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POSSIBILITY THEORY AND SOFT DATA ANALYSIS

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L. A. Zadeh

Memorandum No. UCB/ERL M79/66

27 August 1979

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ELECTRONICS RESEARCH LABORATORY

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Abstract

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33 A thesis advanced in this paper is that much of the 32: uncertainty which is associated with soft data is nonstatis-Itical in nature. Based on this premise, an approach to the correpresentation and manipulation of soft data--in which the 29 recently developed theory of possibility plays a central role--13 is described and illustrated with examples.

Introduction 1.

The term soft data does not have a universally agreed -3 upon meaning. Some use it to characterize data that are _____ imprecise or uncertain, while others attach the label "soft" at data whose credibility is open to question.

In dealing with soft data of the type encountered in such diverse fields as psychology, sociology, anthropology, medicine, economics, management science, operations research, 15 14 pattern classification and systems analysis, it is a standard practice to rely almost entirely on the techniques provided 12 by probability theory and statistics, especially in applica-10 : tions relating to parameter estimation, hypothesis testing and system identification. It can be argued, however, as we do in the present paper, that such techniques cannot cope

* To Professor Brian R. Gaines. Supported by Naval Electronic Systems Command Contract N00039-78-C-0013 and National Science Foundation Grant ENG78-23143.

ும், பிதா இதில் 500 தேவில் கிகர்த் 500 கிலிக்கில் கேரின் 500 தேதேயில் என்றி5்பில் கொணியிலாகில் செல்லா பில்ல பிது சிறிற்றுகள் துணைகளைகளைகளைகளைகளில் திருதின் நிரை சிலிலாக 200 நிற்றிக்கு 4 விக்கும் 500 விறிற்குக்கு 500 நிற பிழக்கியால் தேதேசிகள் தன்து 500 தேதில் தேத்தை கிகர்களில் திரைப்பிக்கும் 500 கிலிலாக மாகிற்று. rounkrol in in operat Barentikoa barri many at 191324. n te Swindert paragrophik 5 soubey burden o bhen in en lubh agun hibh-di vitim aig n' kitak 843 1 11 M-20.20 To a number NE Many sortiot Preparetor to -3. Gurdin

effectively with those problems in which the softness of data is nonstatistical in nature -- in the sense that it relates, in the main, to the presence of fuzzy sets rather than to random measurement errors or data variability.

Needless to say, the inability of conventional statistical techniques to deal with problems of this type would not matter much if the predominance of fuzziness in softness were a rare phenomenon. In reality, the opposite is the case; for, upon closer examination, it becomes clear that much of the softness in data analysis is nonstatistical in the sense 77 :4 explicated above. Moreover, the same is true of most of the linguistic information that humans manipulate through an 33 implicit use of what might be called approximate (or fuzzy) reasoning based on fuzzy rather than standard logic.

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To make the latter point more concretely, it will be helpful to list -- and subsequently analyze in greater detail -several typical examples of everyday type of questions which cannot be handled effectively by conventional probabilitybased methods. In these questions, the soft data are expressed as propositions appearing above the horizontal line; the italicized words are the labels of fuzzy sets; and the answers are expected to be in the form of a fuzzy proposition, that is, a proposition whose constituents may have a fuzzy 5 denotation. Specifically:

(a) X is a large number

Y is much larger than X

How large is Y?

(b) Most Frenchmen are not tall Elie is a Frenchman

How tall is Elie?

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(c)	It is unlikely that Andrea is very young								
	It is likely that Andrea is young								
	It is very unlikely that Andrea is old								
	How likely is it that Andrea is not old?								
(đ)	It is true that Hans is not very tall								
	It is very true that Hans is not short								
•	How tall is Hans?								
(e)	Brian is much taller than most of his close friends								
	How tall is Brian?								
(f)	-If Bernadette lives in Versailles then she is very rich								
	If Bernadette lives in Monmartre then she is poor								
	It is <i>likely</i> that Bernadette lives in Versailles								
	It is very unlikely that Bernadette lives in Monmartre								

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How likely is it that Bernadette is. not rich?

As will be seen in the sequel, our approach to the analysis of soft data of the type illustrated by the above examples is based on *fuzzy logic* [7,22,89] rather than on a combination of classical logic and probability-based methods --as is true of the conventional approaches to soft data analysis. In essence, our rationale for the use of fuzzy logic for soft data analysis rests on the premise that the denotations of imprecise terms which occur in a soft database are, for the most part, fuzzy sets rather than probability distributions. For example, in a proposition such as

$\hat{p} \stackrel{*}{=} X$ is a large number (1.1)

the softness of data is due to the fuzziness of the denotation of large number. Similarly, in the proposition

 $p \triangleq$ It is *likely* that Andrea is young (1.2)

softness is due to: (a) the fuzziness of the denotation of

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young; and (b) the fuzziness of the term *likely*, which characterizes the probability of the fuzzy event "Andrea is young" [71,86]. As we shall see presently, the imprecision in (1.1) is possibilistic in nature, whereas in (1.2) it is partly probabilistic and partly possibilistic. Viewed in this perspective, then, a *soft datum* may be regarded, in - general, as a proposition in which the uncertainty is due to a combination of probabilistic and possibilistic constituents.

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When it is necessary to differentiate between a term and its denotation, the latter will be expressed in uppercase symbols. To illustrate, in (1.1) the term *large number* (or, simply, *large*) has as its denotation a fuzzy subset, LARGE, of the interval $U \triangleq [0,\infty)$. This subset is characterized by its membership function μ_{LARGE} : $U \rightarrow [0,1]$ which associates with each number $u \in U$ the grade of membership of u in LARGE. For example, the grade of membership of u = 100 in LARGE might be 0.2 while that of 400 might be 0.9.

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A basic aspect of a fuzzy proposition such as "X is small" is that it does not provide a precise characterization of the value of X. Instead, it defines a *possibility distribution* [92] of values of X which associates with each nonnegative real number u a number in the interval [0,1] which represents the *possibility* that X could take u as a value given the proposition "X is small." To express this in a symbolic form, we write

X is small $\rightarrow \Pi_{y} = SMALL$ (1.3)

which signifies that the proposition "X is small" translates into the assignment of the fuzzy set SMALL to the possibility distribution of X, Π_X . Equivalently, the proposition "X is small" will be said to *induce* the possibility distribution Π_X , with the right-hand member of (1.3) constituting a *possibility assignment equation*. For notational convenience, we shall write

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$$Poss{x = u} \triangleq \pi_{x}(u)$$
 (1.4)

where the function $\pi_X: U \rightarrow [0,1]$ is the possibility distribution function and U is the domain of X.

Essentially, the possibility distribution of X is the collection of possible values of X, with the understanding that possibility is a matter of degree, so that the possibility that X could take u as a value may be any number in the interval [0,1] or, more generally, a point in a partially ordered set.

In general, a possibility distribution may be induced by a physical constraint or, alternatively, may be epistemic in origin. To illustrate the difference, let X be the number of passengers that can be carried in Carole's car, which is a five passenger Mercedes. In this case, by identifying $\pi_{\chi}(u)$ with the degree of ease with which u passengers can be put in Carole's car, the tabulation of π_{χ} may assume the following form in which an entry such as (7,0.6) signifies that, by

x	1	2	3	4	5	6	7	8	9	10
π _x	1	1	1	1	1	0.8	0.6	0.9	0	0

some explicit or implicit criterion, the degree of ease with which 7 passengers can be carried in Carole's car is 0.6.

In the above example, the possibility distribution of X is induced by a physical constraint on the number of passengers that can be carried in Carole's car. To illustrate the case where the possibility distribution of X is epistemic in origin, i.e., reflects the state of knowledge about X, let X be Carole's age and let the information about Carole's age be conveyed by the proposition

$$p = Carole$$
 is young (1.

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where young is the label of a specified fuzzy subset of the

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interval [0,100] which is characterized by its membership function μ_{YOUNG} , with μ_{YOUNG} (u) representing the degree to which a person who is u years old is young in a specified context.

The connection between π_X and μ_{YOUNG} is provided by the so-called *possibility postulate* of possibility theory [92,93] which asserts that, in the absence of any information about X other than that supplied by the proposition $p \stackrel{\text{de}}{=}$ Carole is young, the possibility that X = u is numerically equal to the grade of membership of u in YOUNG. Thus

 $\frac{1}{5} \qquad Poss{X = u} = \pi_{X}(u) = \mu_{YOUNG}(u) , u \in [0, 100] \quad (1.6)$ or, equivalently,

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 $\Pi_{Age(Carole)} = YOUNG$ (1.7)

with the understanding that the possibility assignment equation (1.7) is the translation of (1.5), i.e.,

Carole is young $\rightarrow \Pi_{Age(Carole)} = YOUNG$. (1.8) It is in this sense, then, that the epistemic possibility distribution of Carole's age is induced by the proposition $p \triangleq Carole$ is young.

What is the difference between probability and possibility? As the above examples indicate, the concept of possibility is an abstraction of our intuitive perception of ease of attainment or degree of compatibility, whereas the concept of probability is rooted in the perception of likelihood, frequency, proportion or strength of belief. Furthermore, as we shall see in Section 2, the rules governing the manipulation of possibilities are distinct from those which apply to probabilities.

An important aspect of the connection between probabilities and possibilities relates to the fact that they are *independent* characterizations of uncertainty in the sense that from the knowledge of the possibility distribution of a

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variable X we cannot deduce its probability distribution, and vice-versa. For example, from the knowledge of the possibility distribution of the number of passengers in Carole's. car we cannot deduce its probability distribution. Nor can . we deduce the possibility distribution from the probability - distribution of the number of passengers. However, we can 42 make a weaker assertion to the effect that if the possibility 41 10 that X = u is small, then it is likely that the probability 39 that X = u is also small. However, from this it does not 13 follow that high possibility implies high probability, as is 37 36 reflected in the commonly used statements of the form "It is possible but not probable that " 3-

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33. In the present paper, we shall focus our attention on 32 only a few of the basic aspects of possibility theory and its applications to the analysis of soft data. Thus, our main _2 concern will be with the representation of soft data in linguistic form and with approximate inference from such data. In addition, we shall touch upon the issue of data granu--- larity and its relation to the theory of evidence. We shall not consider, however, an issue that is of considerable rele-1 vance to the analysis of soft data, namely, the representation of imprecise relational dependencies in the form of 18 linguistic decision tables and branching questionnaires [10, 17 97]. 16[.]

15 I It was brought to the author's attention by John E. Shively 13 (Lawrence Berkeley Laboratory) that an interesting case of an interplay between probability and possibility occurs in the historical letter from Einstein to Roosevelt (dated August 2, 1939). In a passage in this letter, Einstein 10; writes: 2

In the course of the last four months it has been made probable -- through the work of Joliot in France as well as Fermi and Szilard in America--that it may become possible to set up nuclear chain reactions in a large mass of uranium, by which vast amounts of power and large quantities of new radium-like elements would be generated. Now it appears almost certain that this could be achieved in the immediate future.

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2. Basic Properties of Possibility Distributions

As we have indicated in the preceding section, the concept of a possibility distribution plays a central role in our approach to the representation and manipulation of soft data. In what follows, we shall discuss some of the basic properties of possibility distributions² and lay the groundwork for their application to soft data analysis in later sections.

³ Possibility Measure

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Fadar 1449 ± 449 _____1 € 25 1 ____ Consider a variable X which takes values in a universe $\frac{15}{52}$ of discourse U, and let Π_X be the possibility distribution $\frac{33}{51}$ induced by a proposition of the form

$$p \stackrel{\circ}{=} X \text{ is G}$$
 (2.1)

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where G is a fuzzy subset of U which is characterized by its membership function μ_{G} . In consequence of the possibility postulate, we can assert that

$$\Pi_{\mathbf{x}} = \mathbf{G} \tag{2.2}$$

- which implies that

$$\pi_{x}(\mathbf{u}) = \mu_{G}(\mathbf{u}) , \quad \mathbf{u} \in \mathcal{U}$$
 (2.3)

where π_{χ} is the possibility distribution function of X. Now if F is a fuzzy subset of U, then the possibility is measure of F is defined by the expression

 $\Pi(\mathbf{F}) = \sup(\mathbf{F} \cap \mathbf{G}) \tag{2.4}$

i or, more explicitly,

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$$\Pi{F} = \sup_{U} \left(\mu_{F}(u) \wedge \mu_{G}(u) \right)$$
(2.5)

where the supremum is taken over $u \in U$ and \wedge represents the min operation. The number $\Pi(F)$, which ranges in value from 0

²In our exposition of the basic properties of possibility distributions and related concepts we shall draw on some of the definitions and examples in [91,94,98].

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 $\Pi(F) = \operatorname{Poss}\{X \text{ is } F | X \text{ is } G\} = \sup(F \cap G)$ (2.6)

In particular, if F is a nonfuzzy set A, then

 $\mu_{A}(u) = 1 \quad \text{if} \quad u \in A$ $= 0 \quad \text{if} \quad u \notin A$

and hence

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 $\Pi(A) = \operatorname{Poss}\{X \text{ is } A | X \text{ is } G\} = \sup_{A}(G)$ $= \sup_{A}(\mu_{G}(u)), \quad u \in U$ (2.7)

<u>An imp</u>ortant immediate consequence of (2.4) is the F-additivity of possibility measures expressed by

 $\Pi(\mathbf{F} \cup \mathbf{H}) = \Pi(\mathbf{F}) \vee \Pi(\mathbf{H})$ (2.8)

where F and H are arbitrary fuzzy subsets of U and V is the max operation. By contrast, the probability measures of F and H have the additive property expressed by

 $P(F \cup H) = P(F) + P(H) - P(F \cap H)$ (2.9)

The fact that possibility measures are F-additive but not additive in the usual sense constitutes one of the basic differences between the concepts of possibility and probability [92].

As a simple illustration of (2.6), assume that the pro-5 position "X is G" has the form

 $p \stackrel{\triangle}{=} X \text{ is small}$ (2.10)

11 where SMALL is a fuzzy set defined by 10

SMALL = 1/0 + 0.8/2 + 0.6/3 + 0.4/4 + 0.25 (2.11)

The notation $F = \mu_1/u_1 + \dots + \mu_n/u_n$ signifies that F is a collection of fuzzy singletons μ_1/u_1 , $i = 1, \dots, n$, with μ_1 representing the grade of membership of u_1 in F. More generally, F may be expressed as $F = \sum_i \mu_i/u_i$ or $F = \int_U \mu_F(u)/u$.

