LINGUISTIC DECISION ANALYSIS USING FUZZY SETS

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Contents

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1. Introduction

The application of fuzzy set theory to the problem of making a decision when only incomplete or uncertain information is available has been the subject of much research over the last decade (see Kickert[1] for a recent review). The basic premise behind the work is that there are situations where it is more natural to handle uncertainty by fuzzy set theory than by probability theory. Whilst we agree with this, we feel that most published work does not go far enough in its utilization of the theory.

In this paper we present a technique for fuzzy decision making that is based on linguistic approximation and truth qualification. The advantage of our approach is that it generates a linguistic assessment of the decision and thus makes explicit the subjective nature of any choice made using fuzzy information.

Our original motivation for this study came from a paper by Watson et al.[2] in which a single-stage binary-choice decision problem is analyzed using expected utility theory. By allowing the probabilities and utilities to be fuzzy numbers, Watson et al. show how the problem may be made more "realistic." However, this fuzzification makes the final decision less clear cut and some further procedure is required to resolve the ambiguity.

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In the next section we shall examine the solution proposed by Watson et al. and argue that any method that generates a numerical assessment of the decision is inconsistent with the original desire to introduce fuzziness. We show that our technique produces a more satisfactory statement of the decision.

In section three we extend our technique so that we can solve any single-stage multi-choice decision problem. We do not restrict ourselves to problems arising from an expected utility approach, but require only that each alternative has associated with it a fuzzy set of "suitability." At the end of the section we consider a small class of multi-stage decision problems.

Few authors mention the numerical aspects of computer implementation, so in section four we discuss the main details of our technique. Essentially, we are concerned with the manipulation of fuzzy numbers and the tools needed in linguistic approximation.

We present two examples in section five which illustrate our concepts. We analyze in considerable detail the problem given in Watson et al. and a problem concerned with an investment decision.

Finally, in section six we consider some aspects of a user interface with the aim of showing how a computer aided decision package might be developed.

2. Single-stage binary-choice decisions

Consider the decision problem shown diagramatically in Figure 1. This simple binary decision tree depicts the problem of choosing between actions A and B. If we choose A, then there is a probability p of an outcome with utility U_1 and a probability $1-p$ of an outcome with utility U_2.

Conversely, if we choose B, then there is a probability q of an outcome with utility U_3 and a probability $1-q$ of an outcome with utility U_4.

Using the theory of expected utility, we should choose A in preference to B if and only if

$$pU_1 + (1-p)U_2 > qU_3 + (1-q)U_4$$

If p, q, U_1, U_2, U_3 and U_4 are know precisely then the choice is clear. If however some, or all, of them are only imprecisely known then the choice becomes "fuzzy." We can formalize this imprecision by assigning fuzzy numbers† to the probabilities and utilities. As a result, each side of the inequality may be evaluated to get the fuzzy expected utility of A, μ_A, and the fuzzy expected utility of B, μ_B.

Depending upon the choice of fuzzy numbers, the result is likely to be two overlapping fuzzy sets, perhaps as shown in Figure 2. The problem is how to select between A and B on the basis of this fuzzy information. Although the peak in μ_A is to the right of (i.e., at a higher utility than) the peak in μ_B, the long tail in μ_A indicates that there is some possibility that B might be preferred to A.

It is clear that we need a fuzzy assertion about the choice and in the remainder of this section we shall consider ways in which this might be generated.

2.1 The method proposed by Watson et al

Let us examine the method proposed by Watson et al. The first step in their procedure is to form the cartesian product of the expected utilities, $\mu_{A\times B}$, so that

†Strictly, we shall only be concerned with convex normal fuzzy numbers. See section 4.2 for definitions of these terms.

$$\mu_{A\times B}(a,b) = \mu_A(a) \wedge \mu_B(b)$$

where the symbol $\wedge$ stands for the minimum operation. Next, they specify a set of binary fuzzy relations, $\{R_i\}$, which express the preference for A over B, or vice versa. One example being, "A is somewhat preferred to B" which is defined by

$$\begin{aligned} \mu_R(a,b) &= 1 & a &\geq b + 0.2 \\ &= 0.5 + 2.5(a+b) & b + 0.2 &\geq a \geq b - 0.2 \\ &= 0 & a &\leq b - 0.2 \end{aligned}$$

Then they compute a "degree of support," δ_w, for each of the R_i given $A \times B$. This is done by extending the truth table for implication in the two-valued logic (i.e. $X \supset Y = \neg X \cup Y$) to fuzzy sets. The result is

$$\delta_w = \bigwedge_{a,b} [(1-\mu_{A\times B}(a,b)) \vee \mu_R(a,b)]$$

where the term in square brackets is the fuzzy implication, and where the symbol $\vee$ stands for the maximum operation.

Whilst this is quite satisfactory, we feel that this is a rather artificial way of solving a fuzzy decision problem. Far from generating a "fuzzy statement about the extent to which A is preferred to B" (Watson et al. pp. 3-4), the method simply produces a numerical ranking of some pre-specified preference relations.

This criticism applies to most fuzzy decision techniques which have appeared in the literature. The result of any fuzzy decision procedure should surely be a linguistic, rather than a numerical, statement of the choice. Not only would this be an intuitively more reasonable basis for making the decision, but it would also be in accord with the original motivation for introducing fuzziness into the problem.

2.2 Truth qualification and linguisitc approximation

Let us reconsider the problem. Given some fuzzy information about the utility of A and B we wish to choose one of them and to qualify our choice in some linguistic manner. There are several ways in which we might do this. The technique we discuss here is efficient and relies on ideas in truth qualification and linguisitc approximation.

The basic concept is that we shall express our preference for A over B, or B over A, as a truth qualified proposition in the manner suggested by Zadeh.[3] Thus our decision will have the form "A is marginally preferred to B is very true," "B is definitely preferred to A is moreorless true," etc. These should be viewed as the fullest statements we can make about our choice. They have three elements. Firstly, there is the basic binary decision of A over B, or B over A. Secondly, there is the strength of this preference; marginally preferred, definitely preferred, etc. Then thirdly, there is some qualification in terms of degree of truth.

The first step in arriving at our decision is to compute the "difference" between A and B. We do this by forming a fuzzy set, Z, defined by

$$\mu_Z(u) = \bigvee_{\substack{a,b \\ \text{s.t.: } a-b=u}} [\mu_A(a) \wedge \mu_B(b)]$$

which is simply the extension principle (see section 4.1 for a definition) applied to algebraic difference. We note in passing, that there are some interesting computational problems associated with calculating μ_Z. A more detailed discussion is left until section 4.

The fuzzy set Z is thus a measure of the preference we have for one of the two alternatives. If it is "positive" we favor A. If it is "negative" we favor B. Depending upon the problem, Z might be similar to any of the typical fuzzy sets shown in Figure 3.

Intuitively, the difference between Z_3 and Z_4 is that in the former we only slightly prefer A to B whilst in the latter we strongly prefer A to B. Similarly, Z_1 indicates a strong preference for B. The situation represented by Z_2 is one in which we have no preference, that is, we are indifferent to the choice.

This kind of reasoning clearly suggests that the first two levels of our decision are determined by the "sign" and "magnitude" of Z. Thus the second step in the decision process is to define some primary fuzzy sets of strength of preference to which we will give labels such as "indifferent" or "marginally preferred."

The penultimate step is to define what we mean by "true," "moreorless true," etc. Following Zadeh[3] we shall do this by assigning fuzzy sets to each of these labels. In particular, "true" will be the fuzzy set shown in Figure 4.

At first sight, this definition of "true" may seem counter-intuitive. However, it arises quite naturally from the way in which we translate our decision statements. Notice that fuzzy sets of truth are defined on the closed interval [0,1] of truth values.

Zadeh's meaning representation language PRUF allows truth qualified propositions of the form A is P to B is τ where P is our preference and where τ is the linguistic truth value. Such propositions are translated into a fuzzy set with the same meaning, a concept which is called "semantic equivalence." The equivalent set, denoted L, is defined by

$$\mu_L(u) = \mu_\tau(\mu_P(u))$$

So that if τ is the "true" set we get $\mu_L(u) = \mu_P(u)$ $\forall u$. This means that qualifying a statement with the truth value "true" leaves its meaning unchanged.

We have now fixed what we mean by statements such as "A is marginally preferred to B is very true"; we mean the fuzzy set L that is semantically equivalent. Since the universe of discourse of L is the same as that of Z, the final step in our decision process is to find an L which corresponds in meaning to Z.

To do this we shall utilize some recent work by Bonissone[4] on pattern recognition techniques applied to linguistic approximation. Section 4.3 of the paper gives some of the relevant technical details so we will limit ourselves to a sketch of the method.

Firstly, we generate a set of decision statements called the term set. This will contain statements similar to the ones already introduced but will also contain more complex ones like "A is from marginally preferred to strongly preferred to B is very true" where the construction "from P_1 to P_2" is a suitably defined fuzzy set (see appendix 2).

Then we extract those elements of the term set which correspond, in a broad way, to the Z computed for the current problem. The selection is made not by comparing membership functions, but by comparing a small number of "orthogonal features" which characterize them. This is very efficient and greatly simplifies the task.

Finally, we select from amongst this subset the statement whose membership function most closely matches the membership function for Z. This statement is the linguistic solution to the decision problem.

2.3 Other approaches

There are, in fact, several other ways in which we might produce linguistic decisions. We outline two which we feel are most promising.

The first of these is a rule based procedure. For this we need to construct a rule-set which exemplifies the conditions under which we would choose A over B, or B over A. Each rule would be an implication statement of the form $X,Y \supset P$ is τ. Then, with a suitable choice for the definition of fuzzy implication and for compositional inference, we have a fuzzy modus ponens which generates a conclusion L from the given A and B. Linguistic approximation is then used to label L.

The second method is essentially a variation of the method described in section 2.2. Instead of translating "A is P to B is τ" into L, we compute an unlabelled truth set, T, perhaps by using the extension principle. That is, we have something of the form

$$\mu_T(t) = f(\mu_P(u),\mu_Z(u))$$

We then use linguistic approximation to label T from a suitably defined term set of truth values.

3.0 Multi-choice decisions

The technique we have just described may be regarded as a solution to the 'core' decision problem of choice between two alternatives. In this section we develop an extended technique which allows us to make multi-choice, and also multi-criteria, decisions.

3.1 Single-stage decisions

The basic problem is to select from a set of alternatives, $\{a_i: i=1,\ldots,m\}$, given some fuzzy information about the "suitability" of each of them. We shall assume that this information is given by a set of fuzzy sets, $\{S_i: i=1,\ldots,m\}$, where each of the S_i is defined by a membership function which maps the real line onto the closed interval $[0,1]$.

We interpret suitability as a measure of the ability of an alternative to meet our decision criteria. In general, the S_i will be computed from some basic structural relationships defined by the problem and are simply an aggregation of all the relevant information about each alternative. Thus for the binary-choice problem described in section 2, we would have two suitability sets, S_1 and S_2, which correspond to the fuzzy expected utility of A and the fuzzy expected utility of B. (In section 5 we give an example where the S_i are computed from a linguistic assessment of the alternatives with respect to four criteria.)

Given this statement of the problem, we have to rank the a_i on the basis of the S_i and then generate a linguistic statement about our choice. To help illustrate the technique consider the simple example of choosing from three alternatives given s_1, s_2 and s_3 as shown in Figure 5.

It is not clear which alternative we should choose here, but intuitively we would prefer either a_2 or a_3. To help us rank the a_i we shall introduce the concept of "dominance." This is closely related to Zadeh's[5] definition of separation between fuzzy sets.

The separation, σ, between two convex fuzzy sets A and B is given by

$$\sigma(A,B) = 1 - \bigvee_x(\mu_A(x) \wedge \mu_B(x))$$

We now define the dominance, δ, of A over B by

$$\delta(A,B) = \bigvee_x (\mu_{\leq A}(x) \wedge \mu_B(x))$$

where $\leq A$ is the fuzzy set "less than or equal A" formed from A by setting

$$\mu_{\leq A}(x) = 1.0 \qquad x < x^*$$
$$= \mu_A(x) \qquad x \geq x^*$$

with x^* being the leftmost value (i.e. lowest value) of x for which $\mu_A(x) = 1.0$. This is illustrated in Figure 6. Clearly, when the peak in μ_A is to the left of, or coincides with, the peak in μ_B we have $\delta(A,B) = 1 - \sigma(A,B)$. (Notice that whilst $\delta(A,A) = 1$, $\delta(A,B) \neq \delta(B,A)$ in general.)

This is not the only definition of dominance we could have used. Nonetheless, it does reflect our interest in the behavior, with respect to A, of the left slope of B. It is important to realize that δ gives us no information about the overall shape of B. Thus the set B' in Figure 6 is such that $\delta(A,B) = \delta(A,B')$ and yet we might easily prefer B' to B.

Our definition allows us to construct a dominance relation, R_δ, between the S_i. For the example above this would be

$$R_\delta = \begin{bmatrix} 1.0 & 0.8 & 0.6 \\ 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 \end{bmatrix}$$

where $R_\delta(i,j)$ is the dominance of S_i over S_j.

We see that there are two rows with all 1.0's which indicates that S_2 completely dominates S_1 and S_3, and that S_3 completely dominates S_1 and S_2. This confirms our intuition about the choice being between alternatives a_2 and a_3. However, inspection of R_δ shows that the dominance of S_1 over S_2 (= 0.8) is greater than the dominance of S_1 over S_3 (= 0.6), and we interpret this to mean that a_3 should be preferred to a_2. In a more complex situation this might not be so obvious and we need to formalize the idea.

Basically what we do is form a difference fuzzy set, Z, analogous to the one formed in solving the binary-choice problem. It performs two functions. Firstly, it aggregates the information contained in R_δ and secondly, it forms a target set for labelling using the linguisitc approximation techniques.

The actual procedure is as follows. We define a vector of weights, W, such that $W(i) = \bigwedge_j R_\delta(i,j)$. These represent the overall degree to which each of the alternatives is dominated by the others.† Then, for each element, k, of W which equals 1.0, we define Z as the fuzzy set induced by the difference function, g, where g itself is given by

$$g(x_1,\ldots,x_m) = W(k)x_k - \sum_{\substack{i=1\\i\neq k}}^{m} W(i)x_i \Big/ \sum_{\substack{i=1\\i\neq k}}^{n} W(i)$$

Thus when the x_i are replaced by the S_i we get

$$\mu_Z(u) = \bigvee_{\substack{(x_1,\ldots,x_m)\\ \text{s.t. } g(x_1,\ldots,x_m)=u}} [\bigwedge_i^m \mu_{S_i}(x_i)]$$

Notice that when m = 2, g has exactly the same form as the difference function of section 2.

†It is interesting to note the correspondence between W and the vector of preferences, I, derived by Baas and Kawkernaak[6] in their recent paper on fuzzy decision making. Some simple manipulations will easily show that W and I are the same; we leave the demonstration to appendix 1.

As before, we use some suitable technique to compute μ_Z given the μ_{S_i} and then generate a linguistic decision of the form

"a_k is P to all-other-alternatives is τ"

where P is the strength of our preference and τ is the truth qualification.

Of course, when there is more than one dominant alternative we shall have more than one decision. However, if one of these has a higher strength of preference we would, in general, prefer it to the others. This issue is not without ambiguity, though, as we shall see in our second example.

3.2 Multi-stage decisions

The solution to the multi-choice decision problem that we have just given allows us to solve a class of multi-stage decision problems. Specifically, we can address those problems for which there are a finite number of outcomes at each stage in the decision sequence.

Our general procedure is to adopt a dynamic programming approach. That is, we solve all the possible final stage decision problems and then solve all the penultimate ones using the suitability information associated with the actions selected at the final stage. This process is repeated until we have solved all the problems of every stage and have thus formed a complete solution to the multi-stage problem.

There are obviously only a limited number of problems that can be solved efficiently in this way. We do believe, however, that our technique allows us to handle some significant practical problems.

