achieved by limiting the step size of the unconstrained optimization algorithms to be used in the construction of the x_i's. In addition, a special second order unconstrained optimization algorithm was devised, which converges only to strong local minima. Once this was done, we proposed a test for penalty limitation based on the rate of convergence of multiplier methods, established by Bertsekas. The combined result is a globally convergent multiplier method with automatic limitation of penalty growth. Finally, it should be pointed out that similar results can also be developed for multiplier methods using a second order updating formula for the multipliers (e.g. Fletcher [F5]). This extension is advisable especially when one uses a second order unconstrained optimization algorithm; the required proofs, based on a rate of convergence result due to Bertsekas [B4] are essentially routine.

Appendix 1. Rate of Convergence.

The following theorem is a slight extension of results proved by Bertsekas [B3] and Polyak and Tret'yakov [P6].

Theorem A1. Suppose that Assumptions 1 and 2 are satisfied and suppose that $\bar{x}$ is a strong local minimizer for (1) with multipliers $\bar{\lambda}$ and $\bar{\psi}$. Then, given real sequences $\{\gamma_i\}_{i=0}^{\infty}$, $\{c_i\}_{i=0}^{\infty}$ with $\gamma_i > 0$, $c_i \geq 0$, $i = 0,1,2,\ldots$, and $\gamma_i \to 0$, $c_i \to \infty$ as $i \to \infty$, and given $S \subset \mathbb{R}^{m+\ell}$ a compact set, there exists an integer $i_o \geq 0$ and a scalar $\varepsilon > 0$ such that for any $i \geq i_o$, for all $(\lambda,\psi) \in S$ and for all $v \in \mathbb{R}^n$ such that $\|v\| \leq \gamma_i$, there exists a unique point $x_v(\lambda,\psi,c_i)$ in $B(\bar{x},\varepsilon)$ which satisfies

$$\nabla F(x_v(\lambda,\psi,c_i),\lambda,\psi,c_i) = v \tag{A1}$$

Furthermore, for some scalar $M > 0$,

$$\|x_v(\lambda,\psi,c_i)-\bar{x}\|^2 \leq \frac{M}{c_i^2}(\|\lambda-\bar{\lambda}\|^2 + \|\psi-\psi\|^2) + M\gamma_i^2 \tag{A2}$$

and

$$\|\tilde{\lambda}_v(\lambda,\psi,c_i)-\lambda\|^2 + \|\tilde{\psi}_v(\lambda,\psi,c)-\bar{\psi}\|^2 \leq \frac{M}{c_i^2}(\|\lambda-\bar{\lambda}\|^2 + \|\psi-\bar{\psi}\|^2) + M\gamma_i^2 \tag{A3}$$

where we have used the notation

$$\tilde{\lambda}_v(\lambda,\psi,c) = (\lambda+cg(x_v(\lambda,\psi,c)))_+ \tag{A4}$$

$$\tilde{\psi}_v(\lambda,\psi,c) = \psi + ch(x_v(\lambda,\psi,c)) \quad \square \tag{A5}$$

This theorem extends Bertsekas' result in two ways. First, he assumes that $\{\gamma_i c_i\}$ is bounded for all i, whereas we only assume that $\gamma_i \to 0$ as $i \to \infty$. Second, Bertsekas' theorem implies that the ε-ball within which the solution to (A1) is unique depends on i, whereas in our statement this ball is indpendent of i and this insures that any sequence converging to $\bar{x}$ will eventually be captured in that neighborhood. We omit the proof of our theorem because it is essentially identical to Bertsekas' proof provided that one restates the Implicit Function Theorem in the special form below.

<u>Lemma A1</u> (implicit function theorem). Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and let $x_o \in \mathbb{R}^m$, $y_o \in \mathbb{R}^n$ be such that $f(x_o,y_o) = 0$; moreover, suppose that the Jacobian $\frac{\partial f}{\partial y}$ with respect to the last n variables is nonsingular at (x_o,y_o). Then there exist two scalars $\alpha > 0$, $\beta > 0$ and a unique mapping u of $U = B(x_o,\alpha) \subset \mathbb{R}^m$ into $V = B(y,\beta) \subset \mathbb{R}^n$ such that $f(x,u(x)) = 0$ for every $x \in U$. Moreover, u is continuous in U and $u(x_o) = y_o$. Finally, α and β do not depend directly on f but only on an upper bound on $\left\|\frac{\partial f(x_o,y_o)}{\partial y}^{-1}\right\|$ and on a function $\phi : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$\phi(x,y_1,y_2) \triangleq f(x,y_1) - f(x,y_2) - \frac{\partial f}{\partial y}(x_o,y_o)(y_1-y_2). \tag{A6}$$

¤

The importance of this reformulation is demonstrated in the following corollary.