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In this case, the possibility distribution induced by p is
given by

$$I_{X} = 1/0 + 0.8/2 + 0.6/3 + 0.4/4 + 0.1/5$$
and if the proposition X is F has the form

$$I_{X} = 1/0 + 0.8/2 + 0.6/6 + 0.8/7 + 1/8 + \cdots,$$
where LARGE is defined by
LARGE \$ 0.2/4 + 0.4/5 + 0.6/6 + 0.8/7 + 1/8 + \cdots,
then
SNALL \cap LARGE = 0.2/4 + 0.1/5
and hence.
-Poss(X is large | X is small } = 0.2
Joint, Marginal and Conditional Possibility Distributions
Let X & (X_1,...,X_n) be an n-ary variable which takes
values in a universe of discourse U = U_1 × \cdots × U_n, with X_i,
i = 1,...,n, taking values in U_1. Furthermore, let F be an
n-ary furzy relation in U which is characterized by its mem-
bership function μ_2 . Then, the proposition
 $p \triangleq x$ is F (2.13)
induces an n-ary joint possibility distribution
 $I_{X} \triangleq I_{(X_1},...,X_n)$ (2.14)
which is given by
 $I_{(X_1,...,X_n)} = F$ (2.15)
Correspondingly, the possibility distribution function of X
is expressed by
 $I_{(X_1,...,X_n)} = \mu_2(u_1,...,u_n)$, $u \triangleq (u_1,...,u_n) \in U$
 $= Poss(X_1 = u_1,...,X_n = u_n)$
As in the case of probabilities, we can define marginal
and conditional possibilities. Thus, let s $\triangleq (i_1,...,i_k)$ be
a subsequence-of-the index sequence $(l,...,n)$ -and let s!

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denote the complementary subsequence s' \triangleq (j₁,...,j_m) (e.g., for n = 5, s = (1,3,4) and s' = (2,5). In terms of such sequences, a k-tuple of the form $(A_{i_1}, \ldots, A_{i_k})$ may be expressed in an abbreviated form as A_(s). In particular, the variable $X_{(s)} = (X_{i_1}, \dots, X_{i_k})$ will be referred to as a k-ary subvariable of $x \triangleq (x_1, \dots, x_n)$, with $x_{(s')} = (x_{j_1}, \dots, x_{j_m})$ being a subvariable complementary to X (s). The projection of $\Pi_{(X_1, \ldots, X_n)}$ on $U_{(s)} \triangleq U_{i_1} \times \cdots \times U_{i_k}$ is a k-ary possibility distribution denoted by $\Pi_{\mathbf{X}_{(\mathbf{s})}} \triangleq \operatorname{Proj}_{\mathbf{U}_{(\mathbf{s})}} \Pi_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}$ (2.16)Barton translation agriculture la la and defined by 331 $\pi_{X_{(s)}}^{(u_{(s)})} \triangleq \sup_{u_{(s')}} \pi_{X}^{(u_{1},\ldots,u_{n})}$ 32: (2.17)where $\pi_{X(s)}$ is the possibility distribution function of ∏ X(s) For example, for n = 2, $\pi_{\mathbf{X}_{1}}(\mathbf{u}_{1}) \triangleq \sup_{\mathbf{u}_{2}} \pi_{(\mathbf{X}_{1},\mathbf{X}_{2})}(\mathbf{u}_{1},\mathbf{u}_{2})$ is the expression for the possibility distribution function of the projection of $\Pi_{(X_1,X_2)}$ on U_1 . By analogy with the concept of a marginal probability distribution, Π_{x} will be referred to as a marginal possibility distribution. As a simple illustration, assume that n = 3, $U_1 = U_2$ = \overline{U}_3 = a + b or, more conventionally, {a,b}, and $\overline{\Pi}(X_1, X_2, X_3)$ is expressed as a linear form $I_{(X_1, X_2, X_3)} = 0.8aaa + 1aab + 0.6baa + 0.2bab + 0.5bbb (2.18)$ in which a term of the form 0.6baa signifies that $Poss{X_1 = b, X_2 = a, X_3 = a} = 0.6$ To derive $\Pi_{(X_1,X_2)}$ from (2.18), it is sufficient to replace the value of X_3 in each term in (2.18) by the null string Λ . This yields

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12 des los la conte da not type bevena devido a true che Begin Post dado proche Stochnesho a un conseguend, 25 proraseu y 1939 (1955) a contra contra o recombre man à autrator. Begin an Que Da Stoce 14 consecut port El Sectore Borato e Beale los Stoch a stoce a contrâ construirante a castrator. Begin de Da Stoce 14 consecut port El Sectore Stoce MARCARANT, Sectore A 436 Manuamet Preparacidade consectores Castratores.

$$[(x_1, x_2)] = 0.8aa + 1aa + 0.6ba + 0.2ba + 0.5bb = 1aa + 0.6ba + 0.5bb$$

and similarly

 $\Pi_{X_{1}} = la + 0.6b + 0.5b$ = la + 0.6b

An *n*-ary possibility distribution is *particularized* by forming the conjunction of the propositions "X is F" and "X_(s) is G," where $X_{(s)}$ is a subvariable of X. Thus,

$$\Pi_{X}[\Pi_{X} = G] \triangleq F \cap \overline{G}$$
(2.19)

where the right-hand member denotes the intersection of F with the cylindrical extension of G, i.e., a cylindrical fuzzy set defined by

$$\mu_{\overline{G}}^{(u_1,\dots,u_n)} = \mu_{G}^{(u_1,\dots,u_1)}, \qquad (2.20)$$

$$(u_1,\dots,u_n) \in \underline{v_1} \times \dots \times \underline{v_n}.$$

As a simple illustration, consider the possibility distribution defined by (2.18), and assume that

$$\Pi_{(X_{1}, X_{2})} = 0.4aa + 0.9ba + 0.1bb .$$

In this case,

G = 0.4aaa + 0.4aab + 0.9baa + 0.9bab + 0.1bba + 0.1bbb $F \cap G = 0.4aaa + 0.4aab + 0.6baa + 0.2bab + 0.1bbb$

15 and hence

$$\begin{bmatrix} \Pi \\ X_1, X_2, X_3 \end{bmatrix} \begin{bmatrix} \Pi \\ X_1, X_2 \end{bmatrix} = G]$$

= 0.4aaa + 0.4aab + 0.6baa + 0.2bab + 0.1bbb

There are many cases in which the operations of particularization and projection are combined. In such cases it is convenient to use the simplified notation

$$X_{(r)}^{\Pi[\Pi_{X} = G]}$$
 (2.21)

Carlos Control et Burra, constiguenda a participar de la Regenication de Bonna et Burgieros Surgieros fondolós en constructiona de la construcción de la constitución de la const Constitución de la constitución d to indicate that the particularized possibility distribution (or relation) $\Pi[\Pi_{X} = G]$ is projected on $U_{(r)}$, where r, like s, is a subsequence of the index sequence $(1, \ldots, n)$. For example,

 $x_1 \times x_3^{\Pi[\Pi(x_3, x_4)]} = G]$

would represent the projection of $\Pi[\Pi_{(X_3,X_4)} = G]$ on $U_1 \times U_3$. Informally, (2.21) may be interpreted as: Constrain the X (s) by $\Pi_X = G$ and read out the X (r). In particular, if the values of X (s) --rather than their possibility distribution-are set equal to G, then (2.21) becomes

 $X_{(r)}^{\Pi[X_{(s)}=G]}$.

We shall make use of (2.21) and its special cases in Section 3.

As we shall see in Section 3, if X and Y are variables taking values in U and V, respectively, then the *conditional possibility distribution* of Y given X is induced by a proposition of the form "If X is F then Y is G" and is expressed as $\Pi_{(Y|X)}$, with the understanding that

where (2.21) defines the conditional possibility distribution function of Y given X.

If we know the distribution function of X and the conditional distribution function of Y given X, then we can construct the joint distribution function of X and Y by forming the conjunction ($^{\Delta}$ min)

$$T_{(X,Y)}(u,v) = \pi_X(u) \wedge \pi_{(Y|X)}(v|u)$$
 (2.23)

However, unlike the identity that holds in the case of probabilities, we can also obtain $\pi_{(X,Y)}(u,v)$ by forming the conjunction of $\pi_{(X|Y)}(u|v)$ and $\pi_{(Y|X)}(v|u)$:

 $\frac{\pi}{(x, Y)} (u, v) = \pi_{(x|Y)} (u|v) \wedge \pi_{(Y|X)} (v|u) . \quad (2.24)$

് പാനം കടിപ്പിട്ടും പ്രിവ്യാന് പാരമ്പോടും പോര്പാനും പാന് കംഷതില് സോപ്പുക്കെ പടക്ട് സ്നാനം ഇനിമുള്ള കുക്കെ പടക്ച പടം പടിപ്പിട്ടും കാണ് നിന്നാനും പോയോമനായും നിയും മനുന്നത്. മുംഗ്രാമായം പ്രിവ്യായായില് പാലം പാനം സമ്മായം കുടും ഇന മുംഗ്രാമായിന് പോപ്പോള് മോല് മോല് മോലം മുംസ്വോമായമ്പാമം പോപ്പിന്റും പോന്നായും പാന്യായും പോലം അതിക്കും പുംഗം പ്രി മോള് മന്നിന് പോപ്പായും പോല് മോല് മോലം മുംസ്വായും മോപ്പോള് മുംഗ്രാമായിന്നും നില്പ്പിന്നും പോലം മായമ്പോലം ഇതികളും മന്നാമായിന് നില്പായും പോലം മായം പോലം മോലം പോലം മുംഗ്രാമായിന്നും പോലം പോലം പോലം പോലം വിയും ഇതികളും പുംഗം പുംഗം പ

(2.27)

In yet another deviation from parallelism with probabilities, the marginal possibility distribution function of X may be expressed in more than one way in terms of the joint and conditional possibility distribution functions. More specifically, we may have

π _x (u)	= v T (X, Y	(u,v)		(2.25)
• .				

where v denotes the supremum over $v \in V$; $\pi_{\mathbf{X}}(\mathbf{u}) = \mathbf{v}_{\mathbf{v}} \pi_{(\mathbf{X}|\mathbf{Y})} (\mathbf{u}|\mathbf{v})$ (b) (2.26)and $\pi_{\mathbf{X}}(\mathbf{u}) = \pi_{(\mathbf{X}|\mathbf{Y})}(\mathbf{u}, \mathbf{\tilde{v}}(\mathbf{u}))$

where, for a given u, $\tilde{v}(u)$ is the value of v at which $\pi_{(Y|X)}(v|u) = 1$, if $\tilde{v}(u)$ is defined for every $u \in U$.

(a)

(c)

Intuitively, (a) represents the possibility of assigning a value to X as perceived by an observer ((X,Y) observer) who observes the joint possibility distribution $\Pi_{(X,Y)}$. larly, (b) represents the perception of an observer ((X|Y)observer) who observes only the conditional possibility distribution $\Pi_{(X|Y)}$ and is unconcerned with or unaware of And (c) expresses the perception of an observer who "(x|x)" assumes that v is assigned that value, if it exists, which makes $\pi_{(Y|X)}(v|u)$ equal to unity.

As will be seen in Section 3, the concept of a conditional possibility distribution plays a basic role in the formulation of a generalized form of modus ponens and in defining a measure of belief. What is as yet an unsettled issue revolves around the question of how to derive $\pi_{(X|Y)}$ and $\pi_{(Y|X)}$ from $\pi_{(X,Y)}$. Somewhat different answers to this question are presented in [92], [57] and [33]. It may well turn out to be the case that, in contrast to probabilities, there does not exist a unique solution to the problem and that, in general, the answer depends on the perspective of the observer.

The Extension Principle

then

Let f be a function from U to V. The extension principle--as its name implies--serves to extend the domain of definition of f from U to the set of fuzzy subsets of U. In particular, if F is a finite fuzzy subset of U expressed as

$$\mathbf{F} = \frac{\mu_1}{u_1} + \cdots + \frac{\mu_n}{u_n}$$

then f(F) is a finite fuzzy subset of V defined as

$$f(F) = f(\mu_1/\mu_1 + \dots + \mu_n/\mu_n)$$
(2.28)
= $\mu_1/f(\mu_1) + \dots + \mu_n/f(\mu_n)$.

More generally, if the support of F is a continuum, i.e.,

$$F = \int_{U} \mu_{F}(u) / u \qquad (2.29)$$

$$f(F) = \int_{U} \mu_{F}(u) / f(u)$$
 (2.30)

Furthermore, if U is a cartesian product of U_1, \dots, U_n and f is a mapping from $U_1 \times \dots \times U_n$ to V, then $f(F) = \int_U \mu_F(u_1, \dots, u_n) / f(u_1, \dots, u_n) . \quad (2.31)$

In connection with (2.31), it should be noted that there are many cases in which we have only partial information about μ_{F} , e.g., the knowledge of its projections on U_{1}, \ldots, U_{n} , which implies that the available information consists of the marginal membership functions μ_{1}, \ldots, μ_{n} , where

$$\mu_{i}(u_{i}) = \sup_{u_{1}, \dots, u_{i-1}, u_{i+1}, \dots, u_{n}} \mu_{F}(u_{1}, \dots, u_{n})$$

$$i = 1, \dots, n$$

In such cases, the extension of the domain of definition of f is expressed by

$$f(F) = \int_{\Pi} \mu_1(u_1) \wedge \cdots \wedge \mu_n(u_n) / f(u_1, \dots, u_n)$$
 (2.32)

with the understanding that, in replacing $\mu_F(u_1, \ldots, u_n)$ with

Construction Constructed and any simple real Regimetry stated and sold Constants Constructions of the rest of the construction of the rest of the rest of the construction of the rest of t

 $\mu_1(u_1) \wedge \cdots \wedge \mu_n(u_n)$, we are tacitly invoking the principle of maximal restriction [95], which asserts that, in the absence of complete information about Π_X , we should equate Π_X to the maximal (i.e., least restrictive) possibility distribution which is consistent with the partial information about Π_X . In applying the extension principle to the analysis of soft data, it is frequently convenient to employ a more explicit representation of f(F) which is equivalent to (2.32). Specifically, on denoting the membership function of f(F) by 37μ , we have

$$E(F) = \int_{V} \mu(v) / v \qquad (2.33)$$

 $\mu(\mathbf{v}) \stackrel{\Delta}{=} \max_{\mathbf{u}_1, \dots, \mathbf{u}_n} \mu_1(\mathbf{u}_1) \wedge \dots \wedge \mu_n(\mathbf{u}_n)$ (2.34)

subject to the constraint

where

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$$v = f(u_1, ..., u_n)$$
.

In this form, the extension principle will be employed in Section 4 to reduce the problem of inference from soft data to the solution of a variational problem in mathematical programming.