4.0 Numerical and computational considerations

Our discussion so far has concentrated on the theoretical aspects of the division problem. Equally important are the numerical techniques employed to implement the theory. In this section of the paper we shall consider some of the computational aspects of constructing fuzzy sets induced by mappings. We shall also describe, in detail, the linguistic approximation technique we have used.

4.1 Sampled sets and the extension principle

Central to the implementation of the decision techniques we have discussed in the previous sections is the extension principle (see Zadeh[7]). This allows any non-fuzzy function to be fuzzified in the sense that if the function arguments are made fuzzy sets, then the function value is also a fuzzy set whose membership function is uniquely specified.

More formally, if the scalar function, f, takes n arguments $(x_1, x_2, \ldots, x_n)$ denoted $\underline{x}$, and if the membership functions associated with each of these is given by $\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)$ then

$$\mu_{f(\underline{x})}(y) = \bigvee_{\substack{\underline{x} \\ \text{s.t. } f(\underline{x}) = y}} \left[\bigwedge_{i=1}^{n} \mu_i(x_i) \right]$$

Whilst this is straightforward, the linguistic approximation techniques that we use require the membership functions to be sampled. That is, the fuzzy sets should have finite discrete supports. This introduces some problems which are best illustrated by a simple example.

Suppose we wish to fuzzify $y = x_1 \cdot x_2$ and that $\mu_1(x_1)$ and $\mu_2(x_2)$ are as in Figure 7. If we sample $\mu_1(x_1)$ so that we get a finite discrete set, μ_1^*, such that

$$\mu_1^* = \{0.5/1.5,\ 1.0/2.0,\ 0.5/2.5\}$$

and if we sample $\mu_2(x_2)$ to give

$$\mu_2^* = \{0.5/2.0,\ 1.0/3.0,\ 0.5/4.0\} ,$$

then the result of using the extension principle is the finite discrete set shown in Figure 8.

There are several important features of this result. Firstly, there are eight points in the support of $\mu^*_{f(\cdot)}$, more than in either of μ_1^* or μ_2^*. Secondly, only three of the sampled membership values are actually correct (cf. the correct solution shown as a dashed line). Then thirdly, the spacing between the sampled points is no longer regular.

Increasing the number of samples in μ_1^* and μ_2^* is no help since we generate many more sample points in the result. In the worst case, we could have as many points in the support of the result as the product of the number of points in the support of each of the operands. Furthermore, we are not even guaranteed a more accurate representation.

Clearly, we need a more efficient and accurate representation of the fuzzy sets and in the next section we show how the work of Prade and Dubois[8] can be utilised for our problem.

4.2 Fuzzy numbers

A fuzzy number is simply a fuzzy subset of the real line and is completely defined by its membership function such that

$$\mu : \mathbb{R} \rightarrow [0,1] .$$

For our purposes, we further restrict this definition to those fuzzy numbers which are both normal and convex. Thus in addition to the above constraint we have

normality: $\bigvee_x \mu(x) = 1.0 \quad x \in \mathbb{R}$

convexity: $\mu(\lambda x_1 + (1-\lambda)x_2) \geq \mu(x_1) \wedge \mu(x_2) \quad \forall x_1, x_2 \in \mathbb{R}, \ \forall \lambda \in [0,1]$

All such numbers may be characterized by a 4-tuple (a,b,α,β) where, as shown in Figure 9, $[a,b]$ is the closed interval on which the membership function is equal to 1.0, α is the "left-bandwidth" and β is the "right-bandwidth." Notice that crisp numbers are easily represented in this form by writing $(a,a,0,0)$ and that non-fuzzy sets may be written $(a,b,0,0)$. If we now limit the shape of the left and right slopes of the membership function to be an even function, $S(\cdot)$, such that $S(-x) = S(x)$ and $S(0) = 1$, then, if $S(\cdot)$ is also monotonically decreasing on $[0,+\infty)$, the simple algebraic operations can be written as formulae involving the parameters in the 4-tuple.

So, if $\tilde{m} \triangleq (a,b,\alpha,\beta)_S$ and $\tilde{n} \triangleq (c,d,\gamma,\delta)_S$ with the understanding that the left slopes are given by $S(\frac{a-x}{\alpha})$ and $S(\frac{c-x}{\gamma})$, and the right slopes by $S(\frac{x-b}{\beta})$ and $S(\frac{x-d}{\delta})$, then Table 1 gives the formulae for addition, subtraction, multiplication and division.

Note that the last six formulae are only approximate in that the left and right bandwidths of the result are not exact. However, they introduce very little error and in practice have proved themselves to be of great value. (For a more detailed discussion of this point and others pertaining to fuzzy numbers see Bonissone[9].)

We now have a compact way of representing the kind of fuzzy sets in which we are interested. Furthermore, because we can perform algebraic operations with these representations we do not need to use the extension principle but can compute the output set induced by a mapping in a direct way.

Table 1: Basic operations with fuzzy numbers

Operation	Result	Conditions
$\tilde{m} + \tilde{n}$	$(a+c,\ b+d,\ \alpha+\gamma,\ \beta+\delta)$	all $\tilde{m}$, $\tilde{n}$
$\tilde{m} - \tilde{n}$	$(a-d,\ b-c,\ \alpha+\delta,\ \beta+\gamma)$	all $\tilde{m}$, $\tilde{n}$
$\tilde{m} \times \tilde{n}$	$(ac,\ bd,\ a\gamma+c\alpha-\alpha\gamma,\ b\delta+d\beta+\beta\delta)$	$\tilde{m} > 0$, $\tilde{n} > 0$
	$(ad,\ bc,\ d\alpha-a\delta+\alpha\delta,\ -b\gamma+c\beta-\beta\gamma)$	$\tilde{m} < 0$, $\tilde{n} > 0$
	$(bd,\ ac,\ -b\delta-d\beta-\beta\delta,\ -a\gamma-c\alpha+\alpha\gamma)$	$\tilde{m} < 0$, $\tilde{n} < 0$
$\tilde{m} \div \tilde{n}$	$(\frac{a}{d},\ \frac{b}{c},\ \frac{a\delta + d\alpha}{d(d+\delta)},\ \frac{b\gamma + c\beta}{c(c-\gamma)})$	$\tilde{m} > 0$, $\tilde{n} > 0$
	$(\frac{a}{c},\ \frac{b}{d},\ \frac{c\alpha - a\gamma}{c(c-\gamma)},\ \frac{d\beta - b\delta}{d(d+\delta)})$	$\tilde{m} < 0$, $\tilde{n} > 0$
	$(\frac{b}{c},\ \frac{a}{d},\ \frac{-b\gamma-c\beta}{c(c-\gamma)},\ \frac{-a\delta-d\alpha}{d(d+\delta)})$	$\tilde{m} < 0$, $\tilde{n} < 0$

Sampling needs only then to be performed on the output set, at which stage we can fix the number of sample points in accordance with our requirements.

4.3 Linguistic approximation

Given a fuzzy set Z which represents our preference for one action over others, our problem is to find a decision statement which, when interpreted as a fuzzy set, has the same meaning as Z.

The first step towards a solution is to generate a set of possible decision statements which we call the term set, $\mathcal{L}$. Each element in this set will be considered as a sentence in a synthetic language and, as such, can be generated using a context free grammar in the manner of Zadeh.[10] Thus if the grammar G is a 4-tuple (V_N,V_T,S,P) where V_N is the set of non-terminals, V_T is the set of terminals (the vocabulary), S is the starting symbol and P is the set of productions, then our choice of these will determine the size and form of the term set. Obviously, this will be problem dependent, but in general $\mathcal{L}$ should be large enough so that there is a wide choice of possible decisions. It should also be dense so that decisions can be reasonably precise and it should also be understandable.

Having formed $\mathcal{L}$ we need to search amongst its elements, L_i, for the one which is closest in meaning to the unlabelled Z. It is extremely inefficient to make pairwise comparisons for all L_i and so we use a technique commonly employed in pattern recognition. This consists of representing each of the L_i by a pattern, $\underline{P}_i$, of characteristic features, and then searching in this low order pattern space for possible decision statements.

A more formal description of this approach is as follows. Assume that the universe of discourse, U, on which the L_i are defined is finite and discrete and that $|U| = D$. Then, we define a function F such that

$$F: [0,1]^D \rightarrow \mathbb{R}^N \quad , \qquad N << D$$

which maps each L_i onto the N-dimensional pattern space P. Each element in P is a point (denoted by the vector $\underline{P}_i$) corresponding to the values of the characteristic features of L_i. Thus

$$F(\mu_{L_i}(x)) = \underline{P}_i \triangleq (p_i^1, p_i^2, \ldots, p_i^N) \; .$$

Note that whilst $\mu_{L_i}(x)$ fully characterises L_i, $\underline{P}_i$ is not a complete representation in P since it can happen that for some $L_i \neq L_j$, $\underline{P}_i = \underline{P}_j$.

The choice of the components of F is crucial, since the correct selection of features determines the success or failure of almost any pattern recognition process.

In choosing these, we try to have the minimum number consistent with a good representation of the original data. We in fact take note of four features which have proved themselves to be efficient in practice (see Bonissone[4] for a more detailed discussion of this point).

The first feature is the power of the fuzzy set. That is

$$p_i^1 = \sum_{k=1}^{D} \mu_{L_i}(x_k) \; .$$

The second is a measure of the fuzziness of the set. Using a definition proposed by Loo[12]

$$p_i^2 = \left(\sum_{k=1}^{D} H^2(\mu_{L_i}(x_k)) \right)^{0.5}$$

where $H(h) = h$, $0.0 \leq h < 0.5$, and $H(h) = 1 - h$, $0.5 \leq h \leq 1.0$. The third and fourth features are the first moment of the membership distribution of L_i (a measure of "center of gravity") and its skewness (a measure of asymmetry).

We need also to define a distance in P. Since it is a Euclidean space, we use a weighted Euclidean distance, d_1, defined by

$$d_1(\underline{P}_1,\underline{P}_2) = \{\sum_{i=1}^{N} W_i^2(p_1^i - p_2^i)^2\}^{1/2} .$$

The weights W_i play an important role since they allow different features to be emphasized. They are defined as $W_i = I_i/R_i$; R_i is the length of the range of values that p^i takes over all the points in the data set, and I_i is obtained from the user by means of pairwise comparison tests. (The reader should refer to Bonissone[11] for a more thorough discussion of the weights.)

The pattern space search thus consists of finding the set of L_i, denoted LA[Z], which satisfy $d(\underline{P}_i,\underline{P}_Z) < E$ where E is a parameter which defines our tolerance in judging the similarity.

This prescreening process thus yields a small set of decision statements which are close in meaning to Z. If we want a unique answer, we have to apply some further metric between each element of LA[Z] and the unlabelled Z.

The metric we use is a modified form of Bhattacharya distance that is defined by

$$d_2(L_i,Z) = [1 - R(L_i,Z)]^{1/2} , \quad L_i \in LA[Z]$$

where R is called the Bhattacharya coefficient. In the discrete case this is given by

$$R(L_i,Z) = \sum_{k=1}^{D} \left[\frac{\mu_{L_i}(x_k)\mu_Z(x_k)}{\text{power}(L_i)\text{power}(Z)}\right]^{1/2} .$$

Thus finally, the linguistic decision statement corresponding to Z is that L_i for which $d_2(L_i,Z)$ is smallest. We denote this L_Z.

5. Examples

We have chosen two examples to illustrate and further illuminate our ideas. The first is the single-stage binary-choice problem considered by Watson et al. We present several solutions to this so that the reader can get a feel for the main features of our technique. The second example is a single-stage multi-choice, and multi-criteria, investment decision problem. It is somewhat artificial, in the sense that it does not correspond in detail to a real-life situation, but it does show how such problems are naturally solved using the fuzzy approach.

5.1 The naval task force commander's problem

The commander of a small naval task force is cruising just off an enemy coastline. An aeroplane is approaching from the direction of the coastline and the commander has to decide whether or not to fire on the plane. Before he learns if it is a friend or an enemy, he has to decide if he should shoot at it with intent to kill. If it turns out to be an enemy and he either hasn't shot at it or has shot and missed, then there is the danger that the plane might bomb his task force and score a hit.

The problem he faces is represented by the decision tree shown in Figure 10. There are four probabilities associated with this problem, namely

p_1 - the probability that the plane will be shot down (killed) if shot at. This is the same whether it is an enemy or a friend.

p_2 - the probability that the plane is friendly.

p_3 - the probability that an enemy plane would score a hit on the task force after being shot at but not killed.

p_4 - the probability that an enemy plane would score a hit on the task force if it has not been shot at.

These lead to eight possible outcomes, each of which has a different utility. The expected utility of shooting is therefore

$$A = p_2(U_1p_1 + (1-p_1)U_2) + (1-p_2)(U_3p_1 + (1-p_1)(U_4p_3 + U_5(1-p_3)))$$

and the expected utility of not shooting is

$$B = U_6p_2 + (1-p_2)(U_7p_4 + U_8(1-p_4)) \ .$$

The difference function is thus given by $Z = A - B$. If $Z > 0$ then the commander should prefer shooting and if $Z < 0$ then he should prefer not shooting.

Obviously, if the probabilities and utilities are known exactly then the problem is straightforward. If some, or all, of them are only imprecisely known, then we have a fuzzy decision problem. We shall consider five combinations of probability and utility.

Case 1. We assume that the p's and U's are non-fuzzy. The utility values are just those given by Watson et al. and the probability values are the mid-point values of their fuzzy probabilities. Thus

$p_1 = 0.7$	$p_2 = 0.8$	$p_3 = 0.2$	$p_4 = 0.35$
$U_1 = -10.0$	$U_2 = -3.3$	$U_3 = 5.8$	$U_4 = -26.9$
$U_5 = -1.2$	$U_6 = 0.0$	$U_7 = -28.8$	$U_8 = -2.5$

Using these values in the corresponding formulae, we calculate that $Z = -3.6194$. As expected, this is consistent with the solution given by Watson et al., which is that the commander should not shoot at the plane.

Case 2. Here we leave the utility values as in Case 1, but make the probabilities fuzzy. Figure 11 shows the membership distributions. Note that they differ only very slightly from those given by Watson et al. In fact, we have used linear $S(\cdot)$ type fuzzy numbers to simplify the computations.

The resulting fuzzy Z is shown in Figure 12 together with the difference membership distribution, Z_W, computed by Watson et al. Also shown are the corresponding linguistic approximations L_Z and L_W. This clearly illustrates the way in which the linguistic approach to decision making emphasizes basic similarities rather than numerical differences. Indeed, the only difference between L_Z and L_W is in the truth qualification. Recalling that the standard form of the decision is "A is P to B is τ," then L_Z gives

A: not shooting

P: from marginally better than to indifferent with

B: shooting

τ: true

and L_W gives

A: not shooting

P: from marginally better than to indifferent with

B: shooting

τ: very true

Case 3. We shall now see what happens when both the probabilities and utilities are fuzzy. The membership distributions are shown in Figures 13a and 13b. Whilst these are close to those used in Case 2, there are some significant changes. Principally, these are an increase in the utility of killing an enemy plane and a less fuzzy probability that the plane is friendly.

The resulting Z and L_Z are shown in Figure 14. Notice that Z is much "sharper" now (a result of changes in p_2) and that L_Z is correspondingly less fuzzy. The decision is now

A: not shooting

P: between marginally better than and indifferent with

B: shooting

τ: moreorless true

This is as expected. Our decision whilst still favoring not shooting has a sharper but lower strength.

Case 4. The only difference between this case and Case 3 is that p_2 (the probability of the plane being friendly) is now as shown in Figure 15. This reduction in value has an interesting effect as we see in Figure 16. Here Z is almost symmetrical about zero and so L_Z corresponds to

A: not shooting

P: between marginally better than and marginally worse than

B: shooting

τ: very true

Or, in other words, we are indifferent to the two possible courses of action.