<u>Corollary A1</u> Let $\mathcal{F} = \{f_c | c \in C\}$ be a parametric family of functions satisfying the hypotheses of Lemma A1 and suppose that there exists a scalar $M > 0$ such that

$$\left\|\frac{\partial f_c}{\partial y}(x_o,y_o)^{-1}\right\| \leq M \quad \forall c \in C \tag{A7}$$

and suppose that, for some $\phi : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$$f_c(x,y_1) - f_c(x,y_2) - \frac{\partial f_c}{\partial y}(x_o,y_o)(y_1-y_2) = \phi(x,y_1,y_2) \quad \forall c \in C; \tag{A8}$$

Then there exist two scalars $\alpha > 0$, $\beta > 0$ such that for any $c \in C$ there exists a unique mapping u_c of $U = B(x_o,\alpha) \subset \mathbb{R}^m$ into $V = B(y_o,\beta) \subset \mathbb{R}^n$ satisfying $f_c(x,u_c(x)) = 0$ for every $x \subset U$. Furthermore, for all $c \in C$, u_c is continuous in U and $u_c(x_o) = y_o$. ¤

Appendix 2. A Second Order Algorithm

The following algorithm is an extension of an algorithm proposed by Mukai and Polak [M2]. Since the augmented Lagrangian is in general not twice continuously differentiable we have introduced a "smearing" scheme which is an analogous to the use of ε-active constraints in Polak and Zoutendijk methods of feasible directions [P4] [Z1] and is related to the ε-bundles used in nondifferentiable optimization (see Demyanov [D1,2], Polak-Sangiovanni-Vincentelli [P5]. Furthermore we have introduced a limitation of the stepsize, as required in our theory.

In the sequel we shall make use of the following simplified notation

$$g(x) \triangleq \nabla_x F(x,\lambda,\psi,c) \tag{A9}$$

$$H(x) \triangleq H(x,\lambda,\psi,c,\mu) \tag{A10}$$

with $H(x,\lambda,\psi,c,\mu)$ as defined in (43a). For given (λ,ψ,c,μ) we define, as in Mukai-Polak, the function $\phi : \mathbb{R}^n \to \mathbb{R}^1$ by

$$\phi(x) = \min \frac{1}{2}\{\langle e,H(x)e\rangle \mid \|e\| \leq 1\}. \tag{A11}$$

The following algorithm is used to compute a point

$x' = A(x,\lambda,\psi,c,\gamma,\mu,\rho)$ (see (44a-d), (45)).

Algorithm A1.

Data: $\alpha \in (0,1)$, $\beta \in (0,1)$, $0 < \varepsilon_o << 1$, $x_0 = x$, $\mu > 0$, $\gamma > 0$

Step 0: Set $i = 0$.

Step 1: Compute $\phi(x_i)$ and an

$$e_i \in \{e \in \mathbb{R}^n \mid \langle g(x_i),e\rangle \leq 0,\ \|e\| \leq 1,\ \phi(x_i) = \frac{1}{2}\langle e,H(x_i)e\rangle\}. \tag{A12}$$

Step 2: If $\phi(x_i) \geq -\mu$ and $\|g(x_i)\| \leq \gamma$, set $x' = x_i$ and stop; else go to step 3.

Step 3. If $\phi(x_i) < 0$ go to step 6; else go to Step 4.

Step 4. If the least eigenvalue of $H(x_i)$ is less than ε_o, go to step 6; else go to step 5.

Step 5. Compute $h_i = -H(x_i)^{-1}g(x_i)$, set $\nu_o = 1$ and go to step 8.

Step 6. Compute $h_i = -g(x_i) + e_i$

Step 7. If $\langle h_i, H(x_i)h_i\rangle \leq 0$, set $\nu_o = 1$ and go to step 8; else set $\nu_o = \beta^{k_i}$ where $k_i \geq 0$ is the smallest integer satisfying

$$\beta^{k_i} \leq -\langle g(x_i),h_i\rangle / \langle h_i,H(x_i)h_i\rangle \tag{A13}$$

Step 8. Compute the smallest nonnegative integer ℓ_i satisfying

$$\nu_o\beta^{\ell_i}\|h_i\| \leq \rho \tag{A14}$$

and

$$f(x_i+\nu_o\beta^{\ell_i}h_i)-f(x_i) \leq \alpha[\nu_o\beta^{\ell_i}\langle g(x_i),h_i\rangle + \frac{1}{2}(\nu_o\beta^{\ell_i})^2\langle h_i,H(x_i)h_i\rangle] . \tag{A15}$$

Step 9. Set $x_{i+1} = x_i + \nu_o\beta^{\ell_i}h_i$, set $i = i+1$ and go to step 1. ¤

Theorem A2. Either the sequence $\{x_i\}$ constructed by the above algorithm is unbounded or the algorithm terminates after a finite number of steps, yielding $x' = A(x;\lambda,\psi,c,\gamma,\mu,\rho)$. ¤

We omit a proof of this theorem since it follows the same lines as the proof given by Mukai and Polak and is rather tedious.