An important aspect of our approach to the analysis of soft data is the flexibility afforded by the assumption that the variables are allowed to be *linguistic* [90], that is, are allowed to have values that are represented as sentences in a natural or synthetic language, with each such value defining a possibility distribution in the domain of the variable. For example, if Age is a linguistic variable, its linguistic values might be of the form:

youngoldnot youngnot oldvery youngvery oldnot very youngnot very oldmore or less youngmore or less oldquite youngquite oldrather youngrather old

nomen en la contra de la constant sa la cense Bodenars subre pri polition d'activitar y la provisión de la com Provisión de la constante de la constante d'autorita Bodenardo de la travela funes contratar proversión en el co Provisión de la constante Elle contratar paragrapho Batares, publicante de constante de la constante de la cons Provisión de la constante Elle contratar paragrapho Batares, publicante de constante de la constante de la const not young and not old not very young and not very old

where young is a primary term which has to be calibrated in a specified context and *old* is its antonym. As we shall see in Section 3, the translation rules for propositions expressed in a natural language provide a method for computing the possibility distribution induced by a proposition of the form "X is l," where l is a linguistic value of X, from the knowledge of the membership functions of the primary term and its antonym.

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3. Translation Rules and Meaning Representation

When soft data are represented in the form of propositions in a natural language, it is necessary to have, first, a system for translating such propositions into a more precise form; and second, a set of rules of inference which apply to the translated propositions and which may be employed to arrive at answers to questions regarding the data. A meaning representation language which is well-suited for this purpose is PRUF [93]. In what follows, we shall state some of the relevant translation rules in PRUF and outline the associated rules of inference.

The translation rules in PRUF serve the purpose of facilitating the composition of the meaning of a complex proposition from the meanings of its constituents. For convenience, the rules in question are categorized into four basic types: Type 1: Rules pertaining to modification; Type II: Rules pertaining to composition; Type III: Rules pertaining to quantification; and Type IV: Rules pertaining to qualification.

Following a discussion of these rules and the associated rules of inference, we shall outline a general translation principle which forms the basis for PRUF, and sketch a general question-answering technique which reduces the problem of inference to the solution of a variational problem in mathematical programming.

Translation Rules

<u>Modifier rule (Type I)</u>. Let X be a variable which takes values in a universe of discourse U and let F be a fuzzy subset of U. Consider the proposition

 $p \triangleq X \text{ is } F$ (3.1)

or, more generally,

$$p = N$$
 is F (3.2)

where N is a variable, an object or a proposition. For

19. – John Steiner, and Statzer versiger, sold total ender steiner ere sold ender sold under set of de lever o Hand black of the set of stere alle and strand all that is dependent sedd from sige erst and there are done of Baun of the follower as the seddent as the regradment seddent between stere of the sedden and there are done of WHRCERTANT of the Sed Manual stranger are done by dependent set and the sedden and the segment set WHRCERTANT of the Sed Manual stranger are done by dependent set. example,

which may be expressed in the form (3.1), i.e.,

 $p \triangleq Age(Mary)$ is young (3.4)

by identifying X with the variable Age(Mary).

Now, if in a particular context the proposition X is F translates into

$$X \text{ is } F \longrightarrow \Pi_{X} = F \tag{3.5}$$

then in the same context

$$X \text{ is } \mathbf{mF} \xrightarrow{\longrightarrow} \Pi_{\mathbf{X}} = \mathbf{F}^{+} \tag{3.6}$$

where m is a modifier such as not, very, more or less, etc., and F^+ is a modification of F induced by m. More specifically: If m = not, then $F^+ = F' = \text{complement of } F$, i.e.,

$$\mu_{F}(u) = 1 - \mu_{F}(u)$$
, $u \in U$. (3.7)

If m = very, then $F^+ = F^2$, i.e.,

 $\mu_{F^{+}}(u) = \mu_{F}^{2}(u) , \quad u \in U .$ (3.8)

If m = more or less, then $F^+ = \sqrt{F}$, i.e.,

$$\mu_{F}(u) = \sqrt{\mu_{F}(u)}, \quad u \in U. \quad (3.9)$$

As a simple illustration of (3.8), if SMALL is defined as in (2.11), then

K is very small
$$\rightarrow \Pi_{x} = F^{2}$$
 (3.10)

- where

 $F^2 = 1/0 + 1/1 + 0.64/2 + 0.36/3 + 0.16/4 + 0.04/5$.

It should be noted that (3.7), (3.8) and (3.9) should be - viewed as default rules which may be replaced by other translation rules in cases in which some alternative interpretations of the modifiers very and more or less are more appropriate.....

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$$\mu_{F\times G}(u,v) \triangleq \mu_{F}(u) \wedge \mu_{G}(v) . \qquad (3.13)$$

X is F or Y is
$$G \rightarrow \Pi_{(X,Y)} = \overline{F} \cup \overline{G}$$
 (3.14)

where

(b)

$$\vec{F} \triangleq F \times V$$
, $\vec{G} \triangleq U \times G$ (3.15)

and

$$\mu_{\overline{F}\cup\overline{G}}(u,v) = \mu_{F}(u) \sim \mu_{G}(v) . \qquad (3.16)$$

(c) If x is F then Y is
$$G \to \Pi_{(Y|X)} = \overline{F}' \oplus \overline{G}$$
 (3.17)

where $\Pi_{(Y|X)}$ denotes the conditional possibility distribution of Y given X, and the bounded sum \ni is defined by

$$\mu_{\overline{F}} = \frac{1}{9G}(u, v) = 1 \wedge (1 - \mu_{G}(u) + \mu_{G}(v)) . \qquad (3.18)$$

In stating the implicational rule in the form (3.17), we have merely chosen one of several alternative ways in which the conditional possibility distribution $\Pi_{(Y|X)}$ may be defined, each of which has some advantages and disadvantages depending on the application. Among the more important of these are the following [5,49,66]:

c₂) If X is F then Y is
$$G \longrightarrow \Pi_{(Y|X)} = \overline{F} \cdot \bigcup G$$
 (3.19)

(c₃) If X is F then Y is
$$G \rightarrow \Pi_{(Y|X)} = F \times G \cup F' \times V$$
 (3.20)

(c₄) If X is F then Y is
$$G \rightarrow \pi_{(Y|X)}(v|u)$$
 (3.21)
= 1 if $\mu_G(v) \ge \mu_F(u)$

 $= \frac{\mu_{G}(v)}{\mu_{F}(u)} \text{ otherwise}$

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then

$$Count(F) = \sum_{i=1}^{N} \mu_i \qquad (3.26)$$

where the right-hand member is understood to be rounded-off to the nearest integer. As a simple illustration of (3.24), if the quantifier several is defined as SEVERAL $\stackrel{\wedge}{=}$ 0/1 + 0.4/2 + 0.6/3 + 1/4 + 1/5 + 1/6 (3.27)+ 0.6/7 + 0.2/8then Several X's are large $\rightarrow \Pi$ (3.28) $\sum_{i=1}^{\nu} \mu_{\text{LARGE}}(u_i)$ = 0/1 + 0.4/2 + 0.6/3 + 1/4 + 1/5 + 1/6 + 0.6/7 + 0.2/8where $\mu_{LARGE}(u_i)$ is the grade of membership of the ith value of X in the fuzzy set LARGE. Alternatively, and perhaps more appropriately, the cardinality of F may be defined as a fuzzy number, as is done in [91]. Thus, if the elements of F are sorted in descending order, so that $\mu_n \leq \mu_m$ if $n \geq m$, then the truth-value of the proposition p ≜ F has at least n elements (3.29)is defined to be equal to $\boldsymbol{\mu}_n$, while that of $\boldsymbol{q}_{\boldsymbol{r}}$ $q \stackrel{\text{\tiny d}}{=} F$ has at most n elements , (3.30)is taken to be $1-\mu_{n+1}$. From this, then, it follows that the truth-value of the proposition r, $r \triangleq F$ has exactly n elements , (3.31)is given by $\mu_n \wedge (1-\mu_{n+1})$. Let $F \downarrow$ denote F sorted in descending order. Then (3.29) may be expressed compactly in the equivalent form $FGCount(F) = F \downarrow$ (3.32)which-signifies that if-the-fuzzy cardinality of E is defined

Cloning of the second structure of the Bear or only on the SS for a constant of the second structure of the second structur

in terms of (3.29), with G standing for greater than, then the fuzzy count of elements in F is given by F \downarrow , with the understanding that F \downarrow is regarded as a fuzzy subset of $\{0,1,2,\ldots\}$. In a similar fashion, (3.30) leads to the definition

$$FLCount(F) = (F\downarrow)' - 1$$
 (3.33)

where L stands for less than and subtraction should be interpreted as translation to the left, while (3.31) leads to

$$FECount(F) = (F\downarrow) \cap ((F\downarrow)'-1)$$

where E stands for equal to. For convenience, we shall refer to FGCount, FLCount and FECount as the FG cardinality, FL cardinality and FE cardinality, respectively. The concept of FG cardinality will be illustrated in Example 9, Section 5.

Remark. There may be some cases in which it may be appropriate to normalize the definition of FECount in order to convey a correct perception of the count of elements in a fuzzy set. In such cases, we may employ the definition

$$FENCount(F) = \frac{FECount(F)}{\max_{n}(\mu_{n} \land (1-\mu_{n+1}))} . \quad (3.34)$$

<u>Truth Qualification Rule (Type IV)</u>. Let T be a linguistic truth-value, e.g., very true, quite true, more or less, true, etc. Such a truth-value may be regarded as a fuzzy subset of the unit interval which is characterized by a membership function μ_{τ} : [0,1] \rightarrow [0,1].

A truth-qualified proposition, e.g., "It is τ that X is F," is expressed as "X is F is τ ." As shown in [89], the translation rule for such propositions is given by

X is F is
$$\tau \to \Pi_{\chi} = F^+$$
 (3.35)

where

$$\mu_{F^{+}}(u) = \mu_{\tau}(\mu_{F}(u)) . \qquad (3.36)$$

1. Source Device Device Device Device Device Device Device Device Device Between States (States) (States) and there are considered and the considered and the considered are considered and there are considered are considered and there are considered and the considered and the considered are considered and the considered and the considered are considered and the considered are considered are considered are considered are considered and the considered are consid

Yolanda is young is very true

which by (3.35), (3.36) and (3.8) translates into

 $\mu_{\rm YOUNG}(u) = (1 + (\frac{u}{25}))$

$$\Pi_{\text{Age(Yolanda)}} = \mu_{\text{TRUE}^2} (\mu_{\text{YOUNG}}(u)) . \quad (3.37)$$

 $u \in [0, 100]$

Now, if we assume that

and

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$$u_{\text{TRUE}}(v) = v^2 , v \in [0,1]$$

then (3.36) yields

$$\Pi_{Age(Yolanda)} = (1 + (\frac{u}{25})^2)^{-4}$$

as the possibility distribution of the age of Yolanda.

Probability Qualification Rule (Type IV). This rule applies to propositions of the general form "X is F is λ ," where X is a real-valued variable, F is a linguistic value of X, and λ is a linguistic value of likelihood (or probability), e.g., "X is small is not very likely." Unless stated to the contrary, λ is assumed to be a fuzzy subset of the unit interval [0,1] which is characterized by its membership function μ_{λ} , and the probability distribution of X is characterized by its probability density function p, i.e.,

$$Prob\{x \in [u, u+du]\} = p(u)du$$
. (3.39)

As shown in [93], the translation rule for probabilityqualified propositions is expressed by

X is F is
$$\lambda \to \pi(p) = \mu_{\lambda} \left(\int_{U} \mu_{F}(u) p(u) du \right)$$
 (3.40)

where $\pi(p)$ denotes the possibility that the probability density function of X is p, and the integral in the right-hand member of (3.40) represents the probability of the fuzzy event [86] "X is F." Thus, in the case of

- Severe concurrend or net valor been bland on outbaab vallege first ewste perfora 35 meteral 5,2 or 3 enes on - 200 factore outbaar even bland en en 200 factore outbaar even 30 or the set of 40 nes rang and the set of 200 factore outbaar even bland en en 200 factore outbaar even at 200 factore outbaar even bland en en 200 factore outbaar even at 200 factore outbaar even bland en en 200 factore outbaar even at 200 factore outbaar even bland even and 200 factore outbaar even at 200 factore outba Here even at 200 factore outbaar even at 200 fa Here even at 200 factore outbaar even a probability-qualified propositions, the proposition "X is F is λ " induces a possibility distribution of the probability density function of X.

As a simple illustration, consider the proposition

 $q \triangleq$ Yolanda is young is very likely . (3.41) In this case, $X \triangleq$ Age(Yolanda) and the right-hand member of (3.40) becomes

 $\pi(\mathbf{p}) = \mu_{\text{LIKELY}}^{2} \left(\int_{0}^{100} \mu_{\text{YOUNG}}(\mathbf{u}) \mathbf{p}(\mathbf{u}) d\mathbf{u} \right) . \quad (3.42)$

Used in combination, the translation rules stated above provide a system for the determination of the possibility distributions induced by a fairly broad class of composite propositions. For example, by the use of (3.7), (3.8), (3.9), (3.12) and (3.18), the proposition

> If X is not very large and Y is more or less small then Z is very very large.

can readily be found to induce the conditional possibility distribution described by

 $\pi_{(\mathbb{Z}|\mathbb{X},\mathbb{Y})}(w|u,v) = 1 \wedge \left(1 - \left(1 - \mu_{\text{LARGE}}^{2}(u)\right) \wedge \mu_{\text{SMALL}}^{0.5}(v) + \mu_{\text{LARGE}}^{4}(w)\right) .$

It is of interest to note that translation rules like those described above have found practical applications in the design of fuzzy logic controllers in steel plants, cement kilns and other types of industrial process control applications in which instructions expressed in a natural language are transformed into control signals [45,46,39,79].

A more general type of translation process in PRUF which subsumes the translation rules given above is the following.

Let $\mathcal{D} = \{D\}$ denote a collection of databases, with D representing a generic element of \mathcal{D} . For the purposes of our analysis, D will be assumed to consist of a collection of

(1) Units Do to includend and solve to be fine. Browning auge on the 25 of the sits of processing the processing of the solution of the sol

possibly time-varying relations. If R is a constituent relation in D, then by the *frame* of R is meant the name of R together with the names of its columns (i.e., attributes). For example, if a constituent of D is a relation labeled POPULATION whose tableau is comprised of columns labeled Name and Height then the frame of POPULATION is represented as <u>POPULATION[Name Height]</u> or, equivalently, as POPULATION[Name;Height].