Case 5. If we make a further reduction in p_2 so that it is as shown in Figure 17, then Z and L_Z are as shown in Figure 18. The decision now becomes:

A: shooting

P: from indifferent with to marginally better than

B: not shooting

τ: very true

Thus we have a reversal of our original decision brought about simply by reducing the value of p_2. This is not too surprising since an analysis of the formulae for the expected utilities of A and B shows them to be primarily dependent on the probability that the plane is friendly.

In concluding this example we would like to re-emphasize the main advantage of our approach which is that decisions are linguistic rather than numerical. The technique is thus genuinely a fuzzy decision procedure, at least at the highest level, and because of this, small numerical changes in the definition of primary sets and errors introduced in computing Z are relatively unimportant.

5.2 An investment decision problem

A private citizen has a moderately large amount of capital which he wishes to invest to his best advantage. He has selected five possible investment areas, $\{a_1, a_2, a_3, a_4, a_5\}$, and has four criteria, $\{c_1, c_2, c_3, c_4\}$, by which to judge them. These are

a_1 - the commodity market

a_2 - the stock market

a_3 - gold and/or diamonds

a_4 - real estate

a_5 - long term bonds

and

c_1 - the risk of losing the capital sum

c_2 - the vulnerability of the capital sum to modification by inflation

c_3 - the amount of interest received

c_4 - the cash realisability of the capital sum.

His rating of the alternatives with respect to the criteria $\{r_{ij}: i=1,\ldots,5, j=1,\ldots,4\}$ is expressed linguistically as shown in Table 2. His problem is to select one of the a_i with the additional constraint that he does not consider the criteria to be equally important but gives them linguistic weights, $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ as shown in Table 3.

The first step in solving this problem is to compute a suitability set for each of the alternatives. Since the linguistic ratings and weights are appropriately defined fuzzy numbers (see Appendix 3), we just use a fuzzy weighted sum to give

$$S_i = \sum_{j=1}^{4} \alpha_j r_{ij} \; .$$

The results are as shown in Figure 19.

We then compute the dominance relation to be

$$R_\delta = \begin{bmatrix} 1.0 & 1.0 & .81 & .49 & 1.0 \\ 1.0 & 1.0 & .90 & .58 & 1.0 \\ 1.0 & 1.0 & 1.0 & .85 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ .77 & .79 & .63 & .34 & 1.0 \end{bmatrix}$$

Table 2

		Criteria			
		c_1	c_2	c_3	c_4
Alternatives	a_1	high	moreorless high	very high	fair
	a_2	fair	fair	fair	moreorless good
	a_3	low	from fair to moreorless low	fair	good
	a_4	low	very low	moreorless high	bad
	a_5	very low	high	moreorless low	very good

Table 3

α_1	α_2	α_3	α_4
moderately important	moreorless important	very important	moreorless unimportant

and we see that S_4 (real estate) dominates all the others. The weights are thus [.49 .58 .85 1.0 .34] and the corresponding Z and L_Z are as shown in Figure 20. Recalling that the standard form of decision is "a_k is P to all-other-alternatives is τ" then we have

a_k: real estate

P: from indifferent with to marginally better than

τ: moreorless true

It is interesting to compare this linguistic statement with the dominance weights derived from R_δ. In some sense, the W(i) over-emphasize the differences making it seem that real estate is a clear cut choice. The linguistic statement, however, is rather more cautious. It picks out real estate, but makes us realize that the preference is only marginal.

This characteristic feature of the linguistic approach is, we feel, particularly valuable. In any decision problem where there is uncertainty in the data, there must be uncertainty in the decision itself. Obviously, one of the alternatives has to be selected, but we should be aware of the consequences of fuzzifying the problem.

Let us pursue this discussion by slightly modifying the problem. Suppose the investor changes his assessment of the importance of c_4. Instead of being simply "moreorless unimportant" he feels it is "from moreorless unimportant to moderately important." This gives a new set of suitability sets (see Figure 21) and a new dominance relation

$$R_\delta = \begin{bmatrix} 1.0 & 1.0 & .91 & .58 & 1.0 \\ 1.0 & 1.0 & .99 & .72 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & .96 & .81 & .51 & 1.0 \end{bmatrix}$$

The corresponding dominance weights are

$$W = [.58\ .72\ 1.0\ 1.0\ .51]$$

and we see that two alternatives, a_3 and a_4, completely dominate the others. This means that we need to form two difference sets; one for real estate, Z_4, and one for gold/diamonds, Z_3. In fact these are sufficiently similar that they have the same L_Z (see Figure 22). Thus our decision is that the investor is now indifferent to real estate and gold/diamonds and that

a_k: both real estate and gold/diamonds

P: from indifferent with to marginally better than

τ: moreorless true

This is a very interesting result. Inspection of Figure 21 might have led us to believe that real estate would be preferred. However, we see that, once again, the linguistic approach emphasizes basic similarities rather than numerical differences.

If we wish to choose between real estate and gold/diamonds we have to apply other criteria to Z_3 and Z_4 (or S_3 and S_4) such as selecting the one with least fuzziness or the one with the highest first moment. Alternatively, we might treat the problem as a single-stage binary-choice problem and apply the methods of Section 2.

A better approach would be to assess the relative dominance of a_3 and a_4 with respect to the non-dominant alternatives. By that we mean that two sub-problems should be considered; the dominance of a_3 over a_1, a_2 and a_5, and the dominance of a_4 over a_1, a_2 and a_5. Thus we form a dominance relation for each of these reduced problems by striking out the appropriate row and column in R_δ. This gives

R_R = relative dominance of real estate

$$= \begin{bmatrix} 1.0 & 1.0 & .58 & 1.0 \\ 1.0 & 1.0 & .72 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & .96 & .51 & 1.0 \end{bmatrix}$$

and

R_G = relative dominance of gold/diamonds

$$= \begin{bmatrix} 1.0 & 1.0 & .91 & 1.0 \\ 1.0 & 1.0 & .99 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & .96 & .81 & 1.0 \end{bmatrix}$$

The corresponding dominance weights are

$$W_R = [.58\ .72\ 1.0\ .51]$$

and

$$W_G = [.91\ .99\ 1.0\ .81]$$

If we now define two fuzzy differences

$$Z_R = S_4 - \left\{\frac{W_R(1)S_1 + W_R(2)S_2 + W_R(4)S_5}{W_R(1) + W_R(2) + W_R(4)}\right\}$$

and

$$Z_G = S_3 - \left\{\frac{W_G(1)S_1 + W_G(2)S_2 + W_G(4)S_5}{W_G(1) + W_G(2) + W_G(4)}\right\}$$

then we get a substantially different picture of the preference for real estate and gold/diamonds. Figure 23 shows Z_R with L_R which corresponds to

a_k: real estate

P: between indifferent with and marginally better than

τ: moreorless true

and Figure 24 shows Z_G and L_G which correspond to

a_k: gold/diamonds

P: between marginally worse than and marginally better than

τ: true

Obviously, there is now a much stronger decision in favor of real estate.

Even though the difference is clear in this example, we should observe that, in general, this is not necessarily a straightforward problem. It might easily have been the case that L_R and L_G differed in only the smallest detail; the difference between moreorless true and true for example. In such a case we might not be justified in unequivocally choosing the decision with the highest truth value.

To summarize, it does seem that if there are two, or more, dominant alternatives, then we should form a deicsion for each of them without attempting to say which is best. We may well perform additional comparisons but we should certainly not insist that one alternative is to be preferred.

6. Some aspects of a user interface

In this section we want to indicate how the techniques we have been describing might be incorporated in a computer aided design package. At several points in our discussion we have emphasized the subjective, user oriented, nature of fuzzy decision making. Indeed, the techniques depend upon so much user generated information that the obvious way to use them is in an interactive computer program. This should have a modular structure, perhaps like the one shown in Figure 25.

At each stage the user would be interacting with the program providing it with information about the problem. Some indication of the interaction is shown to the right of the figure.

Clearly, an essential part of any such package would be a device for graphical input and output. So too would be an internal structuring of the decision so that the user could discover, in whatever detail required, exactly how the decision was made. Implementation would require a language in which fuzzy sets can be manipulated easily. The computation required by the examples in this report was performed using APL which seems particularly suited to this kind of work.

7. Conclusions

The basic conclusions to be drawn from this study have already been outlined. To summarise, we firmly believe that there are situations in which fuzzy sets are an appropriate way of representing uncertainty. It seems to us self evident that in such cases any decisions taken are inherently fuzzy and it is clearly not appropriate to give the final choice some artificial precision. That is, decisions should be linguistic rather than numerical.

In developing a technique which generates linguistic decisions we have drawn heavily on the notions of linguistic approximation, pattern recognition and fuzzy numbers. This has meant that our procedures are complex, though not complicated, and necessitate the use of a digital computer.

However, the result is an easily used tool for structuring and solving fuzzy decision problems. The decision maker can interact with the decision procedure at every level and, as a consequence, we feel that our method is of direct practical benefit.

Acknowledgments

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Appendix 1: Demonstration of the equivalence of W and I

In their paper on fuzzy decision making Baas and Kwakernaack[6] develop a fuzzy set, I, which characterises the extent to which alternative a_i is the best alternative. In this appendix we shall show that when the suitability sets are convex normal fuzzy numbers I and W are equivalent.

Given the suitability sets $\{S_i : i = 1,\ldots,m\}$ Bass and Kwakernaack form the cartesian product set, P, such that

$$(1) \qquad \mu_P(x_1,\ldots,x_m) = \bigwedge_{i=1}^{m} \mu_{S_i}(x_i) \ .$$

Then they define a conditional fuzzy set $I|P$ such that

$$(2) \qquad \mu_{I|P}(i|x_1,\ldots,x_m) = 1 \quad \text{if} \quad x_i \geq x_j \quad \forall j \neq i$$
$$= 0 \quad \text{otherwise}$$

Then finally, they form I by

$$(3) \qquad \mu_I(i) = \bigvee_{x_1,\ldots,x_m} [\mu_{I|P}(i|x_1,\ldots,x_m) \wedge \mu_P(x_1,\ldots,x_m)] \ .$$

We should note two important aspects of these equations. Firstly, equation (2) simply defines a sequence of subspaces in $\mathbb{R}^m$, $\{R_i : i = 1,\ldots,m\}$, each of which is bounded by a set of m-1 constraints, $x_i \geq x_j \ \forall j \neq i$. That is, R_i is the subspace in which x_i is the largest element. Secondly, equation (3) can then be seen as a way of computing, for each i, a match between P and the corresponding subspace.

Now let us consider two cases: $\mu_I(i) = 1$ and $\mu_I(i) \neq 1$. In the former, we see from equation (3) that $\mu_I(i) = 1$ if and only if some, or all, of the peak in P is within subspace R_i. That is, there is a value

$x_1,\ldots,x_m$ which satisfies the constraints and which makes $\mu_P(x_1,\ldots,x_m) = 1.0$. This implies that the peak in S_i lies to the right of (i.e. is at a higher value than) the peaks in the other suitability sets. This is pictured in Figure A1. Since the notion of dominance is defined by

$$\mu_{R_\delta}(i,j) = \bigvee_x (\mu_{\leq S_i}(x) \wedge \mu_{S_j}(x)) \; .$$

[Note the change in notation from Section 3. We have written $R_\delta(i,j)$ as $\mu_{R_\delta}(i,j)$ to aid the discussion; we shall also write $W(i)$ as $\mu_W(i)$.] It is clear that in this case

$$\mu_{R_\delta}(i,j) = 1.0 \quad \forall j$$

which means that

$$\mu_W(i) = \bigwedge_{j=1}^{m} \mu_{R_\delta}(i,j)$$
$$= 1.0$$

Now consider the case when $\mu_I(i) \neq 1$. This means that the peak in P is not within subspace R_i (i.e. there is no value $x_1,\ldots,x_m$ which satisfies the constraints and makes $\mu_P(x_1,\ldots,x_m) = 1.0$). Since $\mu_P(x_1,\ldots,x_m)$ is a convex normal fuzzy set, we see that μ_P is non-decreasing in R_i. Its maximum value must, therefore, be attained on a boundary of R_i.

The boundary will be defined by those original constraints which are active. These will depend on the actual S_i, but for the case shown in Figure A2, all constraints of the form

$$x_k \leq x_i \; , \quad k > i$$

are active and all those of the form

$$x_j \leq x_i \; , \quad j < i$$

are inactive. Therefore, equation (3) can be rewritten as

(4)
$$\mu_I(i) = \bigvee_{\substack{x_1 \le x_i \\ \vdots \\ x_{i-1} \le x_i \\ x_i = x_{i+1} = \cdots = x_m}} [\mu_P(x_1,\ldots,x_m)]$$

If we also rewrite equation (1) as

(5)
$$\mu_P(x_1,\ldots,x_m) = \left(\bigwedge_{j=1}^{i-1} \mu_{S_j}(x_j)\right) \wedge \left(\bigwedge_{j=i}^{m} \mu_{S_j}(x_j)\right)$$
$$= \mu_A(x_1,\ldots,x_{i-1}) \wedge \mu_B(x_i,\ldots,x_m) \ ,$$

then, because there will be a value of $x_1,\ldots,x_{i-1}$ which satisfies the inactive constraints and makes $\mu_A(x_1,\ldots,x_{i-1}) = 1.0$, equation (4) simply becomes

$$\mu_I(i) = \bigvee_{x_i = x_{i+1} = \cdots = x_m} [\mu_B(x_i,\ldots,x_m)]$$

Rewriting this as

$$\mu_I(i) = \bigvee_x [\bigwedge_{j=i}^{m} \mu_{S_j}(x)]$$
$$= \bigvee_x [\bigwedge_{j=i+1}^{m} (\mu_{S_i}(x) \wedge \mu_{S_j}(x))]$$
$$= \bigwedge_{j=i+1}^{m} [\bigvee_x (\mu_{S_i}(x) \wedge \mu_{S_j}(x))]$$
$$= \bigwedge_{j=i+1}^{m} \delta(S_i,S_j)$$
$$= \bigwedge_{j=i+1}^{m} \mu_{R_\delta}(i,j)$$

and since S_i dominates S_j $\forall j \le i$ (i.e. $\mu_{R_\delta}(i,j) = 1.0$ $\forall j \le i$) we have

$$\mu_W(i) = \bigwedge_{j=1}^{m} \mu_{R_\delta}(i,j)$$

or

$$\begin{aligned}\mu_W(i) &= \left(\bigwedge_{j=1}^{i} \mu_{R_\delta}(i,j)\right) \wedge \left(\bigwedge_{j=i+1}^{m} \mu_{R_\delta}(i,j)\right) \\ &= 1.0 \wedge \mu_I(i) \\ &= \mu_I(i)\end{aligned}$$

Appendix 2: Some fuzzy set definitions

Two common compound fuzzy sets used throughout the paper are "from A to B" and "between A and B". These are defined as follows. If A and B are given by their respective membership distributions $\mu_A(x)$ and $\mu_B(x)$ then

(i) E = "from A to B"

$$\mu_E(x) = \mu_{A\geq}(x) \wedge \mu_{\leq B}(x)$$

where

$$\mu_{A\geq}(x) = \begin{cases} \mu_A(x), & x \leq x_* \\ 1.0, & x > x_* \end{cases}$$

and

$$\mu_{\leq B}(x) = \begin{cases} 1.0, & x < x^* \\ \mu_B(x), & x \geq x^* \end{cases}$$

with x_* being the leftmost value of x for which $\mu_A(x) = 1.0$ and with x^* being the leftmost value of x for which $\mu_B(x) = 1.0$.

(ii) E = "between A and B"

$$\mu_E(x) = \text{norm}[(1 - \mu_{\leq A}(x)) \wedge (1 - \mu_{\geq B}(x))]$$

where norm[ ] denotes the normalised fuzzy set.

These two compound sets are illustrated in Figure A3.

