COEFFICIENTS FOR STIELTJES' CONTINUED FRACTION
FOR THE GAMMA FUNCTION

by

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ABSTRACT

The first seventy-six coefficients of a continued fraction for $\ln\Gamma(z) + z - (z-1/2)\ln z - \ln\sqrt{2\pi}$, are given. The computation, based on Wall's algorithm for converting a function's power series representation to a continued fraction representation, was run on the algebraic manipulation system MACSYMA*.

1. Introduction

Recall Stirling's formula for the gamma function:

$$\ln \Gamma(z) = -z + (z-\frac{1}{2}) \ln z + \ln\sqrt{2\pi} + J(z)$$

where, for real$(z) > 0$,

$$J(z) = \frac{1}{\pi}\int_0^\infty \ln\frac{1}{1-e^{-2\pi u}} \cdot \frac{z}{z^2+u^2}\, du$$

Furthermore, asymptotically

$$J(z) = \sum_{p=0}^{\infty} (-1)^p \frac{c_p}{z^{2p+1}}, \qquad (1)$$

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where

$$c_p = \frac{B_{2p+2}}{(2p+1)(2p+2)}, \qquad p = 0, 1, 2, \cdots$$

and $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, $B_6 = \frac{1}{42}$, $\cdots$, are the Bernoulli numbers. Henrici [2] refers to $J(z)$ as the *Binet function*, and gives the details for the derivation of the above formulae.

Wall [6, pp. 192-202] gives an algorithm for constructing a continued fraction development of power series such as (1), which we summarize below:

Using the symbolic operation on polynomials of *formal integration* with respect to a variable u, and an infinite sequence of numbers $c_0, c_1, c_2, \cdots$, in which the i^{th} power of u is replaced by c_i:

$$\int (k_0 + k_1 u + \cdots + k_n u^n)\, d\varphi_c(u) \equiv k_0 c_0 + k_1 c_1 + \cdots k_n c_n ,$$

one computes a_i, $i = 0,1,2,\ldots$, of

$$\sum_{p=0}^{\infty} \frac{c_p}{z^{p+1}} = \cfrac{a_0}{z + \cfrac{a_1}{z + \cfrac{a_2}{z + \cdots}}}$$

by defining the auxiliary polynomials

$$q_{-1}(u) = 0 \qquad q_0(u) = c_0, \tag{2}$$

initializing

$$a_0 = c_0. \tag{3}$$

and using the recurrence for $p = 1,2,3,\ldots$:

$$q_p(u) = u q_{p-1}(u) - a_{p-1} q_{p-2}(u) , \tag{4}$$

$$a_p = \frac{\int u^p q_p(u)\, d\varphi_e(u)}{\prod_{i=0}^{p-1} a_i} , \tag{5}$$

where

$$e_p = \begin{cases} \text{if } p \text{ is even} & c_{\frac{p}{2}} \\ \text{if } p \text{ is odd} & 0 \end{cases} \qquad p = 1, 2, \cdots \tag{6}$$

Stieltjes, [5, pp. 520-521] gives the first five a_i for $J(z)$, noting that "Le calcul des $[a_i]$ est très pénible ... la loi de ces nombres paraît etrêmement compliqué." However, advances of the last decade in the power of algebraic manipulation languages and systems have made it easy to use the recurrence (2-6) as the basis for a computer program. The MACSYMA system [3,4] was used to compute the first forty-one a_i coefficients.

2. The coefficients

The first seven coefficients computed via MACSYMA agree with those given by Stieltjes, and by Wall [6, p.365]:

$$a_0 = \frac{1}{12} \quad a_1 = \frac{1}{30} \quad a_2 = \frac{53}{210} \quad a_3 = \frac{195}{371} \quad a_4 = \frac{22999}{22737}$$

$$a_5 = \frac{29944523}{19733142} \quad a_6 = \frac{109535241009}{48264275462}$$

The next few numbers in the sequence are:

$$a_7 = \frac{29404527905795295658}{97692142878531557 85}$$

$$a_8 = \frac{455377030420113432210116914702}{113084128923675014537885725485}$$

$$a_9 = \frac{26370812569397719001931992945645578779849}{5271244267917980801966553649147604697542}$$

$$a_{10} = \frac{152537496709054809881638897472985990866753853122697839}{24274291553105128438297398108902195365373879212227726}$$

$$a_{11} = \frac{100043420063777451042472529806266909090824649341814868347109676190691}{13346384670164266280033479022693768890138348905413621178450736182873}$$