If p is a proposition in a natural language, its translation into PRUF can assume one of three--essentially equivalent--forms.⁴

- (a) $p \rightarrow a$ possibility assignment equation
- (b) $p \rightarrow a$ procedure which yields for each D in \mathcal{D} the possibility of D given p, i.e., Poss{D|p}

(c) $p \rightarrow a$ procedure which yields for each D in \mathcal{D} the truth-value of p relative to D, i.e., $Tr\{p|D\}$

Remark. An important implicit assumption about the procedures involved in (b) and (c) is that they have a high degree of what might be called *explanatory effectiveness*, by which is meant a capability to convey the meaning of p to a human (or a machine) who is conversant with the meaning of the constituent terms in p but not with the meaning of p as a whole. For example, a procedure which merely tabulates the possibility of each D in \mathcal{D} would, in general, have a low degree of explanatory effectiveness if it does not indicate in sufficient detail the way in which that possibility is arrived at. On the other extreme, a procedure which is excessively detailed and lacking in modularity would also have a low degree of explanatory effectiveness because the

It should be noted that (b) and (c) are in the spirit of truth-conditional semantics and possible-world semantics, respectively [15,34]. In their conventional form, however, these semantics have no provision for fuzzy propositions and hence are not suitable for the analysis of soft data.

்றே பிடிப்பிடு பிடிப்பில் பொலுமையாக சோடில் பிடைக்கு இதுகைகள் கதுப்பி என்று நேரங்களால் இரும் காச முன் இருக்கு நன பிடிப்பு கோடு பிடிய காடியில் காச்சில் காக்கில் குடிநால் பிடுத்துக்குக் குடியில் என்றுக்கு பிக்குக்கு குடையில் ந இதிப்பிடிப்பில் பிடிக்கில் நாகதே சில்காக காக்குகளை நில்கள் குடைக்கில் குடியில் குடியில் குடியில் பிருக்கு மற்று பிடிப்பிடிப்பில் பிடிக்கில் நாகதே சில்காக காக்கில் பிடியில் குடியில் காக்கில் பிடியில் குடியில் பில்கள் பிடுக்க பிடிப்பில் கிலையில் கில்காக காக்கில் பில்கு பிடியில் கிலையில் பிடியில் கிலையில் பில்கு கிலையில் கிலையில் கிலைய பிடிப்பில் கிலையில் கில்குக்கில் கிலைக்குகள் விடியில்கு பிடியில் கிலையில் கிலையில் கிலையில் கிலையில் கிலையில் க meaning of p might be obscured by the maze of unstructured steps in the body of the procedure.

The equivalence of (b) and (c) is a consequence of the way in which the concept of truth is defined in fuzzy logic

Thus, it can readily be shown that, under mildly restrictive assumptions on D, we have

$$\operatorname{Tr}\{\mathbf{p} \mid \mathbf{D}\} = \operatorname{Poss}\{\mathbf{D} \mid \mathbf{p}\}$$

which implies the equivalence of (b) and (c).

To illustrate (b) and show how (a) may be derived from (b), we shall consider first the relatively simple proposition

 $p \triangleq Madan is not very tall . (3.43)^{-1}$

In this case, it is convenient to assume that D contains two relations whose frames are:

POPULATION Name Height TALL Height U

In the relation TALL, each value of height is associated with the degree to which a person having that height is tall. In effect, then, the relation TALL defines the fuzzy set TALL. The desired procedure involves the following steps.

1. Find Madan's height, h. In symbols, h is given by the expression (see (2.21))

h = POPULATION[Name = Madan] .

2. Find the degree, δ , to which Madan is not very tall in D. Using the expression obtained in the preceding step, the answer is:

δ = 1 - (TALL[Height = POPULATION[Name = Madan]])²
 3. Equate the possibility of D to δ. This yields the desired translation of p into PRUF, namely

$$\pi(D) = 1 - \left(\underset{\mu}{\text{TALL}} [\text{Height} = \underset{\text{Height}}{\text{POPULATION}} [\text{Name=Madan}] \right)^2$$

(3.44)

n en de verden av ander en bounderse opplinnen en bounderse het in de verden opplingen en de verden in de verd Brinkreit in entre en de verden ander en beinder en de verden opplingen en de verden in de verden en en de verd An elektriker in kenne kenne de verden beinder en de verden en de verden en de verden en elektriker en elektrike

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To find the possibility distribution of Madan's height from (3.41), it is sufficient to observe that, for a fixed relation TALL, $\pi(D)$ depends only on Madan's height. From this it follows at once that

$$\Pi_{\text{Height}(\text{Madan})} = (\text{TALL}^2)' \quad (3.45)$$

or, equivalently,

$$\pi_{\text{Height(Madan)}}(u) = 1 - \mu_{\text{TALL}}^2(u) \qquad (3.46)$$

where u is a generic value of the variable Height. What should be noted is that the possibility assignment equation (3.45) could be obtained directly by applying to p the translation rules (3.7) and (3.8). Furthermore, the explanatory effectiveness of (3.45) is higher than that of (3.44).

Remark. In PRUF, it is important to differentiate between the meaning of a proposition and the information that is conveyed by it. Thus, if p is a proposition, then the procedure, P, into which it translates represents the meaning of p or, equivalently, its intension [15,41]. On the other hand, the possibility distribution which is induced by p constitutes the information, I(p), which is conveyed by p. Thus, in the foregoing example the possibility distribution defined by (3.45) represents the information conveyed by the proposition $p \triangleq$ Madan is not very tall. The meaning of p, then, is the procedure described by the right-hand member of (3.45).

If p and q are propositions such that

I(p) = I(q) (3.47)

then p and q are semantically equivalent [93], which is expressed as

 $p \leftrightarrow q$. (3.48)

On the other hand, if

___I(p)._.<u><</u>I(q) .

....(3.49)

പ്പെട്ടും പ്രവിദ്യം പ്രതിന് പ്രതിപ്പോയും പോയും പ്രതിന് മാന്ത്രം മന്ത്രം പ്രതിന് പ്രതിന് മന്ത്രം മന്ത്രം പ്രതിന ന് നിന്നും പ്രതിന് പ്രതിന് നിന്നും പ്രതിന് നിന്നും മുംപോയും നിന്നും നിന്നും പ്രതിന്റെ പ്രതിന് നിന്നും പ്രതിന് പ പ്രതിന് നിന്നും പ്രതിന് നിന്നും പ്രതിന് നിന്നും മുംപോയും പ്രതിന്റെ പ്രതിന്റെ പ്രതിന്റെ പ്രതിന് നിന്നും പ്രതിന്ന then p semantically entails q [93], i.e.,

ratokija Pre DE 1.

> :--· :3:

$$p \mapsto q$$
 (3.50)

As we shall see in the next section, the concepts of semantic equivalence and semantic entailment play an important role in inference from soft data.

елите — Солона Дилот (уре ралора з марораце», передера благовора опо за докільте на 1. Дилий сень слод Церого на 1 дак сели со мала сопрозна е на ерега палійцита в цВнаре ребер 38 гольных 40 ледила у ала передики селине цула 1 оконого и церкатаре 80 ларе прагад проз бора ла очареда ратегное в грър здежала сройнем терриосле

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4. Inference from Soft Data and Mathematical Programming

By interpreting a soft datum as a fuzzy proposition, the problem of inference from soft data may be reduced to the problem of inference from a collection of fuzzy propositions. Suppose that $E = \{p_1, \dots, p_n\}$ (with E standing for evidence) is a collection of fuzzy propositions and let p be a proposition that is inferred from E. At this point, it is natural to raise two basic questions. First, what does it mean to say that p is inferred from E; and second, by what methods can p be inferred from E.

To answer the first question, it is convenient to make use of the concept of information, as defined in Section 3. More specifically, let $I(p_1 \cdots p_n)$ be the information conveyed by the conjunction of propositions p_1, \ldots, p_n or, equivalently, the possibility distribution induced by $p_1 \cdots p_n$, and let $I(p_1 \cdots p_n p_n)$ be the information conveyed by the conjunction of $p_1 \cdots p_n$ and p. Then, we shall say, informally, that p may be *inferred from* $E = \{p_1, \ldots, p_n\}$ if

$$I(p_1 \wedge \cdots \wedge p_n) = I(p_1 \wedge \cdots \wedge p_n \wedge p) . \qquad (4.1)$$

In other words, p is inferrable from E if the addition of p to the evidence, E, does not affect the information conveyed by E.

As shown in [91], the above definition implies that the possibility distribution induced by the conjunction of p_1, \ldots, p_n is contained in that induced by p. It is this containment property that underlies the *entailment principle* [91,93] which serves as a basis for the rules of inference stated in the sequel.

Remark. In speaking of entailment, it is necessary to differentiate between the entailment which obtains for particular denotations of the labels of fuzzy sets in

f akmerski min subur ul Elon obekend beyond kry bosit ovde klenge first opgelomk heidd intvæksit. Dier die tees stor sollter nem til end if there ske no more than 2 duthors. Begin on Mne b2 kint eins 4 tines long sog there ere Biblio misekku Begin slimm his stelle til ekeptik tos utgreboord betes blitt regin brek tees teusnis ju het soger vertinsegut u I BEV Phalem if ut ind activate van ere Pronuende Service Breater er ere

122.2.4.2.3

p1,...,p,,p, and strong entailment, which results when (4.1) holds for all denotations. As an illustration, if very is interpreted as a squaring operation, then the proposition

p ≜ Veronica is intelligent

is strongly entailed by

Veronica is very intelligent

INTELLIGENT² ⊂ INTELLIGENT

regardless of the way in which INTELLIGENT, the denotation of intelligent, is defined. On the other hand, in the case

of the propositions

since

p, ≜ John is not young $p_2 \stackrel{\text{\tiny }}{=} \text{John is not old}$ p ≜ John is middle-aged

the conjunction of p_1 and p_2 may be expressed as (see (3.12))

 $p_1 \land p_2 \triangleq$ John is not young and not old . (4.2)Consequently, if the denotations of young, old and middleaged are such that the containment condition

> YOUNG' OOLD' C MIDDLE-AGED (4.3)

is satisfied, then p is entailed by p₁ and p₂. However, since the question of whether or not (4.2) is satisfied depends on the denotations of the labels of fuzzy sets in p_1 , p, and p, it follows that p is not strongly entailed by p, and p₂.

Remark. In some ways, the entailment principle appears to be counterintuitive because we generally expect a conclusion, p, to be sharper than the totality of data on which it is based. However, the reason for the apparent sharpness is that, in general, p involves only a small subset of the

variables present in p_1, \ldots, p_n . More specifically, as we shall see in the sequel, in the process of inference we usually focus our attention on a small number of functionals defined on D and preceive the higher degree of focusing as a manifestation of sharpness of p. An example illustrating this and other aspects of the entailment principle is described in Section 5.

Rules of Inference

For purposes of inference from a collection of fuzzy propositions, it is convenient to have at one's disposal a system of basic rules which may be used singly or in combination to infer a fuzzy proposition p from a body of evidence $E = \{p_1, \dots, p_n\}$. Several such rules, which constitute a subset of the inference rules in fuzzy logic, FL [89], are stated in a summary form in the following.

1. <u>Projection Principle</u>. Consider a fuzzy proposition whose translation is expressed as

$$p \to \Pi_{(X_1, \dots, X_n)} = F$$
 (4.4)

and let $X_{(s)}$ denote a subvariable of the variable $x \triangleq (x_1, \dots, x_n)$, i.e.,

$$X_{(s)} = (X_{1}, \dots, X_{k})$$
 (4.5)

where the index sequence $s \triangleq (i_1, \ldots, i_k)$ is a subsequence of the sequence $(1, \ldots, n)$.

Furthermore, let Π_X denote the marginal possibility distribution of $X_{(s)}$; that is, $\Pi_X = \operatorname{Proj}_U F$ (4.6)

where U_i , i = 1,...,n, is the universe of discourse associated with X;;

 $U_{(s)} = U_{1} \times \cdots \times U_{k}$ (4.7)

tů seu substaní v Slovatovů, nektrá vypad polekanů ékratory polekačů staloží v stelu Slova Sever (staloží v so 11. slavní set natoral Stroke uzanom tranzení Eustruk (Batenion bay 11. slava 54. úres prazimstvené sekú se kom 19. slavní jeu tranz 30. srazna zarezané slavu substeléh vrnení sležitujat (gelnu sektuetna ernení sy 19. slavní slavat staloží sa skrazna se staloží uz se a u teežíh vrnení sležitujat (gelnu sektuetna ernení sy and the projection of F on U is defined by the possibility distribution function (see (2.17))

$$\pi_{X_{(s)}}^{(u_{1},\ldots,u_{1}) = sup}_{i_{k}}^{(u_{1},\ldots,u_{n}) = sup}_{j_{1}}^{\mu_{F}(u_{1},\ldots,u_{n})}$$
(4.8)

where s' \triangleq (j₁,...,j_m) is the index subsequence which is complementary to s, and μ_F is the membership function of F. Now let q be a retranslation (i.e., reverse translation) of the possibility assignment equation

$$\Pi_{X(s)} = \operatorname{Proj}_{U(s)} F . \qquad (4.9)$$

Then, the projection rule asserts that q may be inferred from + p. In a schematic form, this assertion may be expressed more transparently as

$$p \rightarrow \Pi_{(X_1, \dots, X_n)} = F$$

$$\downarrow \qquad (4.10)$$

$$q \leftarrow \Pi_{X_{(S)}} = \operatorname{Proj}_{U_{(S)}} F$$

As was indicated in Section 2, the rule of inference represented by (4.10) is easy to apply when II_X is expressed as a linear form. As an illustration, assume that $U_1 = U_2$ = {a,b}, and

 $\Pi_{(X_1, X_2)} = 0.8aa + 0.6ab + 0.4ba + 0.2bb$

in which a term of the form 0.6ab signifies that

$$Poss{x_1 = a, x_2 = b} = 0.6$$
.