Appendix 3: Linguistic ratings and weights for example 2

All the ratings and weights used in Example 2 are fuzzy numbers defined on the closed interval [0,1] of the real line. Furthermore, $S(\mathcal{F})$ is linear giving a particularly simple form. Only seven basic set shapes are used to represent the range of linguistic values (see Figure A4). This means that each set has several interpretations. Thus

shape 1 corresponds to:	"very low" "very bad" "very unimportant"
shape 2 corresponds to:	"low" "bad" "unimportant"
shape 3 corresponds to:	"moreorless low" "moreorless bad" "moreorless unimportant"
shape 4 corresponds to:	"fair" "neither unimportant nor important"
shape 5 corresponds to:	"moreorless high" "moreorless good" "moreorless important"
shape 6 corresponds to:	"high" "good" "important"
shape 7 corresponds to:	"very high" "very good" "very important"

List of Figure Captions

Fig. 1

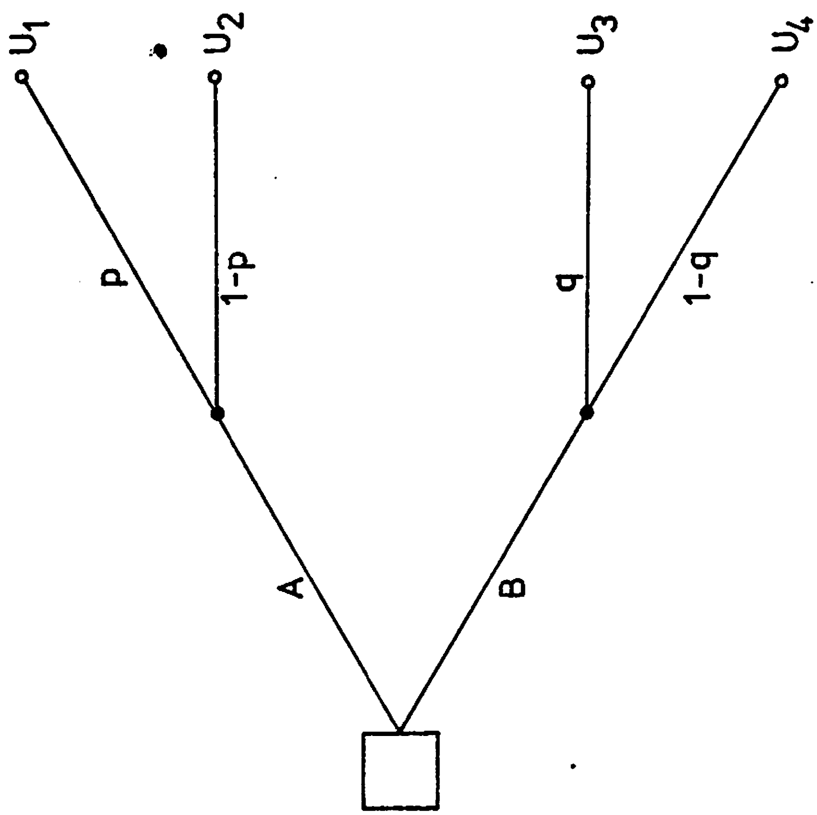


Fig. 2

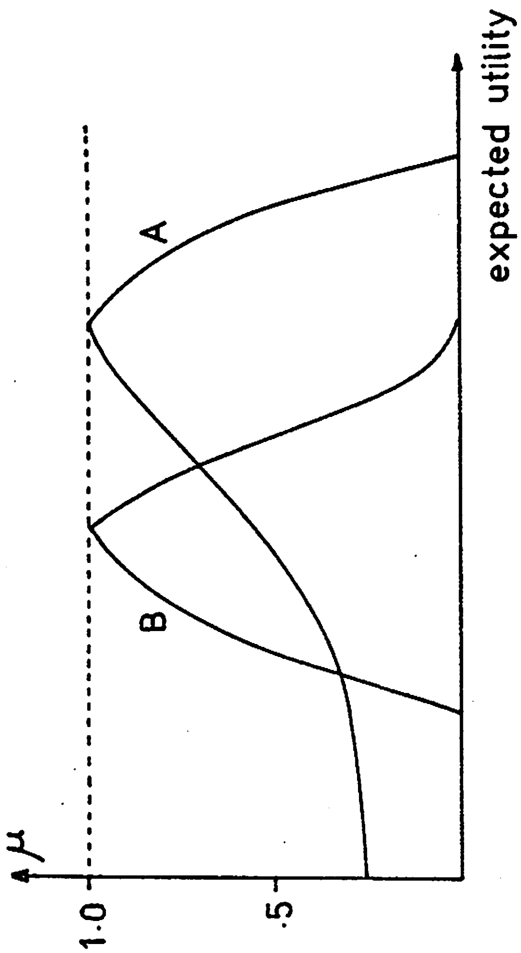


Fig. 3

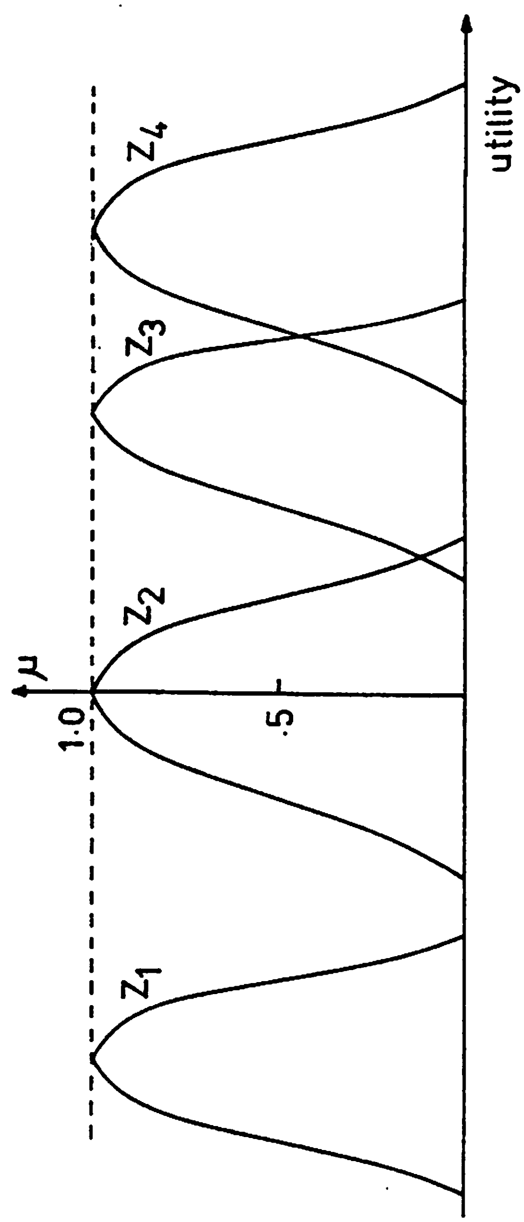


Fig. 4

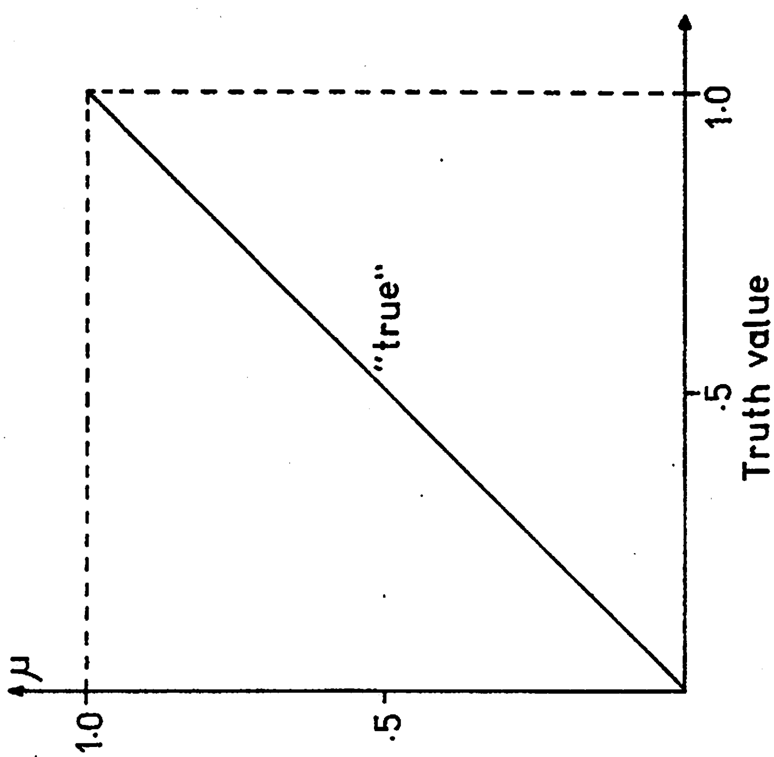


Fig. 5

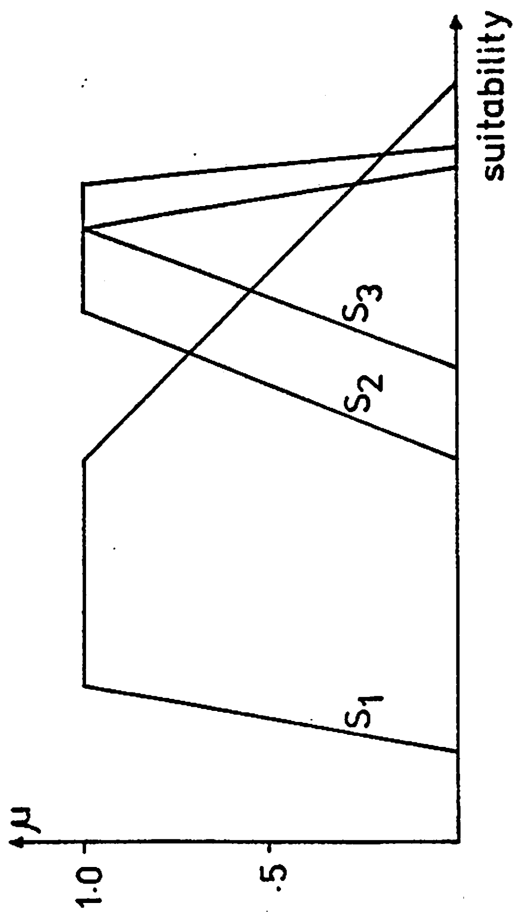


Fig. 6

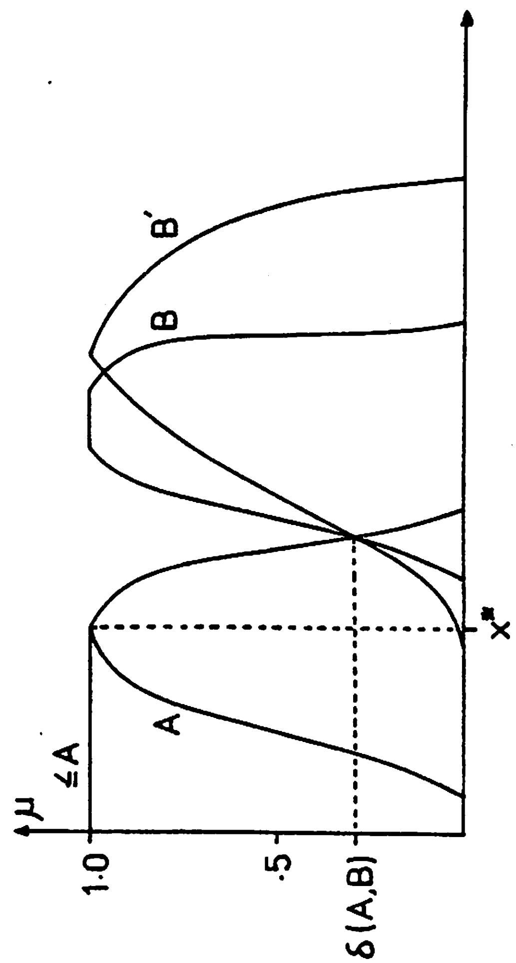


Fig. 7

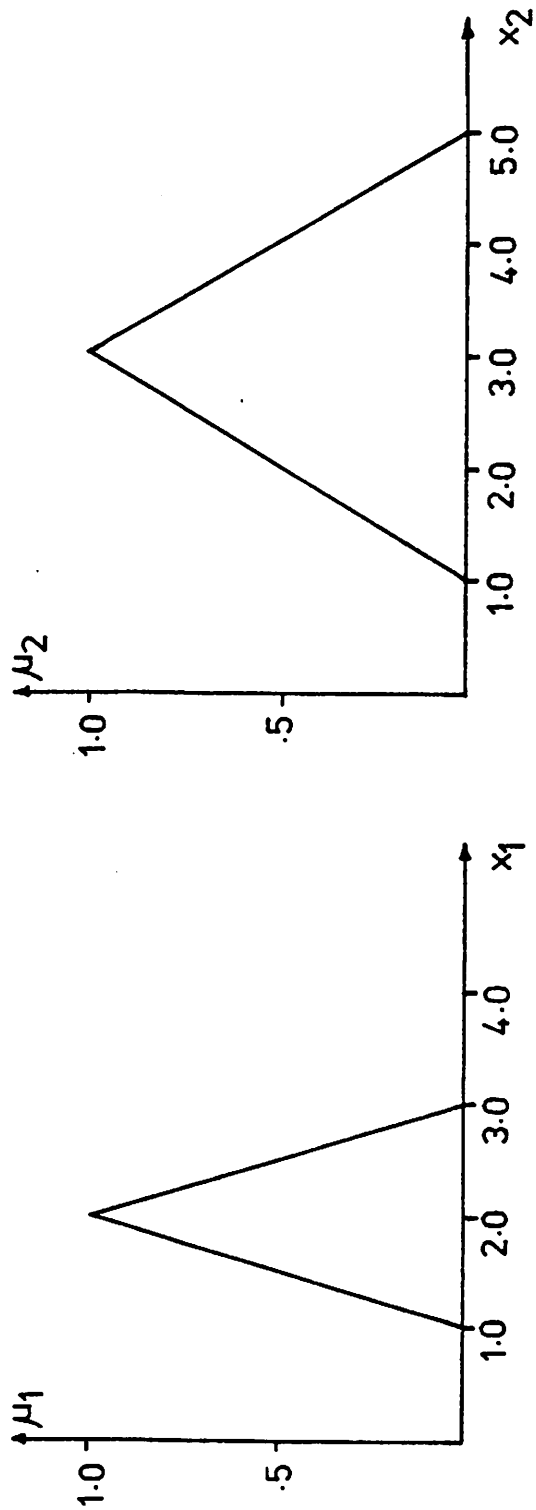


Fig. 8

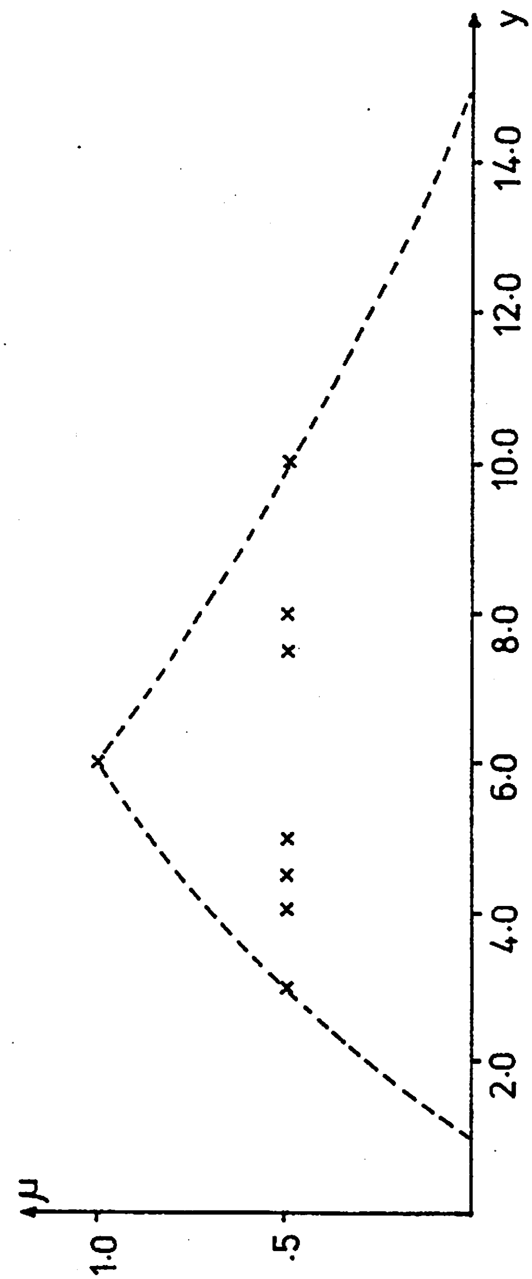


Fig. 9

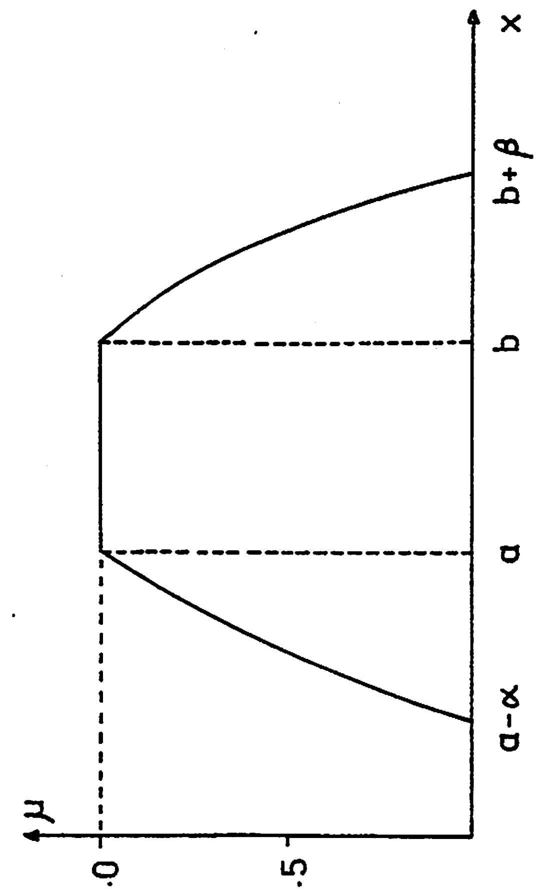
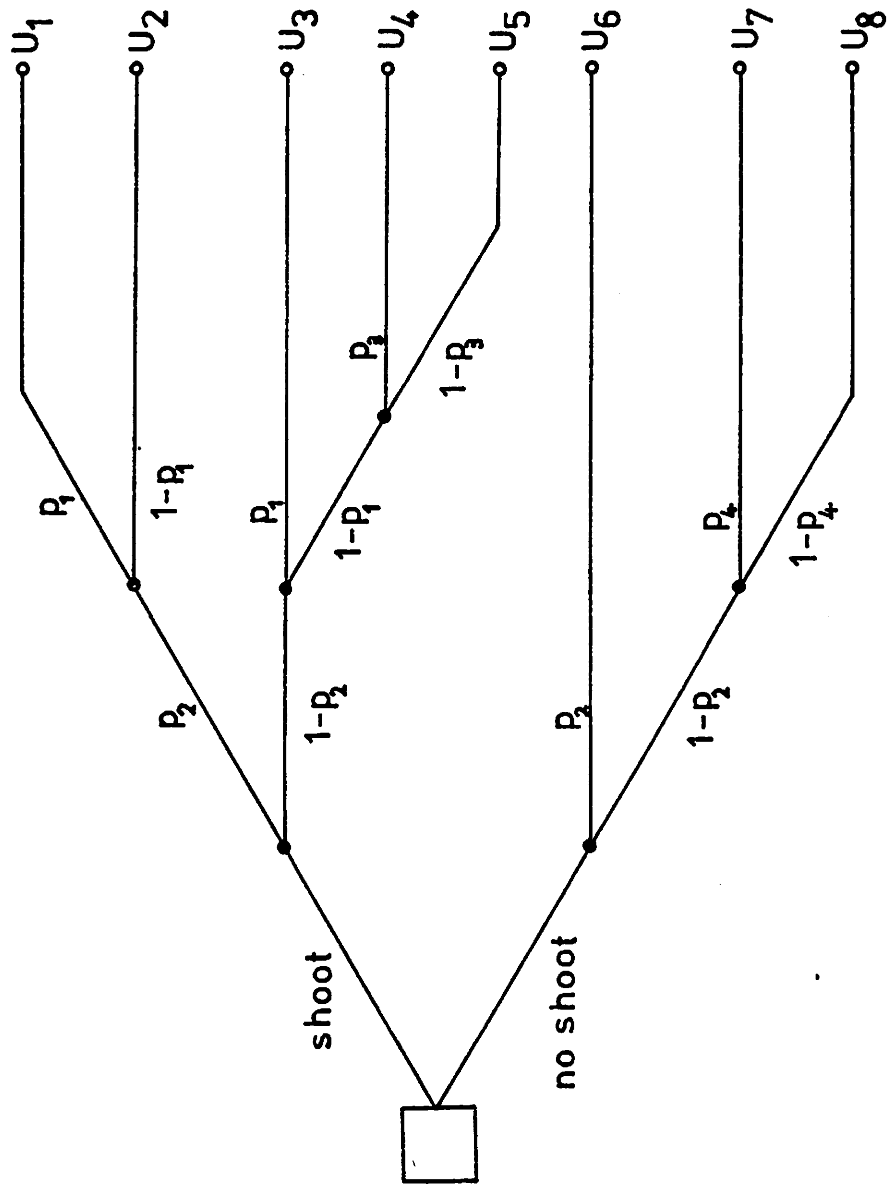


Fig. 10



shoot
no shoot
P_2
$1-P_2$
P_1
$1-P_1$
P_1
$1-P_1$
P_3
$1-P_3$
P_3
$1-P_2$
P_4
$1-P_4$
U_1
U_2
U_3
U_4
U_5
U_6
U_7
U_8

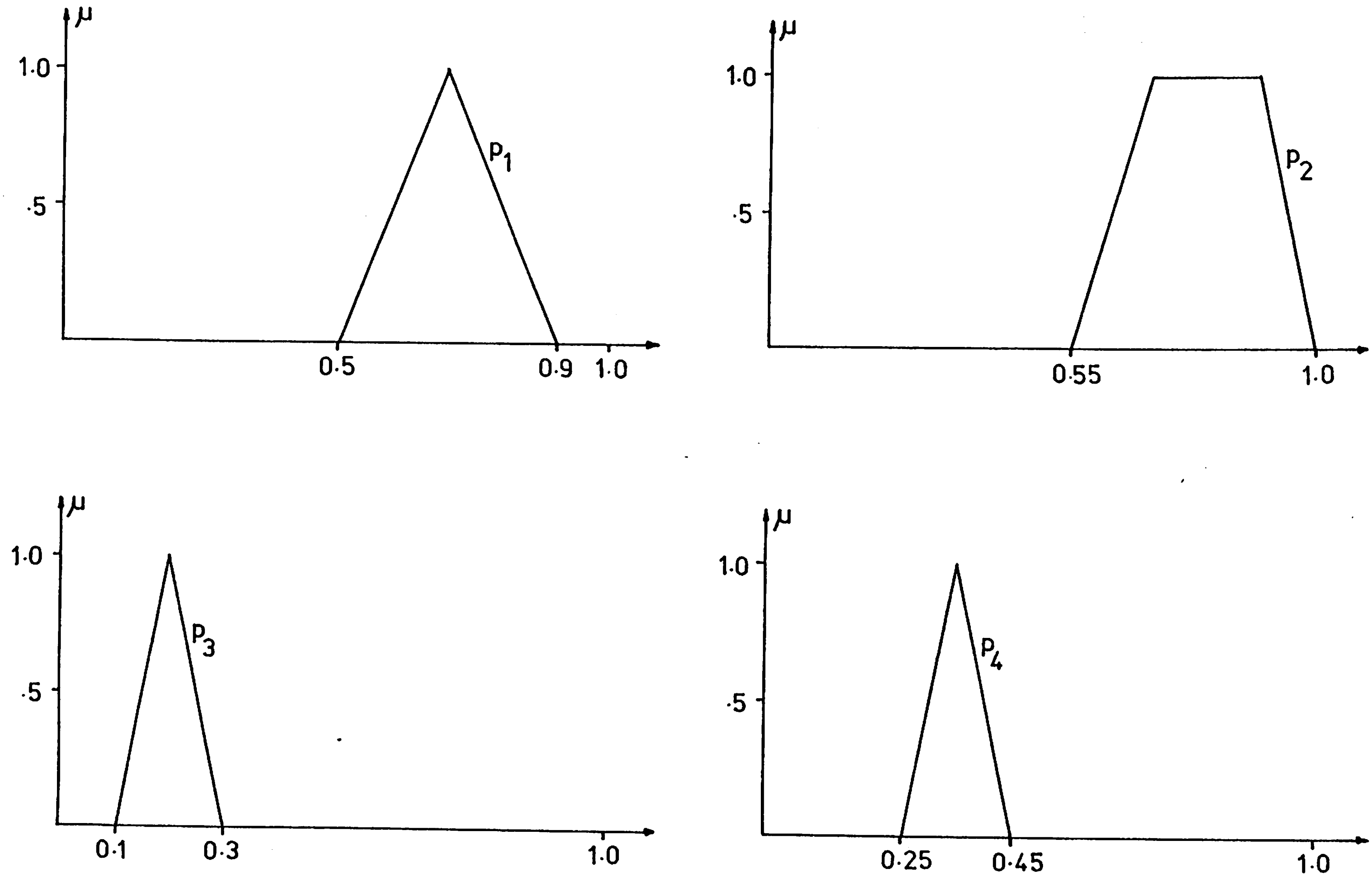
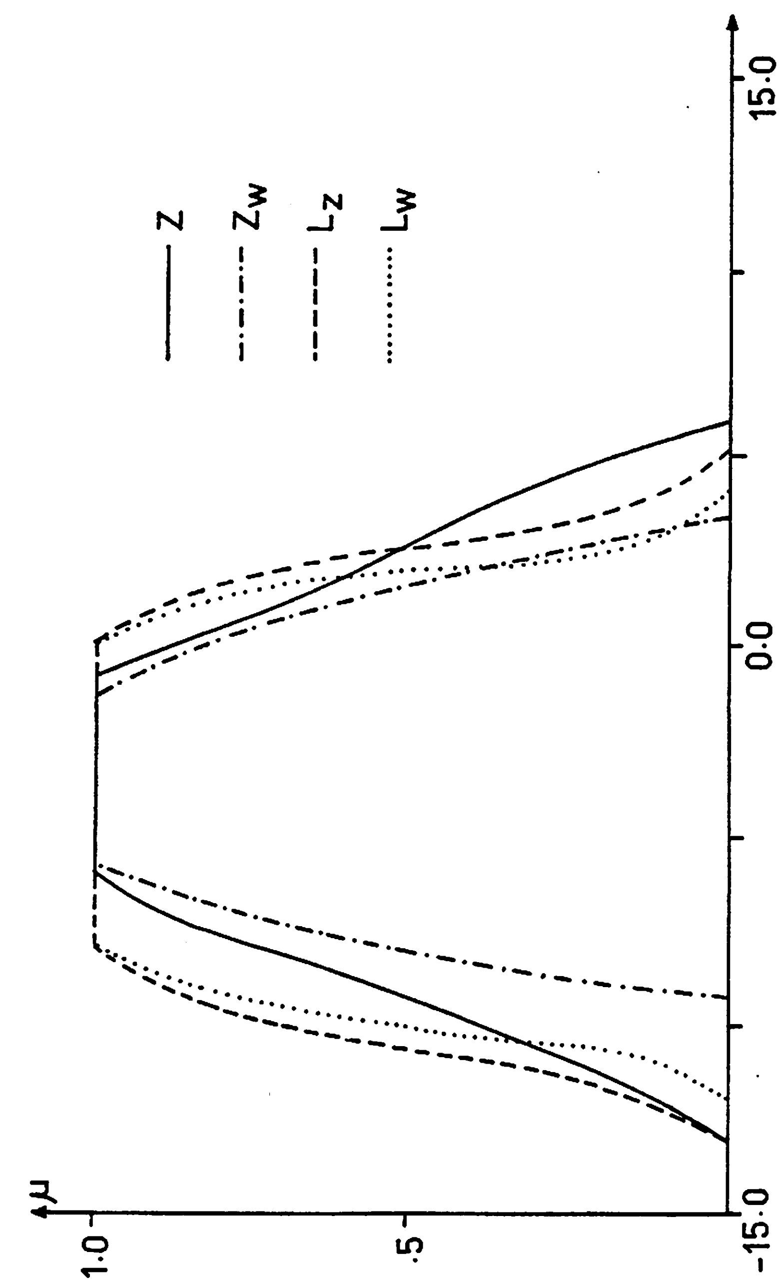