Bibliography

[B1] D. P. Bertsekas, Multiplier methods: a survey, Automatica, 12 (1976), pp. 133-145.

[B2] D. P. Bertsekas, Combined primal-dual and penalty methods for constrained minimization, SIAM Journal on Control and Optimization, 13 (1975), pp. 521-544.

[B3] D. P. Bertsekas, On penalty and multiplier methods for constrained optimization, SIAM Journal on Control, 14 (1976), pp. 216-235.

[B4] D. P. Bertsekas, On the convergence properties of second-order multiplier methods, J. Optimization Theory Appl., 25 (1978), pp. 443-449.

[B5] J. D. Buys, Dual algorithms for constrained optimization, Ph.D. Thesis, Rijksuniversiteit te Leiden, 1972.

[D1] V. F. Demyanov, Algorithm for some minimax problems, Journal of Computer and Systems Sciences, vol. 2 (1968), pp. 342-380.

[D2] V. F. Demyanov, Differentiability of a maximum function: I, USSR Comp. Math. and Math. Phys., 8 (1968), no. 6, pp. 1-15.

[F1] R. Fletcher, A class of methods for nonlinear programming with termination and convergence properties, in Integer and Nonlinear Programming, J. Abadie, ed., North-Holland, Amsterdam, 1970, pp. 157-175.

[F2] R. Fletcher, Methods related to Lagrangian functions, in numerical methods for constrained optimization, P. E. Gill and W. Murray, eds., Academic Press, London, 1974, pp. 219-239.

[F3] R. Fletcher and S. Lill, A class of methods for nonlinear programming II: Computational experience, in Nonlinear Programming, J. B. Rosen, O. L. Mangasarian and K. Ritter, eds., Academic Press, 1971.

[F4] R. Fletcher, A class of methods for nonlinear programming III: Rate of convergence, in Numerical Methods for Nonlinear Optimization, F. A. Lootsma, ed., Academic Press, 1972.

[F5] R. Fletcher, An ideal penalty function for constrained optimization, Journal of the Institute of Mathematics and its Applications, 15 (1975), pp. 319-342.

[G1] T. Glad and E. Polak, A mutliplier method with automatic limitation of penalty growth, Electronics Research Lab. Memo #M/77/54 (1977), also Mathematical Programming, in press.

[H1] M. R. Hestenes, Multiplier and gradient methods, J. Optimization Theory Appl., 4 (1969), pp. 303-320.

[H2] M. R. Hestenes, The Weierstrass E-function in the calculus of variations, Transactions of the American Mathematical Society, 60 (1946), pp. 51-71.

[M1] H. Mukai and E. Polak, A quadratically convergent primal-dual algorithm with global convergence properties for solving optimization problems with equality constraints, Mathematical Programming, 9 (1975), pp. 336-349.

[M2] H. Mukai and E. Polak, A second-order method for the general nonlinear programming problem, J. Optimization Theory Appl., 26 (1978), pp. 515-532.

[P1] M. J. D. Powell, A method for nonlinear constraints in minimization problems, in Optimization, R. Fletcher, ed., Academic Press, New York, 1969, pp. 283-298.

[P2] M. J. D. Powell, Algorithms for nonlinear constraints that use Lagrangian functions, Mathematical Programming, 14 (1978), pp. 224-248.

[P3] D. A. Pierre and M. J. Lowe, Mathematical programming via augmented Lagragians: An introduction with computer programs, Addison-Wesley, 1975.

[P4] E. Polak, Computational methods in optimization, Academic Press, New York, 1971.

[P5] E. Polak and A. Sangiovanni-Vincentelli, Theoretical and computational aspects of the optimal design centering, tolerancing and tuning problem, IEEE Transactions on Circuits and Systems, 26 (1979), no. 5.

[P6] B. T. Polyak and N. V. Tret'yakov, The method of penalty estimates for conditional extremum problems, USSR Comput. Math. and Mathematical Phys., 13 (1974), pp. 42-58.

[R1] R. T. Rockafellar, A dual approach to solving nonlinear programming problems by unconstrained optimization, Mathematical Programming, 5 (1973), pp. 354-373.

[R2] R. T. Rockafellar, Augmented Lagrange multiplier functions and duality in nonconvex programming, SIAM Journal on Control, 12 (1974), pp. 268-285.

[Z1] G. Zoutendijk, Methods of feasible directions, Elsevier, Amsterdam, 1960.