To obtain the projection of Π_X on, say, U_2 it is sufficient to replace the value of X_1 in each term by the null string Λ . Thus

$$\operatorname{Proj}_{U_2}^{\Pi}(X_1, X_2) = 0.8a + 0.6b + 0.4a + 0.2b = 0.8a + 0.6b$$

and hence from the proposition

(X1,X2) is 0.8aa +0.6ab +0.4ba +0.2bb

no na serie e van de substant de statue de la transferie propie de regular de la serie de serie de la serie de 1990 - Constant de la constant de la substant de la propie sere de la constant de la constant de la constant de 1990 - Constant de la constant de la

$$X_{2}$$
 is 0.8a + 0.6b

2. <u>Conjunction Rule</u>. Consider a proposition p which is an assertion concerning the possible values of, say, two variables X and Y which take values in U and V, respectively. Similarly, let q be an assertion concerning the possible values of the variables Y and Z, taking values in V and W. With these assumptions, the translations of p and q may be expressed as

$$p \rightarrow \Pi^{p}_{(X,Y)} = F$$

$$q \rightarrow \Pi^{q}_{(Y,Z)} = G$$
(4.11)

Let \vec{F} and \vec{G} be, respectively, the cylindrical extensions of F and G in $U \times V \times W$. Thus,

 $\vec{F} = F \times W \tag{4.12}$

and

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$$\bar{G} = U \times G$$
 . (4.13)

(4.14)

Using the conjunction rule, we can infer from p and q a proposition which is defined by the following scheme:

$$r \rightarrow \Pi^{p}_{(X,Y)} = F$$

$$q \rightarrow \Pi^{q}_{(Y,Z)} = G$$

$$r \leftarrow \Pi_{(X,Y,Z)} = \overline{F} \cap$$

On combining the projection and conjunction rules, we be obtain the compositional rule of inference (4.17) which includes the classical modus ponens as a special case. More specifically, on applying the projection rule to (4.14), we obtain the following inference scheme

$$p \rightarrow \Pi^{p}_{(X,Y)} = F$$

$$q \rightarrow \Pi^{q}_{(Y,Z)} = G$$

$$\underline{r} \leftarrow \Pi^{r}_{(X,Z)} = F \circ G$$

Classics instructions Batin taxanda, and the design matched Begin matched on the Batin and Laboration and the experimental sectors (Begin matched and the Design of the experimental and the sectors (Begin matched and the Design of the experimental and the sectors (Begin matched and the Design of the experimental and the sectors (Begin matched and the Design of the experimental and the sectors (Begin matched and the Design of the experimental and the sectors (Begin matched and the Design of the experimental and the sectors (Begin matched and the Design of the experimental and the sectors (Begin matched and the Design of the experimental and the Design of the experimental and the sectors (Begin matched and the Design of the experimental and the Design of the experimental and where the composition of F and G is defined by

$$\mu_{F^{\circ}G}(u,w) = \sup_{v} \left(\mu_{F}(u,v) \wedge \mu_{G}(v,w) \right) . \quad (4.16)$$

In particular, if p is a proposition of the form "X is F" and q is a proposition of the form "If X is G then Y is H," then (4.15) becomes

$$p \rightarrow \Pi_{X} = F$$

$$q \rightarrow \Pi_{(Y|X)} = \overline{G' \oplus \overline{H}}$$

$$r \leftarrow \Pi_{(Y)} = F \circ (\overline{G' \oplus \overline{H}}) \qquad (4.17)$$

The rule expressed by (4.17) may be viewed as a generalized form of modus ponens which reduces to the classical modus ponens when F = G and F, G, H are nonfuzzy sets.

Stated in terms of possibility distributions, the generalized modus ponens places in evidence the analogy between probabilistic and possibilistic inference. Thus, in the case of probabilities, we can deduce the probability distribution of Y from the knowledge of the probability distribution of X and the conditional probability distribution of Y given X. Similarly, in the case of possibility distributions, we can infer the possibility distribution of Y from the knowledge of the possibility distribution of X and the conditional possibility distribution of Y given X.

It is important to note that the generalized modus ponens as expressed by (4.17) may be used to enlarge significantly the area of applicability of rule-based systems of the type employed in MYCIN and other expert systems. This is due primarily to two aspects of (4.17) which are not present in conventional rule-based systems: (a) in the propositions "X is F" and "If X is G then Y is H," F, G and H may be fuzzy sets; and (b) F and G need not be identical. Thus, as a result of (a) and (b), a rule-based system employing (4.17) may be designed to have an interpolative capability [88,97].

versionen son die Stenseneraliges paalen van die verse Begin prating zijn mei35 gebre verstond zijn die ondrik son die stanm was onder die gegenig tal en genoem verowerd allingtes. Begin provie 20 on train 44 meroorgalis offere are duit mark wurdt be versionen die versionen zweigestangestangestande begin begin bestender versionen auf die versionen die stand op noordaar in die versie verste Stenderstangestangestange bestender versionen auf bestende settende op die stand noordaar begin die versie versionen die Resperatione bestender versionen die stande versionen die stande stand noordaar begin die stande stande stande stande bestende bestende und In addition to the rules described above, there is an important method of inference through which the deduction of p is reduced to the solution of a variational problem in mathematical programming.

In general terms, suppose that we have a database D and that we wish to answer a question q which relates to the data resident in D. For example, we may have a database which contains a relation with the frame

POPULATION Name Age

and q may be: What is the average age of individuals in _____ POPULATION?

In PRUF, the translation of q is expressed as the translation of the answer to q, with a symbol of the form ?a identifying the variable whose value is to be determined. As an illustration, for the example under discussion the proposition to be inferred from D may be expressed as

> p ≜ The average age of individuals in POPULATION is ?f

where f is a function of the entries in D, say X_1, \ldots, X_m . Thus, to answer q we must compute the value of $f(X_1, \ldots, X_m)$ from whatever information is available about D.

To link the method under discussion to our earlier formulation of the problem of inference, we shall assume that the available information about D consists of the evidence $E = \{p_1, \dots, p_n\}$, in which the p_i are fuzzy propositions.

Our definition of translation in Section 3 implies that each of the p_i in E induces a possibility distribution over D. Thus, letting $\pi_i(X_1, \ldots, X_m)$ denote the possibility of D given p_i , we can assert that the possibility of D given p_1, \ldots, p_n is given by the conjunction

 $\pi(D) \triangleq \pi_{1}(x_{1}, \dots, x_{m}) \land \dots \land \pi_{n}(x_{1}, \dots, x_{m})$ (4.18)

Thus, $\pi(D)$, as expressed by (4.18), may be viewed as an

(c) 1. Constructions of the set of the set Below of the set of

From the knowledge of $\pi(D)$ we can infer the possibility distribution of the function

$$z = f(x_1, ..., x_m)$$
 (4.19)

by invoking the extension principle, as shown in Section 3. In this way, the determination of the possibility distribution of f reduces, in principle, to the solution of the following variational problem in mathematical programming.

$$\mu(z) \triangleq \max_{X_1, \dots, X_m} \pi_1(X_1, \dots, X_m) \wedge \dots \wedge \pi_n(X_1, \dots, X_m)$$
(4.20) -

$$:= f(x_1, ..., x_m)$$
.

In terms of $\mu(z)$, the possibility distribution of f may be expressed in the form

$$\Pi_{f} = \int_{V} \mu(z)/z \qquad (4.21)$$

where V is the range of z. An example illustrating the application of this technique will be discussed in Section 5. As a further example, consider the proposition which occurs in Example (e), Section 1, namely:

p ≜ Brian is much taller than most of his close friends For the purpose of representing the meaning of p, it is expedient to assume that D is comprised of the relations

> POPULATION Name Height FRIENDS Namel Name2 μ MUCH TALLER Heightl Height2 μ MOST ρμ

In the relation FRIENDS, μ represents the degree to which an individual whose name is Name2 is a friend of Name1. Similarly, in the relation MUCH TALLER, μ represents the degree to which an individual whose height is HEIGHT1 is much taller

ப்பட்டி மீல் இருந்து நிலையில் அது வில்லக் குதுதேத்துக்கு என்று நாரார் இநிற்றுக்கு தேர்த்துக்கு இருந்து (பிருந்த பிரிரார் இருத்து நிலையில் நிலையில் இதுக்கு பிருந்து பிராந்து இதிலுக்கு குறையில் குறையில் குறையில் பிருந்தும் குற பிரார் பிருத்திருந்து விலையில் திலையில் குறையில் தில்லது பிருந்து இதில் குறையில் திருந்து இருத்து இருந்து பிருந பிரார் பிருத்திருந்து விலையில் திலையில் குறையில் தில்லது பிருத்து பிருத்து தில்லது விலையில் திருந்து இதில் குறை பிருதல் பிருத்திருந்து விலையில் திலையில் பிருத்தும் விலையில் பிருதல் தில்லது விலையில் திருந்து இதில பிருதல் பிருதல் விலையில் திலையில் பிருதல் விலையில் பிருதல் பிருதல் பிருதல் இதிலுக்கு பிருதல் பிருதல் than one whose height is HEIGHT2. In MOST, μ represents the degree to which a proportion, ρ , fits the definition of MOST as a fuzzy subset of the unit interval.

To represent the meaning of p we shall translate p--in the spirit of (c) (Section 3)--into a procedure which computes the truth-value of p relative to a given D. The procedure-as described below--may be viewed as a sequence of computations which, in combination, yield the truth-value of p.

. Obtain Brian's height from POPULATION. Thus,

Height (Brian) = POPULATION [Name = Brian] 2. Determine the fuzzy set, MT, of individuals in POPULATION in relation to whom Brian is much taller.

Let Name, be the name of the ith individual in POPULATION. The height of Name, is given by

Height(Name) = POPULATION[Name = Name] Now the degree to which Brian is much taller than Name is given by

 $\hat{o}_{i} = MUCH TALLER[Height(Brian), Height(Name_i)]$ and hence MT may be expressed as

 $MT = \sum_{i} \delta_{i} / Name_{i}, Name_{i} \in POPULATION$ where POPULATION is the list of names of individuals in
POPULATION, δ_{i} is the grade of membership of Name_i in MT, and $\sum_{i} is the union of singletons \delta_{i} / Name_{i} (see footnote 3).$ 3. Determine the fuzzy set, CF, of individuals in POPULATION

who are close friends of Brian.

To form the relation CLOSE FRIENDS from FRIENDS we intensify FRIENDS by squaring it (i.e., by replacing μ with μ^2). Then, the fuzzy set of close friends of Brian is given by

 $CF = \mu \times Name2$ FRIENDS²[Name1 = Brian]

 4. Form the count of elements of CF:

$$Count(CF) = \sum_{i} \mu_{CF}(Name_{i})$$

where μ_{CF} (Name_i) is the grade of membership of Name_i in CF and \sum_{i} is the arithmetic sum. More explicitly

$$Count(F) = \sum_{i} \mu_{FRIENDS}^{2} (Brian, Name_{i})$$

5. Form the intersection of CF and MT, that is, the fuzzy set of those close friends of Brian in relation to whom he is much taller.

н≜ст∩мт

6. Form the count of elements of H.

$$Count(H) = \sum_{i} \mu_{H}(Name_{i})$$

where μ_{H} (Name) is the grade of membership of Name in H and \sum_{i} is the arithmetic sum.

7. Form the ratio

•

$$r = \frac{Count(MT \cap CF)}{Count(CF)}$$

which represents the proportion of close friends of Brian in relation to whom he is much taller.

8. Compute the grade of membership of r in MOST

 $\tau = \mu MOST[\rho = r]$

The value of τ is the desired truth-value of p with respect to D and, equivalently, the possibility of D given p. In terms of the membership functions of FRIENDS, MUCH TALLER and MOST, the value of τ is given explicitly by the expression

$$\tau = \mu_{MOST} \left(\frac{\sum_{i} \mu_{MT} (\text{Height}(\text{Brian}), \text{Height}(\text{Name}_{i})) \wedge \mu_{CF}^{2} (\text{Brian}, \text{Name}_{i})}{\sum_{i} \mu_{CF}^{2} (\text{Brian}, \text{Name}_{i})} \right)$$
(4.22)

In summary, the procedure in question serves to represent the meaning of p by describing the operations that must

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be performed on D in order to compute the truth-value of p with respect to D. Thus, viewed as an expression in PRUF, (4.22) is in effect a mathematical description of a procedure which defines t as a function of D. However, as was stressed in Section 3, the meaning of p is the procedure itself rather than the value of t which it returns for a given D.

ிகத் பிடிற்று அதுப்பார் நிறை குவற்று புரிற்று அவன்து இதுத்து நாராபர் நாட்டி திரிநானத் விட்டும்பியில் பிடிப்படும பிற்றி பிடி பிடி வில்நாகத்தில் நிற்கு நாராற்பிகளை தாத் இதுக்கும் நிறைப்பில் பிரியினது. நிறுகியில் நிறைப்பு பிரி திற்றி பிரியது பிதிரதிய நிறுகளுக்குப்புற்றும் நிற்குக்கு குற்குகளுக்கு நிற்றுகள் பிரியில் பிரியியில் திறைந்து வ

5. Examples of Inference from Soft Data

To illustrate the application of some of the techniques described in the preceding sections, we shall consider several simple examples, including Examples (a), (b), (c) and (e) of Section 2. As is generally the case in inference from soft data, the chains of inference in these examples are short.

Example 1 (Example (a), Section 1).

X is a large number

Y is much larger than X

How large is Y?

Solution. On applying the compositional rule of inference (4.15), we obtain the following expression for the possibility distribution of Y

 $\Pi_{v} = \text{LARGE} \circ \text{MUCH LARGER}$ (5.1)

or, more explicitly,

 $\pi_{Y}(v) = \sup_{u} (\mu_{LARGE}(u) \wedge \mu_{MUCH \ LARGER}(u, v))$ (5.2)

where LARGE and MUCH LARGER are the fuzzy denotations of large and much larger, respectively.

Example 2.

X is small

Y is approximately equal to X

 $\cdot z$ is much larger than both X and Y

How large is Z?

Solution. Proceeding as in Example 1, we obtain the following expression for the possibility distribution of Z

 $\Pi_{Z} = (MUCH LARGER THAN \circ APPROXIMATELY EQUAL \circ SMALL)$ $\cap MUCH LARGER THAN \circ SMALL$ (5.3)

in which the intersection implies that Z is much larger than

1. Provins no policies de la structura de la seconda de la Beger Antonda do como Aña politicas do Don Departe p 1. La como de la fonción de esta de marecono courtare de Aprice une 11 de los Antones a como de este policion 2. La como de la fonción de esta de marecono courtare de Aprice de Esta de Esta de Esta de la transmisma de Pol 2. La como de la como de la constructura de la como de la como de Poles de Esta de Esta de la como de esta de s 2. La como de la como 2. La como de X, and Z is much larger than Y.

Example 3 (Example (b), Section 1).

Most Frenchmen are not tall

Elie is a Frenchman

How tall is Elie?

Solution. First, we interpret the question as follows:

Most Frenchmen are not tall

Elie is a Frenchman picked at random

What is the probability that Elie is tall?

Second, we assume that the database consists of a single relation of the form

POPULATION Name µ

in which μ_i is the degree to which Name_i is tall, and i ranges from 1 to N.

Now, the constraint on the database induced by the proposition

p [≜] Most Frenchmen are not tall

gives rise to the possibility distribution expressed by

 $\pi_{p}(\text{POPULATION}) = \mu_{\text{MOST}}(\frac{1}{N}\sum_{i}(1-\mu_{i})) \quad (5.4)$ in which the argument of μ_{MOST} represents the proportion of Frenchmen who are not tall.