Table 1 is a list of the a_i, $i=0, \cdots, 75$ rounded to 40 significant digits, computed from the exact rational coefficients using MACSYMA's "bigfloat" facilities (see [1]).

i	a_i
0	8.333333333333333333333333333333333333333B-2
1	3.333333333333333333333333333333333333333B-2
2	2.523809523809523809523809523809523809524B-1
3	5.256064690026954177897574123989218328841B-1
4	1.0115230681268417117473721247306152966 53B0
5	1.51747364915328739842849151949547618924 6B0
6	2.26948897420495996090915067220987758417 7B0
7	3.00991738325939817007314073420771572737 4B0
8	4.02688719234390122616887595318144283632 7B0
9	5.00276808075403005168850241227618857481 2B0
10	6.28391137081578218007286315495164726427 8B0
11	7.49591912238403392975235470826750546574 6B0
12	9.04066023436772669953113936026048174933 6B0
13	1.04893036545094822771883713045926295221 0B1
14	1.22971936103862058639894371400919176597 4B1
15	1.39828769539924301882597606512787300859 1B1
16	8.02909947194640583606548940791886623874 3B0
17	1.79766073998702775925694723076715543993 2B1
18	2.03097620274416537438054147204948968937 0B1
19	2.24704716399331324955179415715079221090 0B1
20	2.50658465489459720291634003225063053682 4B1
21	2.74644518250291336091755589826462226732 3B1
22	3.03218212316730471268825993064057869944 9B1
23	6.59170678599459759244651699720695252588 2B1
24	3.60776989312992426451533209008554523367 8B1
25	3.89527066823115557345443904104810462991 6B1
26	4.23334900435769572113818539488560973399 1B1
27	4.54469608500616210144241757375414510828 5B1
28	4.90892031290125977081648833502750872924 5B1
29	5.24412887514153373125698560469961084271 5B1
30	5.63448453453418435384203659474761135421 3B1
31	5.99356839071658582078525834927521121101 3B1
32	6.41004227559203545279066118922379177529 1B1
33	6.79301407880182211863677027451985358165 2B1
34	7.23559405552117019696800529632362179107 5B1
35	7.64246546268296897525850904222875264035 7B1
36	8.11114032372479654848142309856834609745 0B1
37	8.54192212764109726145856387173486827269 9B1
38	9.03668147238641085955135745816833777807 9B1
39	9.49138371000098879530762312919869274587 7B1
40	1.00122178463929197488990746834466834979 9B2
41	1.04908498885700660194790168477869060685 3B2
42	1.10377497511259522718618577574506205542 4B2
43	1.15403203778565004332604427852194656658 9B2
44	1.21132774586592246272656674300848605264 4B2
45	1.26397949226322100914791186933331368356 3B2

Table 1.

i	a_i
46	1.3238801212745777124745067023484310876 84B2
47	1.3789273293526790555462030709153772541 64B2
48	1.4414321232873779390474948140912495475 54B2
49	1.4988755283421789101553742881865647388 64B2
50	1.5639837717596460187103076754693773104 62B2
51	1.6238240704473507909517351861565508811 29B2
52	1.6915350847289936957098317088198800912 37B2
53	1.7537729385634627364221689930070747952 18B2
54	1.8240860786455476260152767220538529606 29B2
55	1.8887221170575546109739623689945883055 35B2
56	1.9616367685655539878456793127442155644 31B2
57	2.0286715915928879855049813352780968214 67B2
58	2.1041871683152878077037073065498319076 17B2
59	2.1736213489797373981774160361037681011 71B2
60	2.2517372906306894266947578895304062477 37B2
61	2.3235713770478166101286964369603677471 04B2
62	2.4042871472770125545645731488027721337 33B2
63	2.4785216645366000174047783003950119858 40B2
64	2.5618367491518982282482570820493841674 57B2
65	2.6384722010005432867246727264316545105 66B2
66	2.7243861063746172576789278222201378601 05B2
67	2.8034229767267850471613273819826506711 79B2
68	2.8919352283637005230990206062848452543 92B2
69	2.9733739826633640102480985942414377048 78B2
70	3.0644841239047655448600293071218890602 41B2
71	3.1483252103563431640407485753911538578 84B2
72	3.2420328012100224652078520125367867093 44B2
73	3.3282766518945168473647422043787911968 14B2
74	3.4245812679706832212428923436862964087 56B2
75	3.5132282998606041003930378676572649207 94B2

Table 1, continued.