Fig. 11

Fig.12



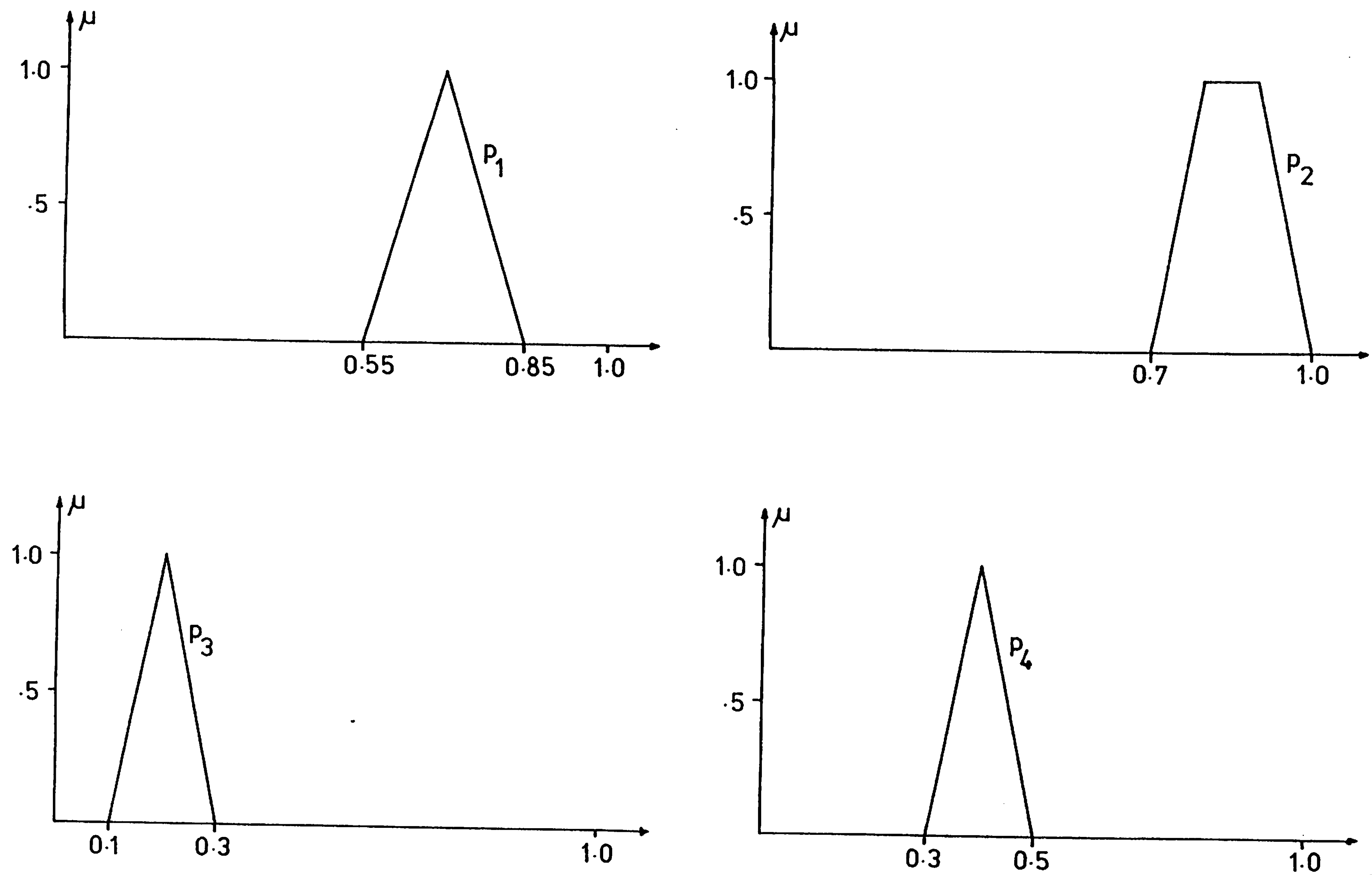
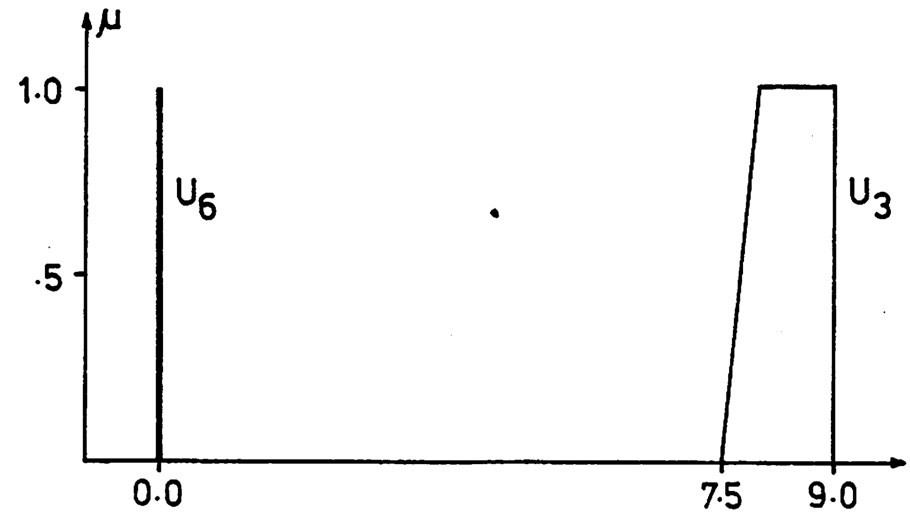
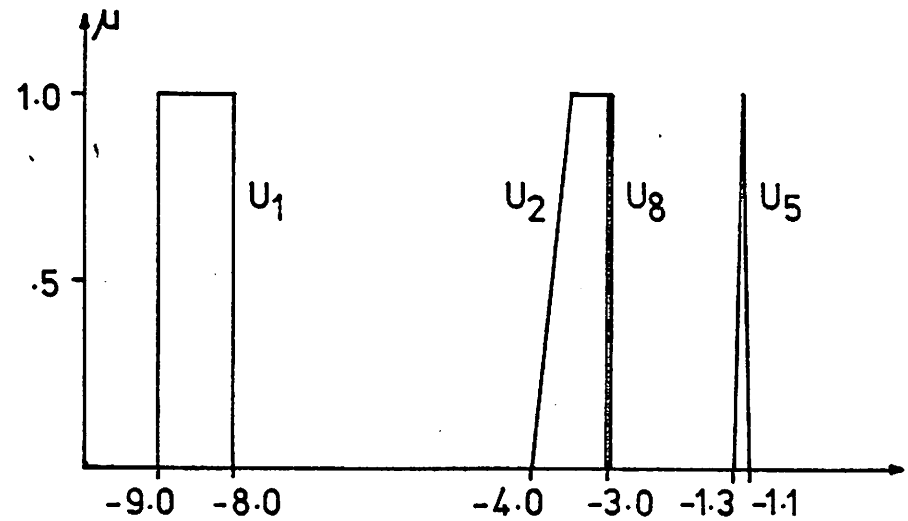