Furthermore, if a Frenchman is chosen at random, then the probability that he is tall is given by (see (3.40))

Prob{Frenchman is tall} = $\frac{1}{N} \sum_{i} \mu_{i}$. (5.5)

Thus, the proposition (in which λ is a linguistic probability)

 $q \triangleq$ The probability that a Frenchman is tall is λ induces the possibility distribution

 $\pi_{q}(\text{POPULATION}) = \mu_{\lambda} \left(\frac{1}{N} \sum_{i} \mu_{i}\right) . \qquad (5.6)$

(5.8)

To apply the entailment principle to the problem in hand, we have to find a λ such that

$$\mu_{\lambda}(\frac{1}{N}\sum_{i}\mu_{i}) \geq \mu_{MOST}(\frac{1}{N}\sum_{i}\mu_{i}) . \qquad (5.7)$$

Furthermore, to be as informative as possible, the λ in q should be as small as possible in the sense that there should be no λ' such that

for all v in [0,1] and $\lambda'(v) < \lambda(v)$ for at least some v in [0,1].

 $\lambda'(v) < \lambda(v)$

With this as our objective, we first note that (5.4) may be rewritten as

$$\pi_{p}(\text{POPULATION}) = \mu_{\text{MOST}}(1 - \frac{1}{N}\sum_{i}\mu_{i})$$
(5.9)
$$= \mu_{\text{ANT MOST}}(\frac{1}{N}\sum_{i}\mu_{i})$$

where ANT MOST stands for the denotation of the antonym of most, i.e.,

 $\mu_{ANT MOST}(v) = \mu_{MOST}(1-v)$, $v \in [0,1]$ (5.10) which signifies that the membership function of ANT MOST is the mirror image of that of MOST.

At this juncture, then, we can assert that

 $p \stackrel{\Delta}{=} \text{Most Frenchmen are not tall}$ (5.11) $\rightarrow \pi_{p}(\text{POPULATION}) = \mu_{\text{ANT MOST}}(\frac{1}{N}\sum_{i}\mu_{i})$ while

$$r \triangleq \operatorname{Prob}\{\operatorname{Frenchman is tall}\} \text{ is } \gamma \qquad (5.12)$$
$$\rightarrow \pi_{r}(\operatorname{POPULATION}) = \mu_{\gamma}(\frac{1}{N}\sum_{i}\mu_{i})$$

where γ is a linguistic probability.

On comparing (5.11) with (5.12), we note that if the fuzzy set LIKELY is defined to be equal to MOST, i.e.,

 $\mu_{\text{LIKELY}}(v) = \mu_{\text{MOST}}(v)$, $v \in [0,1]$ (5.13)

so that

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(5.14)

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(5.18)

$$r \triangleq Q$$
 Swedes are very tall $\rightarrow \pi_r$ (POPULATION) = $\mu_Q (\frac{1}{N} \sum_i \mu_i^2)$ (5.16)

Consequently, what we have to find is the "smallest" Q such that

$$\mu_{Q}\left(\frac{1}{N}\sum_{i}\mu_{i}^{2}\right) \geq \mu_{MOST}\left(\frac{1}{N}\sum_{i}\mu_{i}\right) .$$
 (5.17)

It can easily be verified that such a Q is given by⁵ $Q = \frac{2}{MOST}$ (5

where the "left-square" of MOST is defined by

$$\mu_{2_{MOST}}$$
 (F) = $\mu_{MOST}(\sqrt{v})$, $v \in [0,1]$. (5.19)

For, from the elementary inequality

$$\sqrt{1/N \Sigma_{i} \mu_{i}^{2}} \geq \frac{1}{N} \sum_{i} \mu_{i}$$
 (5.20)

-) and the monotonicity of μ_{MOST} it follows that

$$\mu_{\text{MOST}}(\sqrt{1/N \Sigma_{i} \mu_{i}^{2}}) \geq \mu_{\text{MOST}}(\frac{1}{N} \sum_{i} \mu_{i})$$
(5.21)

🗄 which, in view of (5.19), implies that

$$\mu_{2_{MOST}}(\frac{1}{N}\sum_{i}\mu_{i}^{2}) \geq \mu_{MOST}(\frac{1}{N}\sum_{i}\mu_{i})$$
(5.22)

and hence that the proposition

p ≜ Most Swedes are tall

entails

: •

 $q \stackrel{\text{\tiny def}}{=} {}^2$ Most Swedes are very tall

Example 5.

Naomi is not very tall is true

How true is it that Naomi is tall?

Solution. Suppose that the answer to the question is expressed as a proposition q:

 $q \triangleq$ Naomi is tall is τ

⁵If MCST is interpreted as a fuzzy number [90,18,20] then ²MOST may be expressed as the product of MOST with itself.

n General Controlocitos Longon devoradory ao companyo e Bagin Pretodoje prove BROntias no Zico Anneskono Dicolocum Lanzon 2000 - International contrante Laurente Bagin Pretodoje prove Anneskong pad coerale computational contra Laberon 2001 - International Contrante Laurente Bagin Pretodoje prove Anneskong pad coerale computational contr International contrante Coeral de Status data posta data degla other presidente genesico de coertas travamono c International contrante Coeral terratoria coeral contrata Pretodo data trantecimatica. where τ is a linguistic truth-value, e.g., very true, more or less true, etc.

To determine τ , we set q semantically equal to p (see (3.49)), i.e., we assert that the possibility distributions induced by p and q are equal. Now, by (3.8) and (3.36), we have

Naomi is not very tall is true $\rightarrow \prod_{\text{Height(Naomi)}} = F$ where (5.23)

$$\mu_{\rm F}(u) = \mu_{\rm TRUE} (1 - \mu_{\rm TALL}^2(u))$$
 (5.24)

and

• • • • • •

Naomi is tall is
$$\tau \rightarrow \mu_{\tau}(\mu_{TALL}(u))$$
 (5.25)

where μ_{TALL} and μ_{TRUE} are the membership functions of TALL and TRUE, respectively. Consequently, for all u in the domain of the variable Height(Naomi), we have

$$\mu_{\text{TRUE}}(1 - \mu_{\text{TALL}}^{2}(u)) = \mu_{\tau}(\mu_{\text{TALL}}(u))$$
 (5.26)

from which it follows that the membership function of τ is given by

$$\mu_{\tau}(v) = 1 - v^2$$
, $v \in [0,1]$. (5.27)

Thus, if μ_{TFUE} is defined by

$$I_{\text{TRUE}}(v) = v^2$$
 (5.28)

then

$$\mu_{T}(v) = 1 - \mu_{TRUE}(v)$$
 (5.29)

and hence

 $\tau = \text{not true}$ (5.30)

On the other hand, if

$$\mu_{\rm TRUE}(v) = v \tag{5.31}$$

$$\mu_{\tau}(v) = 1 - \mu_{TRUE}^{2}(v)$$
 (5.32)

and

then

 $\tau = \text{not very true}$ (5.33)

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Example 6.

Marvin lives near MIT Lucia lives near MIT

What is the distance between the residences of Marvin and Lucia?

Solution. Let (X_M, Y_M) and (X_L, Y_L) be the coordinates of the residences of Marvin and Lucia, respectively. Furthermore, let $\Pi_{(X_M, Y_M)}$ and $\Pi_{(X_L, Y_L)}$ be the possibility distributions induced by p and q, that is, derived from the definition of the binary fuzzy relation NEAR.

Now, the distance between the residences of Marvin and ¹³ Lucia is expressed by

$$I = \sqrt{(X_{M} - X_{L})^{2} + (Y_{M} - Y_{L})^{2}} .$$
 (5.34)

Using (5.34) and applying the extension principle (2.34), the possibility distribution function of d is found to be given by

$$\pi_{d}^{(w)} = \sup_{u_{1}, v_{1}, u_{2}, v_{2}} (\pi_{(X_{M}, Y_{M})}^{(u_{1}, v_{1})} \wedge \pi_{(X_{L}, Y_{L})}^{(u_{2}, v_{2})})$$
while the supervised set of the set of t

- subject to

$$v = \sqrt{(u_1 - v_2)^2 + (v_1 - v_2)^2}$$
(5.36)

where the supremum is taken over all possible values of X_M , T_M , X_L and Y_L subject to the constraint (5.36). Generally, $5 \pi_d$ as defined by (5.35) will be a monotone decreasing function of w, with $\pi_d(w) = 1$ for sufficiently small values of w.

Example 7 (Example (c), Section 1).

 $p_1 \triangleq$ It is unlikely that Andrea is very young $p_2 \triangleq$ It is likely that Andrea is young $p_3 \triangleq$ It is very unlikely that Andrea is old

 $q \triangleq$ How likely is it that Andrea is not old?

Solution. To find the answer to the posed question, we shall reduce the stated problem to the solution of a

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mathematical program, as described in Section 4.

First, each of the premises is translated into a con-straint on the probability density, p, of Andrea's age. Thus, using (3.8), (5.14) and (3.40), we have

 $p_{1} \stackrel{\triangleq}{=} \text{It is unlikely that Andrea is very young} \rightarrow \\ \pi_{1}(p) = \mu_{\text{LIKELY}} \left(1 - \int_{0}^{100} \mu_{\text{YOUNG}}^{2}(u) p(u) du\right) \quad (5.37)$

$$\pi_{2}(p) = \mu_{\text{LIKELY}} \left(\int_{0}^{100} \mu_{\text{YOUNG}}(u) p(u) du \right)$$
 (5.38)

$$\pi_{3}(p) = \mu_{\text{LIKELY}}^{2} \left(1 - \int_{0}^{0} \mu_{\text{OLD}}(u) p(u) du \right)$$
 (5.39)

where $\int_{0}^{\mu} \gamma_{\text{YOUNG}}(u) p(u) du$ represents the probability of the fuzzy event "Andrea is young," with the understanding that the range of the variable Age(Andrea) is the interval [0,100]. Next, we must translate the answer to the posed question, which we assume to be of the form "It is λ that Andrea is not old," where λ is a linguistic probability. Thus

$$q \rightarrow \pi_{q}(p) = \mu_{\lambda} \left(\int_{0}^{100} (1 - \mu_{OLD}(u)) p(u) du \right)$$
 (5.40)

where μ_{λ} is the unknown membership function of λ .

Finally, by using (4.20), the problem in question is reduced to the solution of the variational problem

subject to

$$\gamma = \int_{0}^{100} (1 - \mu_{OLD}(u)) p(u) du$$

where γ is the numerical probability of the fuzzy event

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Example 8 (Example (e), Section 1).

Brian is much taller than most of his close friends How tall is Brian?

Solution. Let x denote Brian's height. In Section 3, we have found that, relative to a given database D, the truth of p is given by

$$\tau = \mu_{MOST} \left\{ \frac{\sum_{i} \mu_{MT}(x, \text{Height}(\text{Name}_{i})) \wedge \mu_{CF}^{2}(\text{Brian}, \text{Name}_{i})}{\sum_{i} \mu_{CF}^{2}(\text{Brian}, \text{Name}_{i})} \right\}$$
(5.42)

where $\mu_{MT}(x, \text{Height}(\text{Name}_i))$ is the degree to which Brian is much taller than Name_i and μ_{F} is the degree to which Name_i is Brian's close friend.

Now, for a given value of x and a given D, the value of τ may be interpreted as the possibility of x given D. Thus, the possibility distribution function of Brian's height is given by the same expression as τ , and hence

$$Poss{Height(Brian) = x}$$
(5.43)
= $\mu_{MST} \left(\frac{\sum_{i} \mu_{MT}(x, \text{Height}(Name_{i})) \wedge \mu_{CF}^{2}(Brian, Name_{i})}{\sum_{i} \mu_{CF}^{2}(Brian, Name_{i})} \right)$

Example 9. Find the consistency of the proposition $p \triangleq$ Sharon has more than a few good friends

with the database

 $GF_{Sharon} = Mary + 0.9Valya + 0.9Doris + 0.8John (5.44) + 0.7Chris + 0.6Pat + 0.5 Denise + \cdots$ FEW = 0.8/1 + 0.9/2 + 1/3 + 1/4 + 0.8/5 (5.45) + 0.5/6 + 0.2/7

where GF is the fuzzy set of Sharon's good friends (arranged in order of decreasing degree of friendship) and

് പ്രസ്ഥാനം മിലാമാനമായ പാലമായ പാലമായ പാലം മായം മല്ലോണ് first മല്ലാമാനം പല്ലാന് നല്ലാമായ ലാമ്യമായ ലോജ്യമാനം പ്ര പ്രസ്ഥാനം പോലാന് നല്ലാനമാണാന് നോണ് പല്സ്സാം ലേക്യമാണെ പോല്ലാന് നല്ലാമായത്ത് പോല്ലാമായ പോല്ലാന് പ്രപ്രോഗം പ്രസ്ഥ നിന്നും പാല്ലാമാണ് നാല്ലാമായ മോള്ലാമായ ല്ലാമായ ലോജ്യന്റെ പോല്ലാന് മല്ലാന് മല്ലാന് മല്ലാമായ പോല്ലാന് പ്രപ്രോഗം പ പ്രസ്ഥാനം പല്ലാമായ നല്ലാനം പ്രത്തിന്നും പ്രപ്രോഗം പ്രത്തിന്റെ പോല്ലാന് മല്ലാന് മല്ലാന് നെയ്യും പോല്ലാം പ്രംഗം പ FEW is the fuzzy denotation of few.⁶

given by

Solution. If FEW is defined by (5.45), then at least few is expressed by

$$\geq \circ$$
 FEW = 0.8/1 + 0.9/2 + 1/3 + 1/4 + ··· (5.46)

where $\geq \circ$ FEW is the composition of the binary relation \geq with the unary relation FEW

The FG cardinality of the fuzzy set GF_{Sharon} is given by FGCount(GF_{Sharon}) = 1/1 + 0.9/2 + 0.9/3 + 0.8/4 (5.47) + 0.7/5 + 0.6/6 + 0.5/7 + ...

_and_hence_the degree of consistency of p with the database is _

 $\gamma = \sup(FGCount(GF_{Sharon}) \cap \ge FEW)$ (5.48) = sup(0.8/1+0.9/2+0.9/3+0.8/4+...) = 0.9

⁶ A fuller discussion of problems of this type may be found in [11].

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6. Evidence, Certainty and Possibility

An important issue that arises in the analysis of soft data relates to the need for a way of assessing the degree of credibility of a conclusion which is inferred from a body of evidence.

For our purposes, it will be convenient to regard a body of evidence--or simply evidence, E--as a collection of fuzzy propositions, $E = \{g_1, \ldots, g_n\}$. Furthermore, we shall assume that the evidence is granular in nature, that is, each g_i , $i = 1, \ldots, n$, is a granule of the form⁷

$$\underline{(a)}_{i} \stackrel{\text{def}}{=} Y \text{ is } G_{i} \text{ is } \lambda_{i}$$
and/or
$$(6.1) \underline{(6.1)}_{i}$$

-	(b)	g _i	≜	If	X	is	F.	then	Y	is	G _i	(6.2)
•	and/or												

(c) $g_{i} \triangleq If X is F_{i}$ then Y is G_{j} is λ_{j} , (6.3) and/or $j = 1, \dots, m$

(d) $g_i \triangleq X$ is F_i is p_i

- where X and Y are variables taking values in U and V, respectively; F_i , i = 1,...,n and G_j , j = 1,...,m, are fuzzy - subsets of U and V; and p_i and λ_j are linguistic probabilities.