Tables 2 and 3 list the coefficients for an alternative representation of $J(z)$

$$J(z) = \cfrac{z}{12z^2 + b_0 - \cfrac{c_1}{12z^2 + b_1 - \cfrac{c_2}{\ddots}}}$$

where

$$b_0 = .4$$

$$b_i = 12(a_{2i+1}+a_{2i}) \qquad c_i = 144(a_{2i}\, a_{2i-1}), \qquad i=1,2,3,\cdots$$

i	b_i
0	0.4000000000000000000000000000000000000B0
1	9.335849056603773584905660377358490566038B0
2	3.034796060736154932211036373071309783078B1
3	6.335287628957229757178749687701111973860B1
4	1.083558632771751753342885403854915769337B2
5	1.653579659183977933179002143586298327603B2
6	2.343595666665265077206341279782373352572B2
7	3.153608467725436328269903734964477729477B2
8	4.083619057989025567474210640721337749320B2
9	5.133628040084974348718802755040338280324B2
10	6.303635804877012676600675116618303364976B2
11	7.593642619397524121625199890938145024174B2
12	9.003648673633295805563725357360379836312B2
13	1.053365410723662938709672356236770581073B3
14	1.218365902565135220248816872767254348635B3
15	1.395366351030092420955275393282738707827B3
16	1.584366762527262908571291775649237442833B3
17	1.785367142184496700667181720626284931772B3
18	1.998367494163907257192798436436385724418B3
19	2.223367821886487958583077670484043662875B3
20	2.460368128195558292205370983748030747982B3
21	2.709368415477894324614676065120410346415B3
22	2.970368685754972166249374334810159683448B3
23	3.243368940752708121624851727916570010218B3
24	3.528369181955468219043442922733377143701B3
25	3.825369410648396171594451433951113829909B3
26	4.134369627950947718558400842192345863746B3
27	4.455369834843722684387086909258129519396B3
28	4.788370032190130368020792777626774863078B3
29	5.133370220754030247057348011184320010546B3
30	5.490370401214207244188145191788928793809B3
31	5.859370574176335086363221739037340943488B3
32	6.240370740182929817967515770177246413627B3
33	6.633370899721682765808306245043346237540B3
34	7.038371053232477440016543040631539551124B3
35	7.455371201113330450680933459015651501750B3
36	7.884371343725447175087113060298693487390B3
37	8.325371481397544785963116253612273595459B3

Table 2.

i	c_i
1	1.2114285714285714285714285714285714285714285710B0
2	7.655948181402079586486848857799373997309B1
3	4.959201190175930198012731830988393121128B2
4	1.745366077535117757619059316742423241255B3
5	4.526920976861447519967723398164662704254B3
6	9.758600341887450877062551736971890545280B3
7	1.857441569437939445625405361351134960819B4
8	3.232437611535013974272020050544990982625B4
9	5.257448904270412865298085575428104525217B4
10	8.110676073729689939046113370665169803584B4
11	2.398384046644839173431363193227693132782B5
12	1.712258012653309827013301548743936282793B5
13	2.374565789528654189666941851377114595008B5
14	3.212575333581539453319148834075522970890B5
15	4.254906678348833346879164760370787860141B5
16	5.532339854324943018853742608913421289652B5
17	7.077814889421611826753306967832360241589B5
18	8.926431809334035617413846745442872391397B5
19	1.111545063783674123903065444955124887838B6
20	1.368429139702583147492112697282782107420B6
21	1.667453410751961172748906829281358663300B6
22	2.012991878862645875375037017012355252439B6
23	2.409634545848657555098756809056189019312B6
24	2.862187413419268980322101892522655018082B6
25	3.375672483189359776188841369995228453305B6
26	3.955327756688360085189128911967977864693B6
27	4.606607235368024141816075201672086629019B6
28	5.335180920609225728095720532581572424814B6
29	6.146934813727930076868092213441542397005B6
30	7.047970915980467745105382901781339712041B6
31	8.044607228568213201141129879386533699858B6
32	9.143377752641752842652268925368107443311B6
33	1.035103248930461277485546062600863256837B7
34	1.167453743961660510552887411681205064000B7
35	6.560537302299852639729957872052729330972B6
36	1.469804198522646017978971060723595695003B7
37	1.641305358245108845029552876209333653768B7
38	1.827393939718309328902428058553116170524B7

Table 3.