Fig. 13a

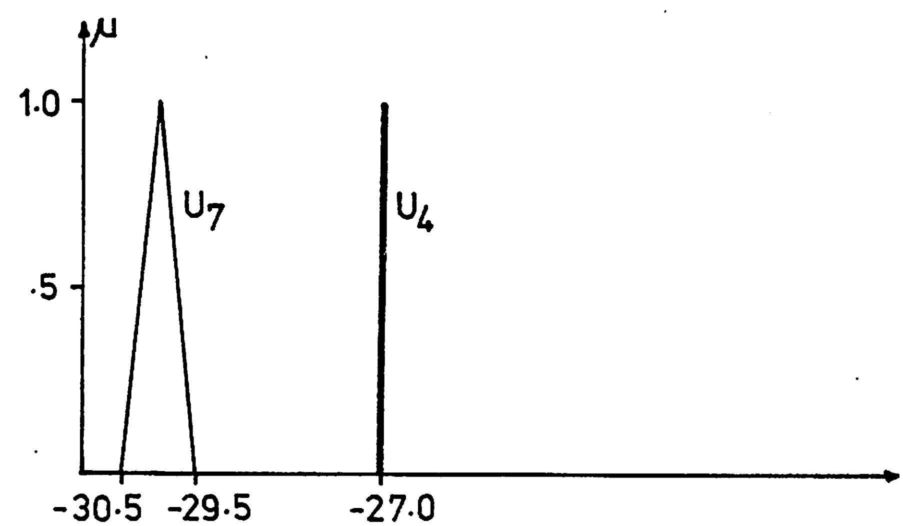
Fig. 13b



μ
1.0
.5
U6
U3
0.0
7.5
9.0



μ
1.0
.5
U1
U2
U8
U5
-9.0
-8.0
-4.0
-3.0
-1.3
-1.1



μ
1.0
.5
U7
U4
-30.5
-29.5
-27.0

Fig. 14

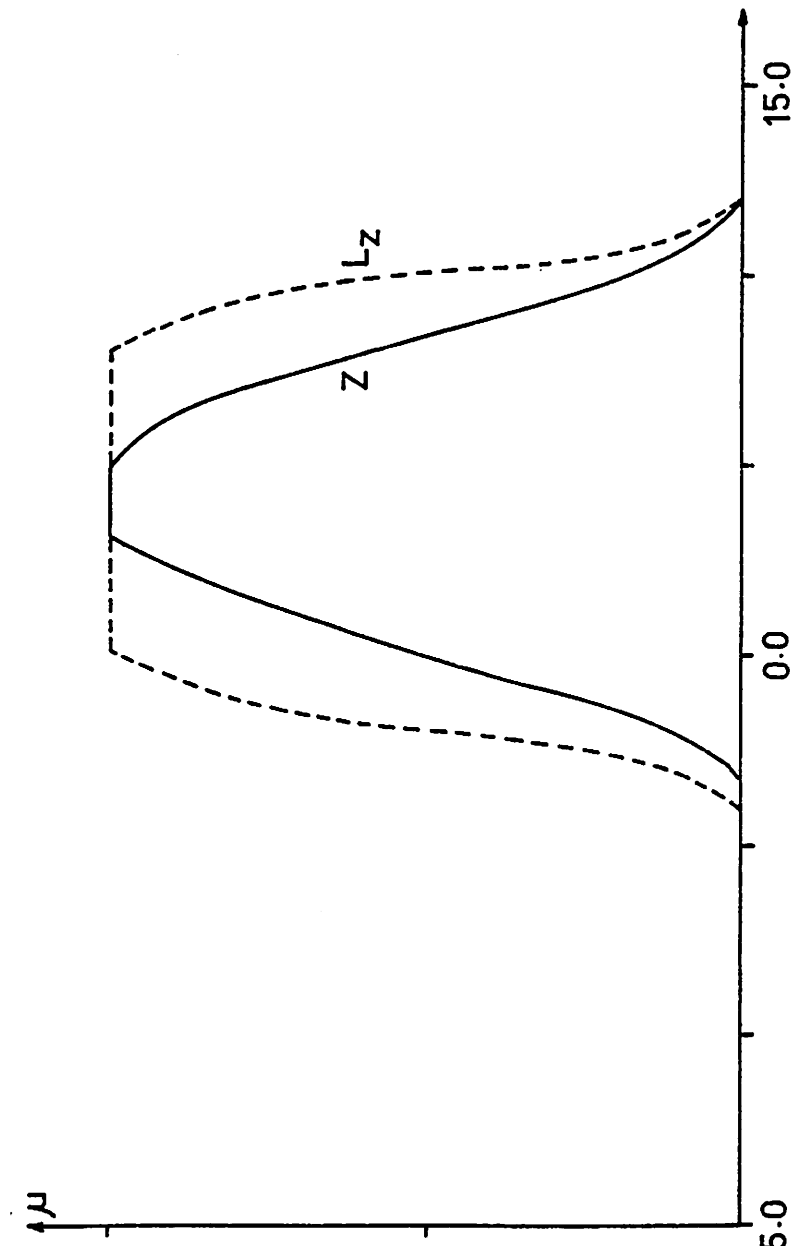


Fig. 15

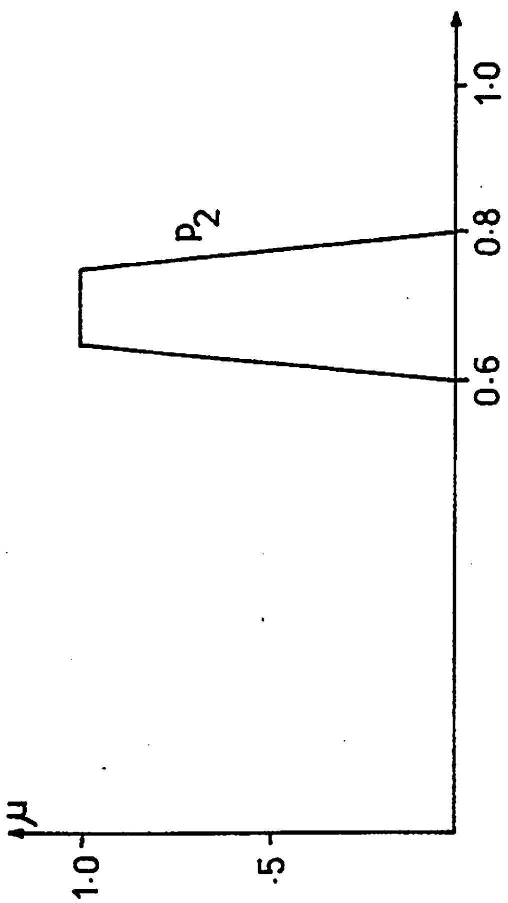


Fig. 16

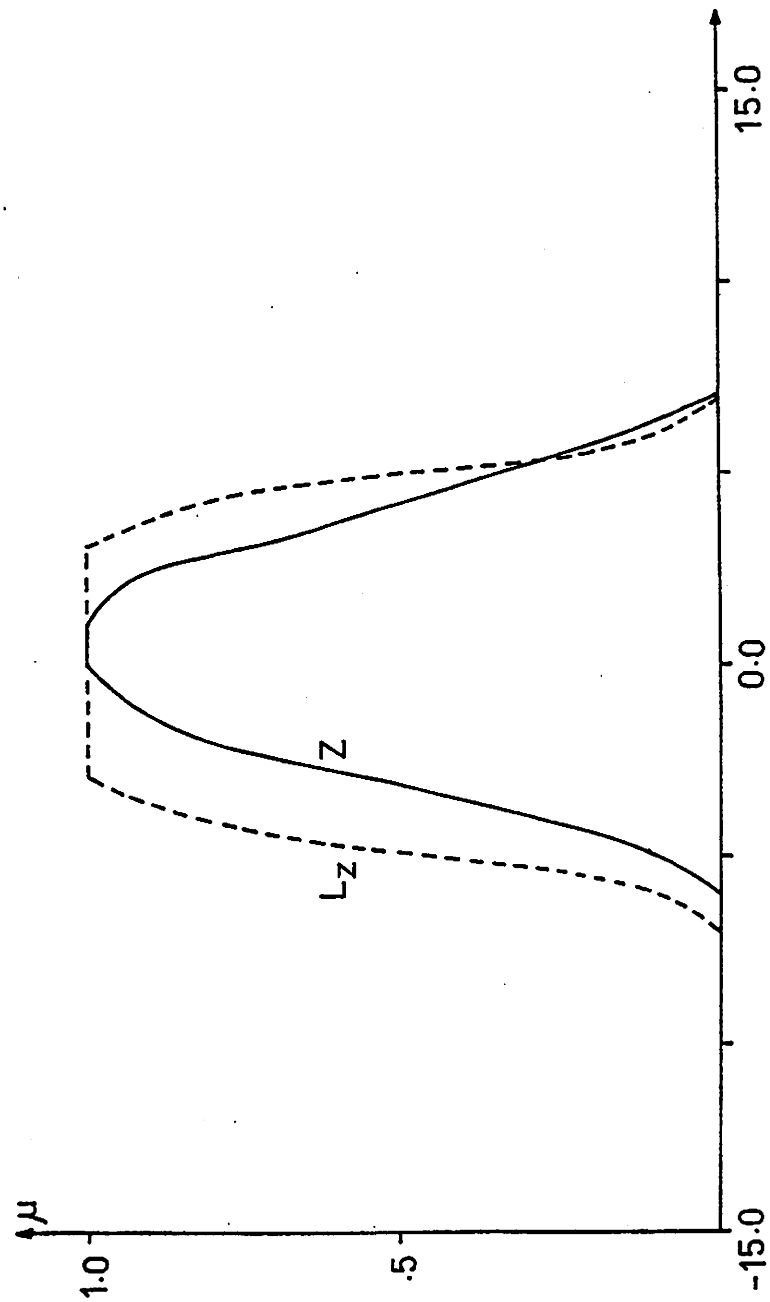


Fig. 17

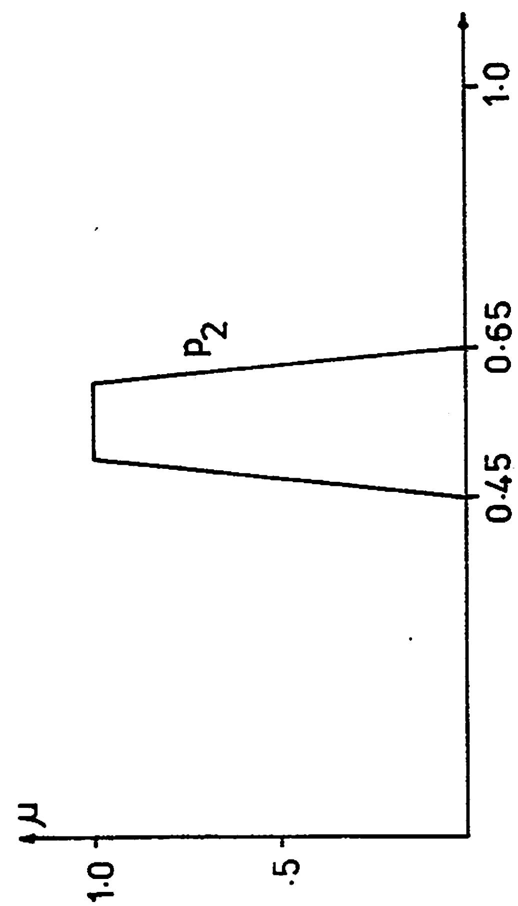


Fig. 18

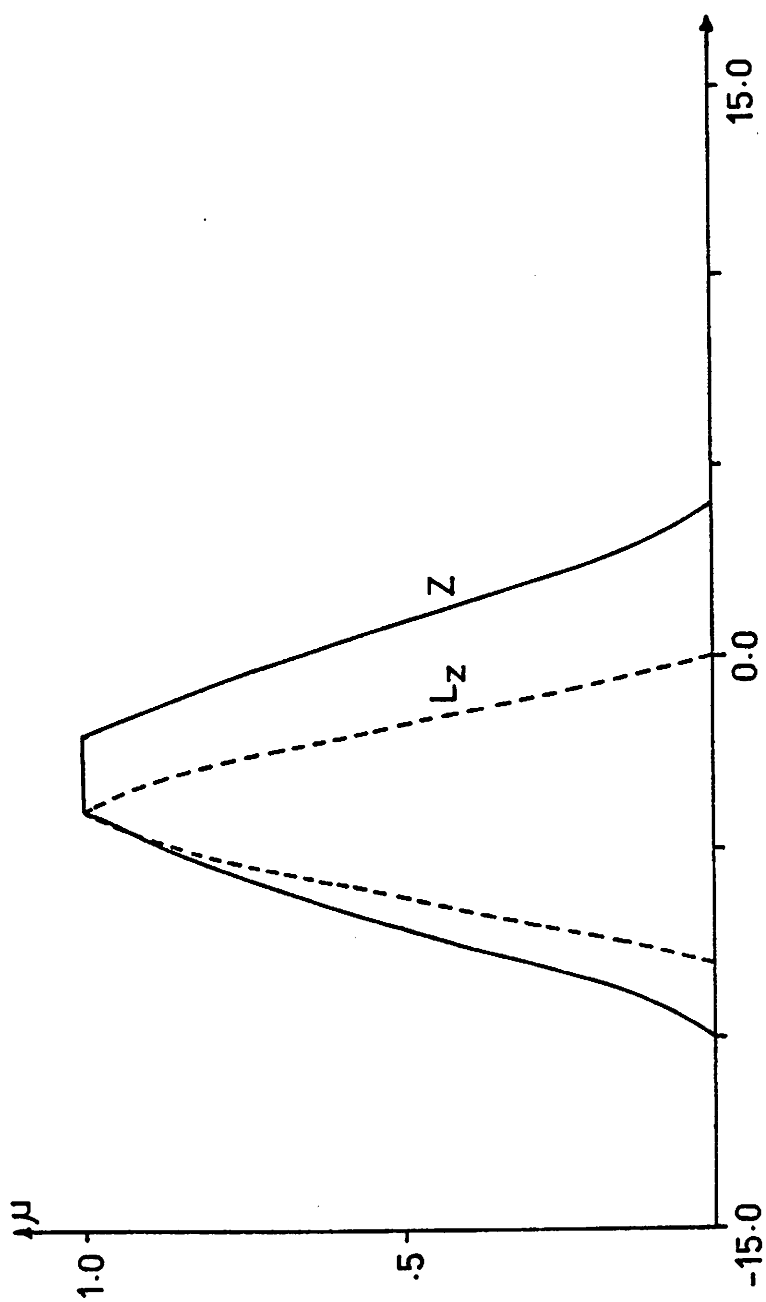


Fig. 19

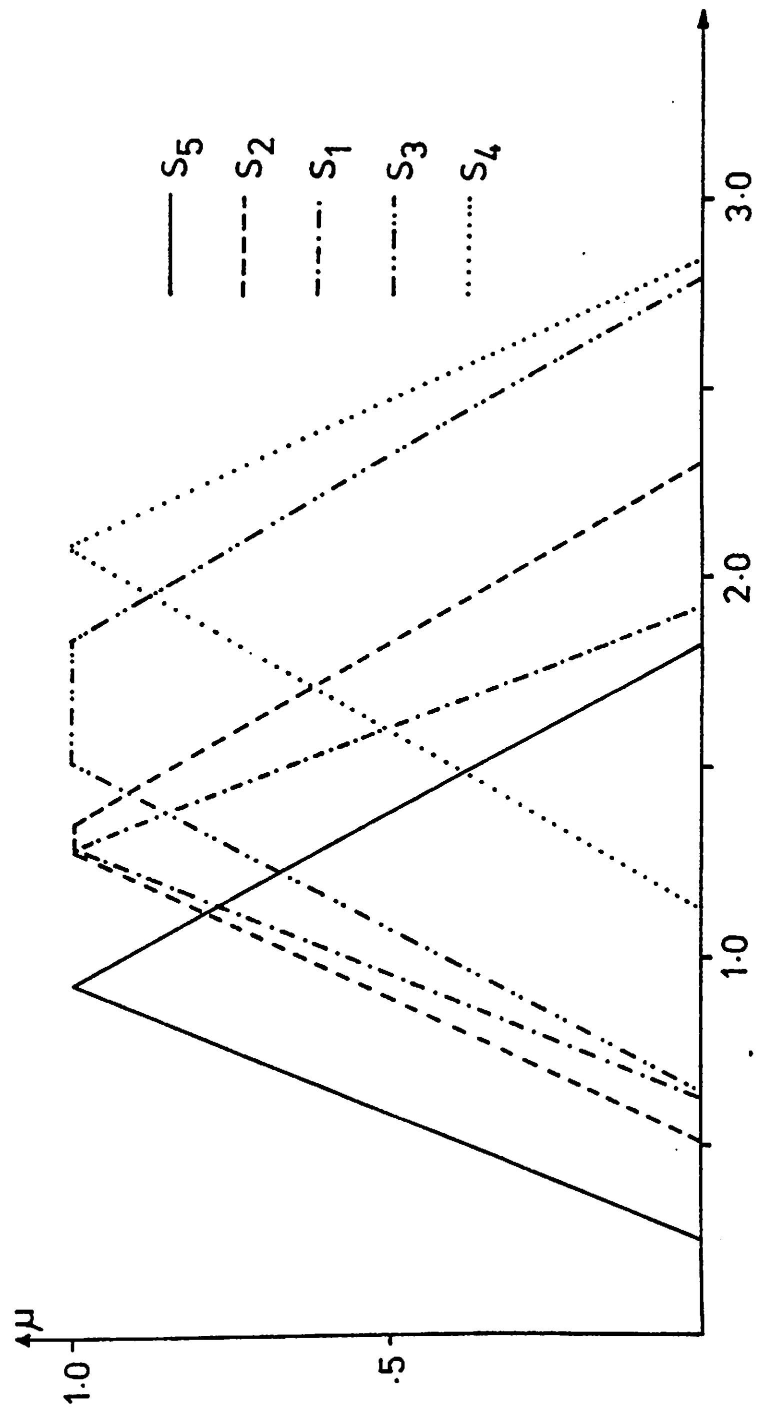


Fig. 20

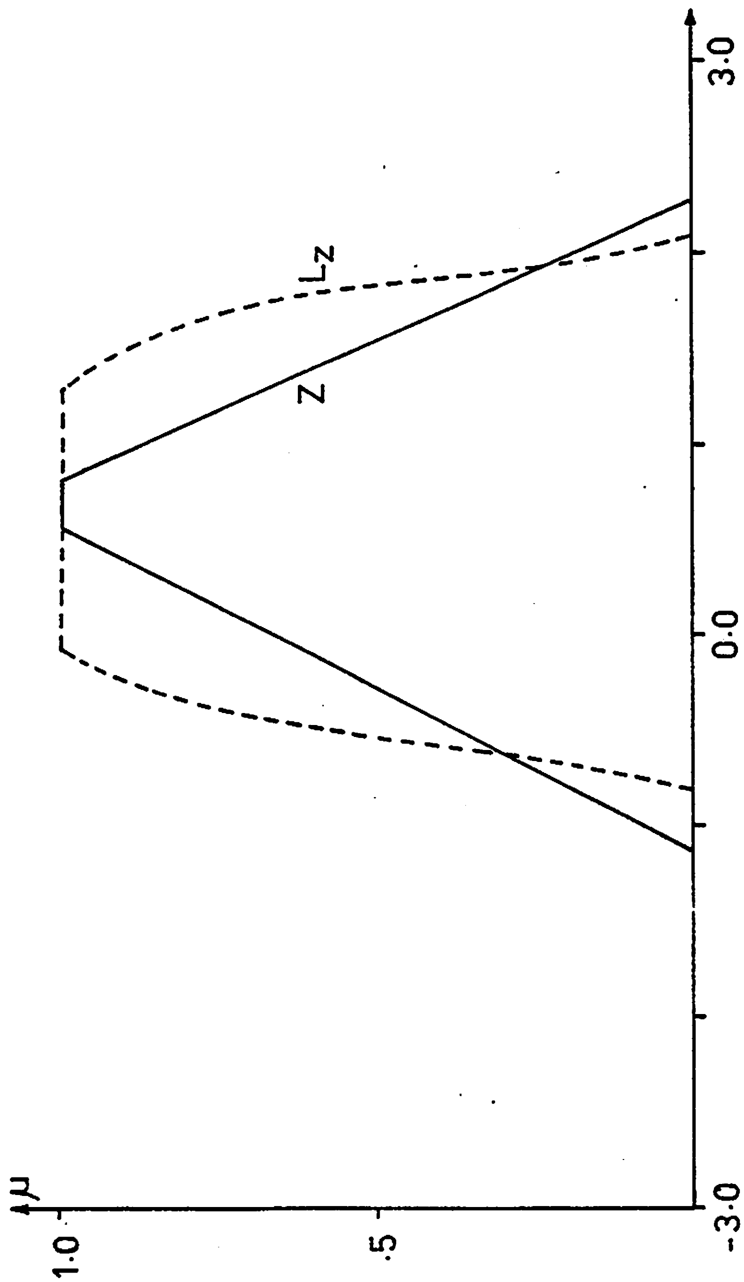


Fig. 21

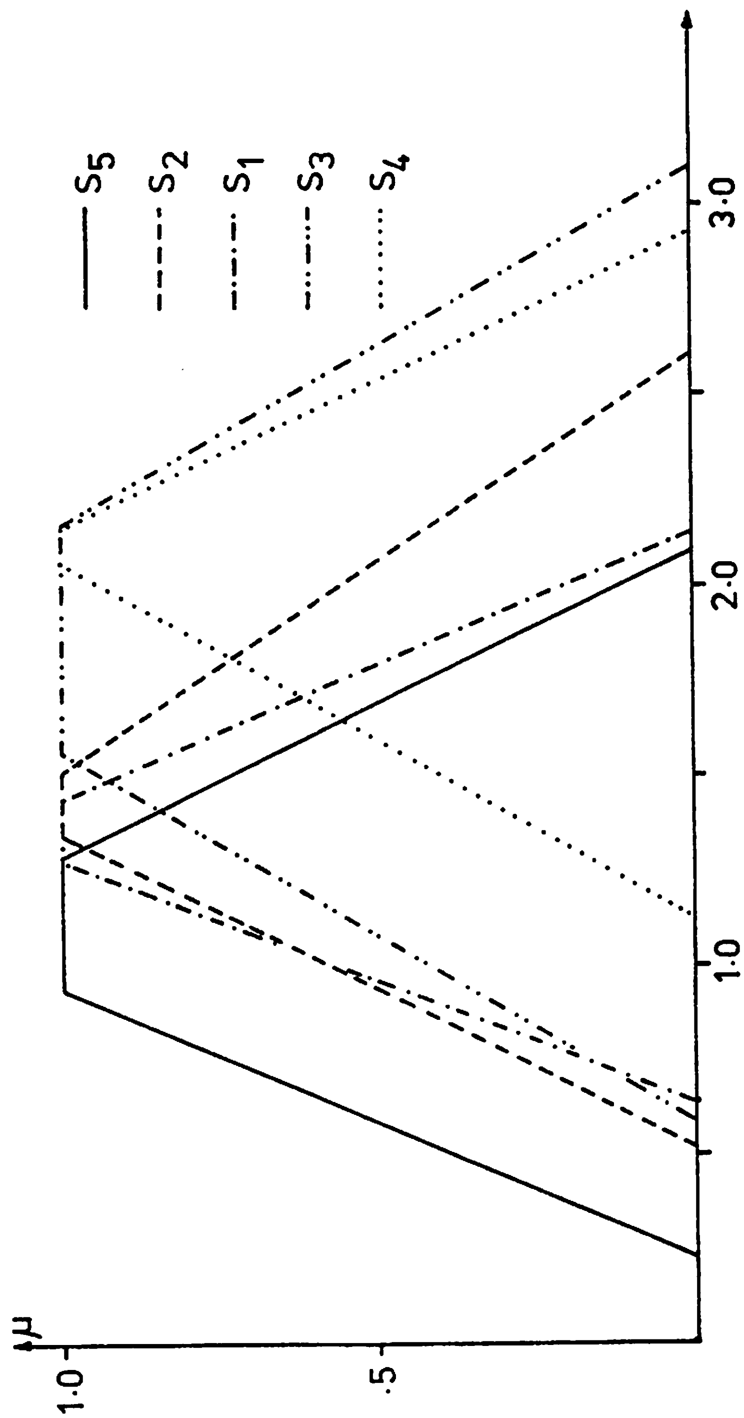


Fig.22

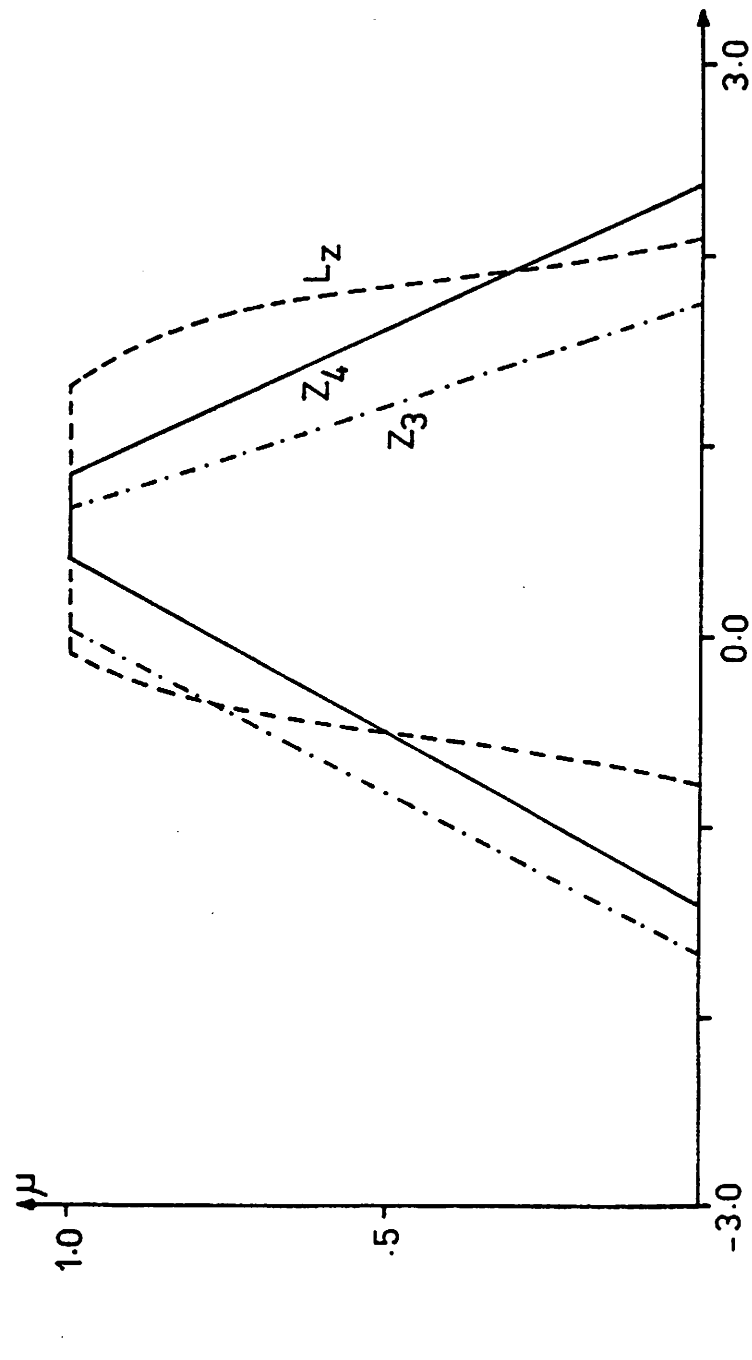


Fig. 23

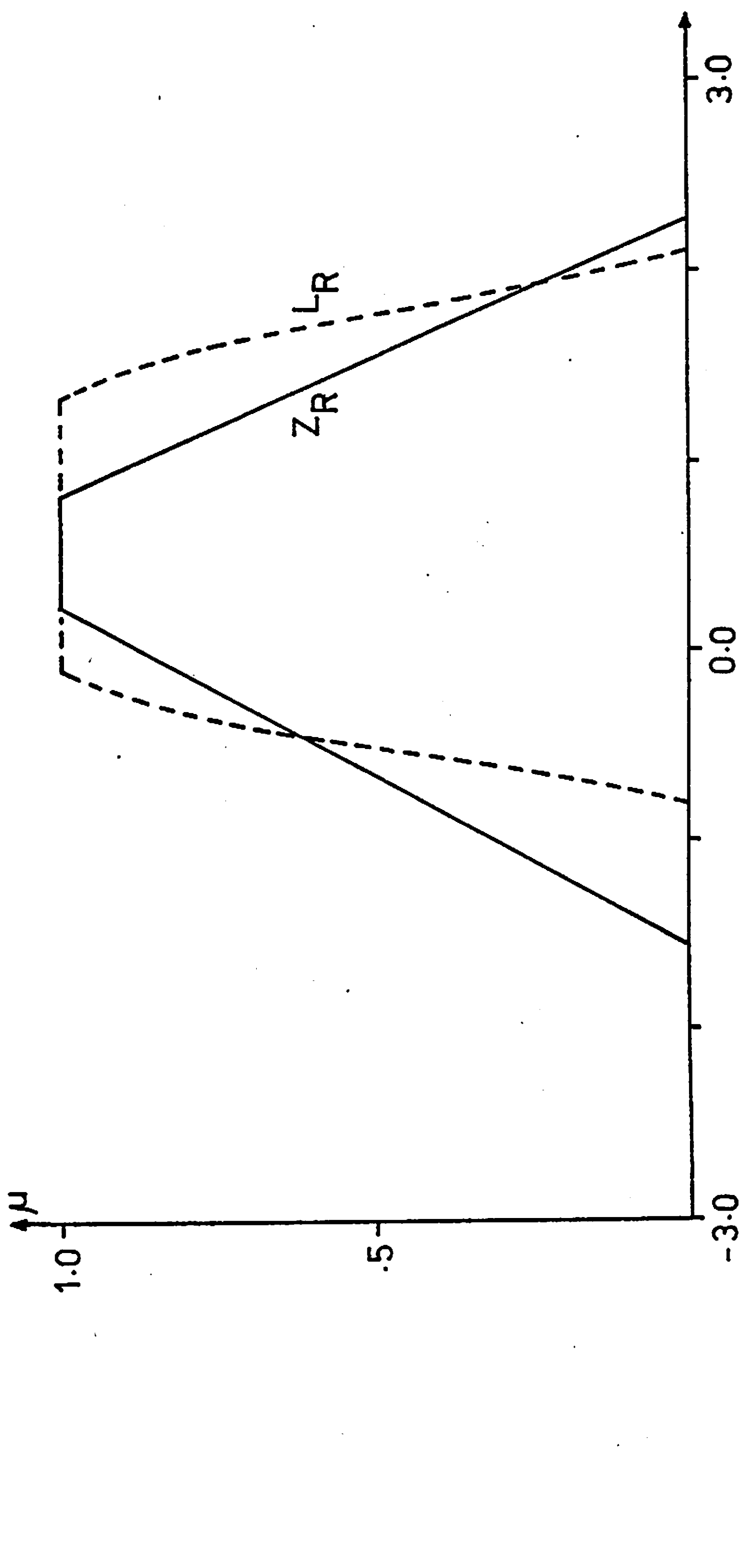


Fig. 24

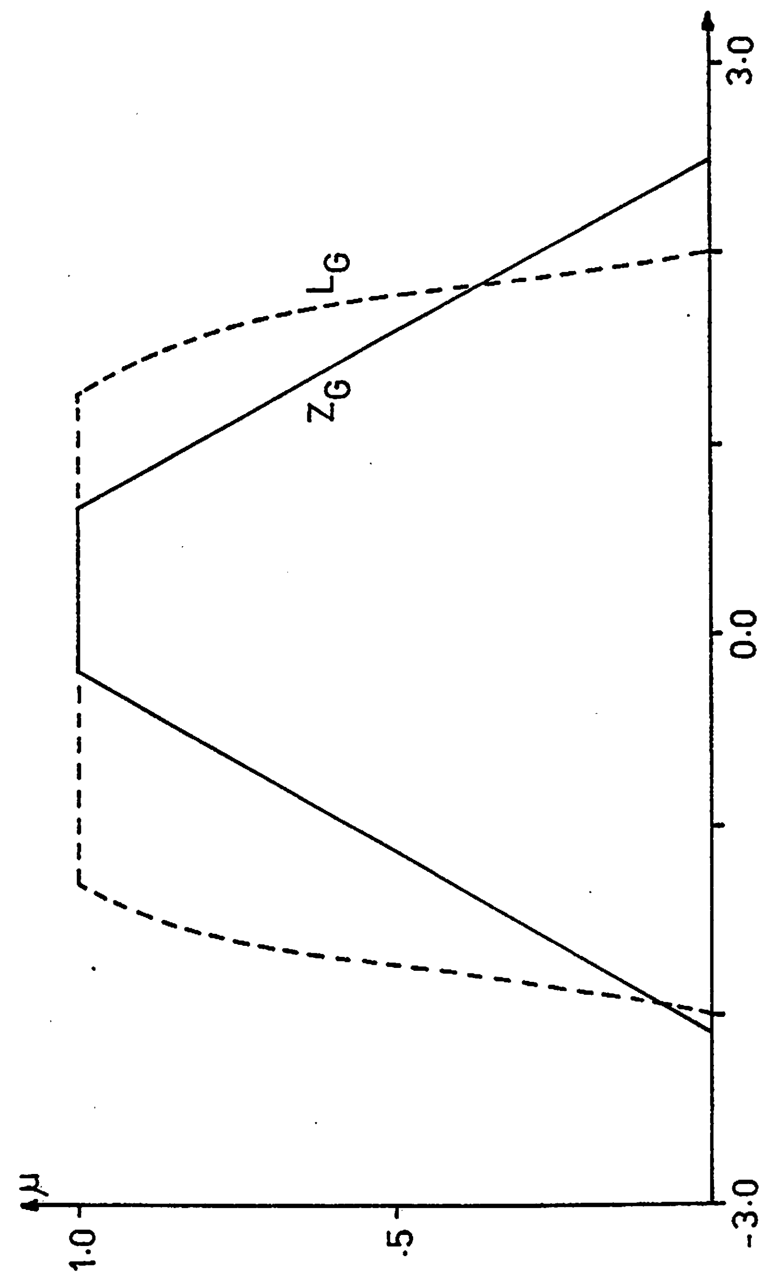