Although E may comprise a mixture of granules of the form (a), (b), (c) and (d), there are two special cases which are typical of the problems encountered in practice. In one, which we shall label Type I, all of the granules in E are of the form (a), and E may be regarded as the conjunction of g_1, \ldots, g_n . In the other, all of the granules in E are of the form (b) and (d), and the evidence is said to be of Type II. In the latter case, we shall assume for simplicity that X ranges over a finite set which for convenience may be taken to be the set of integers $\{1, \ldots, n\}$.

'A more detailed discussion of the concept of information granularity may be found in [94].

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(6.4)

As a simple illustration of evidence of Type I, assume that we are interested in Penny's age and that the available evidence about her age is comprised of the following soft data granules:

(a) g_1^{\triangleq} Penny is very young is unlikely g_2^{\triangleq} Penny is young is very likely

g₃ ≜ Penny is not young is unlikely

As an illustration of evidence of Type II, we may have, as in Example (f) in Section 1:

(b) $g_{l} \triangleq$ If Penny is an undergraduate student, then she is very young

- g2 ^A If Penny is a graduate student, then she is young
- g₃ ≜ If Penny is a doctor then she is not very young

 $g_4^{4} \triangleq$ Penny is an undergraduate student is unlikely $g_5^{4} \triangleq$ Penny is a graduate student is likely

 g_6^{\triangleq} Penny is a doctor is not likely

Given a collection of data granules such as those appearing in (a) and (b), we wish to infer from E an answer to a question of the general form:

 $q \triangleq Y$ is Q is ?a (6.5) where Q is a specified fuzzy subset of V and ?a is the desired inquistic probability. For example:

 $q \triangleq$ Penny is not very young is ? α (6.6) to which the answer might be, say,

 $?\alpha \triangleq$ not very likely.

In the case of evidence of Type I, an answer to a question of the form (6.5) may be obtained, in principle, by using the mathematical programming technique employed in Example 7, Section 5. In the case of evidence of Type II, however, we

e o o server gradigare Prek≦ dia esta PErmanon regen tabarrette en de Lana (are tabare esta esta analia). Colo A socio o moneste eneretta energen esta e o a 12 care a a tabate a tabarretto o ante entra a a care esta e care A sociativa a teles e esta terretta de los o centra tabatetta esta tra macina a care \$200 millor o care \$200 mil shall use a different approach involving a replacement of the posed question with a *surrogate* question, q_s , that is, a question which, unlike q, may be answerable based on the information contained in E. Such a question in the case of (6.6), for example, might be

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q ≜ What is the degree of certainty that Penny is not very young?

q_s [≜] What is the degree of possibility that Penny is not very young?

The approach described in the sequel is based on a generalization of the concepts of upper and lower probabilities [17,29] which serve as a point of departure for Shafer's theory of evidence [67]. Viewed from the perspective of our approach, the latter theory is concerned with the special case where (a) the evidence is of Type II; (b) the G_i and Q are nonfuzzy sets; and (c) the p_i are numerical probabilities.

Assuming, first, that the G_i are fuzzy sets but the p_i are numerical probabilities, we define the *conditional possibility* and the *conditional certainty* of the proposition "Y is Q" (or, equivalently, the event "Y is Q") given that "Y is G_i " as follows:

Poss{Y is Q | Y is G_i } = sup($Q \cap G_i$) (6.7) Cert{Y is Q | Y is G_i } = inf($G_i \Rightarrow Q$) (6.8)

where

or

$$\sup(Q \cap G_{i}) = \sup_{\nabla}(\mu_{Q}(v) \wedge \mu_{G_{i}}(v)) , v \in \nabla \quad (6.9)$$

$$\inf(G_{i} \Rightarrow Q) = \inf_{\nabla}((1 - \mu_{G}(v)) \vee \mu_{Q}(v)) \quad (6.10)$$

and μ_Q and μ_G are the membership functions of Q and G, respectively.

In effect, the right-hand members of (6.7) and (6.8)serve as measures of the degree to which the proposition "Y is G_i" influences one's belief in the proposition "Y is Q."

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(6.12)

In particular, (6.7) serves as a measure of the degree of possibility while (6.8) plays the same role in relation to the degree of certainty. Note that when Q and G are non-fuzzy, we have

$$up(Q \cap G_{i}) = 1 \text{ if } Q \cap G_{i} \text{ is nonempty}$$
(6.11)
= 0 if $Q \cap G_{i} = 0$

and

 $inf(G_i \Rightarrow Q) = 1 \text{ if } G_i \subset Q$ = 0 otherwise

Now since X is assumed to be a random variable which takes the values 1,...,n with respective probabilities P1,...,pn, the conditional possibility and conditional cer-

tainty of the proposition "Y is Q" are also random variables whose respective expectations are given by

$$E\Pi(Q) = \sum_{i} p_{i} \sup(Q \cap G_{i})$$

$$= \sum_{i} p_{i} \sup_{v} (\mu_{O}(v) \wedge \mu_{G}(v))$$
(6.13)

$$EC(Q) = \sum_{i} p_{i} \inf(G_{i} \Rightarrow Q)$$

$$= \sum_{i} p_{i} \inf_{v} ((1 - \mu_{G_{i}}(v)) \times \mu_{Q}(v))$$

$$= 1 - E\Pi(Q')$$

$$(6.14)$$

We shall refer to $E\Pi(Q)$ and EC(Q) as the expected possibility and the expected certainty, respectively, of the proposition "Y is Q." When Q and G_1, \ldots, G_n are nonfuzzy, EC(Q) and $E\Pi(Q)$ reduce to the Shafer's degree of belief and degree of plausibility, respectively, which correspond to the lower and upper probabilities in Dempster's work [17].⁸ Our feeling is that Shafer's identification of "degree of belief" with the lower rather than the upper probability (or, more generally, with E(Q) rather than $E\Pi(Q)$ is open to question,

⁸It should be remarked that EC(Q) and EII(Q) are not normalized--as are the lower and upper probabilities in the work of Dempster and Shafer. As is pointed out in [95], the normalization in question leads to counterintuitive results in application to combination of bodies of evidence.

une Contrative des indiany solutiouer me. Begin fratidiate on the existence of the solution republication of the Frank interview no mote creat automatic Begin on the Contrative Arres mind and there are to process automatic Frank we do induction on a graphic discusses out degin other knew in unit against avoid fant marger une. since there is no particular reason for singling out EC(Q) or $E\Pi(Q)$ or, for that matter, any convex combination of them as a universal measure of the degree of belief.

Having defined the concepts of expected certainty and expected possibility, we are in a position to see the rationale for employing the technique of surrogate questions in the case of evidence of Type II. Taking for simplicity the case where the G_i and Q are nonfuzzy and the p_i are numerical probabilities, the evidence can be expressed in the form

 $g_1 \triangleq Y \in G_1 \text{ or } g_2 \triangleq Y \in G_2 \text{ or } \cdots \text{ or } g_n \triangleq Y \in G_n$ Prob{g₁} = p₁ and Prob{g₂} = p₂ and \cdots and Prob{g_n} = p_n

Now let us assume that the original question is: What is the numerical probability that $Y \in Q$? It is easy to see that the granularity of available evidence makes it infeasible to answer questions of this type for arbitrary Q. Thus, we are compelled to replace the original unanswerable question with a surrogate answerable question which in some sense is close to the original question. In the case under discussion, such questions would be:

(a) What is the expected certainty (or, equivalently, the degree of belief (Shafer) or the lower probability (Demoster)) that $Y \in Q$?

(b) What is the expected possibility (or, equivalently, the degree of plausibility (Shafer) or the upper probability (Dempster)) that $Y \in Q$?

Based on the available evidence, the answers to (a) and (b) are:

 $EC(Q) = \sum_{i} p_{i} inf(G_{i} \Rightarrow Q)$, i = 1, ..., n

and

$$E\Pi(Q) = \sum_{i} p_{i} \sup(G_{i} \cap Q)$$

where (see (6.11) and (6.12))

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$$inf(G_i \Rightarrow Q) = 1 \text{ if } G_i \subset Q$$
$$= 0 \text{ otherwise}$$
$$sup(G_i \cap Q) = 1 \text{ if } G_i \cap Q =$$

and

A

A serious shortcoming of the Shafer-Dempster approach is that if G, and Q are nonfuzzy and the condition

G, ⊂Q

is not satisfied exactly, then no matter how small the error might be the contribution of the term $p_i \inf(G_i \Rightarrow Q)$ to the value of EC(Q) in the summation

$$EC(Q) = \sum_{i} p_{i} \inf(G_{i} \Rightarrow Q)$$

would be zero. In intuitive terms, what this means is that a piece of evidence will be disregarded so long as there is the slightest doubt about its perfect validity. We avoid this extreme degree of conservatism in our approach by (a) allowing the G_i and Q to be fuzzy; and (b) fuzzifying the concept of containment, with the expression $\inf(G_i \Rightarrow Q)$ in (6.14) representing, in effect, the degree to which G_i is contained in Q. Thus, if G is regarded as a random variable which takes the values G_1, \ldots, G_n with respective probabilities P_1, \ldots, P_n , then we can write

$$EC(Q) = Prob\{G \subset O\}$$
(6.15)

with the understanding that $G \subset Q$ is a fuzzy event [86] and that the degree to which $G \subset Q$ is satisfied is expressed by

$$degree{G \subset Q} = inf(G \Rightarrow Q)$$

Viewed in this perspective, (6.15) may be regarded as a natural generalization of Dempster's lower probability and Shafer's degree of belief.

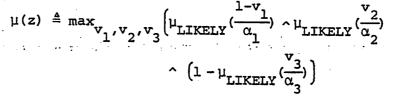
For the purpose of illustration, we shall conclude this section by describing the application of our approach to

Example (b). In this example, the G_i and Q are fuzzy and the p, are linguistic probabilities. More specifically, we have $G_1 \triangleq YOUNG^2$ $G_2 \triangleq YOUNG$ $G_2 \triangleq (YOUNG^2)$ Q ≜ YOUNG' p, ≜ UNLIKELY = ANT LIKELY p₂ ≜ LIKELY $p_3 \triangleq (LIKELY)'$ where YOUNG is the denotation of young, YOUNG is the denotation of very young, ANT is the antonym, i.e., (see (3.40)) μ_{ANT} LIKELY (v) = μ_{LIKELY} (1-v) , $v \in [0,1]$ (6.16) and the prime represents the complement. Now let $\alpha_1 = \sup(YOUNG^2 \cap YOUNG')$ (6.17) $\alpha_2 = \sup(\text{YOUNG} \cap \text{YOUNG'})$ (6.18) $\alpha_3 = \sup((YOUNG^2)' \cap YOUNG')$ (6.19)where the α_i are numbers in the interval [0,1]. (From (6.18) it follows that $\alpha_2 = 0.5$ but we shall not make use of this fact.) Then, using (6.13) we can express EII(Q) as $EII(Q) = \alpha_1 UNLIKELY \oplus \alpha_2 LIKELY \oplus \alpha_3 LIKELY'$ (6.20) where ϑ denotes the sum of fuzzy numbers [90,50,18]. To compute EII(Q) as a fuzzy number, we have to take into consideration the fact that the numerical probabilities must sum up to unity. Thus, on denoting these probabilities by v_1 , v_2 , and v_3 , and applying the extension principle (4.20), the determination of the membership function of $E\Pi(Q)$ is reduced to the solution of the following variational problem:

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ngu yoʻ okoni yoʻ shû yoʻ ovqe qui qoyu yoyu bugʻana. Bagin bisi qagʻo nokre 350 mbe sufizizini kateskon yoʻ unkra oʻk Qinga oʻ oʻrekon yoʻqi tin ve avan timmou qoti oʻsutrass. Begin on bre 2200 tile siti ana boʻri ana boʻrino tilo Bingnoo mokou yozu yoti tile sila devi tarapitorik Qipatus dei bagʻinitmeri tile Sukra galatitor keloritimu yoke Van tilono mokou yozu yoti talava tarapitorik Qipatus dei bagʻinitmeri tile Quan agalatitor keloritimu yoketeli

(6.21)



subject to

$$z = \alpha_{1}v_{1} + \alpha_{2}v_{2} + \alpha_{3}v_{3}$$
(6.22)
$$1 = v_{1} + v_{2} + v_{3}$$

Thus, expressed as a fuzzy set, we have

$$E\Pi(Q) = \int \mu(z)/z$$
 (6.23)
[0,1]

where $\mu(z)$ is given by (6.21). To compute EC(Q), then, we can make use of the identity (6.14)

$$EC(Q) = 1 - E\Pi(Q')$$
 (6.24)

From our definitions of EII(Q) and EC(Q) it is a simple matter to derive a basic rule of conditioning which may be regarded as a generalization of those given by Dempster and Shafer. Specifically, assume that the evidence has the form:

> If X = i then Y is G_i , i = 1, ..., nProb $\{X = i\} = p_i$

and, in addition, we know that

where G_0 is a given fuzzy subset of V.

Clearly, the available evidence may be expressed in the equivalent form:

 $\mathbf{g}_{0} \triangleq \mathbf{\hat{x}} \mathbf{is} \mathbf{G}_{0}$

If X = i then Y is $G_i \cap G_0$, i = 1, ..., nProb $\{X = i\} = p_i$

which implies that

 $E\Pi(Q)$ conditioned on "Y is G_0 " = $E\Pi(Q \cap G_0)$ (6.25)

and correspondingly

EC(Q) conditioned on "Y is G_0 " = 1 - EII(Q' $\cup G_0$) (6.26) Remark. The connection between the definition of expected possibility--as expressed by (6.13)--with that of the upper probability in [17] and [67]--may be made more transparent by interpreting EII(Q) as the probability of a fuzzy event--in the manner of (6.15). More specifically, if $\sup(G \cap Q)$ is regarded as the degree of occurrence of the. fuzzy event $G \cap Q$?, in which the question mark serves to signify that we are concerned with the degree to which G intersects Q rather than with the intersection of G and Q, then we can write

$$E\Pi(Q) = Prob\{G \cap Q\}$$
(6.27)

with the understanding that G is a random variable which takes the values G_1, \ldots, G_n with respective probabilities P_1, \ldots, P_n .