3. Details of the computation

The computation for a_0 through a_{40} was carried out on the MACSYMA Consortium Decsystem 10 (KL model) computer, located at the Massachusetts Institute of Technology's Laboratory for Computer Science in Cambridge, MA. The programming was done in the MACSYMA language [4], whose mathematical features allowed a few lines of code to completely specify the computation. The computation of the coefficients a_0 through a_{40}, as well as their 40 digit approximations, took approximately twenty minutes of central processing unit time (including the time spent on list "garbage collection"). The limiting constraint to continuing the computation is that the space requirements of the simple data representations used in the program exhaust available memory (approximately 1 million bytes) after the computation of a_{40}.

The computation for a_{41} through a_{75} was performed on a version of MACSYMA running on an MIT Lisp Machine [7], a computer designed for Lisp computations involving large amounts of memory. This part of the computation required approximately five hours of computation on the Lisp machine.

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Appendix 1

MACSYMA program for computing a_i coefficients

```
        /*The power series for Binet's function*/

c[p]:=if evenp(p) then abs(bern(p+2))/((p+1)*(p+2)) else 0$

        /*Wall's algorithm 51.5 for computing a continued J fraction
given c[i] from a powerseries.*/

        /*Auxiliary definitions*/

aiprod[n]:=a[n]*aiprod[n-1]$

phiop(qp,n):=qp.(makelist(c[n+i],i,0,length(qp)-1))$

        /*Initialize*/

a[0]:c[0]$

qpvec[-1]:[]$

qpvec[0]:[1]$

aiprod[0]:a[0]$


        /*Recursive definitions for b[n],qpvec[n],a[n]*/


qpvec[n]:=append([0],qpvec[n-1])
                -a[n-1]*append(qpvec[n-2],[0,0])$

a[n]:=phiop(qpvec[n],n)/aiprod[n-1]$

       /*Instructions to compute and print out 40 digits of a0 through a40*/

fpprec(40)$

for i:0 thru 40 do print (bfloat(a[i]));
```

Appendix 2

MACSYMA Output for a_0 through a_{11}

```
(C45) FOR I:0 THRU 25 DO LDISP('A[I]=A[I],BFLOAT(A[I]));
                                       1
(E45)                            A  =  --
                                  0    12

(E46)                       8.333333333333333333333333B-2

                                       1
(E47)                            A  =  --
                                  1    30

(E48)                       3.333333333333333333333333B-2

                                       53
(E49)                            A  =  ---
                                  2    210

(E50)                        2.52380952380952380952381B-1

                                       195
(E51)                            A  =  ---
                                  3    371

(E52)                       5.256064690026954177897574B-1

                                      22999
(E53)                           A  =  -----
                                 4    22737

(E54)                        1.011523068126841711747372B0

                                    29944523
(E55)                         A  =  --------
                               5    19733142

(E56)                        1.517473649153287398428492B0

                                  109535241009
(E57)                       A  =  ------------
                             6    48264275462

(E58)                        2.269488974204959960909151B0

                              29404527905795295658
(E59)                   A  =  --------------------
                         7    9769214287853155785

(E60)                        3.009917383259398170073141B0
```

(E61) $$A_8 = \frac{45537703042011343221011691470 2}{11308412892367501453788572548 5}$$

(E62) 4.026887192343901226168876B0

(E63) $$A_9 = \frac{26370812569397719001931992945645578779849}{5271244267917980801966553649147604697542}$$

(E64) 5.002768080754030051688503B0

(E65) $$A_{10} = \frac{1525374967090548098816388974729859908667538531226978 39}{242742915531051284382973981089021953653738792122277 26}$$

(E66) 6.283911370815782180072664B0

(E67) $$A_{11} =$$

$$\frac{1000434200637774510424725298062669090908246493418148683471096761906 91}{1334638467016426628003347902269376889013834890541362117845073618287 3}$$

(E68) 7.495919122384033929752355B0