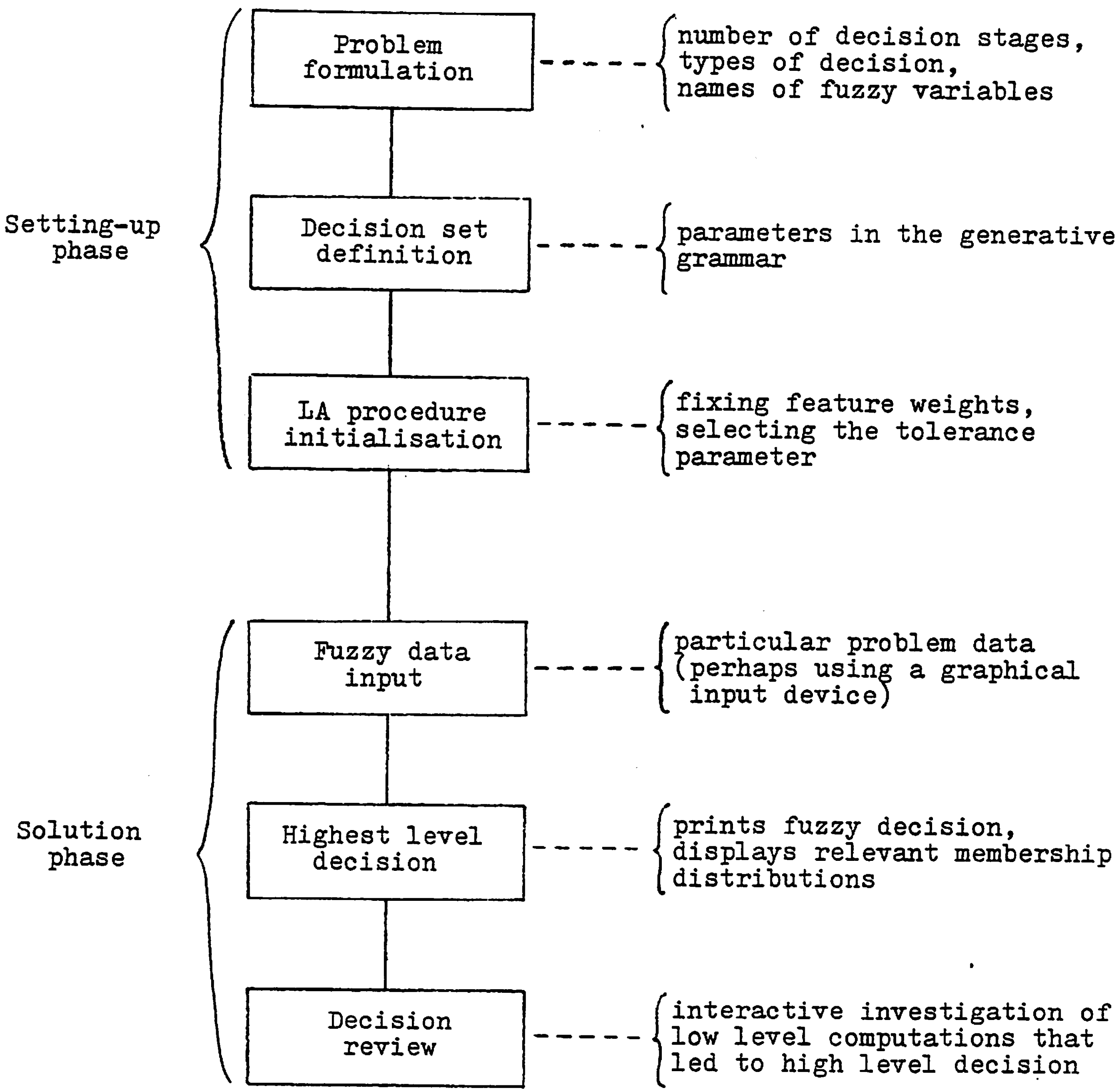


Fig. 25



Setting-up phase
Problem formulation
number of decision stages, types of decision, names of fuzzy variables
Decision set definition
parameters in the generative grammar
LA procedure initialisation
fixing feature weights, selecting the tolerance parameter
Solution phase
Fuzzy data input
particular problem data (perhaps using a graphical input device)
Highest level decision
prints fuzzy decision, displays relevant membership distributions
Decision review
interactive investigation of low level computations that led to high level decision

Fig. A1

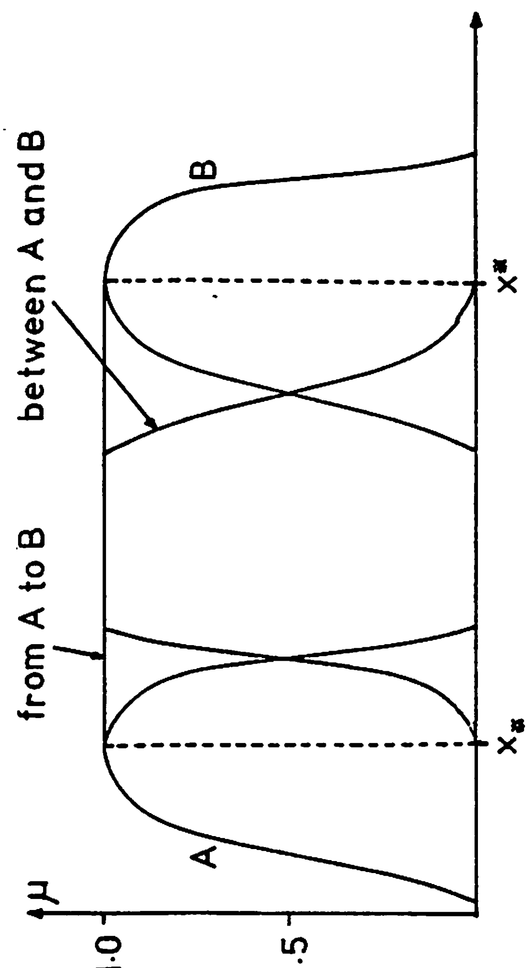


Fig. A2

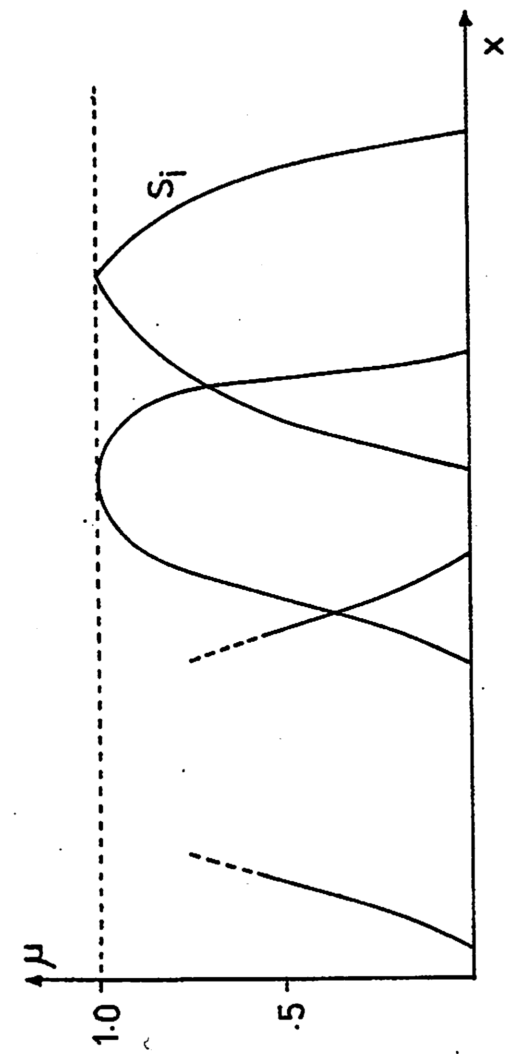


Fig. A3

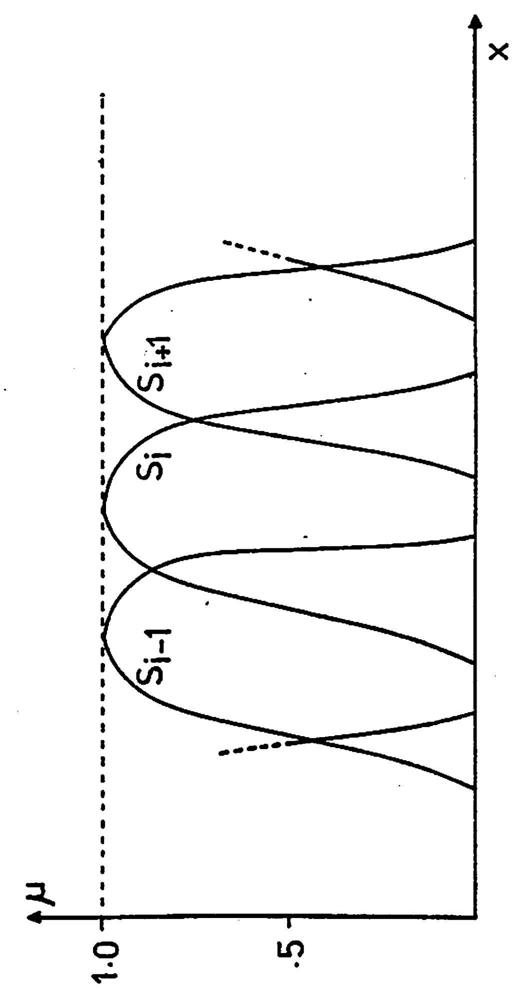


Fig. A4