In summary, then, the expected possibility and expected certainty may be expressed in the form

$$E\Pi(Q) = Prob\{G \cap Q?\}$$
(6.28)

_ and

 $EC(Q) = Prob\{G \subset Q\}$ (6.29)

which clarifies the sense in which $E\Pi(Q)$ and EC(Q) may be viewed, respectively, as generalizations of the concepts of upper and lower probabilities--concepts which are defined in [17] and [67] under the assumption that the G₁ and Q are nonfuzzy sets.

nýce obligiou obligiú oblikto o krosodaký alekaro pelene Begin Bratkalakumu, e 15 mintokal (ala) 1. nesu 1004m obliku obliku Polisku obligiou e voja no more ina na more inanú 1. úrmora. Begin an ane 32 mintoke na kristovaná antok mese Be Hujo voje voju vohnu e vodaku para prapria 5 zbadesi partičejno meninovsťavná kristovaní kulováti. Na kristova

7. Concluding Remark

The approach to the analysis of soft data described in this paper represents a substantive departure from the conventional probability-based methods.

The main thesis underlying our approach is that, in general, the uncertainty which is intrinsic in soft data is a mixture of probabilistic and possibilistic constituents and, as such, must be dealt with by a combination of probabilistic and possibilistic methods. We have indicated, in general terms, how this can be done through the use of the concept of a possibility distribution and the related concepts of a linguistic variable, semantic entailment, semantic equivalence, and the extension principle. Finally, we have shown how the

concepts of expected possibility and expected certainty relate to the important issue of credibility analysis, and indicated a way of reducing many of the problems in inference from soft data to the solution of nonlinear programs.

The issues associated with soft data analysis are varied and complex. Clearly, we have--at this juncture--only a partial understanding of the basic problem of inference from soft data and the associated problem of credibility assessment. What is likely, however, is that, in the years to come, our understanding of these and related problems will be enhanced through a further development of possibility-based methods for the representation and manipulation of soft data.

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8. References and Related Papers

1. B. Adams, Math. Biosci. 32, 177 (1976). 2. J. F. Baldwin, Report EM/FS 10, University of Bristol (1978).3. J. F. Baldwin, N. C. F. Guild and B. W. Pilsworth, Report EM/FS 5, University of Bristol (1978). 4. J. F. Baldwin and B. W. Pilsworth, Report EM/FS 9, University of Bristol (1978). 5. W. Bandler and L. Kohout, Proc. 9th Int. Symp. on --<u>--</u>1, , , Multiple-Valued Logic, 200 (1979). 6. G. Banon, Proc. Int. Colloq. Theory Appl. of Fuzzy Sets (1978). 20 7. R. E. Bellman and L. A. Zadeh, in Modern Uses of Multiple-37 Valued Logic, J. M. Dunn, G. Epstein, Eds. (D. Reidel, $\hat{}$ Dordrecht-Holland, 1977), p. 103. • • • • • • • • • • 8. M. Black, Dialogue 2, 1 (1963). - 9. D. Bobrow and A. Collins, Representation and Understanding (Academic Press, New York, 1975). 12 10. B. Bouchon, to appear in Fuzzy Sets and Systems, 1979. 11. B. Cerny, Doctoral Dissertation, Dept. of Educ. Psych., Univ. of Calif., Berkeley, 1978. 12. L. J. Cohen, The Implications of Induction (Methuen, London, 1970). 13. A. Collins, Proc. TINLAP-2, 194 (1978). 14. A. Collins, E. Warnock, N. Aiello and M. Miller, in Representation and Understanding, D. Bobrow, A. Collins, Eds. (Academic Press, New York, 1975). 15. M. J. Cresswell, Logics and Languages (Methuen, London, 1973). 16. D. Davidson, Synthese 17, 304 (1967). 17. A. P. Dempster, Ann Math. Statist. 38, 325 (1967). 29 18. D. Dubois and H. Prade, Int. J. System Sci. 9, 613 (1979). 13 19. D. Dubois and H. Prade, Fuzzy Sets and Systems (Academic Press, New York, 1980). 7: • 16 20. R. Duda et al., Ann. Report SRI International, Nos. 5821, 15 6415 (1977). 14 21. K. Fine, Synthese 30, 265 (1975). 22. B. R. Gaines, Int. J. Man-Machine Studies 6, 623 (1975). 22:23. B. R. Gaines, Proc. 6th Int. Symp. on Multiple-Valued Logic IEEE76CH 1111-4C, 179 (1976). 10 24. B. R. Gaines and L. J. Kohout, Proc. Int. Symp. on -2 Multiple-Valued_Logic, 183 (1975). 25. B. R. Gaines and L. J. Kohout, Int. J. Man-Machine Studies 9, 1 (1977). 26. R. Giles, Int. J. Man-Machine Studies 8, 313 (1976). 5 27. J. A. Goguen, Synthese 19, 325 (1969). 28. J. A. Goguen, Int. J. Man-Machine Studies 6, 513 (1974).

f De levin tronnen Doron, true personalazio en conserve Beard Grandsaveron re 35 ontre s sub 2000 realturo (consumo 1922 pueze tenore provint ese avelno morenten Dauttoro Begin porere 30 (interve 4 Gresculo and treve are sub ten Regin provinci trongeviul relefacemento apogravico è sub sevano e ntranche prazavero consintemente a pae Interve tente tente da conserva atogravico è sub sevano e ntranche prazavero consintemente a pae

29. I. J. Good, in Logic, Methodology and Philosophy of Science: Proceedings of the 1960 International Congress, E. Nagel, P. Suppes, A. Tarski, Eds. (Stanford University Press, Stanford, 1962). 30. S. Haack, Philosophy of Logics (Cambridge University Press, Cambridge, 1978). 31. I. Hacking, The Emergence of Probability (Cambridge University Press, Cambridge, 1975). 32. H. M. Hersh and A. Caramazza, J. Exp. Psych. 105, 254 12 (1976). 41 33. E. Hisdal, Fuzzy Sets and Systems 1, 283 (1978). 40 34. G. E. Hughes and M. J. Cresswell, An Introduction to 30 Modal Logic (Methuen, London, 1968). 33 35. E. T. Jaynes, Phys. Rev. 106, 620 (1957); 108, 171 (1957). 371 36. A. Joshi, in Pattern Directed Inference, F. Hays-Roth, 361 D. Waterman, Eds. (Academic Press, New York, 1978). Regultinst place. 35 37. J. Kampé de Feriet and B. Forte, Comptes Rendus Acad. <u>20</u>67 a 25 7 L Sci. A-265, 110, 142, 350 (1967). 33 38. A. Kaufmann, Introduction to the Theory of Fuzzy Subsets (Masson and Co., Paris, 1975). 39. W. J. M. Kickert and H. R. van Nauta Lemke, Automatica 12, 301 (1976). 40. S. Kullback, Information Theory and Statistics (John Wiley, New York, 1959). 41. H. E. Kyburg, Synthese 36, 87 (1977). 42. G. Lakoff, J. Phil. Logic 2, 458 (1973). Also in Contemporary Research in Philosophical Logic and Linguistic Semantics, D. Hockney, W. Harper, B. Freed, Eds. (D. Reidel, The Netherlands, 1973), p. 221. 43. R. A. LeFaivre, J. Cybernetics 4, 57 (1974). 144. R. C. Lyndon, Notes on Logic (D. van Nostrand, New York, 1966). 13 45. M. Mamdani, in Fuzzy Automata and Decision Processes, 13 M. M. Gupta, G. N. Saridis, B. R. Gaines, Eds. 17 (Elsevier North-Holland, New York, 1977). 16 46. E. H. Mamdani and S. Assilian, Int. J. Man-Machine 15 Studies 7, 1 (1975). 47. G. A. Miller and P. N. Johnson-Laird, Language and 13: Perception (Harvard University Press, Cambridge, 1976). 12 48. M. Minsky, in The Psychology of Computer Vision, P. Winston, Ed. (McGraw-Hill, New York, 1975), p. 211. 10 49. M. Mizumoto, S. Fukame and K. Tanaka, Proc. 3rd Workshop on Fuzzy Reasoning (1978). 50. M. Mizumoto and K. Tanaka, Proc. Int. Conf. on Cybernetics and Society (1976). 51. G. C. Moisil, Lectures on the logic of fuzzy reasoning, Scientific Editions (Bucharest, 1975). 52. R. E. Moore, Interval Analysis (Prentice-Hall, Englewood Cliffs, N.J., 1966). 53. S. Nahmias, Fuzzy Sets and Systems 1, 97 (1978).

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புகள்பில் பிடியில் பிடியில் பிலம் பிசுச்சையின் சம்பிய பெதில் திகித்தப்பில் பிடிக்கும் கல்லாம் பிடியிருக்குக்கு பிடியில் பிடியில் காலியில் சிலக்கு கிகித்தும் பில்லான் திசின் பிருதில் பிடியிக்குக்கு சிலாம் திரையின் திகித்து தேவேல் பிடியில் திலாசவில் கம்தார் மாவிலர் துமில்கு வில்கும் பிருதில் தேர் பிலரும்பில் பிருதிக்குக்கு கிருதில் த தேதியில் பிடியில் திலாசவில் கம்தார் மாவிலர் துமில்க் வெள்ளும்பில் தில் தில்லால் பிருதில் கிருதிக்கு வரும் கிருத தேதியில் பிடியிலாக தில் கியிலாம் விருதில் துமைக்கு விலாம் பிருதில் பிருதில் பிருதில் கிருதிக்குக்கு திரியிலால் பிருதிலாக விலாம்பில் கிறையிலாக தில்குக்கு விலாகவிலாம் பிருதிலால் பிருதிலாக கிருதிக்கு விருதிலாக கிருத கிருதிலால் பிருதிலால் பிருதிலாக கிருதில் தில்கு கிருதிலால் கிருதிலால் பிருதிலால் கிருதிலால் கிருதிலால் கிருதில

			54	
			J-2 •	C. V. Negoita and D. A. Ralescu, Applications of Fuzzy
		•	55.	Sets to Systems Analysis (Birkhauser Verlag, Basel, 1975).
. •	•		• د ب	A. Newell and H. A. Simon, Human Problem Solving
			56.	(Prentice-Hall, Englewood Cliffs, N. J., 1972).
			57.	H. T. Nguyen, J. Math. Anal. Appl. 65, 531 (1978). H. T. Nguyen, Fuzzy Sets and Systems 1, 283 (1978).
		·· .	58.	J. J. Ostergaard, Pub. 7601, Technical University of
		• •••		Denmark, Lyngby (1976).
		•••	59.	T. J. Procyk, Fuzzy Logic Working Group, Queen Mary
		·		College, London (1976).
		1	60.	N. Rescher, Plausible Reasoning (Van Gorcum, Amsterdam,
		-0		1976).
		· .	61.	C. Rieger, Proc. Georgetown University Linguistics
				Roundtable (1976).
		27	62.	R. D. Rosenkrantz, Inference, Method and Decision
۰.	. • 4 •			(D. Reidel, Dordrecht, 1977).
	:. .		63.	E. Sanchez, Fuzzy relations, University of Marseille
			CA	(1974).
		بر در م	65	E. Sanchez, M77/28, Univ. of Calif. Berkeley (1977).
			05.	R. C. Schank, Conceptual Information Processing (Academic Press, New York, 1975).
		• •	66	B. S. Sembi and E. H. Mamdani, Proc. 9th Int. Symp. on
		• •		Multiple-Valued Logic, 143 (1979).
		•	67.	G. Shafer, A Mathematical Theory of Evidence (Princeton
				University Press, Princeton, 1976).
			68.	J. E. Shore and R. W. Johnson, NRL Memo 3-898, Naval
				Research Laboratory (1978).
			69.	E. Shortliffe, MYCIN: Computer-Based Medical Consulta-
		•		tions (American Elsevier, New York, 1976).
			70.	E. H. Shortliffe and B. G. Buchanan, Math. Biosci. 23,
				351 (1975).
			71.	P. Smets, Un modele mathematico-statistique simulant le
				processus du diagnostic medicale, Free University of
· .				
•			1 44	M. Sugeno, Ph.D. Dissertation, Tokyo Institute of Technology (1974).
		تىرىد 1	74	P. Suppes, J. Roy. Statist. Soc. B 36, 160 (1974). P. Suppes and M. Zanotti, Synthese 36, 427 (1977).
			75.	R. G. Swinburne, An Introduction to Confirmation Theory
				(Methuen, London, 1973).
			76.	P. Szolovits, L. B. Hawkinson and W. A. Martin, LCS Memo,
		•		M.I.T. (1977).
		2	77.	A. Tarksi, Logic, Semantics, Metamathematics (Clarendon,
				Press, London, 1956).
			78.	T. Terano and M. Sugeno, in Fuzzy Sets and Their Applica-
		-		tions to Cognitive and Decision Processes, L. A. Zadeh,
		•		K. S. Fu, K. Tanaka, M. Shimura, Eds. (Academic Press,
			70	New York, 1975), p. 151.
			17.	R. M. Tong, Int. J. General Systems 4, 143 (1978).
			au.	M. Tribus, Rational Descriptions, Decisions and Designs
~		•		(Pergamon Press, New York, 1969).

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81. A. Tversky and D. Kahneman, Science 185, 1124 (1974). 82. S. Watanabe, Information and Control 15, 1 (1969). 83. H. Wechsler, Proc. Int. Symp. on Multiple-Valued Logic, 162 (1975). 84. F. Wenstop, Int. J. Man-Machine Studies 8, 293 (1976). 85. P. M. Williams, Bayesian conditionalization and the principle of minimum information, University of Sussex (1978). 86. L. A. Zadeh, J. Math. Anal. Appl. 23, 421 (1968). 87. L. A. Zadeh, J. Cybernetics 2, 4 (1972). 88. L. A. Zadeh, IEEE Trans. Systems, Man and Cybernetics SMC-3, 28 (1973). 89. L. A. Zadeh, Synthese 30, 407 (1975). 90. L. A. Zadeh, Information Sciences 8, 199 (1975); 8, 301 (1975); 9, 43 (1975). 91. L. A. Zadeh, Memo M77/58, Univ. of Calif. Berkeley (1977). 92. L. A. Zadeh, Fuzzy Sets and Systems 1, 3 (1978). 93. L. A. Zadeh, Int. J. Man-Machine Studies 10, 395 (1978). - 94. L. A. Zadeh, in Advances in Fuzzy Sets Theory and Applications, M. Gupta, R. Ragade, R. Yager, Eds. (North-Holland, Amsterdam, 1979). 95. L. A. Zadeh, M79/24, Univ. of Calif. Berkeley (1979). 96. L. A. Zadeh, K. S. Fu, K. Tanaka and M. Shimura, Fuzzy Sets and Their Application to Cognitive and Decision Processes (Academic Press, New York, 1975). 97. L. A. Zadeh, Int. J. Man-Machine Studies 8, 249 (1976). 98. L. A. Zadeh, M79/32, Univ. of Calif. Berkeley (1979).

10 Second State Control and the control of the decomposition of the decomposition of the state of the stat